LARGE DEVIATION PROBABILITIES FOR MAXIMUM LIKELIHOOD ESTIMATOR AND BAYES ESTIMATOR OF A PARAMETER FOR MIXED FRACTIONAL ORNSTEIN-UHLENBECK TYPE PROCESS

Mishra, Mahendra Nath
Institute of Mathematics and Applications, Bhubaneswar

Rao, Bhagavatula Lakshmi Surya Prakasa
CR Rao Advanced Institute for Mathematics, Statistics and Computer Science, Hyderabad

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LARGE DEVIATION PROBABILITIES FOR MAXIMUM LIKELIHOOD ESTIMATOR AND BAYES ESTIMATOR OF A PARAMETER FOR MIXED FRACTIONAL ORNSTEIN-UHLENBECK TYPE PROCESS

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Mahendra Nath Mishra[*] and **Bhagavatula Lakshmi Surya Prakasa Rao**[†]

Abstract

We investigate the probabilities of large deviations of the maximum likelihood estimator and Bayes estimator of the drift parameter for a mixed fractional Ornstein-Uhlenbeck type process.

Key Words and Phrases: Large deviation, Maximum likelihood estimator, Bayes estimator, Mixed fractional Ornstein-Uhlenbeck type process, Fractional Brownian motion.

1. Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, \ t \geq 0. \tag{1}$$

They investigated the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and proved that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. More general classes of stochastic processes satisfying linear stochastic differential equations driven by a fractional Brownian motion were studied and asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes were investigated in Prakasa Rao (2003, 2005).

[*] Institute of Mathematics and Applications, Bhubaneswar, India.
[†] CR Rao Advanced Institute for Mathematics, Statistics and Computer Science, Hyderabad, India.

Prakasa Rao (2010) gives a comprehensive discussion on problems of estimation for processes driven by a fractional Brownian motion.

Brownian motion has been used traditionally as the driving force for modeling log returns based on the movements of the stock prices in a share market. It has been noticed that there might be long range dependence in the behaviour of movement of stock prices in a share market and the log returns may possibly have heavy tailed distributions. It was suggested by some researchers that the driving force for modeling of prices of movement may be modeled by stochastic differential equations driven by a fractional Brownian motion of suitable Hurst index H. Bjork and Hult (2005) and Kuznetsov (1999) observed that the use of fractional Brownian motion as the driving force is not justifiable as it allows arbitrage opportunities which is contrary to the fundamental ideas of rational market behaviour. In order to avoid this problem, Cheridito (2000, 2003) suggested the use of a mixed fractional Brownian motion as a suitable model to capture the fluctuations in stock price movement. The mixed fractional Brownian motion (mfBm) is a Gaussian process that is a linear combination of the Brownian motion and a fractional Brownian motion with Hurst index H. Cheridito (2001) proved that, for $H \in (3/4, 1)$ the mfBm is equivalent to a Brownian motion and hence modeling price fluctuations via mfBm allows arbitrage-free market. Pricing geometric Asian options under mixed fractional Brownian motion was studied in Prakasa Rao (2015b). Option procing for processes driven by a mixed fractional Brownian motion with superimposed jumps was discussed in Prakasa Rao (2015a). Rudomino-Dusyatska (2003) and more recently Prakasa Rao (2009, 2017) investigated problems of statistical inference for processes modeled via stochastic differential equations driven by a mixed fractional Brownian motion among others. Mixed fractional Brownian models were also studied in Mishura (2008) and Prakasa Rao (2010). Cai et al. (2016) presented a new approach via filtering for analysis of mixed processes of type $\{X_t = B_t + G_t, 0 \leq t \leq T\}$ where $\{B_t, 0 \leq t \leq T\}$ is a Brownian motion and $\{G_t, 0 \leq t \leq T\}$ is an independent Gaussian process. Statistical Analysis of mixed fractional Ornstein-Uhlenbebeck process was investigated in Chigansky and Kleptsyna (2015). Large deviations for drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process were studied by Marushkevych (2016) following the ideas in Bercu et al. (2010) as an application of the Gärtner -Ellis theorem (cf. Dembo and Zeitouni (1998)). His results deal with a limit theorem leading to an exact large deviation result involving a rate function for a specific value of the parameter.

Our aim in this paper is to give an alternate approach for obtaining large deviation probabilities, valid uniformly over compact sets of the parameter, for maximum likelihood and Bayes estimators for the drift parameter involved in a fractional Ornstein-Uhlenbeck type process driven by a mixed fractional Brownian motion following the ideas from Ibragimov and Khasminskii (1981) and generalizing our earlier work in Mishra and Prakasa Rao (2006) for processes driven by a fractional Brownian motion.

2. PRELIMINARIES

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the P-completion of the filtration generated by this process. Let $\{W_t, t \geq 0\}$ be a standard Wiener process and $W^H = \{W_t^H, t \geq 0\}$ be an independent normalized fractional Brownian motion with Hurst parameter $H \in (0, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2}$$

Let

$$\tilde{W}_t^H = W_t + W_t^H, t \geq 0.$$

The process $\{\tilde{W}_t^H, t \geq 0\}$ is called the mixed fractional Brownian motion with Hurst index H. We assume here after that Hurst index H is known. Following the results in Cheridito (2001), it is known that the process $\tilde{W}^H$ is a semimartingale in its own filtration if and only if either $H = 1/2$ or $H \in (\frac{3}{4}, 1]$. We will assume here after that $H \in (\frac{3}{4}, 1]$.

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \tilde{W}_t^H, t \geq 0 \tag{3}$$

where the process $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$dY_t = C(t)dt + d\tilde{W}_t^H, t \geq 0 \tag{4}$$

driven by the mixed fractional Brownian motion $\tilde{W}^H$. Following the recent works by Cai et al. (2016) and Chigansky and Kleptsyna (2015), one can construct an integral transformation that transforms the mixed fractional Brownian motion $\tilde{W}^H$ into a martingale M^H. Let $g_H(s,t)$ be the solution of the integro-differential equation

$$g_H(s,t) + H\frac{d}{ds}\int_0^t g_H(r,t)|s-r|^{2H-1} sign(s-r)dr = 1, 0 < s < t. \tag{5}$$

Cai et al. (2016) proved that the process

$$M_t^H = \int_0^t g_H(s,t)d\tilde{W}_s^H, t \geq 0 \tag{6}$$

is a Gaussian martingale with quadratic variation

$$< M^H >_t = \int_0^t g_H(s,t)ds, t \geq 0. \tag{7}$$

Furthermore the natural filtration of the martingale M^H coincides with that of the mixed fractional Brownian motion $\tilde{W}^H$. Suppose that, for the martingale M^H defined by the equation (6), the sample paths of the process $\{C(t), t \geq 0\}$ are smooth enough in the sense that the process

$$Q_t = \frac{d}{d < M^H >_t}\int_0^t g_H(s,t)C(s)ds, t \geq 0 \tag{8}$$

is well defined in the sense that

$$\int_0^t Q_s d < M^H >_s = \int_0^t g_H(s,t)C(s)ds, t \geq 0.$$

Define the process

$$Z_t = \int_0^t g_H(s,t)dY_s, t \geq 0. \tag{9}$$

As a consequence of the results in Cai et al. (2016), it follows that the process Z is a fundamental semimartingale associated with the process Y in the following sense.

Theorem 2.1: *Let $g_H(s,t)$ be the solution of the equation (5). Define the process Z as given in the equation (9). Then the following relations hold.*
(i) The process Z is a semimartingale with the decomposition

$$Z_t = \int_0^t Q_s d < M^H >_s + M_t^H, t \geq 0 \tag{10}$$

where M^H is the martingale defined by the equation (6).
(ii) The process Y admits the representation

$$Y_t = \int_0^t \hat{g}_H(s,t)dZ_s, t \geq 0 \tag{11}$$

where

$$\hat{g}_H(s,t) = 1 - \frac{d}{d < M^H >_s} \int_0^t g_H(r,s)dr. \tag{12}$$

(iii) The natural filtrations $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ of the processes Y and Z respectively coincide.

Applying Corollary 2.9 in Cai et al. (2016), it follows that the probability measures μ_Y and $\mu_{\tilde{W}^H}$ generated by the processes Y and $\tilde{W}^H$ on an interval $[0,T]$ are absolutely continuous with respect to each other and the Radon-Nikodym derivative is given by

$$\frac{d\mu_Y}{d\mu_{\tilde{W}^H}}(Y) = \exp[\int_0^T Q_s dZ_s - \frac{1}{2}\int_0^T Q_s^2 d < M^H >_s] \tag{13}$$

which is also the likelihood function based on the observation $\{Y_s, 0 \leq s \leq T\}$. Since the filtrations generated by the processes Y and Z are the same, the information contained in the families of σ-algebras $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ is the same and hence the problem of the estimation of the parameters involved based on the observations $\{Y_s, 0 \leq s \leq T\}$ and $\{Z_s, 0 \leq s \leq T\}$ are equivalent.

3. Maximum likelihood estimation and Bayes estimation

Let us consider the stochastic differential equation

$$dX_t = \theta X_t dt + d\tilde{W}_t^H, t \geq 0; X_0 = 0 \tag{14}$$

where $\theta \in \Theta \subset R, W = \{\tilde{W}_t^H,\ t \geq 0\}$ is a mixed fractional Brownian motion with Hurst parameter H. In other words $X = \{X_t, t \geq 0\}$ is a stochastic process satisfying the stochastic integral equation

$$X_t = \theta \int_0^t X_s ds + \tilde{W}_t^H, \quad t \geq 0. \tag{15}$$

Let

$$Q_H(t) = \frac{d}{d < M^H >_t} \int_0^t g_H(s,t) X_s ds, \quad t \geq 0 \tag{16}$$

be well-defined where $< M^H >_t$ and $g_H(t,s)$ are as defined earlier. Suppose the sample paths of the process $\{Q_H(t), 0 \leq t \leq T\}$ belong almost surely to $L^2([0,T],\, d < M^H >_t)$. Define

$$Z_t = \int_0^t g_H(t,s) dX_s,\ t \geq 0. \tag{17}$$

Then the process $Z = \{Z_t,\ t \geq 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

$$Z_t = \theta \int_0^t Q_H(s)\, d < M^H >_s + M_t^H \tag{18}$$

where M^H is the fundamental martingale defined earlier and the process X admits the representation

$$X_t = \int_0^t \hat{g}_H(t,s)\, dZ_s \tag{19}$$

where the function $\hat{g}_H(t,s)$ is as defined by (12). Let P_θ^T be the measure induced by the process $\{X_t, 0 \leq t \leq T\}$ when θ is the true parameter. Following Corollary 2.9 in in Cai et al. (2015), we get that the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp\left[\theta \int_0^T Q_H(s)\ dZ_s - \frac{1}{2}\theta^2 \int_0^T Q_H^2(s) d < M^H >_s\right]. \tag{20}$$

Maximum likelihood estimation

The problem of maximum likelihood estimation of the parameter θ is discussed in Kleptsyna and Le Breton (2002) for fractional Ornstein-Uhlenbeck type process and by Prakasa Rao (2003, 2005) for more general processes. Let $L_T(\theta)$ denote the Radon-Nikodym derivative $\frac{dP_\theta^T}{dP_0^T}$. The maximum likelihood estimator (MLE) $\hat{\theta}_T$ is defined by the relation

$$L_T(\hat{\theta}_T) = \sup_{\theta \in \Theta} L_T(\theta). \tag{21}$$

We assume that there exists a measurable maximum likelihood estimator. Sufficient conditions can be given for the existence of such an estimator (cf. Lemma 3.1.2, Prakasa Rao (1987)). Then

$$\log L_T(\theta) = \theta \int_0^T Q_H(t) dZ_t - \frac{1}{2}\theta^2 \int_0^T Q_H^2(t) d < M^H >_t \tag{22}$$

and the likelihood equation is given by

$$\int_0^T Q_H(t) dZ_t - \theta \int_0^T Q_H^2(t) d < M^H >_t = 0. \tag{23}$$

Hence the MLE $\hat{\theta}_T$ of θ is given by

$$\hat{\theta}_T = \frac{\int_0^T Q_H(t) dZ_t}{\int_0^T Q_H^2(t)\, d < M^H >_t}. \tag{24}$$

Let θ_0 be the true parameter. Using the fact that

$$dZ_t = \theta_0 Q_H(t)\, d < M^H >_t + dM_t^H, \tag{25}$$

it can be shown that

$$\frac{dP_\theta^T}{dP_{\theta_0}^T} = \exp\left[(\theta - \theta_0)\int_0^T Q_H(t) dM_t^H - \frac{1}{2}(\theta-\theta_0)^2 \int_0^T Q_H^2(t) d < M^H >_t\right]. \tag{26}$$

Following this representation of the Radon-Nikodym derivative, we obtain that

$$\hat{\theta}_T - \theta_0 = \frac{\int_0^T Q_H(t) dM_t^H}{\int_0^T Q_H^2(t)\, d < M^H >_t}. \tag{27}$$

This result was also obtained by Marushkevych (2016) by similar arguments and asymptotic properties of the MLE were studied in Theorem 2 of Marushkevych (2016). Marushkevych (2016) investigated large deviation properties of the maximum likelihood estimator of the drift parameter θ as an application of the Gärtner-Ellis theorem following similar work of Bercu et al. (2010) for fractional Ornstein-Uhlenbeck process. We will obtain inequalities for large deviation probabilities for MLE following the work in Mishra and Prakasa Rao (2006) and Ibragimov and Khasminskii (1981).

Bayes estimation

Suppose that the true parameter $\theta \in K \subset \Theta \subset R$ where K is a compact set. Suppose that Λ is a prior probability measure on the parameter space $\Theta \subset R$. Further suppose that the probability measure Λ has a density $\lambda(.)$ with respect to the Lebesgue measure and the prior density function is continuous and positive on the set K and has a polynomial majorant. Fix $\theta_0 \in \Theta$. The posterior density of θ given the observation $X^T \equiv \{X_s, 0 \le s \le T\}$ is given by

$$p(\theta|X^T) = \frac{\frac{dP_\theta^T}{dP_{\theta_0}^T}\lambda(\theta)}{\int_\Theta \frac{dP_\theta^T}{dP_{\theta_0}^T}\lambda(\theta)\, d\theta}. \tag{28}$$

We define the Bayes estimate (BE) $\tilde{\theta}_T$ of the parameter θ based on the path X^T and the prior density $\lambda(\theta)$, to be the minimizer of the function

$$B_T(\phi) = \int_\Theta L(\theta,\phi)\, p(\theta|X^T)\, d\theta,\ \phi \in \Theta$$

where $L(\theta,\phi)$ is a given loss function defined on $\Theta \times \Theta$. In particular, for the quadratic loss function $|\theta - \phi|^2$, the Bayes estimator is the posterior mean given by

$$\tilde{\theta}_T = \frac{\int_\Theta u p^T(u|X^T) du}{\int_\Theta p^T(v|X^T) dv}.$$

Suppose the loss function $L(\theta,\phi) : \Theta \times \Theta \subset R$ satisfies the following conditions (cf. Ibragimov and Khasminskii (1981)):

D(i) $L(\theta,\phi) = L(|\theta-\phi|)$; for some real-valued function $L(.)$ defined on R with properties given below.

D(ii) $L(.)$ is non-negative and continuous on R; L(0)=0;

D(iii) $L(.)$ is symmetric, that is, $L(u) = L(-u), u \in R$;

D(iv) the sets $\{\theta : L(\theta) < c\}$ are convex sets for all $c > 0$, and are bounded for all $c > 0$ sufficiently small;

D(v) the function $L(u)$ has a polynomial majorant, that is, $L(u) \leq f(u)$ where $f(.)$ is a polynomial; and

D(vi) there exists numbers $\gamma > 0$, $\eta_0 \geq 0$ such that for $\eta \geq \eta_0$,

$$\sup\{L(\theta) : |\theta| \leq \eta^\gamma\} \leq \inf\{L(\theta) : |\theta| \geq \eta\}.$$

It is easy to check that the loss function of the form $L(\theta,\phi) = |\theta - \phi|^2$ satisfies the conditions D(i) - D(vi).

Let E_θ^T denote the expectation with respect to the probability measure P_θ^T. Define

$$\psi_T^H(\theta;a) = E_\theta^T[\exp(-a\int_0^T Q_H^2(t)d<M^H>_t)]$$

for $a > 0$. Marushkevych (2016) proved that

$$\lim_{T\to\infty}\frac{1}{T}\log\psi_T^H(\theta;a) = \frac{\theta}{2} - \sqrt{\frac{\theta^2}{4}+\frac{a}{2}}$$

for all $a > -\frac{\theta^2}{2}$. In view of this observation, we make the following assumption.

Condition (A): Let K be a compact subset of Θ. Suppose that

$$\sup_{\theta\in K} E_\theta^T[T^{-1}\int_0^T Q_H^2(t)d<M^H>_t] = O(1)$$

and

$$\sup_{\theta\in K} E_\theta^T[\exp(-u^2T^{-1}\int_0^T Q_H^2(t)d<M^H>_t)] = O(\exp(-Cu^8))$$

as $T \to \infty$ for some positive constant C.

4. Probabilities of large deviations

We now prove the following theorems giving the large deviation probabilities for the MLE and BE discussed in Section 3.

Theorem 4.1: *Under the condition* (A) *, there exist positive constants* C_1 *and* C_2 *such that for every* $T \geq T_0$*, and for every* $\epsilon > 0$,

$$\sup_{\theta\in K} P_\theta^T\{|T^{1/2}(\hat\theta_T - \theta)| > \epsilon\} \leq C_1 e^{-C_2\epsilon^2}$$

where $\hat{\theta}_T$ is the MLE of the parameter θ.

Theorem 4.2: *Under the conditions (A) and $D(i) - D(vi)$ stated earlier, there exist positive constants C_3 and C_4 such that for every $T \geq T_0$, and for every $\epsilon > 0$,*

$$\sup_{\theta \in K} P_\theta^T\{|T^{1/2}(\tilde{\theta_T} - \theta)| > \epsilon\} \leq C_3 e^{-C_4 \epsilon^2}$$

where $\tilde{\theta}_T$ is the BE of the parameter θ with respect to the prior $\lambda(.)$ and the loss function $L(.,.)$ satisfying the conditions $D(i) - D(vi)$.

Fix $\theta \in K$. For proofs of theorems stated above, we need the following lemmas. Define

$$Z_T(u) = \frac{dP^T_{\theta+uT^{-1/2}}}{dP^T_\theta}.$$

Lemma 4.3 : *Under the condition (A) stated above, there exist positive constants C_5 and C_6 independent of T such that*

$$sup_{\theta \in K} E_\theta^T[Z_T^{\frac{1}{2}}(u)] \leq C_5 e^{-C_6 u^2}$$

for $-\infty < u < \infty$.

Lemma 4.4 : *Under the condition (A) stated above, there exists a positive constant C_7 independent of T such that*

$$\sup_{\theta \in K} E_\theta^T \left\{ Z_T^{\frac{1}{2}}(u_1) - Z_T^{\frac{1}{2}}(u_2) \right\}^2 \leq C_7 (u_1 - u_2)^2$$

for $-\infty < u_1, u_2 < \infty$.

Lemma 4.5 : *Let $\xi(x)$ be a real valued random function defined on a closed subset F of the Euclidean space R^k. Assume that random process $\xi(x)$ is measurable and separable. Assume that the following conditions are fulfilled : there exists numbers $m \geq r > k$ and a positive continuous function on $G(x) : R^k \to R$ such that for all $x, h \in F, x + h \in F$,*

$$E|\xi(x)|^m \leq G(x), \ E|\xi(x+h) - \xi(x)|^m \leq G(x)\|h\|^r.$$

Then, with probability one, the realizations of $\xi(t)$ are continuous functions on F. Moreover let

$$\omega(\delta, \xi, L) = \sup |\xi(x) - \xi(y)|$$

where the supremum is taken over $x, y \in F$ with $\|x - y\| \leq h, \ \|x\| \leq L, \ \|y\| \leq L$; then

$$E(\omega(h, \xi, L)) \leq B_0 \left(\sup_{\|x\| \leq L} G(x) \right)^{\frac{1}{m}} L^{k/m} h^{\frac{r-k}{m}} \log(h^{-1})$$

where the constant B_0 depends on m, r and k.

We will use this lemma with $\xi(u) = Z_T^{1/2}(u), m = 2, r = 2, k = 1, G(x) = e^{-cx^2}$ and $L = H + r + 1$. For proof of this lemma, see Ibragimov and Khasminskii (1981) (Correction, cf. Kallianpur and Selukar (1993)).

Proof of Lemma 4.3: Fix $\theta \in K$. We know that

$$
\begin{aligned}
E_\theta^T(Z_T^{1/2}(u)) &= E_\theta^T(\frac{dP_{\theta+uT^{-1/2}}}{dP_\theta^T})^{1/2} \\
&= E_\theta^T[\exp\{\frac{uT^{-1/2}}{2}\int_0^T Q_H(t)dM^H(t) - \frac{1}{4}u^2T^{-1}\int_0^T Q_H^2(t)d<M^H>_t\}] \\
&= E_\theta^T[\exp\{\frac{uT^{-1/2}}{2}\int_0^T Q_H(t)dM^H(t) - \frac{u^2T^{-1}}{6}\int_0^T Q_H^2(t)d<M^H>_t\} \\
&\quad \times\exp\{-\frac{u^2T^{-1}}{12}\int_0^T Q_H^2(t)d<M^H>_t\}] \\
&\leq \left[E_\theta^T\left\{\exp\left(\frac{1}{2}uT^{-1/2}\int_0^T Q_H(t)dM^H(t)\right.\right.\right. \\
&\quad \left.\left.\left.-(1/6)u^2T^{-1}\int_0^T Q_H^2(t)d<M^H>_t\right)\right\}^{4/3}\right]^{3/4} \\
&\quad \times\left[E_\theta^T\left\{\exp\left(-(1/12)u^2T^{-1}\int_0^T Q_H^2(t)d<M^H>_t\right)\right\}^4\right]^{1/4} \\
&\quad \text{(by Holder's inequality)} \\
&= \left\{E_\theta^T\exp\left(\frac{2}{3}uT^{-1/2}\int_0^T Q_H(t)dM^H(t)\ -(2/9)u^2T^{-1}\int_0^T Q_H^2(t)d<M^H>_t\right)\right\}^{3/4} \\
&\quad \times\{E_\theta^T\exp(-(1/3)u^2T^{-1}Q_H^2(t)d<M^H>_t)\}^{1/4} \\
&\leq \left[E_\theta^T\left\{\exp\left(-\frac{1}{3}u^2T^{-1}\int_0^T Q_H^2(t)d<M^H>_t\right)\right\}\right]^{1/4}
\end{aligned}
$$

(since the first term is less than or equal to one (cf. Gikhman and Skorokhod (1972)). The last term is bounded by $C_5e^{-C_6u^2}$ uniformly for $\theta \in K$ for some positive constants C_5 and C_6 by the condition (A) which completes the proof of Lemma 4.3.

We now prove Lemma 4.4.

Proof of Lemma 4.4: Fix $\theta \in K$. Note that

$$
\begin{aligned}
&E_\theta^T\left\{Z_T^{\frac{1}{2}}(u_1) - Z_T^{\frac{1}{2}}(u_2)\right\}^2 \\
&= E_\theta^T\{Z_T(u_1) + Z_T(u_2)\} - 2E_\theta^T\left\{Z_T^{\frac{1}{2}}(u_1)Z_T^{\frac{1}{2}}(u_2)\right\} \\
&= 2\left[1 - E_\theta^T\left\{Z_T^{\frac{1}{2}}(u_1)Z_T^{\frac{1}{2}}(u_2)\right\}\right] \\
&\quad (\text{since } E_\theta^T Z_T(u) = E_\theta^T[\exp\{uT^{-1/2}\int_0^T Q_H(t)dM^H(t) \\
&\quad -\frac{1}{2}u^2T^{-1}\int_0^T Q_H^2(t)d<M^H>_t\}]) = 1.
\end{aligned}
$$

Denote

$$\begin{aligned} V_T &= \left(\frac{dP^T_{\theta_2}}{dP^T_{\theta_1}}\right)^{1/2} \text{ where } \theta_1 = \theta + u_1 T^{-1/2} \text{ and } \theta_2 = \theta + u_2 T^{-1/2} \\ &= \exp\{\frac{1}{2}(u_2 - u_1)T^{-1/2}\int_0^T Q_H(t)dM^H(t) \\ &\quad -\frac{1}{4}(u_2-u_1)^2 T^{-1}\int_0^T Q_H^2(t)d < M^H >_t\}. \end{aligned}$$

Now

$$\begin{aligned} &E^T_\theta\{Z_T^{\frac{1}{2}}(u_1)Z_T^{\frac{1}{2}}(u_2)\} \\ &= E^T_\theta\{(\frac{dP^T_{\theta+u_1T^{-1/2}}}{dP^T_\theta})^{1/2}(\frac{dP^T_{\theta+u_2T^{-1/2}}}{dP^T_\theta})^{1/2}\} \\ &= \int(\frac{dP^T_{\theta_1}}{dP^T_\theta})^{1/2}(\frac{dP^T_{\theta_2}}{dP^T_\theta})^{1/2}dP^T_\theta \\ &= \int(\frac{dP^T_{\theta_2}}{dP^T_{\theta_1}})^{1/2}dP^T_{\theta_1} = E^T_{\theta_1}(V_T) \\ &= E^T_{\theta_1}[\exp\{\frac{1}{2}(u_2-u_1)T^{-1/2}\int_0^T Q_H(t)dM^H(t) \\ &\quad -\frac{1}{4}(u_2-u_1)^2T^{-1}\int_0^T Q_H^2(t)d < M^H >_t\}]. \end{aligned}$$

Thus

$$\begin{aligned} &2\{1 - E^T_\theta(Z_T^{\frac{1}{2}}(u_1)Z_T^{\frac{1}{2}}(u_2))\} \\ &= 2[1 - E^T_{\theta_1}(\exp\{\frac{1}{2}(u_2-u_1)T^{-1/2}\int_0^T Q_H(t)dM^H(t) \\ &\quad -\frac{1}{4}(u_2-u_1)^2T^{-1}\int_0^T Q_H^2(t)d < M^H >_t\})] \\ &\leq 2[1 - \exp E^T_{\theta_1}\{\frac{1}{2}(u_2-u_1)T^{-1/2}\int_0^T Q_H(t)dM^H(t) \\ &\quad -\frac{1}{4}(u_2-u_1)^2T^{-1}\int_0^T Q_H^2(t)d < M^H >_t\}] \text{ (by Jensen's inequality)} \\ &= 2[1 - \exp\{-\frac{(u_2-u_1)^2}{4}T^{-1}E^T_{\theta_1}(\int_0^T Q_H^2(t)d < M^H >_t)\}] \\ &\leq 2[\frac{(u_2-u_1)^2}{4}T^{-1}E^T_{\theta_1}(\int_0^T Q_H^2(t)d < M^H >_t)] \text{ (since } 1 - e^{-x} \leq x) \\ &= C_7(u_2-u_1)^2 \end{aligned}$$

for some positive constant C_7 uniformly $\theta \in K$.

Proof of Theorem 4.1: Fix $\theta \in K$. Denote $U = \{u : \theta + u \in \Theta\}$. Let Γ_r be the interval

$L + r \leq |u| \leq L + r + 1$. We use the following inequality to prove the theorem:

$$P_\theta^T\{\sup_{\Gamma_r} Z_T(u) \geq 1\} \leq C_8(1 + L + r)^{\frac{1}{2}} e^{-\frac{1}{4}(L+r)^2} \tag{29}$$

for some positive constant C_8 uniformly for $\theta \in K$. Observe that

$$\begin{aligned}
P_\theta^T\{|T^{1/2}(\hat{\theta}_T - \theta)| > L\} &\leq P_\theta^T\{\sup_{|u|>l, u\in U} Z_T(u) \geq Z_T(0)\} \\
&\leq \sum_{r=0}^{\infty} P_\theta^T\{\sup_{\Gamma_r} Z_T(u) \geq 1\} \\
&\leq C_9 \sum_{r=0}^{\infty} e^{-C_{10}(L+r)^2} \\
&\leq C_1 e^{-C_2 L^2}
\end{aligned}$$

uniformly for $\theta \in K$ for some positive constants C_1 and C_2. This proves Theorem 4.1. We now prove the inequality (29). Fix $\theta \in K$. We divide the interval Γ_r into N sub-intervals $\{\Gamma_r^{(j)}, 1 \leq j \leq N\}$ each with length at most h. The number of such sub-intervals $N \leq [\frac{1}{h}] + 1$. Choose $u_j \in \Gamma_r^{(j)}$, $1 \leq j \leq N$. Then

$$\begin{aligned}
P_\theta^T\{\sup_{u\in\Gamma_r} Z_T(u) \geq 1\} \quad &\leq \quad \sum_{j=1}^{N} P_\theta^T\{Z_T(u_j) \geq \frac{1}{2}\} \\
&\quad + P_\theta^T\{\sup_{|u-v|\leq h, |u|,|v|\leq L+r+1} |Z_T^{\frac{1}{2}}(u) - Z_T^{\frac{1}{2}}(v)| \geq \frac{1}{2}\}.
\end{aligned} \tag{30}$$

From the Chebyshev's inequality and in view of Lemma 4.3 , it follows that

$$\sup_{\theta\in K} P_\theta^T\{Z_T^{\frac{1}{2}}(u_j) \geq \frac{1}{2}\} \leq C_{10} e^{-(L+r)^2}, 1 \leq j \leq N$$

for some positive constant C_{10}. Applying Lemma 4.5 with $\xi(u) = Z_T^{1/2}(u)$, and using Lemma 4.4, we obtain that

$$E_\theta^T[\sup_{\substack{|u-v|\leq h \\ |u|,|v|\leq (L+r+1)}} |Z_T^{1/2}(u) - Z_T^{1/2}(v)|] \leq C_{11}(L + r + 1)^{\frac{1}{2}} h^{1/2} \log(h^{-1})$$

for some positive constant C_{11} uniformly for $\theta \in K$. Hence

$$P_\theta^T\{\sup_{u\in\Gamma_r} Z_T(u) \geq 1\} \leq C_{12}\{\frac{1}{h} e^{-(L+r)^2} + (L + r + 1)^{\frac{1}{2}} h^{1/2} \log(h^{-1})\}$$

for some positive constant C_{12} uniformly for $\theta \in K$ by using the inequality (30). Choosing $h = e^{-(L+r)^2/2}$, we prove the inequality in Theorem 4.1.

Proof of Theorem 4.2 : Observe that the conditions (1) and (2) in Theorem 5.2 of Ibragimov and Khasminskii (1981) are satisfied by Lemmas 4.3 and 4.4. In view of the conditions on the loss function mentioned in Section 3, we can prove Theorem 4.2 by using Theorem 5.2 in Ibragimov and Khasminskii (1981) with $\alpha = 2$ and $g(u) = u^2$. We omit the details.

Remarks: Bahadur (1960) suggested measuring the asymptotic efficiency of an estimator δ_T of a parameter θ by the magnitude of concentration of the estimator in an interval of a fixed length independent of T, that is, by the magnitude of the probability $P_\theta^T(|\delta_T - \theta| \leq \gamma)$ for some fixed $\gamma > 0$. From the result obtained in Theorem 4.1 proved above, we note that the probability $P_\theta^T(|\hat{\theta}_T - \theta| > \gamma)$ is bounded above by $C_1 e^{-C_2\gamma^2 T}, C_1 > 0, C_2 > 0$ for the maximum likelihood estimator $\hat{\theta}_T$. This bound in turn decrease exponentially to zero as $T \to \infty$ for any fixed $\gamma > 0$. Following the techniques in Theorem 9.3 in Ibragimov and Khasminskii (1981), it can be shown that the MLE is Bahadur efficient under some additional conditions. Similar result follows for the Bayes estimator $\tilde{\theta}_T$ following Theorem 4.2.

References:

Bahadur, R.R. (1960) On the asymptotic efficiency of tests and estimates, *Sankhya*, **22**, 229-252.

Bercu, B., Coutin, L., and Savy, N. (2010) Sharp large deviations for the fractional Ornstein-Uhlenbeck process, *Theory Probab. Appl.*, **55**, 575-610.

Bjork, T. and Hult, H. (2005) A note on Wick products and the fractional Black-Scholes model, *Finance Stoch.*, **9**, 197-209.

Cai, C., Chigansky, P. and Kleptsyna, M. (2016) Mixed Gaussian processes, *Ann. Probab.*,**44**, 3032-3075.

Cheridito, P. (2000) Regularizing fractional Brownian motion with a view towards stock price modeling (Ph.D. Dissertation) ETH, Zurich.

Cheridito, P. (2001) Mixed fractional Brownian motion, *Bernoulli*, **7**, 913-934.

Cheridito, P. (2003) Arbitrage in fractional Brownian motion models, *Finance Stoch.*, **7**, 533-553.

Chigansky, P. and Kleptsyna, M. (2015) Statistical analysis of the mixed fractional Ornstein-Uhlenbeck process, arXiv:1507.04194.

Dembo, A. and Zeitouni, O. (1998) *Large Deviations Techniques and Applications*, 2nd Ed., Springer, New York.

Gikhman, I. and Skorokhod, A. V. (1972) *Stochastic Differential Equations,* Springer, Berlin.

Ibragimov I. A. and Khasminskii R. Z. (1981) *Statistical Estimation:Asymptotic Theory,* Springer-Verlag, New York.

Kalianpur G. and Selukar R. S. (1993) *Estimation of Hilbert-space valued parameters by the method of Sieves*, *Statistics and Probability*, Eds. J. K. Ghosh, S. K. Mitra, K. R. Parthasarthy and B. L. S. Prakasa Rao, Wiley Eastern, New Delhi, pp.235-347.

Kleptsyna, M. L. and Le Breton, A. (2002) Statistical analysis of the fractional Ornstein-Uhlenbeck type process, *Statist.Infer. for Stoch. Proc.*, **5**, 229–248.

Kuznetsov, Yu. (1999) The absence of arbitrage in a model with fractional Brownian motion, *Russian Math. Surveya*, **54**, 847-848.

Le Breton, A. (1998) Filtering and parameter estimation in a simple linear model driven by a fractional Brownian motion, *Statist. Probab. Lett.*, **38**, 263–274.

Marushkevych, Dmytro (2016) Large deviations for drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process, *Modern Stochastics: Theory and Applications*, 3:107-117.

Mishura, Y. (2008) *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Springer, Berlin.

Mishra, M.N. and Prakasa Rao, B.L.S. (2006) Large deviation probabilities for maximum likelihood estimator and Bayes estimator of a parameter for fractional Ornstein-Uhlenbeck type process, *Bull. Inform. and Cyber.*, **38**, 71-83.

Prakasa Rao, B. L. S. (1987) *Asymptotic Theory of Statistical Inference*, Wiley, New York .

Prakasa Rao, B. L. S. (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London and Oxford University Press, New York.

Prakasa Rao, B. L. S. (2003) Parameter estimation for linear stochastic differential equations driven by fractional Brownian motion, *Random Operators and Stochastic Equations*, **11**, 229-242.

Prakasa Rao, B. L. S. (2005) Berry-Esseen bound for MLE for linear stochastic differential equations driven by fractional Brownian motion, *J. Korean Statist. Soc.*, **34**, 281-295.

Prakasa Rao, B.L.S. (2009) Estimation for stochastic differential equations driven by mixed fractional Brownian motions, *Calcutta Statistical Association Bulletin*, **61**, 143-153.

Prakasa Rao, B.L.S. (2010) *Statistical Inference for Fractional Diffusion Processes*, Wiley, London.

Prakasa Rao, B.L.S. (2015a) Option pricing for processes driven by mixed fractional Brownian motion with superimposed jumps, *Probability in the Engineering and Information Sciences*, **29**, 589-596.

Prakasa Rao, B.L.S. (2015b) Pricing geometric Asian power options under mixed fractional Brownian motion environment, *Physica A*, **446**, 92-99.

Prakasa Rao, B.L.S. (2017) Instrumental variable estimation for a linear stochastic differential equation driven by a mixed fractional Brownian motion, *Stoch. Anal. Appl.*, doi.org/10.1080/07362994.2017.1338577 (to appear).

Rudomino-Dusyatska, N. (2003) Properties of maximum likelihood estimates in diffusion and fractional Brownian models, *Theor. Probab. Math. Statist.*, **68**, 139-146.

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