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Review on Quantile Regressions

Yuya Katafuchi[†]

Abstract

Quantile regressions enable researchers to investigate the effects of covariates on the non-central position of outcome distribution unlike the ordinary least squares (OLS). This paper provides an overview of the conditional quantile regression (CQR) and the unconditional quantile regression (UQR). We focus on their technical issues including the specification of the models, the inference of them, additional models and the endogeneity issues in covariates. We also illustrate how to interpret the coefficients estimated by both of these frameworks, and summarise recent empirical applications of them. Furthermore, we discuss the comparison between two frameworks focusing on the difference of interpretation and the difference of magnitudes of estimated coefficients.

Keywords: Conditional Quantile Regression, Unconditional Quantile Regression

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1 Introduction

Linear regression is the standard empirical tool in most social science research that seeks to expose the relationship between a dependent variable and a set of predictor variables. Traditional regression analysis is focused on the mean to summarise the behaviour of the response affected by explanatory variables using the conditional mean of the response. The idea of conditional mean models has been applied broadly in the social sciences, even though these have some limitations, e.g. focusing just on central locations, ignoring the whole distribution of response, violating model assumptions.

The first alternative to the conditional mean model, the median regression can be used to solve some of the concerns above by replacing the least-squares objective function with the least- absolute-distance objective function. Specifically, in the case of skewed distribution of the response variable, the conditional median modelling is considered to be more useful. Median regression, however, remains problematic since it steered researchers away from whole distribution. This is because it estimates the effect of covariates on central tendency, even though much research in social science investigates the causal effect focusing on the specific tail of the distribution, e.g. research of inequality.

To address the concern of focusing just on central tendencies, i.e. to take a look at the non- central position of distribution, conditional quantile regression (CQR), introduced by Koenker and Bassett (1978), is much suited to answer the research questions related to changes in the distribution of a response variable. The CQR model can estimate the effect of explanatory variables on the specific conditional quantile, e.g. .05th quantile and third quartile. Thus, the CQR models the effect of variables on the non-central part of response distribution conditional on covariates while linear regression model identifies the change only in the conditional mean of a response variable. The conditional median regression is a special case of quantile regression modelling conditional 50th percentile as a function of explanatory variables.

Although CQR has become common and widely used in empirical analyses, the interpretation of such effects becomes limited since the estimated effects do not translate to policy questions linked to these covariates. Thus, many researchers have misinterpreted conditional quantile regression coefficients as if they were the effect on the quantile of response rather than quantile of response within groups linked to their set of covariates (Borah and Basu, 2009; Killewald and Bearak, 2014; Porter, 2015; Wenz, 2018). Resolving this concern of misuse, unconditional quantile regression (UQR) introduced by Firpo et al. (2009b) is becoming the popular choice among empirical researches to overcome the limitation of interpretation of CQR.

To date, the review on both CQR and UQR is limited. Furthermore, the reviews on quantile regression focus only on CQR (Yu et al., 2003; Buhai, 2005; Koenker, 2017). Hence, this article aims to offer a review of the theory and application of CQR and UQR, and provide comparisons of them. This paper specifically makes two contributions; the first one is it surveys the literature on endogeneity issues in quantile regression in detail and the second one is that it provides a comprehensive review of UQR including its application.

The remainder of the paper is structured as follows. Section 2 introduces the overview of CQR, its inference and additional models consisting of nonparametric CQR, nonlinear CQR and CQR for longitudinal data. It also reviews endogeneity issues, how to interpret a result of CQR and empirical applications of CQR. Section 3 represents the summary of influence function (IF), recentered influence function (RIF) and three estimation methods of UQR, and this chapter also focuses on a method on endogeneity issues, interpretation of UQR and empirical applications. Section 4 describes the comparison between CQR and UQR. The final section gives concluding remarks of our review.

2 Conditional Quantile Regression

This section describes the definition of quantile, conditional quantile function, the model overview of CQR, its inference and additional issues in CQR.

2.1 Quantile and Conditional Quantile Function

To characterize the location and shape of distributions, quantile and conditional quantile function should be proposed. Let $F(y)$ denote the cumulative distribution function (cdf) of random variable Y, that is, $F(y)=P(Y\leq y)$. When $F(y)$ is continuous and strictly increasing function, the inverse function of $F(Y)$ is defined as $F^{-1}(y)$. In this case, τ-th unconditional quantile of Y for $\tau\in[0,\ 1]$ is defined as

$$q_\tau\equiv F^{-1}(\tau).$$

Specifically, median is straightforwardly given by $q_{.5}=F^{-1}(.5)$. Furthermore, it follows the general form especially in the case that there is no inverse function,

$$q_\tau\equiv\inf\{y : F(y)\geq\tau\}.$$

The problem of finding the τ-th quantile of Y is written as

$$q_\tau=\arg\min_{\xi}\mathbf{E}[\rho_\tau(Y-\xi)],$$

where $\rho_\tau(\cdot)$ is an asymmetric absolute loss function defined as

$$\begin{aligned}\rho_\tau(u)&\equiv[\tau-\mathbf{1}\{u<0\}]u \qquad (1)\\ &=[\tau\mathbf{1}\{u>0\}+(1-\tau)\mathbf{1}\{u\leq 0\}]|u|\\ &=\tau\mathbf{1}\{u>0\}|u|+(1-\tau)\mathbf{1}\{u\leq 0\}|u|\\ &=\begin{cases}\tau|u| & \text{if } u>0\\ (1-\tau)|u| & \text{otherwise}.\end{cases}\end{aligned}$$

The $\rho_\tau(\cdot)$ is also called check function, weighting absolute deviation $Y-\xi$ in the above equation where a τ weight is assigned for the positive deviations and a $1-\tau$ weight is given to the negative deviations. Especially, for the τ-th sample quantile, the minimization problem can be written as

$$\begin{aligned}\hat{q}_\tau &= \arg\min_{q_\tau} \sum_{i=1}^{n} \rho_\tau(y_i - q_\tau)\\ &= \arg\min_{q_\tau} \sum_{i=1}^{n} (\tau - \mathbf{1}\{y_i - q_\tau < 0\})(y_i - q_\tau),\end{aligned}$$

where q_τ is the true τ-th quantile of Y.

We consider the τ-th conditional quantile function

$$\begin{aligned}Q_Y(\tau \mid X=x) &\equiv F_Y^{-1}(\tau \mid X=x)\\ &= \inf\{y : F_{Y|X=x}(y \mid X=x) \geq \tau\},\end{aligned}$$

which specifies τ-th quantile of Y using an arbitrary function of explanatory variables x. Therefore, the minimization problem for quantile of Y at τ is rewritten by

$$q_\tau = \arg\min_{Q_Y(\tau|X=x)} \mathbf{E}[\rho_\tau(Y - Q_Y(\tau \mid X=x))]. \tag{2}$$

In the case that $Q(\cdot)$ is linearly additive function with parameters

$$Q_Y(\tau \mid X=x) = x'\beta(\tau). \tag{3}$$

Koenker and Bassett (1978) define the problem for the coefficient of conditional quantile regression at τ, $\beta(\tau)$ as

$$\beta(\tau) = \arg\min_{\beta(\tau)} \mathbf{E}[\rho_\tau(Y - x'\beta(\tau))]. \tag{4}$$

Thus, the estimator of $\beta(\tau)$ is given by

$$\hat{\beta}(\tau) = \arg\min_{\beta(\tau)} \sum_{i=1}^{n} \rho_\tau(y_i - x_i'\beta(\tau)).$$

Specifically, the coefficient at median is estimated by following minimization problem

$$\begin{aligned}\hat{\beta}(1/2) &= \arg\min_{\beta(1/2)} \sum_{i=1}^{n} \rho_{1/2}(y_i - x_i'\beta(1/2))\\ &= \arg\min_{\beta(1/2)} \sum_{i=1}^{n} [1/2 - \mathbf{1}\{y_i - x_i'\beta(1/2) < 0\}](y_i - x_i'\beta(1/2))\\ &= \arg\min_{\beta(1/2)} \left[1/2 \sum_{y_i > x_i'\beta(1/2)} |y_i - x_i'\beta(1/2)| + 1/2 \sum_{y_i \leq x_i'\beta(1/2)} |y_i - x_i'\beta(1/2)|\right]\\ &= \arg\min_{\beta(1/2)} \sum_{i=1}^{n} |y_i - x_i'\beta(1/2)|,\end{aligned}$$

which is identical with the least absolute deviation estimator.

2.2 Inference of CQR

The elementary asymptotics of CQR are based on the assumption of independently and identically distributed (iid) error and classical large sample theory. Based on the equivalent form of linear conditional quantile function in equation (3)

$$y_i = \beta(\tau)x_i + \varepsilon_i^\tau, \tag{5}$$

where the ε_i^τ have a common distribution whose τ-th quantile is zero, Koenker and Bassett (1978) show that asymptotic distribution of $\hat{\beta}(\tau)$ is given by

$$\sqrt{n}(\hat{\beta}(\tau) - \beta(\tau)) \xrightarrow{d} \mathcal{N}(0, \ \Omega^\tau),$$

where

$$\Omega^\tau = \tau(1-\tau)\mathbf{E}[X_i X_i' f_{\varepsilon^\tau|X=x}(0\,|\,X_i = x_i)]^{-1}\mathbf{E}[X_i X_i']\mathbf{E}[X_i X_i' f_{\varepsilon^\tau|X=x}(0\,|\,X_i = x_i)]^{-1} \tag{6}$$

is the asymptotic variance and $f_{\varepsilon^\tau|X=x}(0\,|\,X_i = x_i)$ is the probability density of the error term ε conditional on X evaluated at the τ-th quantile of the error distribution. Specifically, in the case that the covariates and the disturbance term are independent i.e. $X_i \perp\!\!\!\perp \varepsilon_i^\tau$, asymptotic covariance of $\hat{\beta}(\tau)$ is simplified as

$$\Omega^\tau = \frac{\tau(1-\tau)}{f_{\varepsilon^\tau}^2(0)}\mathbf{E}[X_i X_i']^{-1}, \tag{7}$$

where $f_{\varepsilon^\tau}(\cdot)$ is density function of error term for τ. In the both case of equations (6) and (7), asymptotic covariance Ω^τ depends on a density function of the error term which is generally unknown. Conducting statistical inference on CQR, calculating standard error or confidence interval for parameters require nonparametric estimation of density function of the error term in equation (5), even though there are several problems such as the choice of smoothing parameter, unstability of estimate, the accuracy, etc. Various types of bootstrap become popular to avoid the nonparametric estimation of the error density (Hahn, 1995; He, 2017) e.g. paired bootstrap (Horowitz, 1998) and residual-based (wild) bootstrap (Feng et al., 2011).

Estimation of $\beta(\tau)$ in equation (4) i.e. calculating $\hat{\beta}(\tau)$ in parametric linearly additive model of Koenker and Bassett (1978), can be numerically achieved based on linear programming method since the derivative with respect to parameters of objective function is discontinuous. The recent development of computer power and the development of interior point method in the 1980s enable researchers to solve large-scale linear programming problems with fairly less time consumption. The simplex method of Barrodale and Roberts (1973) is first adapted for CQR with a moderate scale of data by Koenker and D'Orey (1987) while interior-point based linear programming for regression quantile of Portnoy and Koenker (1997) is widely used for large dataset. Details of computation are discussed in Koenker (2005), Davino et al. (2014) and Furno and Vistocco (2018).

2.3 Additional Models on CQR

Given its characteristics, CQR can be applied to a wide variety of ways. Unlike the classical regression model, it can be used in several frameworks. This section presents the main enhancement of CQR: nonparametric CQR, nonlinear CQR, and application of CQR to panel data.

2.3.1 Nonparametric CQR

In equation (4), Koenker and Bassett (1978) assume linearity in conditional quantile function. In real data analysis, such an assumption might be strong and has a possibility of misidentification of the model. Thus, researchers develop several nonparametric methods of CQR by relaxing linearity.

The locally polynomial model for CQR introduced by Chaudhuri (1991), which is one of the nonparametric models for CQR, assumes that in the simplest specification

$$y_i = f(x_i) + \varepsilon_i,$$

where $f(\cdot)$ is a polynomial function of a given degree. The choice of degree of polynomial is derived from a tradeoff between the computational cost and how fit is good i.e. bias and variance.

Not only relaxing model assumption of linearity, several nonparametric models for CQR is also considered as a tool for dimensional reduction of design matrix, i.e. variable selection. In the LASSO proposed by Tibshirani (1996), the L_1 penalty can be used for CQR. The LASSO penalised linear quantile regression has a minimisation problem

$$\min_{\beta(\tau)} \sum_{i=1}^{n} \rho_\tau(y_i - x_i'\beta(\tau)) + \lambda \sum_{j=1}^{k} |\beta(\tau)_j|,$$

where k is the number of features in x_i. Related penalties for model selection problem are discussed in Wu and Liu (2009).

2.3.2 Nonlinear CQR

Consider the nonlinear specification of conditional quantile function in equation (3)

$$Q_Y(\tau \mid X = x) = f(x, \beta(\tau)), \tag{8}$$

where $f(\cdot)$ is unknown. In the case of nonlinear specification in equation (8), at least one parameter has higher power than one or it can be divided or multiplied by another parameter.

Researchers propose several approaches to nonlinear quantile regression. Censored model in a latent specification of conditional quantile function and its estimation is presented in Powell (1986). The Box-Cox power transformation of conditional quantile function is defined in Powell (1991), modified by Fitzenberger et al. (2010) which suggest the easier way to implement.

2.3.3 CQR for panel data

Usage of panel data, combining time series and cross sections, became a common tool in the empirical analysis of economics. For the conditional quantile function considering unobserved individual effect

$$Q_{Y_{it}}(\tau_k \mid X_{it} = x_{it}) = x'_{it}\beta(\tau) + \alpha_i, \tag{9}$$

for $k=1, \ldots, p$, $i=1, \ldots, n$, and $t=1, \ldots, T_i$, where α_i is the individual fixed effect expressing a pure location shift on the conditional quantiles of the response. Koenker (2009) suggests the minimization problem for equation (9)

$$\min_{\alpha,\beta(\tau)} \sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{t=1}^{T_i} w_k \rho_{\tau_k}[y_{it}-(x'_{it}\beta(\tau_k)+\alpha_i)], \tag{10}$$

where w_k represents the weights assigned to the quantiles performing L-statistics regression in Mosteller (1946) and Koenker (1984). One problem of solving equation (10) is that it could be impractical when the dimensions of features and response are large. Koenker (2009) proposes modification of the problem (10) by imposing L_1 penalty for α_i following Tibshirani (1996)

$$\min_{\alpha,\beta(\tau)} \sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{t=1}^{T_i} \rho_{\tau_k}[y_{it}-(x'_{it}\beta(\tau_k)+\alpha_i)]+\lambda\sum_{i=1}^{n}|\alpha_i|, \tag{11}$$

where λ is the shrinkage parameter regulating the penalization to govern the dimension of α_i. The λ in equation (11) values towards zero reset to the fixed effect estimator in equation (10) while increasing the λ we obtain the model shrinking the fixed effect α_i. The overview of shrinkage parameter and strategy of choosing it is given in Lamarche (2010).

2.4 Endogeneity Issues in CQR

In a real application, especially in microeconometrics, endogeneity is often the main issue to be dealt with to clarify the causal effect of the explanatory variable to the response variable. Endogeneity of the covariates is mainly caused by omitted variable bias, simultaneous equation bias, measurement error, or self-selection bias. When the model has the endogeneity problem, i.e. covariates of the model are correlated to its error term, CQR fails to consistently estimate the causal effect of variables on the quantile of the result of interest. The instrumental variable (IV) could be a tool to investigate the causal effects in the presence of endogenous covariates, even in the framework of CQR.

The first paper dealing with endogeneity in CQR, Abadie et al. (2002) adopt the concept of local average treatment effect (LATE) of Imbens and Angrist (1994) in potential outcome framework, in the case of binary treatment with binary instrumental variable. Given the vector of covariates x, a treatment $D=\{0, 1\}$, a instrumental variable $Z=\{0, 1\}$, Abadie et al. (2002) model

$$Q_\tau(Y\,|\,X=\mathrm{x},\ D=d,\ D_1>D_0)=\alpha(\tau)d+\mathrm{x}'\beta(\tau), \tag{12}$$

where D_1 when $Z=1$ and D_0 when $Z=0$. Under the monotonicity, $P(D_1\geq D_0|\mathrm{X}=\mathrm{x})=1$. The interest of LATE is to specify the treatment effect only on the compliers, not always-takers, never-takers, defiers, i.e. the subpopulation with $D_1>D_0$. They define the estimator

$$(\hat{\alpha}(\tau),\ \hat{\beta}(\tau))=\arg\min_{\alpha(\tau),\beta(\tau)}\mathbf{E}[\kappa_\nu\rho_\tau(Y-\alpha(\tau)d-\mathrm{x}'\beta(\tau))],$$

where

$$\kappa_\nu=1-\frac{D(1-\nu_0)}{1-\pi_0(\mathrm{x})}-\frac{(1-D)\nu_0}{\pi_0(\mathrm{x})}$$

is conditional weight finding compliers, $\nu_0=\mathbf{E}(Z\,|\,Y,\ D=d,\ \mathrm{X}=\mathrm{x})$, and $\pi_0(\mathrm{x})=P(Z=1|\mathrm{X}=\mathrm{x})$. For further datails and identification issues, see Melly and Wüthrich (2017).

The framework in Chernozhukov and Hansen (2004), Chernozhukov and Hansen (2005), Chernozhukov and Hansen (2006) and Chernozhukov and Hansen (2008), which is called as instrumental variable quantile regression (IVQR), develops the structural simultaneous equation model with nonadditive errors as a solution of endogeneity in CQR. IVQR model is elaborated within the potential outcome framework, Y_d when $D=d$. There is thus a conditional quantile function $Q_{Y_d}(\tau|D=d,\ \mathrm{X}=\mathrm{x})=q(d,\ \mathrm{x},\ \tau)$. Conditional on X=x, the potential outcome can be modeled by its conditional quantile function $Y_d=q(d,\ \mathrm{x},\ U_d)$ whare $U_d\sim\mathcal{U}[0,\ 1]$ is the rank variable characterizing heterogeneity of outcome for individuals and independent of the instrumental variable Z given X=x. Let the treatment D be determined as $D=\delta(Z,\ \mathrm{x},\ \mathrm{v})$ for random vector v which is an unobserved information component that could be related to U_d. Chernozhukov and Hansen (2005) consider the linear specification such as equation (12) under above assumptions

$$q(d,\ \mathrm{x},\ \tau)=d'\alpha(\tau)+\mathrm{x}'\beta(\tau). \tag{13}$$

To estimate the parameters in equation (13), Chernozhukov and Hansen (2006) and Chernozhukov and Hansen (2008) propose the following transformation of conditional quantile function by the exclusion restriction of Z,

$$Q_{Y-D'\alpha|\mathrm{X}=\mathrm{x},\ \mathrm{Z}=\mathrm{z}}(\tau|\mathrm{X}=\mathrm{x},\ \mathrm{Z}=\mathrm{z})=\mathrm{x}'\beta(\tau)+z'\gamma(\tau,\ \alpha), \tag{14}$$

which concentrates out the coefficients on exogenous variables X and proposes a nonconvex optimization problem over only the parameters of treatment, α. Chernozhukov and Hansen (2006) recommend using a grid search to estimate the parameter in equation (14). The estimation of parameters in equation (13) can be also conducted in the framework of generalized method of moment by making use of the method in Chernozhukov and Hong (2003). One could see further detail of estimation of IVQR in Chernozhukov and Hansen (2013) and Chernozhukov et al. (2017).

2.5 Interpretation of CQR

The term "conditional" of CQR comes from the linear specification

$$Q_Y(\tau|X=x)=x'\beta(\tau) \tag{15}$$

Unlike the analysis on average by OLS, interpretation of the coefficients in CQR is the marginal effect of the regressor on the quantile defined by the covariates. The conventional OLS models the conditional mean of response Y given covariates X as

$$\begin{aligned}&\mathbf{E}[Y|X=x]=X\beta\\ \Rightarrow&\mathbf{E}[\mathbf{E}[Y|X=x]]=\mathbf{E}[X\beta]\\ \Leftrightarrow&\int\mathbf{E}[Y|X=x]f_X(x)\mathrm{d}x=\mathbf{E}[X]\beta.\end{aligned}$$

The left-hand side follows

$$\begin{aligned}\int\mathbf{E}[Y|X=x]f_X(x)\mathrm{d}x&=\iint yf_{Y|X=x}(y|X=x)f_X(x)\mathrm{d}y\mathrm{d}x\\ &=\int y\left[\int f_{Y|X=x}(y|X=x)f_X(x)\mathrm{d}x\right]\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \int y\left[\int f_{X,Y}(X,\ Y)\mathrm{d}x\right]\mathrm{d}y \\
&= \int y f_Y(y)\mathrm{d}y \\
&= \mathbf{E}[Y], \qquad (16)
\end{aligned}
$$

that is, OLS consistently estimates the effect of increasing the mean value of X on the unconditional mean of Y due to the law of iterated expectations. In the case of τ-th CQR, generally the conditional quantile model does not yield the effect on the unconditional quantile as

$$
\begin{aligned}
&Q_\tau[Y|X=x]=X\beta(\tau) \\
&\Rightarrow \mathbf{E}[Q_\tau[Y|X=x]]=\mathbf{E}[X\beta(\tau)] \\
&\not\Leftrightarrow q_\tau=\mathbf{E}[X]\beta(\tau),
\end{aligned}
$$

where q_τ is the τ-th unconditional quantile of Y. Whether the effect estimated by CQR can be generalized to the effect on the unconditional quantile depends on the number of covariates X.

First, we consider the case where we have a single covariate, e.g. the response variable is the wage, and the independent variable is the education dummy $D^e=\{0,\ 1\}$ which equals one if an individual graduated from university. By CQR, we could thus obtain the difference between the first quartile (0.25-th quantile) of the wage of the individual who has the degree of university and first quartile of the wage of the individual who did not graduate from university, i.e. we could estimate the effect of educational attainment in university at the first quartile of unconditional distribution as

$$
\begin{aligned}
\beta(0.25) &= Q_Y(0.25|D^e=1)-Q_Y(0.25|D^e=0) \\
&= F_Y^{-1}(0.25|D^e=1)-F_Y^{-1}(0.25|D^e=0).
\end{aligned}
$$

In this case, without any covariates than single treatment, estimated parameter $\beta(0.25)$ at the first quartile can explain the effect of university graduation on the first quartile of unconditional distribution of wage (Frölich and Melly, 2013; Melly and Wüthrich, 2017; Wenz, 2018).

On the other hand, we suppose the case that there are multiple covariates, e.g. incorporating gender dummy $D^g=\{0,\ 1\}$ which equals one if an individual graduated is male as an additional covariate with the previous setting. In this model, we could estimate the difference between the first quartile in the wage distribution conditional on gender for each status of university graduation, i.e. the coefficient of educational attainment estimated by CQR explains the effect of university graduation at the first quartile of conditional distribution. For instance, if the coefficient of education at the first quartile is 0.5 and that at the third quartile is 0.25, it then means that the wage inequality caused by educational status in lower wage (first quartile) group conditional on their gender is larger than that in higher wage (third quartile) group conditional on gender; it does not mean that the education-wage inequality in lower-wage group in the unconditional wage distribution is smaller than that in its higher counterpart.

Killewald and Bearak (2014), Porter (2015), and Wenz (2018) discuss the possibility of misinterpretation of a result of estimation by CQR and its example.　As we reviewed, in the case of multiple covariates, we are not able to obtain the effect of the treatment variable on the unconditional quantile of the response variable by CQR while we can estimate it by UQR which is discussed in later sections.

2.6 Application of CQR

Empirical analysis of CQR in economics literature is becoming popular to show the complete picture of the causal effect of interest on the conditional distribution of response rather than the effect just for the conditional mean which can be conducted by conventional linear regression.

One of the earliest and seminal works is Buchinsky (1994) which provides practice of how to apply CQR to the study of wages, and then suggests there is the heterogeneity in the effects of schooling and experience on the conditional quantiles of wages in the U.S., i.e. returns to schooling and experience differ across quantiles of the conditional wage distribution.　Likewise, Chamberlain (1994) examines the effect of union membership on the entire conditional distribution of wages.　They find that union membership yields a higher return at the lower tail of the wage distribution than at the mean of it.

Researchers of labour economics devote more effort to investigate the labour market by conducting CQR analysis to clarify the different effect of various type of interest on the income distribution. Fitzenberger and Kurz (2003) conduct the quantile regression analysis of the labour market in Germany and find that the effects of skill and industry dummies on wages are different across conditional quantiles.　Budd and McCall (2001) investigate the wage distribution of the specific industry, grocery stores by CQR.　García et al. (2001) suggest that increasing effect of educational attainment holds for men; nevertheless does not hold it for women by estimating CQR separately to gender.　attributed to different returns to characteristics increase over the wage distribution.　Lemieux (2006) uses CQR to look at how wage differentials differ at different points of wage distribution focusing on postsecondary educational attainment.

CQR is widely used not only in labour economics, in other microeconometrics fields.　Levin (2001) discusses the topic of the effect of class size reduction and peer effects on scholastic achievement and reveals that there are positive peer effects specifically in the lower tail of the outcome distribution. Abrevaya (2001) investigates the effects of demographic and maternal information on the birth weights by quantile regression and shows the evidence that these effects appear higher impact at lower conditional quantiles.　Katafuchi and Delgado (2018) conduct penalised CQR with L_1 and Elastic Net (Zou and Hastie, 2005) penalty based on hedonic price model with geographic information system data including geodesic distance information.　They show the evidence that some characteristics of the evaluation point of land are not significant in higher conditional quantile of land price distribution.

Empirical analyses using CQR also expand to other fields than econometrics. In ecology, Bowen et al. (2015) highlight the changes in quantiles of water quality regarding chloride concentrations. In the field of medicine, Wei et al. (2006) illustrate nonparametric CQR for estimation of reference growth curves of children's height and weight. In biology, Eilers and de Menezes (2005) give the penalised quantile regression analysis of comparative genomic hybridisation data.

3 Unconditional Quantile Regression

As we reviewed in Section 2.1, CQR measures the effect of explanatory variables on the conditional quantiles. The CQR, however, is not able to investigate the effect of a variable to a quantile of whole distribution, e.g. to identify the effect of the ratio of union participation on the unconditional income distribution. Firpo et al. (2009b) elaborate the method called UQR or RIF (recentered influence function) regression to measure the effect of predictors on the unconditional quantiles. In this section, we present a review of the methods which enable us to clarify the causal effect of a variable on the unconditional quantiles.

3.1 IF, RIF and Unconditional Quantile Partial Effect

UQR or RIF-regression is the method to estimate the covariates effect on the unconditional quantiles based on the concepts of influence function (IF) and RIF which are used in the robust statistic literature (Hampel et al., 2011). The IF measures the influence of an individual observation on a distributional statistic $v(F_Y)$ of interest without having to recalculate the statistic. Let the mixing distribution be $F_{Y,t,G_Y}=(1-t)F_Y+tG_Y=t(G_Y-F_Y)+F_Y$. Thus, taking differentiation of $v(F_{Y,t,G_Y})$ with respect to t yields

$$\begin{aligned}\lim_{t\downarrow 0}\frac{v(F_{Y,t,G_Y})-v(F)}{t}&=\left.\frac{\partial v(F_{Y,t,G_Y})}{\partial t}\right|_{t=0}\\&=\int\left.\frac{\partial v(F_{Y,t,\Delta_Y})}{\partial t}\right|_{t=0}\mathrm{d}(G_Y-F_Y)(y),\end{aligned}\tag{17}$$

where Δ_Y is the probability function which equals 1 at the value y in F_{Y,t,G_Y}. Hampel (1974) defines the IF as the argument of integral in equation (17) i.e.

$$\begin{aligned}\mathrm{IF}(y\,;\,v,\ F_Y)&\equiv\left.\frac{\partial v(F_{Y,t,\Delta_Y})}{\partial t}\right|_{t=0}\\&=\lim_{t\downarrow 0}\frac{v(F_{Y,t,\Delta_Y})-v(F_Y)}{t}.\end{aligned}\tag{18}$$

On the other hand, RIF is derived from a first order von Mises linear approximation (von Mises, 1947) of $v(G_Y)$:

$$v(G_Y)\approx v(F_Y)+\int\mathrm{IF}(y;v,\ F_Y)\mathrm{d}G_Y(y).\tag{19}$$

When we consider equation (19) for the particular case $G=\Delta_Y$, we define the RIF

$$\begin{aligned}\mathrm{RIF}(y;v,\ F_Y)&\equiv v(F_Y)+\int \mathrm{IF}(y;v,\ F_Y)\mathrm{d}\Delta_Y(y)\\&=v(F_Y)+\mathrm{IF}(y;v,\ F_Y).\end{aligned} \tag{20}$$

The RIF enables us to assess the covariates effect on the distributional statistic $v(\cdot)$. By the property of IF that

$$\begin{aligned}\mathbf{E}[\mathrm{IF}(y;v,\ F_Y)]&=\int \left.\frac{\partial v(F_{Y,t,\Delta_Y})}{\partial t}\right|_{t=0}\mathrm{d}F_Y(y)\\&=0,\end{aligned}$$

and equation (20),

$$\begin{aligned}\mathbf{E}[\mathrm{RIF}(y;v,\ F_Y)]&=\int \mathrm{RIF}(y;v,\ F_Y)\mathrm{d}F_Y(y)\\&=\int [v(F_Y)+\mathrm{IF}(y;v,\ F_Y)]\mathrm{d}F_Y(y)\\&=v(F_Y).\end{aligned} \tag{21}$$

It follows

$$\begin{aligned}v(F_Y)&=\int \mathrm{RIF}(y;v,\ F_Y)\mathrm{d}F_Y(y)\\&=\int\int \mathrm{RIF}(y;v,\ F_Y)\mathrm{d}F_{Y|X}(y|X=x)\mathrm{d}F_Y(x)\quad \left(\because F_Y(y)=\int F_{Y|X}(y|X=x)\mathrm{d}F_X(x)\right)\\&=\int \mathbf{E}[\mathrm{RIF}(y;v,\ F_Y)|X=x]\mathrm{d}F_X(x),\end{aligned} \tag{22}$$

where $\mathbf{E}[\mathrm{RIF}(y;v,\ F_Y)|X=x]=\int \mathrm{RIF}(y;v,\ F_Y)\mathrm{d}F_{Y|X}(y|X=x)$. Equation (22) states that any functional statistic $v(F_Y)$ can be written as the mean of the conditional mean of RIF-regression $\mathbf{E}[\mathrm{RIF}(y;v,\ F_Y)|X=x]$. By this fact, Corollary 2 in Firpo et al. (2009b) defines the unconditional partial effect. Consider a change of the distribution of covariate X from F_X to G_X. It corresponds to the change of the distribution of Y from F_Y to

$$G_Y^*(y)=\int F_{Y|X}(y|X=x)\mathrm{d}G_X(x). \tag{23}$$

In this case, taking differentiation of $v(F_{Y,\ t,\ G_Y^*})$, where $F_{Y,\ t,\ G_Y^*}=(1-t)F_Y+tG_Y^*$, yields

$$\begin{aligned}\left.\frac{\partial v(F_Y,t,G_Y^*)}{\partial t}\right|_{t=0}&=\lim_{t\downarrow 0}\frac{v(F_Y,t,G_Y^*)-v(F)}{t}\\&=\int \mathrm{RIF}(y;v,\ F_Y)\mathrm{d}(G_Y^*-F_Y)(y)\\&=\int\int \mathrm{RIF}(y;v,\ F_Y)\mathrm{d}(F_{Y|X}(y|X=x)\mathrm{d}G_X(x)-F_{Y|X}(y|X=x)F_X(x))\\&=\int \mathbf{E}[\mathrm{RIF}(y;v,\ F_Y)|X=x]\mathrm{d}(G_X-F_X)(x),\end{aligned} \tag{24}$$

which captures the marginal effect of a change in the distribution of covariates. By equations (22) and (24), the unconditional partial effect of the change of covariates from X to $X+t$ is expressed as the average derivative of the RIF with respect to covariates:

$$\alpha(v,\ F_Y)\equiv\lim_{t\downarrow 0}\frac{v(F^*_{Y,t})-v(F_Y)}{t}$$
$$=\int\frac{\partial\mathbf{E}[\mathrm{RIF}(y;v,\ F_Y)|X=x]}{\partial x}\mathrm{d}F_X(x), \tag{25}$$

where $v(F^*_{Y,t})=\int F_{Y|X}(y\,|\,X=x)\mathrm{d}F_X(x-t)$.

The fact of IF and RIF which we reviewed in this section can be applied to the unconditional quantile as following Firpo et al. (2009b). Our interest is the τ-th unconditional quantile for $v(\cdot)$ i.e. $v(F_Y)=q_\tau$. For the IF in equation (18),

$$\mathrm{IF}(y\ ;\ q_\tau)=\frac{\tau-\mathbf{1}\{y\leq q_\tau\}}{f_Y(q_\tau)}$$
$$=\begin{cases}\dfrac{\tau-1}{f_Y(q_\tau)} & \text{if } y\leq q_\tau\\[2ex] \dfrac{\tau}{f_Y(q_\tau)} & \text{otherwise,}\end{cases}$$

where $f_Y(q_\tau)$ is the value of density function of Y evaluated at q_τ. Furthermore for the RIF in equation (20),

$$\mathrm{RIF}(y\ ;\ q_\tau)=q_\tau+\mathrm{IF}(y\ ;\ q_\tau)$$
$$=q_\tau+\frac{\tau-\mathbf{1}\{y\leq q_\tau\}}{f_Y(q_\tau)} \tag{26}$$
$$=\frac{\mathbf{1}\{y>q_\tau\}}{f_Y(q_\tau)}+q_\tau-\frac{1-\tau}{f_Y(q_\tau)}. \tag{27}$$

Our main interest of UQR is modeling the conditional expectation of RIF in the integral of equation (25), i.e. $\mathbf{E}[\mathrm{RIF}(y\ ;\ q_\tau)\,|\,X=x]=m_\tau(X)$. From the representation of the RIF in (27), we have the probability model as follows:

$$\mathbf{E}[\mathrm{RIF}(y\ ;\ q_\tau)|X=x]=\frac{1}{f_Y(q_\tau)}P[Y>q_\tau|X=x]+q_\tau-\frac{1-\tau}{f_Y(q_\tau)}. \tag{28}$$

The estimation issues of the parameters in the equation (28) are covered in the next section.

Firpo et al. (2009b) derive the unconditional quantile partial effect (UQPE) which applies the unconditional quantile to the unconditional partial effect in equation (25)

$$UQPE(\tau)\equiv\alpha(q_\tau)=\int\frac{\partial\mathbf{E}[\mathrm{RIF}(y\ ;\ q_\tau)|X=x]}{\partial x}\mathrm{d}F_X(x)$$
$$=\int\frac{\partial}{\partial x}\left[\frac{1}{f_Y(q_\tau)}P[Y>q_\tau|X=x]+q_\tau-\frac{1-\tau}{f_Y(q_\tau)}\right]\mathrm{d}F_X(x)$$
$$=\frac{1}{f_Y(q_\tau)}\int\frac{\partial}{\partial x}P[Y>q_\tau|X=x]\mathrm{d}F_X(x). \tag{29}$$

The UQPE in equation (29) indicates that the effect of a small change of distribution of covariates on Y equals to the impact on the probability of Y is more than q_τ times the scaling factor $1/f_Y(q_\tau)$.

Previous discussion to equation (29) does not refer to any specific functional form like the structural model

$$Y=h(X,\ \varepsilon), \tag{30}$$

where ε be the disturbance term and unknown mapping $h(\cdot)$. In the real analysis, we often assume the specific structure of model of the conditional mean, $\mathbf{E}[\mathrm{RIF}(y\ ;\ q_\tau)\,|\,X=x]=m_\tau(X)$.

Consider the simplest case, $Y=h(X,\ \varepsilon)=X'\beta+\varepsilon$, linear and additively separable case. Under the assumption of the exogeneity of covariates, we have the UQPE

$$\begin{aligned}
UQPE(\tau)=\alpha(q_\tau)&=\frac{1}{f_Y(q_\tau)}\int\frac{\partial}{\partial x}P[Y>q_\tau|X=x]\mathrm{d}F_X(x)\\
&=\frac{1}{f_Y(q_\tau)}\int\frac{\partial}{\partial x}P[\varepsilon>q_\tau-x'\beta]\mathrm{d}F_X(x)\\
&=\frac{1}{f_Y(q_\tau)}\int\frac{\partial}{\partial x}[1-F_\varepsilon(q_\tau-x'\beta)]\mathrm{d}F_X(x)\\
&=\frac{1}{f_Y(q_\tau)}\int[-\beta(-f_\varepsilon(q_\tau-x'\beta)]\mathrm{d}F_X(x)\\
&=\frac{\beta}{f_Y(q_\tau)}\mathbf{E}[f_{Y|X}(q_\tau|X=x)]\\
&=\frac{\beta}{f_Y(q_\tau)}\int f_{Y|X}(q_\tau|X=x)\mathrm{d}F_X(x)\\
&=\frac{\beta}{f_Y(q_\tau)}f_Y(q_\tau)\\
&=\beta,
\end{aligned}\tag{31}$$

that is, the effect of the marginal change of covariates equals β in the linearly additively separable structure.

In the simplest case, the connection between the UQPE and the parameter in the structural form is easily drawn in equation (31). In the general case, fortunately, Firpo et al. (2009b) show that the useful link between the structural form and UQPE can be derived. In Proposition 1 in Firpo et al. (2009b), we assume that $Y=h(X,\ \varepsilon)$ is strictly monotonic in ε and that the covariates and the disturbance term are independent. The UQPE is thus defined as

$$UQPE(\tau)=\alpha(q_\tau)=\mathbf{E}\left[\frac{f_{Y|X}(q_\tau|X=x)}{f_Y(q_\tau)}\ \frac{\partial h(X,\ h^{-1}(x,\ q_\tau))}{\partial x}\right], \tag{32}$$

which is also written as

$$\begin{aligned}
UQPE(\tau)=\alpha(q_\tau)&=\mathbf{E}\left[\frac{f_{Y|X}(q_\tau|X=x)}{f_Y(q_\tau)}\ \frac{\partial Q_{F_{Y|X}(Y|X=x)}[h(X,\ \varepsilon)|X=x]}{\partial x}\right]\\
&\equiv\mathbf{E}\left[\frac{f_{Y|X}(q_\tau|X=x)}{f_Y(q_\tau)}CQPE(F_{Y|X}(q_\tau|X=x),\ \ X)\right],
\end{aligned}\tag{33}$$

where

$$CQPE(\tau,\ X)=\lim_{t\downarrow 0}\frac{Q_\tau[h(X+t,\ \varepsilon)|X=x]-Q_\tau[Y|X=x]}{t}$$
$$=\frac{\partial Q_\tau[h(X,\ \varepsilon)|X=x]}{\partial x} \tag{34}$$

and $Q_\tau[Y|X=x]$ is the τ-th quantile of Y conditional on X. The proofs of equations (33) and (34) are derived in the appendix of Firpo et al. (2009b). Equation (33) describes that the UQPE corresponds to a weighted average of the CQPE (conditional quantile partial effect) in equation (34), by the analogy with the UQPE, measured at $F_{Y|X}(q_\tau|X=x)$ which indicates the case that the unconditional quantile q_τ falls in the conditional distribution of response $F_{Y|X}$.

3.2 Inference of UQR

RIF-regression would be achieved by the estimation of the $UQPE(\tau)$ reviewed in Section 3.1. This section discusses the several method developed in Firpo et al. (2009b).

3.2.1 Probability Response Model

As we reviewed in Section 3.1, one should estimate the parameter in the left-hand side of the conditional mean of RIF in equation (28) to obtain the effect of the covariates on the unconditional quantile of the response. In this sense, our model is written as a probability response model

$$m_\tau(x)\equiv\mathbf{E}[\mathrm{RIF}(Y\ ;\ q_\tau)|X=x]=\frac{1}{f_Y(q_\tau)}P[Y>q_\tau|X=x]+q_\tau-\frac{1-\tau}{f_Y(q_\tau)}, \tag{35}$$

and $UQPE(\tau)$ is also written as

$$UQPE(\tau)\equiv\int\frac{\partial m_\tau(x)}{\partial x}\mathrm{d}F_X(x)$$
$$=\int\frac{\partial}{\partial x}\left\{\frac{1}{f_Y(q_\tau)}P[Y>q_\tau|X=x]+q_\tau-\frac{1-\tau}{f_Y(q_\tau)}\right\}\mathrm{d}F_X(x)$$
$$=\frac{1}{f_Y(q_\tau)}\int\frac{\partial}{\partial x}P[Y>q_\tau|X=x]\mathrm{d}F_X(x)$$
$$=\frac{1}{f_Y(q_\tau)}\mathbf{E}\left[\frac{\partial}{\partial x}P[Y>q_\tau|X=x]\right],$$

where the right hand side is the average derivative of the probability that Y is greater than the τ-th quantile of Y conditional on the covariates X.

The remainder part of this section discusses three methods to estimate the $UQPE(\tau)$ proposed by Firpo et al. (2009b). All of them require the estimation of unknown quantities $\hat{q}_\tau$ and $\hat{f}_Y(\hat{q}_\tau)$ in the feasible version of the RIF

$$\widehat{\mathrm{RIF}}(y\ ;\ \hat{q}_\tau)=\hat{q}_\tau+\widehat{\mathrm{IF}}(y\ ;\ \hat{q}_\tau)$$
$$=\hat{q}_\tau+\frac{\tau-\mathbf{1}\{y\leq\hat{q}_\tau\}}{\hat{f}_Y(\hat{q}_\tau)}$$

$$=\frac{\mathbf{1}\{y>\hat{q}_\tau\}}{\hat{f}_Y(\hat{q}_\tau)}+\hat{q}_\tau-\frac{1-\tau}{\hat{f}_Y(\hat{q}_\tau)}. \tag{36}$$

For the $\hat{q}_\tau$ in equation (36), it seems useful to employ the estimator of q_τ by the minimization problem presented by Koenker and Bassett (1978)

$$\hat{q}_\tau=\arg\min_{q_\tau}\sum_{i=1}^{n}\rho_\tau(y_i-q_\tau)$$
$$=\arg\min_{q_\tau}\sum_{i=1}^{n}(\tau-\mathbf{1}\{y_i-q_\tau<0\})(y_i-q_\tau),$$

while for the $\hat{f}_Y(\hat{q}_\tau)$ in equation (36), Firpo et al. (2009a) introduce the kernel density estimator of Rosenblatt (1956)

$$\hat{f}_Y(\hat{q}_\tau)=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{h}\kappa_Y\left(\frac{Y_i-\hat{q}_\tau}{h}\right), \tag{37}$$

where h is a positive scalar bandwidth and $\kappa_Y(\cdot)$ is a kernel function which is symmetric real-valued function around zero.

3.2.2 RIF-Logit Estimation

The RIF-Logit estimation would be achieved by directly exploiting the expression of the conditional mean model of RIF probability response model in equation (35). Firpo et al. (2009b) describe the logistic model for the probability part of equation (35) as

$$P[Y>q_\tau|X=x]=P[\mathbf{1}\{Y>q_\tau\}=1|X=x]$$
$$=\Lambda(X'\beta(\tau)_{logit}),$$

where $\Lambda(z)=\frac{1}{\exp(-z)}$ is the cdf of a logistic distribution and $\beta(\tau)_{logit}$ is coefficients for a τ-th quantile of a response variable. One could estimate the parameter $\beta(\tau)_{logit}$ using maximum likelihood as following estimator

$$\hat{\beta}(\tau)_{logit}=\arg\max_{\beta(\tau)_{logit}}\sum_{i=1}^{N}[\mathbf{1}\{Y>\hat{q}_\tau\}X_i'\beta(\tau)_{logit}+\log(1-\Lambda(X_i'\beta(\tau)_{logit})].$$

Since equation (32), the estimator of $UQPE(\tau)$ of RIF-Logit is derived as

$$\widehat{UQPE}(\tau)_{logit}=\frac{\hat{\beta}(\tau)_{logit}}{\hat{f}_Y(\hat{q}_\tau)}\frac{1}{N}\sum_{i=1}^{N}\Lambda(x_i'\hat{\beta}(\tau)_{logit})(1-\Lambda(x_i'\hat{\beta}(\tau)_{logit}))$$

allowing heterogenous marginal effects. Firpo et al. (2009a) show the asymptotic distribution of $\widehat{UQPE}(\tau)_{logit}$ under the condition of no-endogeneity of covariates as follows

$$\sqrt{Nh}(\widehat{UQPE}(\tau)_{logit}-UQPE(\tau)_{logit}\xrightarrow{d}\mathcal{N}(0,\ V_{logit}(q_\tau,\ \kappa)),$$

where h is a bandwidth of kernel density estimator and $V_{logit}(q_\tau,\ \kappa)$ is the asymptotic variance discussed in Firpo et al. (2009a) in detail.

3.2.3 RIF-Nonparametric Estimation

The RIF-Nonparametric (RIF-NP) estimation does not assume the specific form in the probability $P[Y>q_\tau|X=x]$. Firpo et al. (2009b) present the estimation method of RIF-NP by using a polynomial series (Newey, 1994) for the probability. One could estimate the parameter vector $\beta(\tau)_{NP}$ by

$$\hat{\beta}(\tau)_{NP}=\frac{1}{N}\sum_{i=1}^{N}\frac{\mathrm{d}R_K(X_i)}{\mathrm{d}X}\hat{\lambda}_K(q_\tau),$$

where

$$\hat{\lambda}_K(q_\tau)=\left(\sum_{i=1}^{N}R_K(X_i)R_K(X_i)'\right)^{-1}\left(\sum_{i=1}^{N}[R_K(X_i)\mathbf{1}\{Y_i>q_\tau\}]\right),$$

and $R_\kappa(X_i)$ is a K-dimensional vector of polynomials of X. Firpo et al. (2009a) derive nonpara- metric estimator of $UQPE(\tau)$ of RIF-NP as

$$\widehat{UQPE}(\tau)_{NP}=\frac{\hat{\beta}(\tau)_{NP}}{\hat{f}_Y(\hat{q}_\tau)} \tag{38}$$

and its asymptotic distribution under the exogeneity condition of X as

$$\sqrt{Nh}\,(\widehat{UQPE}(\tau)_{NP}-UQPE(\tau)_{NP}\xrightarrow{d}\mathcal{N}(0,\ \ V_{NP}(q_\tau,\ \ \kappa)),$$

where h is a bandwidth of kernel density estimator and $V_{NP}(q_{\tau,}\ \kappa)$ is the asymptotic variance noted in Firpo et al. (2009a).

3.2.4 RIF-OLS Estimation

In the case that the probability $P[Y>q_\tau|X=x]$ is linear in covariates, i.e. the RIF is linear in X, RIF-OLS presents consistent estimates of the average marginal effect of covariates. One can obtain the parameter by the following estimator using OLS

$$\begin{aligned}\hat{\beta}(\tau)_{OLS}&=\arg\min_{\beta(\tau)_{OLS}}\sum_{i=1}^{N}[\widehat{\mathrm{RIF}}(Y_i\,;\,\hat{q}_\tau)-X_i'\beta(\tau)_{OLS}]\\&=\left(\sum_{i=1}^{N}X_iX_i'\right)^{-1}\sum_{i=1}^{N}X_i\,\widehat{\mathrm{RIF}}(Y_i\,;\,\hat{q}_\tau).\end{aligned}$$

Firpo et al. (2009a) discuss the estimator of $UQPE(\tau)_{OLS}$ and its asymptotic distribution:

$$\widehat{UQPE}(\tau)_{OLS}=\hat{\beta}(\tau)_{OLS}$$

and

$$\sqrt{Nh}\,(\widehat{UQPE}(\tau)_{OLS}-UQPE(\tau)_{OLS})\xrightarrow{d}\mathcal{N}(0,\ \ V_{OLS}(q_\tau,\ \ \kappa))$$

under the condition of no-endogeneity of covariates. Firpo et al. (2009a) discuss the detail of asymptotic variance $V_{OLS}(q_\tau,\ \kappa)$. The estimators suggest that practitioners could perform the two-step procedure to estimate the UQPE. The first step is to estimate the feasible version of RIF using the kernel density estimation, while the second step is the OLS estimation for $\beta(\tau)_{OLS}$. Even though the RIF-OLS strongly assumes the linearity in the probability, most of the applications of UQR employ it to estimate the effect of covariates on the specific part of the unconditional distribution of the response variable because one can straightforwardly estimate the $UQPE(\tau)$ by the two-step procedure.

For all of three estimation methods, it is crucial that we should identify the density of Y by the kernel density estimation regardless of the existence of the uncertainty. Firpo et al. (2009b) propose the usage of the Gaussian kernel and suggest other consistent estimators of the density could be used as well. Instead, Lubrano and Ndoye (2014) present a Bayesian approach to estimate the density function of the response for the RIF-OLS estimation in RIF-regression.

3.3 Additional Models on UQR

As we reviewed in the previous part in this section, UQR can let us assume the flexible form of structural model especially when we implement the RIF-NP estimation of the average marginal effect. For instance, Ghosh (2013) proposes Box-Cox Unconditional Quantile Regression Model by transforming equation (30) to

$$Y = h^{\dagger}(X'\beta(\tau),\ \lambda(\tau),\ \varepsilon), \tag{39}$$

where $h^{\dagger}$ is strictly monotone in $X'\beta(\tau)$, $\lambda(\tau)$ is the shape parameter.

3.4 Endogeneity Issues in UQR

In all of three estimation methods, Firpo et al. (2009b) assume that there is no endogeneity between the covariates and disturbance term, i. e. $X \perp\!\!\!\perp \varepsilon$. It suggests that the presence of endogenous regressors has UQR fail to propose the consistent estimate of the UQPE. As well as we discussed in Section 2.4, there are several tools to investigate the causal effects dealing with the endogeneity in covariates.

Frölich and Melly (2013) discuss how to estimate the causal effect of endogeneous regressors in UQR. They proposes unconditional IV quantile treatment effect approach which extend the framework of LATE (Imbens and Angrist, 1994) even though it only deals with cases where both endogeneous treatment and IV are dichotomous. The idea is based on the framework developed by Imbens and Angrist (1994) and Abadie et al. (2002). Frölich and Melly (2013) estimate effect of treatment across the distribution for compliers by the estimator as follows. Let $D_i = \{0,\ 1\}$ be the binary treatment, $Z_i = \{0,\ 1\}$ be the binary instrument variable, and $\Delta(\tau) = q_{\tau}^{Y1} - q_{\tau}^{Y0}$ be the distributional impact of a treatment where q_{τ}^{Yd} is the τ-th quantile of Y^d. Thus, the estimator can be written as

$$(\hat{\alpha}(\tau),\ \hat{\Delta}(\tau)) = \arg\min_{\alpha(\tau),\ \Delta(\tau)} \sum_{i=1}^{N} \Omega_i \rho_\tau (Y_i - \alpha(\tau) - D_i \Delta(\tau))$$

where

$$\Omega_i = \frac{Z_i - P[Z_i=1|X_i=x_i]}{P[Z_i=1|X_i=x_i](1-P[Z_i=1|X_i=x_i])}(2D_i-1),$$

is the modified inverse propensity score weighting.

On the other hand, Rothe (2010) and Imbens and Newey (2009) develop the methodology to estimate the unconditional effects of endogenous regressors in the case of continuous treatment variable by identifying the unconditional partial effect using the control function approach (Blundell and Powell, 2003; Chernozhukov et al., 2015; Wooldridge, 2015). Imbens and Newey (2009) assume the endogenous subvector X_1 of covariates X can be defined as

$$X_1 = h_1(Z,\ \eta),$$

where the covariates vector $X=(X_1',\ X_2')'$ is in the structural model $Y=h(X, \varepsilon)$, exogeneous vector $Z=(Z_1',\ X_2')$ consists of the instrument Z_1 and the exogenous counterpart X_2, η is disturbance term, and $h_1(\cdot)$ is strictly monotonic with respect to η. Under the orthogonality assumption of the instruments to error terms ε and η, Imbens and Newey (2009) show that there exists a control variable V such that

$$V = F_{X_1|Z}(X_1,\ Z) = F_\eta(\eta),$$

where X and ε are independent conditional on V. The control variable V is a latent unobserved regressor that accounts for the possible endogeneity of X_1 which is needed to be estimated in a first stage, i.e. when V is added in the model, X_1 is no longer endogeneous. Using the control function approach, Ghosh (2016) presents the control variable unconditional quantile partial effect $CVUQPE(\tau)$

$$\begin{aligned}
CVUQPE(\tau) &= \mathbf{E}\left[\frac{\partial[\mathrm{RIF}(Y\ ;\ q_\tau)|X=x,\ V=v]}{\partial x}\right] \\
&= \int \frac{\partial \mathbf{E}[\mathrm{RIF}(y\ ;\ q_\tau)|X=x,\ V=v]}{\partial x} \mathrm{d}F_{X|V}(x) \\
&= \int \frac{\partial}{\partial x}\left[\frac{1}{f_Y(q_\tau)} P[Y>q_\tau|X=x,\ V=v] + q_\tau - \frac{1-\tau}{f_Y(q_\tau)}\right] \mathrm{d}F_{X|V}(x) \\
&= \frac{1}{f_Y(q_\tau)} \int \frac{\partial}{\partial x} P[Y>q_\tau|X=x,\ V=v]\mathrm{d}F_{X|V}(x) \\
&= \frac{1}{f_Y(q_\tau)} \int -\mathbf{E}\left[\frac{\partial}{\partial X} F_{Y|X,V}(q_\tau,\ x,\ v)\right]\mathrm{d}F_{X|V}(x) \\
&= -\frac{1}{\frac{\partial}{\partial q_\tau} F_Y(q_\tau)} \mathbf{E}\left[\frac{\partial}{\partial X} F_{Y|X,V}(q_\tau,\ x,\ v)\right].
\end{aligned} \tag{40}$$

Its estimator and asymptotic properties are discussed by Ghosh (2016) in detail. This identification is not specific for a model with endogeneous regressors only; it can be also applied to exogenous regressors when there is no need for a control variable. Equation (40) thus results a generalization of the $UQPE(\tau)$ in Firpo et al. (2009b) in the presence of endogeneous regressors.

As well as in the linear regression with instrumental variable method (Sawa, 1969; Hall et al., 1996), the problem of weak-instruments, which is caused by the case that the instruments are weakly correlated with the endogenous regressors, is not trivial in the context of UQR. Katafuchi (2019) confirms highly biased estimates of covariates in the case of a weak-relevance instrument by a simulation experiment of $CVUQPE(\tau)$.

3.5 Interpretation of UQR

As we discussed in Section 2.5, coefficients estimated by CQR is not able to explain the causal effect of independent variable on the unconditional quantile of the distribution of the dependent variable when we incorporate multiple covariates into the model while coefficients estimate by UQR can do.

Similarly to the law of iterated expectation applied to the conditional mean model of OLS in equation (16), we consider the linear RIF-regression model as

$$\begin{aligned}
&\mathbf{E}[\mathrm{RIF}(y\,;\,q_\tau)|X=x]=m_\tau(X)=X\beta(\tau)\\
\Rightarrow&\mathbf{E}[\mathbf{E}[\mathrm{RIF}(y\,;\,q_\tau)|X=x]]=\mathbf{E}[X\beta(\tau)]\\
\Leftrightarrow&\int\mathbf{E}[\mathrm{RIF}(y\,;\,q_\tau)|X=x]f_X(x)\mathrm{d}x=\mathbf{E}[X]\beta(\tau).
\end{aligned}$$

The left-hand side follows

$$\begin{aligned}
\int\mathbf{E}[\mathrm{RIF}(y\,;\,q_\tau)|X=x]f_X(x)\mathrm{d}x&=\iint\mathrm{RIF}(y\,;\,q_\tau)f_{Y|X=x}(y|X=x)f_X(x)\mathrm{d}y\mathrm{d}x\\
&=\int\mathrm{RIF}(y\,;\,q_\tau)\left[\int f_{Y|X=x}(y|X=x)f_X(x)\mathrm{d}x\right]\mathrm{d}y\\
&=\int\mathrm{RIF}(y\,;\,q_\tau)f_Y(y)\mathrm{d}y\\
&=\mathbf{E}[\mathrm{RIF}(y\,;\,q_\tau)]\\
&=q_\tau,
\end{aligned}$$

that is, estimates of coefficient by UQR consistently explain the effect of increasing the mean value of X on the unconditional quantile of Y.

To illustrate as in Section 2.5, suppose we run RIF-regression and obtain the coefficient of educational attainment at the first quartile and the third quartile; which are 0.5 and 0.25. It thus means that, regardless of what covariates we additionally incorporate into the model of UQR, the wage inequality caused by university graduation in the lower-wage group in whole distribution of wage is larger than that in its higher wage counterpart.

3.6 Application of UQR

UQR or RIF-regression has lately received significant attention from researchers who would like to analyse the covariates effect on the unconditional quantile of the dependent variable, not on the conditional counterpart by CQR or not on the unconditional mean by OLS.

In the seminal work, Firpo et al. (2009b) estimate the impact of the declining union on the logarithm of U.S. wage inequality using individual data of the Current Population Survey. Reporting the UQR and the CQR, they emphasise the difference between the UQPE and the CQPE. Further-more, they confirm the robustness of the RIF-OLS estimation by showing the result of RIF-Logit and RIF-NP. Similarly, Ghosh (2014) presents the unionisation effect on the wage inequality in the U.S. using the Box-Cox UQR model in Ghosh (2013) and suggests that Box-Cox UQR is more applicable than UQR in the case that the distribution of wage has a fat-tail.

As well as in CQR, there is a growing literature on the application of RIF-regression to labour economic issues to investigate the heterogeneous effect of various type of interest on the unconditional distribution of income. Chi and Li (2008) present the increase of gender earning inequality of the Chinese urban area in 1987-2004 is greater at the lower quantiles. Schmutte (2015) clarifies the greater effect of the network quality of the block, which is measured by the average wage premium across all employed workers living in the same block, on the higher quantile of the unconditional wage premium distribution.

RIF-regression is becoming popular not only in labour economics but also in other fields of microeconometrics. In education economics, for example, Webber and Ehrenberg (2010) find that the service-expenditure of a student has higher marginal effects on the lower part of the unconditional distribution of entrance test scores. Alejo et al. (2014) suggest the heterogeneity in the effect of education on the earnings, which is explained especially by the greater effect on higher uncondi- tional quantile of the wage distribution. In the field of health economics, Pryce et al. (2018) show the effect of price increase of alcohol is smaller on the higher unconditional quantile of the expendi- ture distribution on alcohol, i.e., the heavier drinkers respond less to price. Jolliffe (2011) suggests that the effect of income on the unconditional distribution of the body mass index is magnified at the higher quantile of the BMI. In regional economics, Katafuchi and Chen (2018) investigate the distributional effects of land prices on the unemployment rate in Japan by using a prefecture-level panel dataset and find the significant positive relationship between the land price and the left part of the unconditional distribution of the unemployment rate.

The UQR approach is becoming popular not only in the field of applied econometrics. For instance, Grossman (2015) shows that the empowerment zone program, which is a federal program aimed to improve infrastructure, has a significant and greater effect on the lower quantiles of the birth weight than the higher quantiles. Gaskin et al. (2016) find that the patients' racial composition does not have the significant effect on the upper part of the unconditional distribution of the quality of surgical and pneumonia process of care measure. Arora (2016) indicates that parental dementia diagnosis has a significant and negative effect on the quantile above the median of the unconditional distribution of wealth accumulation among unmarried adult children.

4 Comparison between CQR and UQR

The difference between CQR and UQR could be categorised into the two parts; the first one, as reviewed in Section 2.5 and Section 3.5, is how to interpret estimated coefficients. The second one is the magnitude of coefficients.

Coefficients estimated by CQR, which depends on the model of conditional quantile function, can be difficult to interpret in many applied settings since these should be interpreted as effects on quantiles of the conditional distribution of the response variable given the covariates. It thus means that we could not interpret coefficients estimated by CQR as effects estimated by the conventional linear regression, OLS. Of course, CQR could be applied in empirical studies which aim to examine the heterogeneity in effects according to the level of outcome variable within groups defined by other covariates like Chamberlain (1994). The interpretation of coefficients of CQR is, however, more complex when we incorporate more covariates into the model of conditional quantile function. Therefore, many researchers devote their effort to develop the method to measure effects on the unconditional quantile of outcome variable (Firpo et al., 2009b; Frölich and Melly, 2013; Chernozhukov et al., 2013). UQR by Firpo et al. (2009b), which is the most straightforward approach to estimate effects on the part of the unconditional distribution, can model effects of covariates on quantiles of response variable unconditional on covariates. Coefficients estimated by UQR could be interpreted as effects estimated by OLS, regardless how many additional covariates practitioner incorporate into the model.

Even though whether we should use CQR or UQR depends on the main research question and objective, several papers compare coefficients of CQR and UQR for the same dataset. Borah and Basu (2009) conduct a simulation experiment for various situations such as single covariate and multiple covariates and confirm that estimated coefficients of CQR and UQR are same only in the case of single covariate of the model without any additional covariates. Firpo et al. (2009b) estimate the $UQPE(\tau)$ of the unionization on the U.S. wage distribution in same setting as Chamberlain (1994) and conclude that the difference between the $UQPE(\tau)$ and the $CQPE(\tau)$ is large for the union effect since the conditional and unconditional distribution of wage in the U.S are different. Killewald and Bearak (2014) show the motherhood penalty on log wages by CQR and UQR and report that the motherhood penalty decrease monotonically according to the log wage distribution conditional on covariates while it does not decrease monotonically in the case of coefficients by RIF- regression. Katafuchi and Chen (2018) investigate the effects of land price on the unemployment rate and confirm the heterogeneous and significant effects of land price on the unemployment distribution conditional on covariates while the land price does not have the significant effect on the higher part of the unconditional distribution of unemployment rate.

5 Conclusion

This review provides an overview of CQR and UQR consisting of the technical introduction of models, review of various applications of them and the comparison of their estimation methods.

We illustrate that many researchers have developed the CQR to achieve the goal of analysing effects not only on central tendencies, especially in the case of incorporating endogenous regressors into the model. Whereas the CQR was developed with the aim of explaining the effect of a covariate on quantiles of outcome to complete the regression picture (Koenker and Hallock, 2001) beyond the conditional-mean framework of conventional linear regression, practitioners have to carefully interpret coefficients estimated by CQR since these could be explained as effects on quantiles of the distribution within a group defined by other covariates.

We also demonstrate that the UQR can be the more generalised alternative to the CQR since coefficients estimated by RIF-regression can be interpreted as effects on unconditional quantiles of outcome distribution such as the effect estimated by OLS. We note that UQR has the much more flexible specification in the structural model than CQR and there is an increasing number of papers for UQR developing the approach to consistently estimate the coefficients in the case that there is an endogeneity of covariates. Thanks to the properties of UQR, especially its ease of interpretation, there is an emerging number of researches which apply the RIF-regression to the empirical analysis as we have reviewed.

In Section 4, we compare CQR and UQR from the two viewpoints: their interpretation and magnitude of coefficients. The articles we reviewed in the comparison note that whether UQR or CQR should be applied depends on the research question. We strongly hope the readers could be motivated to apply these approaches, but do not present the misconception, which pertains whether the effect on conditional or unconditional quantiles on covariates, to other readers.

Finally, there are several unanswered questions regarding these approaches. Can we construct the model of RIF-regression according to the variable selection procedure like in the conventional linear regression or the penalised CQR? How can we evaluate the prediction accuracy of UQR? How do we choose the best instrumental variable in the framework of UQR when there are several useless instrumental variables? These issues should be addressed in future work.

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