A study on holomorphic mappings from Kähler manifolds

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Abstract : We study holomorphic mappings from Kähler manifolds relating to the hyperbolicity or parabolicity of the domain manifolds. First, we study the existence of bounded harmonic functions and bounded holomorphic functions on Kähler Cartan-Hadamard manifolds. Next, we survey the Schwarz lemma on Kähler manifolds, and raise some problems to estimate the gradient of holomorphic mappings. In the last, we prove 2 types of the general Schwarz lemma on a Kähler CH manifold as a partial answer to the above problem.

Summary

In this thesis, we study holomorphic mappings from Kähler manifolds relating to the existence of bounded holomorphic functions and to the hyperbolicity or parabolicity of the domain manifolds. All over this work, under the theory of several complex variables, evaluations of the Hessian and Laplacian of the distance function play important roles in the proof of our results.

In Chapter 1, we study the existence of bounded harmonic functions and bounded holomorphic functions on Kähler Cartan-Hadamard(CH) manifolds. In the first, we solve the Dirichlet problem at infinity on a Riemann manifold whose sectional curvature is bounded above by a quadratic decaying function and below by a negative constant. Next, we apply the above solution to Complex Analysis on a Kähler CH manifold, whose metric restricted to every geodesic sphere is conformal to that of the standard sphere. And we show that there exists a holomorphic extension from the sphere at infinity and it coincides with the solution of the Dirichlet problem at infinity, if the problem is solvable. So we see that such a manifold admits many bounded holomorphic functions. Moreover we show that a Kähler CH manifold of the same type whose sectional curvature is bounded above by a quadratic decaying function is biholomophic to a bounded strictly pseudoconvex domain in $\mathbf{C}^n$.

On the other hand, it is well known that the Schwarz-type lemma is closely related to the Liouville's theorem which shows the nonexistence of bounded holomorphic functions. So we survey the Schwarz lemma on Kähler manifolds tracing back to the classical Schwarz-Pick lemma in Chapter 2 . And we raise some problems relating to the Schwarz lemma, that is, let $F(r)$ be a negative monotone-decreasing or monotone-increasing function of r , the distance from a point, and if the Ricci curvature of a

Kähler CH manifold is bounded below by $F(r)$, does the Schwarz type lemma hold up to $F(r)$?

As a partial answer to the above problem we give 2 types of the general Schwarz lemma. One is the general schwarz lemma on a Kähler CH manifold whose Ricci curvature is bounded below by a quadratic decaying function of r , and the other is by a quadratic growing function. Using these results, we give some formula to estimate a growth of a bounded holomorphic function.

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Takashi YASUOKA

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Contents

Chapter 1

The Dirichlet Problem at Infinity and Complex Analysis on Cartan-Hadamard Manifolds

1.1 Introduction

In this paper we shall study hyperbolicity of Cartan-Hadamard manifolds.

In Section 1.1 we shall define and solve the Dirichlet problem at infinity for Laplacian Δ, which gives a partial extension of the result of Anderson [1] and Sullivan [15] in Theorem 1.1(cf.[4]). In Section 1.2 we apply the solution of the Dirichlet problem at infinity to a complex analysis on a Kähler Cartan-Hadamard manifold whose metric restricted to every geodesic sphere is conformal to that of the standard sphere. It seems that the sphere at infinity of such a manifold admits a CR-structure. In fact we can define a CR-function at infinity on the sphere at infinity. We shall show in Theorem 1.2 that there exists a holomorphic extension from the sphere at infinity and it coincides with the solution of the Dirichlet problem at infinity, if the Dirichlet problem at infinity is solvable. So we see that such a manifold admits many bounded holomorphic functions. By the similar method we shall show in Theorem 1.3 that such a manifold is biholomorphic to a strictly pseudoconvex domain in $\mathbf{C}^n$, if the holomorphic sectional curvature $K_h(x)$ is less than $-1/(1+r(x)^2)$, where $r(x)$ is a distance function from a pole. Theorem 1.3 is a partial answer to a conjecture raised by Green and Wu[8].

1.2 Dirichlet problem at infinity

Let M be a Riemannian manifold of dimension n with metric g_{ij} . We denote by T_pM the tangent space at $p \in M$. For a C^2 function u , we define the Hessian D^2u of u at p by

$$D^2u(X,Y) = X(Yu) - (D_XY)u$$

for $X, Y \in T_pM$, where D_X is the covariant derivative. The Laplacian Δu of u is the trace of D^2u , which is expressed by

$$\Delta u = \Sigma_{ij} g^{-\frac{1}{2}} \frac{\partial}{\partial x_j}(g^{\frac{1}{2}} g^{ij} \frac{\partial u}{\partial x_i})$$

in a local coordinates $(x_1, \cdots, x_n)$, where $g = det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. By the definition, for an orthonormal basis $X_1, \cdots, X_n$ of T_pM we see that $\Delta u_p = \Sigma_i (D^2u)(X_i, X_i)$.

A C^2 function u on M is said to be *harmonic* if $\Delta u = 0$. u is *subharmonic* if $\Delta u \geq 0$, and u is *superharmonic* if $\Delta u \leq 0$. A continuous function u is subharmonic if it is everywhere a subsolution of the Dirichlet problem [7]. The maximum principle and the Harnack's principle are valid for harmonic functions globally on M [2,3].

Let M be a simply connected complete Riemannian manifold of nonpositive sectional curvature, M is called a *Cartan-Hadamard manifold.* By the well known theorem of Cartan-Hadamard, for any $p \in M$ exp: $T_pM \to M$ is a diffeomorphism. We can construct the boundary of M following Everlein and O'Neil [5].

DEFINITION. Two normal geodesic rays $\gamma_1(t), \gamma_2(t)(t > 0)$ in M are said to be *asymptotic* if there is a constant $c > 0$ such that dist $(\gamma_1(t), \gamma_2(t)) < c$ for all $t > 0$.

We see that the asymptotic relation is an equivalence relation.

DEFINITION. *Sphere at infinity* $S(\infty)$ is the set of asymptotic classes of geodesic rays in M .

Let $\overline{M} = M \cup S(\infty)$ and fix a point $o \in M$. For $v \in T_oM$ we define the *cone* around v of angle δ by

$$C(v,\delta) = \{x \in M : \prec_o (v, \dot{\gamma}_x(0)) < \delta\},$$

where $\gamma_x(t)$ is the normal geodesic rays through x starting from o, and $\prec_o$ denotes angle in T_pM. Let $T(v,\delta,r) = C(v,\delta) \setminus B_o(r)$ be the *truncated cone* of radious r, where $B_o(r)$ is the geodesic r-ball around o. The set of all $T(v,\delta,r)$, for all $v \in T_oM$, and $r > 0$, and $B_q(r)$, for all $q \in M$ and $r > 0$, defines a local basis of topology on $\overline{M}$ [5]. It is called the *cone topology*. The cone topology is independent of the choice of the origin $o \in M$. In this topology M is homeomorphic to a closed ball $\overline{B}$ in $\mathbf{R}^n$, and $S(\infty)$ is h meomorphic to the boundary ∂B.

Dirichlet problem at infinity. Given a continuous function f on $S(\infty)$, find $u \in C^0(\overline{M})$ satisfying $\Delta u = 0$ on M and $u = f$ on $S(\infty)$.

The maximum principle implies that if the Dirichlet problem at infinity is solvable, then there are many bounded harmonic functions on such a manifold. Anderson [1] and Sullivan [15] showed that the Dirichlet problem at infinity is solvable if the sectional curvature $K(x)$ satisfies $-a^2 \le K(x) \le -b^2$, where a and b are positive constants. Theorem 1.1 is a partial extension of the result of Anderson[1] and Sullivan[15], and the proof is based on Anderson-Sullibvan[2]. The second inequality of (1.1) in Theorem 1.1 is a little similar to the inequality: $curvature(x) < r(x)^{-2}$, in fact the condition: $curvature(x) < r(x)^{-2}$ implies several properties relating to hyperbolicity (cf.Greene-Wu[9]).

THEOREM 1.1. *Let M be a Cartan-Hadamard manifold and $K(x)$ be the sectional curvature at $x \in M$. Suppose relative to some $o \in M$,*

$$-a^2 \le K(x) \le -\frac{1}{1+r(x)^{2-\varepsilon}} \quad for \ x \in M \tag{1.1}$$

for two constants $a > 0$ and $2 > \varepsilon > 0$, then the Dirichlet problem at infinity is uniquely solvable, where $r(x) = \mathrm{dist}(o,x)$.

In the following of this section M always denotes a Cartan-Hadamard manifold with metric $g = (g_{ij})$, and $o \in M$ is fixed.

LEMMA 1.1. If the sectional curvature $K(x)$ satisfies

$$K(x) \leq -\frac{1}{1 + r(x)^{2-\varepsilon}} \quad for \ x \in M \tag{1.2}$$

for a constant $2 > \varepsilon > 0$, then for any two normal geodesic rays $\gamma_1(t), \gamma_2(t)$ starting from $o \in M$ with angle $\theta = \sphericalangle_o (\dot\gamma_1(0), \dot\gamma_2(0)) < \pi/4$, we have

$$\mathrm{dist}(\gamma_1(t), \gamma_2(t)) > 2t + 2(2 + t)^{1-\varepsilon/2}(\log\theta - 1). \tag{1.3}$$

Proof. For every integer m , we see that $K(x) < -1/(1+m)^{2-\varepsilon}$ on $B_o(m)$. Comparing (1.2) with the space of constant curvature $-1/(1+m)^{2-\varepsilon}$, by the Rauch's comparison theorem we obtain

$$\mathrm{dist}(\gamma_1(t), \gamma_2(t)) > 2t + 2(1 + m)^{1-\varepsilon/2}(\log\theta - 1) \quad for \ \ 0 < t < m.$$

Define the function $f(t)$ on $t \in [0, \infty)$ by

$$f(t) = 2t + 2(1 + m)^{1-\varepsilon/2}(\log\theta - 1), \quad if \ \ t \in [m - 1, m).$$

We get

$$\mathrm{dist}(\gamma_1(t), \gamma_2(t)) > f(t) \quad on \ \ t \in [0, \infty)$$

. On the other hand

$$f(t) \geq 2t + 2(2 + t)^{1-\varepsilon/2}(\log\theta - 1) \quad on \ \ t \in [0, \infty)$$

since $\theta < \pi/4$. Then we have (1.3) for all $t > 0$.

LEMMA 1.2. *If $K(x)$ satisfies (1.2) on M , then for any positive constant δ with $1 > \delta > 1 - \varepsilon/2$ there exist positive constants r_1 ,and C_1 such that*

$$\Delta \exp(-r(x)^{1-\delta}) < -C_1 \frac{\exp(-r^{1-\delta})}{r(x)^2} \tag{1.4}$$

on $M \setminus B_o(r_1)$.

Proof. If $K(x) \le -C^2$ for a positive constant C , then the Hessian comparison theorem of Greene-Wu [9] implies

$$D^2 r(x) \ge C \ \coth(Cr(x)) \cdot G$$

, where $G = g - dr \otimes dr$. By the same reason of the proof of Lemma 1.1, we have

$$D^2 r(x) \ge (1+m)^{\varepsilon/2-1} \cdot G$$

if $m - 1 \le r < m$. All of the above inequalities on each interval $[m, m+1)$ implies

$$D^2 r(x) \ge \frac{1}{(2+r(x))^{1-\varepsilon/2}} \cdot G \tag{1.5}$$

on M . Direct computations give

$$\Delta \exp(-r(x)^{1-\delta}) < (1-\delta)\exp(-r(x)^{1-\delta}) \frac{-\Delta r(x) + r(x)^{-\delta}}{r^{\delta},} \qquad for \quad r(x) > 1.$$

By (1.5) we have

$$\Delta \exp(-r(x)^{1-\delta}) < (1-\delta) \frac{\exp(-r(x)^{1-\delta})}{r(x)^{2\delta}} [1 - \frac{Cr(x)^{\delta}}{(2+r(x))^{1-\varepsilon/2}}].$$

Since $1 - \varepsilon/2 < \delta$ we obtain (1.4) for sufficiently large r_1 .

Let h be a continuous function on the geodesic unit sphere $S_o(1)$ in M with center at $o \in M$. We extend h radially along rays from o to a function h_o on $M \setminus o$ with boundary values h on $S(\infty)$. Let $\lambda : [0, \infty) \to [0,1]$ be a C^2 function satisfying

$$\lambda(t) = \begin{cases} 1, & t \in [0,1] \\ 0, & t \in [2, \infty). \end{cases}$$

We define a C^2 function $H(x)$ on M by

$$H(x) = \frac{\int_M \lambda(r(x,y)^2) h_o(y) dy}{\int_M \lambda((r(x,y)^2) dy}, \tag{1.6}$$

where $r(x,y) = \mathrm{dist}(x,y)$ and the integral is with respect to the volume form on M . We see that $H(x)$ is continuous on $\overline{M}$ and $H = h$ on $S(\infty)$. If we put $\lambda_1(t) = \lambda(t^2)$, we have $D^2\lambda(r^2) = \ddot{\lambda}_1 dr \otimes dr + \dot{\lambda}_1 D^2 r$. If $K(x)$ satisfies $0 > K(x) \geq -a^2$, then we have

$$0 < D^2 r_y(x) \leq a \cdot \coth(a r_y(x)) \cdot G$$

for any $x, y \in M$ by the Hessian comparison theorem [9], where $r_y(x) = r(x,y)$. Thus we obtain

$$-C_2 g \leq D^2(r_y(x)^2) \leq C_2 g, \qquad x, y \in M,$$

for a positive constant C_2 . We see that

$$\begin{aligned} \Delta H(x_0) &= \Delta[H - h_o(x_0)](x_0) \\ &= \Delta\left[\frac{\int_M \lambda(r(x_0,y)^2)(h_o(y) - h_o(x_0))dy}{\int_M \lambda(r(x_0,y)^2)dy}\right]. \end{aligned}$$

The curvature bounds imply that the volumes of $B_x(1)$ and $B_x(2)$ are bounded from below and above for any $x \in M$. Then we have the following lemma.

LEMMA 1.3. *If* $0 > K(x) \geq -a^2$ *on* M *, then we have*

$$|\, \Delta H(x) \,| < C_3 \sup_{y \in B_x(2)} |\, h_o(y) - h_o(x) \,| \qquad for \quad x \in M, \tag{1.7}$$

where C_3 *is a positive constant.*

Proof of the theorem. We identify $S(\infty)$ with the set of geodesic rays starting from o . We can approximate h of $C^0(S(\infty))$ by Lipshitz continuous functions on $S_o(1) \sim S(\infty)$. By the maximum principle and the Harnack's principle, if a sequence of harmonic functions $u_k \in C^0(\overline{M})$ converges uniformly on $S(\infty), u_k$ converges uniformly on $\overline{M}$ to a harmonic function $u \in C^0(\overline{M})$. Thus we may assume that h is Lipshits continuous on $S_o(1)$. We extend h radially on M . Define $H(x)$ by(1.6). From Lemma 1.3 for a positive constant C_4 we get

$$H(x) < C_4 \max_{y \in B_x(2)} \prec_o (x,y)$$

since H is Lipshitz continuous with respect to $\prec_o (x,y)$. By Lemma 1.1 we obtain

$$\max_{y\in B_x(2)} \prec_o (x,y) < \exp(5-(2+r(x))^{\varepsilon/2})$$

if $r(x)>2$. Then there exits a positive constant C_5 such that

$$|\,\Delta H(x)\,| < C_5 \exp(-(2+r)^{\varepsilon/2}), \quad r(x)>2. \tag{1.8}$$

Choose a constant δ with $1>\delta>1-\varepsilon/2,$ and for arbitraly positive constant C_6 we define function $F^+(x)$ and $F^-(x)$ on M by

$$F^+(x) = H(x) + C_6 \exp(-r(x)^{1-\delta}),$$
$$F^-(x) = H(x) - C_6 \exp(-r(x)^{1-\delta}).$$

From (1.4)and (1.8) we have

$$\Delta F^+(x) < C_5 \exp(-(2+r(x))^{\varepsilon/2}) - C_1 C_6 \frac{\exp(-r(x)^{1-\delta})}{r(x)^{2\delta}},$$
$$\Delta F^-(x) > -C_5 \exp(-(2+r(x))^{\varepsilon/2}) + C_1 C_6 \frac{\exp(-r(x)^{1-\delta})}{r(x)^{2\delta}}$$

on $x\in M\setminus B_o(r_1)$. If we fix a constant r_2 with $r_2>r_1$, F^+ and F^- is superharmonic and subharmonic respectively on $M\setminus B_o(r_2)$ since $\varepsilon/2>1-\delta$. Moreover we choose C_6 such that

$$\max_{x\in\overline{M}} H(x) - \min_{x\in\overline{M}} H(x) < C_6 \exp(-r_2^{1-\delta}) \tag{1.9}$$

Now we define $G^+(x)$ and $G^-(x)$ by

$$G^+(x) = \min\{\inf_{x\in B_o(r_2)} H(x) + C_6\exp(-r_2^{1-\delta}),\ F^+(x)\}$$
$$G^-(x) = \max\{\sup_{x\in B_o(r_2)} H(x) - C_6\exp(-r_2^{1-\delta}),\ F^-(x)\}$$

We have that $G^+(x)$ and $G^-(x)$ are continuous on $\overline{M}$ and constant on $B_o(r_2)$. Then $G^+(x)$ is superharmonic and $G^-(x)$ is subharmonic on $M.$ By (1.9) we can check $G^+(x)>G^-(x)$ on $M,$ moreover we can find a constant $r_3>r_2$ such that

$$C_6\exp(-r_2^{1-\delta}) - (\max_{x\in\overline{M}} H(x) - \min_{x\in\overline{M}} H(x)) > C_6 \exp(-r_3^{1-\delta}). \tag{1.10}$$

(1.10)implies

$$F^+(x) < \inf_{x\in B_o(r_2)} H(x) + C_6 \exp(-r_2^{1-\delta})$$
$$F^-(x) > \sup_{x\in B_o(r_2)} H(x) - C_6 \exp(-r_2^{1-\delta})$$

for $r \in \overline{M} \setminus B_o(r_3)$. The above inequalities mean $F^+(x) = G^+(x)$ and $F^-(x) = G^-(x)$ on $\overline{M} \setminus B_o(r_3)$. Hence $G^+(x) = G^-(x) = h(x)$ on $S(\infty)$. $G^+(x)$ and $G^-(x)$ are barrier functions to solve the Dirichlet problem at infinity by the Perron method. Consequently there is the Perron solution which is exactly the solution of the Dirichlet problem at infinity. The uniqueness follows from the maximum principle. This completes the proof.

Remark. Professor H. Wu informed the author that H. Wu and R. Schoen proved that *if* $-a \cdot r(x)^2 \leq K(x) \leq -\frac{b}{r(x)^2}$ $(b \geq 2)$, *then the Dirichlet problem at infinity is solvable.*

1.3 Complex analysis on Kähler Cartan-Hadamard manifold

Now we prove the existence of bounded holomorphic functions on Kähler Cartan-Hadamard manifold (in short *Kähler CH manifold*) M in a special class. For this purpose we will consider the Dirichlet problem at infinity for $\overline{\partial}$ like that for Δ . If the sphere at infinity $S(\infty)$ should admit a CR-structure and M should be hyperbolic in a sense, there would be a holomorphic extension to M . However, in general $S(\infty)$ admits no differentiable structure. We shall define a CR-function on $S(\infty)$ for a special class of Kähler CH manifolds, and extend to a holomorphic function on M . The boundedness of the extended function follows from the absolute maximum principle.

Furthermore we shall show in Theorem 1.3 that a manifold in the special class is biholomorphic to a bounded domain in $\mathbf{C}^n$ under some curvature condition.

Let M be a complex manifold of dimension n , $n \geq 2$. Let J be the complex structure of M. For a real C^∞ hypersurface N of M and a point p of N , we define the vector subspace $\overline{H}_p(N)$ of $T_pN \otimes \mathbf{C}$ by

$$\overline{H}_p(N) = \{Z \in T_pN \otimes \mathbf{C} : JZ = -\sqrt{-1}Z\}.$$

We see that $\dim_{\mathbf{C}} \overline{H}_p = n - 1$. Let h be a complex valued function on N . If $Zh = 0$ for every $Z \in \overline{H}_p(N)$, we call that h satisfies the tangential Cauchy-Riemann equation at p . If h satisfies the tangential Cauchy-Riemann equation at every point of N , we call h a *CR-function* on N .

In the following let M be a Kähler CH manifold of complex dimension $n, n \geq 2$. Suppose that the metric of M is of the form

$$ds^2 = dr^2 + g(r,\theta)^2\{d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_2 \cdots \sin^2\theta_{2n-1} d\theta_{2n}^2\}, \qquad (1.11)$$

where in terms of the geodesic polar coordinates at $o, \theta = (\theta_2, \cdots, \theta_{2n})$ is a spherical angle of $S_o(1)$, and r denotes the distance from o , i.e. each geodesic sphere with center at o is conformal to the standard sphere in $\mathbf{R}^{2n}$. The Dirichlet problem at infinity on such manifolds is studied by Choi [4]. For example, every rotationally symmetric manifold satisfies this condition (cf. Milnor [12], Shiga [14]). Identifying $S_o(1)$ with $S(\infty)$, for any $h \in C^0(S(\infty))$ we define a continuous function h_o on $\overline{M} \setminus o$ by

$$h_o(r,\theta) = h(\theta) \quad for \quad \theta \in S(\infty) \sim S_o(1).$$

DEFFINITION. We call $h \in C^0(S(\infty))$ a *CR-function at infinity with respect to* $o \in M$, if $h_o(\gamma,\theta)$ is differentiable on $M \in o$ and $h_o(1,\theta)$ is a CR-function on $S_o(1)$.

The following lemma shows that our definition is natural for the above manifolds. We denote by $CR_o(\infty)$ the set of all CR-functions at infinity with respect to $o \in M$.

Note that there exists a bijection between $CR_o(\infty)$ and the set of all CR-functions on $S_o(1)$. Regarding $B_o(2)$ a domain in $\mathbf{C}^n$, we see that $CR_o(\infty)$ is not empty.

LEMMA 1.4. *Let M be a Kähler CH manifold of complex dimension $n (n \geq 2)$. Assume that the Kähler metric in terms of the geodesic polar coordinates at o is of the form (1.11). If $h \in CR_o(\infty)$, then $h_o \mid S_o(t)$ is a CR-function on $S_o(t)$ for all $t > 0$.*

Proof. It is sufficient to show that for any rays $\gamma(t)$ starting from $o \in M, Zh = 0$ at $\gamma(t)$ for all $Z \in \overline{H}_{\gamma(t_0)}(S_o(t))$ and $t > 0$. Then we fix a ray $\gamma(t)$ and $t_0 > 0$. In the geodesic polar coordinates we denote $\gamma(1)$ by $(1, \theta)$, and we may assume that $\sin\theta'_2, \cdots, \sin\theta'_{2n}$ are not 0.

For any $Z_0 \in \overline{H}_{\gamma(t_0)}(S_o(t_0))$ we denote by $Z(t)$ the parallel vector field along $\gamma(t)$ with $Z(t_0) = Z_0$. Since J is parallel and $Z(t)$ is always orthogonal to $\gamma(t)$, we see that $Z(1) \in \overline{H}_{\gamma(1)}(S_o(1))$.

We define the vector field $X_i(t)$ along $\gamma(t)$ by

$$X_i(t) = \frac{1}{g(t, \theta') \sin\theta'_2 \cdots \sin\theta'_{i-1}} \frac{\partial}{\partial\theta_i}, \tag{1.12}$$

$i = 2, \cdots, 2n$. Therefore

$$\begin{aligned} \nabla_{\dot\gamma(t)} X_i(t) &= -\frac{\partial g}{\partial t} \frac{1}{g^2 \sin\theta'_2 \cdots \sin\theta'_{i-1}} \frac{\partial}{\partial\theta_i} \\ &+ \Sigma_{k=2}^{2n} \frac{\Gamma_{1i}^k}{g \sin\theta'_2 \cdots \sin\theta'_{i-1}} \frac{\partial}{\partial\theta_k} + \frac{\Gamma_{1i}^1}{g \sin\theta'_2 \cdots \sin\theta'_{i-1}} \frac{\partial}{\partial r}, \end{aligned}$$

where we put

$$\nabla_{\dot\gamma(t)} \frac{\partial}{\partial\theta_i} = \Gamma_{1i}^1 \frac{\partial}{\partial r} + \Sigma_{k=2}^{2n} \Gamma_{1i}^k \frac{\partial}{\partial\theta_k}.$$

We see that $\Gamma_{1i}^1 = 0$ and $\Gamma_{1i}^i = f^{-1} \partial f / \partial r$. Since the metric tensor is diagonal with respect to the polar coordinates, other Γ_{1i}^k 's are vanished. Then $\nabla_{\dot\gamma(t)} X_i(t) = 0$, that is, $X_i(t)$ is parallel for all $i \geq 2$.

$\{X_i(t_0)\}$ is an orthonormal frame of $T_{\gamma(t_0)}(S_o(t))$. So we may set

$$Y(t) = \Sigma_{k=2}^{2n} a^k X_k(t),$$

$$J(Y(t) = \Sigma_{k=2}^{2n} b^k X_k(t)$$

. Thus we have

$$Z(t) = \Sigma_{k=2}^{2n}\{a^k X_k(t) + \sqrt{-1} b^k X_k(t)\} \tag{1.13}$$

$h \in CR_o(\infty)$ implies $Z(1)h_o = 0$ at $\gamma(1)$. In the geodesic polar coordinates we have

$$g^{-1}\Sigma_{k=2}^{2n}\{\frac{a^k}{\sin\theta'_2\cdots\sin\theta'_{k-1}}\frac{\partial h_o(1,\theta')}{\partial\theta_k} + \sqrt{-1}\frac{b^k}{\sin\theta'_2\cdots\sin\theta'_{k-1}}\frac{\partial h_o(1,\theta')}{\partial\theta_k}\} = 0 \tag{1.14}$$

by(1.12) and (1.13). Similarly

$$Z(t_0)h_o(t_0,\theta') = g^{-1}\Sigma_{k=2}^{2n}\{\frac{a^k}{\sin\theta'_2\cdots\sin\theta'_{k-1}}\frac{\partial h_o(t_0,\theta')}{\partial\theta_k} + \sqrt{-1}\frac{b^k}{\sin\theta'_2\cdots\sin\theta'_{k-1}}\frac{\partial h_o(t_0,\theta')}{\partial\theta_k}\} = 0$$

Recall that $h_o(t_0,\theta) = h_o(1,\theta)$, hence $Z(t_0)h_o = 0$ at $\gamma(t_0)$ by (1.14). This completes the proof.

THEOREM 1.2. *Let M be a Kähler CH manifold of complex dimension $n, n \geq 2$. Assume that the Dirichlet problem at infinity is solvable on M , and the Kähler metric in terms of the geodesic polar coordinates at $o \in M$ is of the form*

$$ds^2 = dr^2 + g(r,\theta)^2\{d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_2\cdots\sin^2\theta_{2n-1}d\theta_{2n}^2\}.$$

Then for any $h \in CR_o(\infty)$, there exists a holomorphic function H on M with boundary values h , and H coincides with the solution of the Dirichlet problem at infinity.

Remark. Trivial examples of Kähler manifolds as above are $\mathbf{C}^n$ and the unit ball B in $\mathbf{C}^n$ with the invariant metric. For the ball B we may identify a CR-function at infinity with respect to the origin as a CR-function on ∂B , hence we can extend it to a holomorphic function on B by the well known method (Hörmander[10, Theorem 2.3.2']). On one hand by Liouville's theorem we see that any CR-function at infinity on the sphere at infinity of $\mathbf{C}^n$ can not be extended to a holomorphic function on $\mathbf{C}^n$. So in order to extend a function of $CR_o(\infty)$ to a holomorphic functin, we need some hypothesis on N relating to hyperbolicity. The hypothesis that the Dirichlet problem at infinity be solvable is fulfilled if, for example, the sectional curvature $K(x)$ satisfies $-a^2 \leq K(x) \leq -1/(1+r(x)^{2-\varepsilon})$ by Theorem 1.1.

Proof. We denote h by $h = h^1 + \sqrt{-1}h^2$, where $h^1 = \text{Re } h, h^2 = \text{Im } h$. Since the Dirichlet problem at infinity is solvable on M, there exist harmonic functions H^1 and H^2 on M with $H^1 = h^1$ and $H^2 = h^2$ on $S(\infty)$. Thus we have only to show that $H = H^1 + \sqrt{-1}H^2$ is holomorphic on M.

It is shown in Greene-Wu[9] that a Kähler CH manifold is a Stein manifold. By Lemma 1.4 h_o is a CR-function on $S_o(r)$ for all $r > 0$. We see that the boundary $S_o(r)$ of $B_o(r)$ is connected and $B_o(r)$ is relatively compact in M. Then we can find a holomorphic function H_r on $B_o(r)$ with $H_r = h_o$ on $S_o(r)$ (Shiga [13, Theorem 2-5]). So we have a sequence of holomorphic functions $\{H_k\}$ with $H_k = h_o$ on $S_o(k)$ for $k \in N$. Put $H_k^1 = \text{Re } H_k$ and $H_k^2 = \text{Im } H_k$. Then H_k^1 and H_k^2 are harmonic on $B_o(r)$ since M is Kähler. In the polar coordinates we have $h_o^1(k,\theta) = H_k^1(k,\theta)$, and $h_o^2(k,\theta) = H_k^2(k,\theta)$ on $S_o(k)$. Since H^1 and H^2 are continuous on M, for any $\varepsilon > 0$ there is a large integer k_0 such that

$$| H_k^j(k,\theta) - H^j(k,\theta) | < \varepsilon \quad for \quad j = 1,2$$

on $S_o(k)$ for all $k > k_0$. The maximum principle implies

$$| H_k^j - H^j | < \varepsilon \quad for \quad j = 1,2$$

on $B_o(k)$ for all $k > k_0$. This means that $\{H_k\}$ converges to H uniformly on every compact subset of M. Then H is holomorphic since $\{H_k\}$ is a sequence of holomorphic functions.

If the Dirichlet problem at infinity is solvable on a Cartan-Hadamard manifold M , then we see that there is the harmonic measure μ^x on $S(\infty)$ from the Riesz representation theorem. Then we have the following corollary (cf.Anderson[1] and Anderson-Sullivan[2]).

COROLLARY. *Let* M *be a Kähler CH manifold of complex dimension* $n, n \geq 2$ *. Assume that the Dirichlet problem at infinity is solvable on* M *, and the Kähler metric in terms of the geodesic polar coordinates at* $o \in M$ *is of the form*

$$ds^2 = dr^2 + g(r,\theta)^2\{d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_2 \cdots \sin^2\theta_{2n-1} d\theta_{2n}^2\}.$$

And let μ^x *is the harmonic measure on* $S(\infty)$ *. Then for every* $h \in CR_o(\infty)$ *,*

$$h(x) = \int_{s(\infty)} h d\mu^x$$

is a holomorphic function on M *with boundary values* h *.*

Let M_1 and M_2 be complex manifolds of comples dimension $n, n \geq 2$. Let D_1 and D_2 be bounded domains with smooth boundaries $\partial D_1, \partial D_2$ respectively. We call a C^∞ mapping f of ∂D_1 to ∂D_2 a *CR-mappping* if $f_*(\overline{H}_p(\partial D_1)) \subset \overline{H}_{f(p)}(\partial D_2)$ for all $p \in \partial D_1$. Note that f is a CR-mapping if and only if for any CR-function h on $\partial D_2, f \circ h$ is a CR-function on ∂D_1 Shiga[13].

Let M be a complex manifold and d_M the Kobayashi pseudodistance. If d_M is a distance and M is complete with respcet to d_M , M is said to be *complete hyperbolic.*

Let D be a domain in $\mathbf{C}^n$. D is called a *strictly pseudoconvex domain with* C^k *boundary* if there exist an open neighborhood U of D and a strictly plurisubharmonic function $r(z)$ on U of class C^k such that $D = \{z \in U : r(z) < 0\}$ and $\operatorname{grad} r(z) \neq 0$ for all $z \in \partial D$.

THEOREM 1.3. *Let* M *be a Kähler CH manifold of complex dimension* $n, n \geq 2$ *. Assume that the Kähler metric in terms of the geodesic polar coordinates at* o *is of* *the form*

$$ds^2 = dr^2 + g(r,\theta)^2\{d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_2 \cdots \sin^2\theta_{2n-1} d\theta_{2n}^2\},$$

and the holomorphic sectional curvature $K_h(x)$ *satisfies* $K_h(x) < -1/(1+r(x)^2)$ *. Then* M *is biholomorphic to a strictly pseudoconvex domain in* $\mathbf{C}^n$ *.*

Remark. It is shown in Shiga [14] that if $g(r,\theta) = g(r)$, and the holomorphic radial curvature $K(x)$ satisfies $K(x) < -\frac{1+\varepsilon}{r(x)^2 \log r(x)}$, then M is biholomorphic to the unit ball in $\mathbf{C}^n$ (cf. Milnor [12]).

The following lemma is given in Fridman [6].

LEMMA 1.5. *Let* $D \supset \mathbf{C}^n$ *be a bounded strictly pseudoconvex domain with* C^3 *boundary, and* M *is a completely hyperbolic manifold of complex dimension n. Suppose that* M *can be exhausted by biholomorphic images of* D, *that is ,for any compact* $K \subset M$ *there is a biholomorphic imbedding* $F_k : D \to M$ *such that* $F_k(D) \subset K$. *Then* M *is biholomorphically equivalent either to* D *or to the unit ball in* $\mathbf{C}^n$ *.*

Proof of the theorem. Recall that M is a Stein manifold. Choose a holomorphic coordinate neighborhood U of M such that $B_o(\varepsilon) \subset\subset U$ for a positive ε . By the Hessian comparison theorem Greene-Wu[9], $r(x)^2$ is strictly plurisubharmonic on M since M is Kähler. We see that $\operatorname{grad} r(x)^2 \neq 0$ on $M \setminus o$. Then we may regard $B_o(\varepsilon)$ as a strictly pseudoconvex domain with C^∞ boundary in $\mathbf{C}^n$. We define a diffeomorphism f_k from $S_o(\varepsilon)$ to $S_o(k) \in N$ by

$$f_k(\varepsilon,\theta) = (k,\theta)$$

where (r,θ) is the polar coordinates at o . Lemma 1.4 implies that f_k is a CR-diffeomorphism. We see that $S_o(\varepsilon)$ and $S_o(k)$ are connected. From the Bochner-Hartogs' theorem on Stein manifolds (Shiga [13]) we see that $B_o(\varepsilon)$ is biholomorphic to $B_o(k)$ for all integer k. For any compact set K in M , there exists an integer k so that $B_o(k) \supset K$ since $\exp_o : T_oM \to M$ is a diffeomorphism. So M is exhausted by biholomorphic images of the strictly pseudoconvex domain $B_o(\varepsilon)$. Since $K_h(x) < -1/(1+r(x)^2)$ and M is complete, M is complete hyperbolic from the theorem of Green and Wu ([9], Theorem E).

It follows that M is biholomorphically equivalent either to the unit ball B in $\mathbf{C}^n$ or to $B_o(\varepsilon)$ from Lemma 1.5. Both $B_o(\varepsilon)$ and B are strictly pseudoconvex, then the theorem is proved.

Bibliography

[1] M. Anderson, The Dirichlet problem at infinity for manifolds of negative curvature, *J. Differential Geom.*,**18**(1983), 701-721.

[2] M. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvature, *Ann. of Math.*, **121**(1985), 429-461.

[3] S. Y. Cheng and S. T. Yau, Differential equations on Riemannian manifolds and their applications, *Comm. Pure Appl. Math.*, **28**(1975), 333-354.

[4] H. Choi, Asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds, *Trans. Amer. Math. Soc.*,**281**(1984), 691-716.

[5] P. Eberlein and B. O'Neil, Visibility manifolds, *Pacific J. Math.*, **46**(197), 45-109.

[6] B. L. Fridman, Biholomorphic invariants of a hyperbolic manifold and some applications, *Trans. Amer. Math. Soc.*, **276**(1983), 685-698.

[7] R. E. Greene and H. Wu, Approximation theorems. C^∞ convex exhaustions and manifolds of positive curvature, *Bull. Amer. Math. Soc.*, **81**(1975), 101-104.

[8] R. E. Greene and H. Wu, Analysis on noncompact Kähler manifolds, *Proc. Symn. Pure Math.*, **30** *AMS Prov. R. I.*, (1977), 69-100.

[9] R. E. Greene and H. Wu, Function theory on manifolds which posses a pole, Lecture Notes in Math. **699**, Springer-Verlag, New York, 1979.

[10] L. Hörmander, An introduction ton complex analysis in several variables. Van Norstrand 1966.

[11] S. Kobayashi, Hyperbolic manifolds and holomorphic mappings, New York; Dekker 1970.

[12] J. Milnor, On deciding whether a surface is parabolic or hyperbolic. *Amer. Math. Monthly*, **84**(1977), 43-46.

[13] K. Shiga, On holomorphic extension from the boundary, *Nagoya Math. J.*, **42** (1971), 57-66.

[14] K. Shiga, A geometric characterization of $\mathbf{C}^n$ and open balls, *Nagoya Math. J.*, **75** (1979), 145-150.

[15] D Sullivan, The Dirichlet problem at infinity for a negatively curved manifold, *J. Differential Geom.*, **18**(1983), 723-732.

Chapter 2

Gradient Estimates of Holomorphic Maps and A General Schwarz Lemma on Kähler CH Manifolds

2.1 Introduction

In this chapter we shall survey the Schwarz lemma on Kähler manifolds, and raise some problems to estimate the gradient of holomorphic mappings. In the latter part of this chapter we prove 2 types of the general Schwarz lemma on a Kähler CH manifold as a partial answer to the above problem.

The classical Schwarz-Pick lemma states that every holomorphic map from unit disc into itself is distance-decreasing with respect to the Poincaré metric. From the geometrical viewpoint , the distance-decreasing or volume-decreasing property of a holomorphic map has been studied succesfully in Ahlfors [1], Chern [3], Kobayashi [6], Lu [7] and Yau [12]. From the analytical viewpoint, we are interested in that the gradient of a holomorphic map is estimated by curvature conditions. This was also studied in Ahlfors [1] and Yau [12].

Relating to these works, we recall the following question raised by Greene-Wu [4]. *What is the largest metric on the unit disc for which a Schwarz-type lemma holds ?*

In this connection the following question is raised naturally for a Kähler CH manifold M . *Let* $F(r)$ *be a negative monotone-decreasing or monotone-increasing* *function of* r *, where* r *is the distance function on* M *from a point of* M *. If the Ricci curvature is bounded below by* $F(r)$ *, does the Schwarz-type lemma hold up to* $F(r)$*?*

In a more explicit style we can describe the above as follows : *If* f *is a holomorophic map from* M *to a Hermitian manifold whose curvature is bounded above by a constant* $-\beta^2$ *, and if the Ricci curvature of* M *satisfies* $\mathrm{Ric} \geq -\alpha^2 r(x)^\delta$ *, then does* $f^* ds_N^2 \leq \frac{\alpha^2 r(x)^\delta}{\beta^2} ds_M^2$ *holds ?* In particular if δ is negative, it may give more exact estimates of the gradient of holomorphic maps. Moreover if δ is less than -2, it may give some information for studying hyperbolicity or parabolicity of M . On one hand if δ is positive, it means that Schwarz-type lemma holds under unbounded Ricci curvature conditions. Here we shall give two theorems which are corresponding to the cases that δ is equal to -2 and equal to 2 of the above problem. So we may expect that these results show the possibility that the above problem be solved affirmatively.

2.2 Preliminaries

Let M be a Riemannian manifolds of dimension m and p be a point of M . We denote by T_pM the tangent space at p , and by X a tangent vector of T_pM . Let $\langle\ ,\ \rangle$ be the Riemannian inner product of M , and $|X|$ be a length of the vector $X \in T_pM$. For a C^2 function u on M , we define the Hessian D^2u of u at $p \in M$ by

$$D^2u(X,Y) = X(Yu) - (D_XY)u$$

for $X, Y \in T_pM$, where D_XY is the covariant derivative associated with the Riemann connection. The Laplacian Δu of u is defined as the trace of D^2u . In other words, if $\{ X_i \}$ is an orthonormal basis of T_pM , then

$$\Delta u = \Sigma_i\ D^2u(X_i, X_i).$$

The curvature tensor R is defined by

$$R(X,Y)Z = -D_XD_YZ + D_YD_XZ + D_{[X,Y]}Z,$$

so that $\langle R(X,Y)X,Y\rangle$ has the same sign as the sectional curvature of the plane spanned by X and Y . The Ricci tensor $\mathrm{Ric}(Y,Z)$ is defined by

$$\mathrm{Ric}(Y,Z) = \Sigma_i \langle R(X_i,Y)Z,X_i\rangle \ .$$

We say that *the Ricci curvature is bounded below* $-\alpha^2$ if $\mathrm{Ric(X,X)} \geq -\alpha^2 |\mathrm{X}|^2$ for all X .

The well-known theorem of Cartan-Hadamard says that if M is simply connected, complete Riemannian manifold of nonpositive curvature, then the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism. In this case we call M a *Cartan-Hadamard manifold.*

Let M and N be a Riemann manifold of dimension m and n respectively. And we denote by ds_M^2 and ds_N^2 a Riemannian metric of M and N respectively. Let f be a differential map from m to N . We define the 2-form $f^*ds_N^2$ on M by

$$f^*ds_N^2(X,Y) = ds_N^2(f_*X, f_*Y)$$

for arbitraly $p \in M$ and $X, Y \in T_pM$. For a positive number C , the inequality $f^*d_N^2 < C\ ds_M^2$ at p means that the inequality $f^*d_N^2(X,Y) < C\ ds_M^2(X,Y)$ is valid for any $X, Y \in T_pM$. In this case we say that *the gradient of* f *is estimated by* C in this paper. Moreover if there is a real-valued function $C(x)$ on M such that $f^*d_N^2 < C(x) ds_M^2$ for all $x \in M$, we say that f is *distance-decreasing up to* $C(x)$ in this paper.

Also, we shall give some definitions with respect to a complex manifold. Let M be a Hermitian manifold, and let $\langle\ ,\ \rangle$ be a Hermitian inner product. X denotes a tangent vector of type (1,0) of holomorphic tangent space T_pM . If R is the curvature tensor of the canonical Hermitian connection on M , *the holomorphic bisectional curvature* determined by X, Y is defined by

$$\frac{\langle R(X,\overline{X})Y,\overline{Y}\rangle}{|X|^2 \cdot |Y|^2} \quad for \quad |X|, |Y| \neq 0$$

We say that *the holomorphic bisectional curvature is bounded above* $-\beta^2$ if

$$\langle R(X,\overline{X})Y,\overline{Y}\rangle \leq -\beta^2 |X| \cdot |Y|$$

for all X, Y . Similarly as in a Riemannian case, the Ricci tensor is defined as

$$\mathrm{Ric}(Y,Z) = \Sigma_i \langle R(X_i,\overline{Y})Z,\overline{X}_i\rangle \ .$$

We say *the Ricci curvature is bounded below* $-\alpha^2$ if $R(X,X) \geq -\alpha^2 |X|^2$ for all X.

If a complete Kähler mainfold M is simply connected with nonpositive sectional curvature, we call M a *Kähler CH manifold.* A Kähler CH manifold is also diffeomorphic to its tangent space.

2.3 Survey of the Schwarz lemma

In the first, we recall the classical Schwarz-Pick lemma:

Let D be the unit disc in the complex plane $\mathbf{C}$, and f be a holomorphic map from D into itself. Then for the Poincaré metric ds_D^2 , it follows that $f^*ds_D^2 \leq ds_D^2$ and the inequality holds everywhere unless f is biholomorphic.

This means every holomorphic map from D into itself decreases distance with respect to the Poincaré metric. And the distance-decreasing property of holomorphic map has been generalized to various forms . For example, it is generalized to higher dimensional case by Bochner-Martin [2].

However Ahlfors [1] was the first to generalize Schwarz lemma by considering the curvature conditions. His result is stated as follows:

Let D be the unit disc in $\mathbf{C}$ with the invariant metric ds_D^2 whose Gaussian curvature is equal to a negative constant $-\alpha^2$. And let M be a Riemann surface with hermitian metric ds_M^2 whose Gaussian curvature is bounded above by a negative constant $-\beta^2$. Then every holomorphic map f from D to M satisfies $f^*ds_M^2 \leq \frac{\alpha^2}{\beta^2}ds_D^2$

Moreover this was generalized by Kobayashi [5] and by others to higher dimensional case. Most of these generalizations of Schwarz Lemma originate from Ahlfors's generalization.

Next, we consider the higher dimensional case for general manifolds. Let M and N be same dimensional manifolds, and f be a holomorphic map from M to N . Now we define the general elementary symmetric function u associated with f according to Chern [3] and Lu [7]. So u is defined by $u = |\det(df)|^2$ according to Chern [3]. We can see that u means the volume ratio of f . The following result is shown in Chern [3]:

Let R be the scalar curvature of M , and Ric be the Ricci curvature of N . Then it follows that $\frac{1}{2}\Delta \log u \geq R - \mathrm{Tr}(f^*(\mathrm{Ric}))$, where $\mathrm{Tr}(f^*(\mathrm{Ric}))$ means the trace of the inverse image of the Ricci form of N .

Using this formula, Chern [3] generalized the Schwarz lemma as follows:

Let B be the n-dimensional unit ball in C^n with the standard Kähler metric, and N be an n-dimensional hermitian Einstein manifold with scalar curvature $\leq -2n(n+1)$. Then every holomorphic map $f : B \to N$ is volume-decreasing.

This was also generalized by Kobayashi [6] to more general manifolds .

After that Lu [7] showed a similar formula relating to a distance ratio. Let M and N be a Hermitian manifods of complex dimension m , and n respectively. And let f be a holomorphic map from M to N . According to Lu, the general symmetric function is defined by $u = \Sigma_i |df|^2$ The following result is shown in Lu [7].

Suppose that the Ricci curvature of M satisfies $\mathrm{Ric} > \alpha$ at a point $p \in M$, and the holomorphic bisectinal curvature of N is bounded above by β at $f(p)$. Then, at p , we have $\Delta u \geq 2(\alpha u - \beta u^2)$ and $\Delta \log u \geq 4(\alpha - \beta u)$.

About ten years later Yau showed the Schwarz lemma in the most general style [12] using Lu's formula. It is mentioned as follows:

Let M be a complete Kähler manifold with Ricci curvature bounded below by $-\alpha^2$ ($\alpha > 0$), and N be a Hermitian manifold with holomorphic bisectional curvature bounded from above by $-\beta^2$ ($\beta > 0$). Then every holomorphic map f from M to N satisfies $f^* ds_N^2 \leq \frac{\alpha^2}{\beta^2} ds_M^2$.

The points of this result are that the domain manifold is a general manifold, the gradient of f is estimated by the Ricci curvature of domain manifold, and moreover this lemma immediately implies the Liouville's theorem, that is:

A complete Kähler manifold with non-negative Ricci curvature does not admit any bounded holomorphic functions.

Before introducing new Schwarz-type lemma, we would like to mention some results related to Kähler CH manifolds.

2.4 Kähler CH manifolds

Here we turn to study Kähler CH manifold. From 1970's a Kähler CH manifold has been studied succesfully. And it has been always studied that: *When M is hyperbolic or when M is parabolic?* Note that we frequently see various partial differential equations in considering these problems. Now we shall make reference to the most typical results relating to these problems .

Let o be a fixed point of M, and we denote by $r(x)$ the distance function from o to x in M. The following result is shown in Siu-Yau [10].

If M is a Kähler CH manifold of complex dimension n with sect. curv. $\geq$

$-\dfrac{\alpha^2}{r(x)^{2+\varepsilon}}$ *for positive constants* α *and* ε *, then* M *is biholmorphic to* $\mathbf{C}^n$ *.*

Note that the proof of this theorem uses the L^2 estimates of the $\overline{\partial}$ problem. And the curvature condition of this theorem was improved by Greene-Wu [4].

On the other hand Greene-Wu proved the following in [4]:

If M *is a Kähler CH manifold of complex dimension* n *with sect. curv.* $\leq -\dfrac{\alpha^2}{(1+r(x)^2)}$ *then* M *is complete hyperbolic in the sense of Kobayashi.*

Moreover we recall some results for restricted cases in this direction. For a rotationally symmetric case the following is shown by Milnor [8] and Shiga [9]:

Let M *be a Kähler CH manifold of complex dimension* n *. Suppose that the Kähler metric of* M *is of the form* $ds^2 = dr^2 + g(r)^2 d\theta^2$ *, where* (r, θ) *be the geodesic polar coordinates on* M *. We denote by* $K(r)$ *the holomorphic radial curvature. Then*
(1) If $K(r) \geq -\dfrac{1}{r^2 \log r}$ *,then* M *is biholomorphic to* $\mathbf{C}^n$ *.*
(2) If $K(r) \leq -\dfrac{1+\varepsilon}{r^2 \log r}$ *for positive* ε *, then* M *is biholomorphic to the unit ball in* $\mathbf{C}^n$ *.*

And we have the following result. However we have few results in this direction.

Let M *be a Kähler CH manifold of complex dimension* n *. Suppose that the Kähler metric of* M *is of the form* $ds^2 = dr^2 + g(r,\theta)^2 d\theta^2$ *with respect to the geodesic polar coordinates on* M *. If sect. curv.* $\leq -\dfrac{1}{1+r(x)^2}$ *for positive* ε *, then* M *is biholomorphic to a strictly pseudoconvex domain in* $\mathbf{C}^n$ *.*

Proof. M is a Stein manifold since a Kähler CH manifold is always a Stein manifold by Greene-Wu[4] . Choose a holomorphic coordinate neighborhood U of M such that $B_o(\varepsilon) \subset\subset U$ for a positive ε . By the Hessian comparison theorem of Greene-Wu[4], $r(x)^2$ is strictly plurisubharmonic on M since M is Kähler. We have that $\mathrm{grad}\, r(x)^2 \neq 0$ on $M \backslash o$. Then we may regard $B_o(\varepsilon)$ as a strictly pseudoconvex domain with C^∞ boundary in $\mathbf{C}^n$. We define a diffeomorphism f_k from $S_o(\varepsilon)$ to $S_o(k) \in N$ by

$$f_k(\varepsilon, \theta) = (k, \theta)$$

where (r, θ) is the polar coordinates at o. Lemma 1.4 in Chapter 1 implies that f_k is a CR-diffeomorphism. We see that $S_o(\varepsilon)$ and $S_o(k)$ are connected. From the Bochner-Hartogs' theorem on Stein manifolds we see that $B_o(\varepsilon)$ is biholomorphic to $B_o(k)$ for all integer k. For any compact set K in M, there exists an integer k so that $B_o(k) \supset K$ since $\exp_o : T_oM \to M$ is a diffeomorphism. So M is exhausted by biholomorphic images of the strictly pseudoconvex domain $B_o(\varepsilon)$. Since $K_h(x) < -1/(1 + r(x)^2)$ and M is complete, M is complete hyperbolic from the theorem of Greene-Wu ([4], Theorem E).

It follows that M is biholomorphically equivalent either to the unit ball B in $\mathbf{C}^n$ or to $B_o(\varepsilon)$ from Lemma 1.5. Both $B_o(\varepsilon)$ and B are strictly pseudoconvex, then the theorem is proved.

2.5 Schwarz lemma on Kähler CH manifolds

Now we return to estimate the gradient of holomorphic maps from a Kähler CH manifold. In this section let o be a fixed point of M, and we denote by $r(x)$ the distance function from o to x in M.

Here we notice that the ordinary Schwarz lemma does not tell us a local estimate of a holmorphic map, in other words, the gradient is bounded by a very constant on a manifold. Hence the following question is raised naturally:

Can we estimate the gradient of holomorphic map locally by a local ratio of the domain's curvature and the object's curvatute?

On one hand, in connection with the Liouville's theorem, the following conjecture has been raised for about twenty years (cf. Wu [13]):

Let M be a complete Kähler CH manifold, For positive numbers α and ε suppose that $\mathrm{Ric} \geq -\dfrac{\alpha^2}{r(x)^{2+\varepsilon}}$ *, then does M admit non-constant bounded holomorphic functions?*

However we know few results in this direction. In connection with these questions and the Schwarz lemma, the following question is raised for a Kähler CH manifold M .

Let $F(r)$ be a negative monotone-decreasing or monotone-increasing function of r . If the Ricci curvature of M is bounded below by $F(r)$ on M , does the Schwarz type lemma hold up to $F(r)$ on M ?

This formulation means that the gradient of a holomorphic map can be estimated locally by its Ricci curvature. We expect this formulation will show more presice estimates than ordinary Schwarz-type lemma. In a more explicit style we can reform the above question as follows :

If f is a holomorophic map from M to a Hermitian manifold whose curvature is bounded above by a constant $-\beta^2$, and if the Ricci curvature of M satisfies $\mathrm{Ric} \geq -\alpha^2 r(x)^\delta$ *for constants α and δ , then does* $f^* ds_N^2 \leq \dfrac{\alpha^2 r(x)^\delta}{\beta^2} ds_M^2$ *holds ?*

In particular if this problem is solved affirmatively for a negative number δ , it may give more exact estimates of the gradient of holomorphic maps for large r . Furthemore if δ is less than -2, we expect that it may give some information for deciding hyperbolicity or parabolicity of M . On one hand if δ is positive, it means that the Schwarz-type lemma holds under unbounded Ricci curvature conditions. And it also assurts that every holomorphic map from such a manifold is destance-decreasing up to its Ricci curvature. Moreover these two theorems suggest us that we may expect a good many various forms of the Schwarz-type lemma for a general manifold.

In the following we shall give two theorems which will be an affirmative answers to the above problem for the cases: $\delta = 2$ and $\delta = -2$.

2.6 Laplacian estimates of the distance function

First, we shall estimate the Laplacian of the distance function on our manifolds.

LEMMA 2.1. *Let M be a Cartan-Hadamard manifold of dimension m . Let o be a point of M , and $r(x) = \mathrm{dist}(x, o)$. Then we have the followings :*

(A) $\Delta r(x) \geq \frac{m-1}{r(x)}$ *on M .*

(B) Let p be a point of M such that $\mathrm{dist}(o, p) \geq 1$, and $l(x) = \mathrm{dist}(x, p)$. If the Ricci curvature satisfies $\mathrm{Ric} \geq \frac{-\alpha^2}{1 + r(x)^2}$ for a positive constant α on M , then we have

$$\Delta l(x) \leq \frac{m - 1 + 2\alpha^2(2 + A^2)}{l(x)}$$

for all $x \in M$,where $A = dist(o, p)$.

Proof. (A) follows directly from the Laplacian comparison theorem Greene-Wu[4] since M is a nonpositive curved manifold and $\Delta r(x) = \frac{m-1}{r(x)}$ for the Euclidean space $\mathbf{R}^n$.

For arbitrarily point $x \in M$, let $\gamma(t)$ be a minimal geodesic joining p and x . And we denote by $\dot{\gamma}(t)$ the tangent vector of γ . Once again from Siu-Yau[10] we have

$$\Delta l(x) \leq \frac{m-1}{l(x)} - \frac{1}{l(x)^2} \int_0^{l(x)} t^2 \, \mathrm{Ric}(\dot{\gamma}(t)) dt. \tag{2.1}$$

The trigonometric inequality implies that $|l(\gamma(t)) - A| < r(\gamma(t))$ for all $t > 0$. Note that $l(\gamma(t)) = t$. Then the curvature condition leads

$$\mathrm{Ric}(\dot{\gamma}(t)) \geq \frac{-\alpha^2}{1 + r(\gamma(t))^2} \geq \frac{-\alpha^2}{1 + (t - A)^2}.$$

If $l(x) > 2A$, we get

$$\begin{aligned}\int_0^{l(x)} t^2 \operatorname{Ric}(\dot\gamma(t))dt &= \int_0^{2A} t^2 \operatorname{Ric}(\dot\gamma(t))dt + \int_{2A}^{l(x)} t^2 \operatorname{Ric}(\dot\gamma(t))dt \\ &\geq \int_0^{2A} t^2(-\alpha^2)dt + \int_{2A}^{l(x)} \frac{-t^2\alpha^2}{1+(t-A)^2}dt\end{aligned}$$

since $\operatorname{Ric} \geq -\alpha^2$ on M .

Direct computations give

$$\begin{aligned}\frac{d}{dt}\{\frac{t^2}{\{1+(t-A)^2\}}\} &= \frac{2t(1+A^2-tA)}{\{1+(t-A)^2\}^2} \\ &\leq \frac{2t(1-A^2)}{\{1+(t-A)^2\}^2} \\ &\leq 0\end{aligned}$$

because $t \geq 2A$ and $A > 1$. Therefore $\frac{t^2}{1+(t-A)^2}$ is monotone decreasing and less than 4. Then we have $\frac{t^2}{1+(t-A)^2} \leq 4$ for $t \geq 2A$. Hence we obtain

$$\int_0^{l(x)} t^2 \operatorname{Ric}(\dot\gamma(t))dt \geq -\frac{8}{3}\alpha^2 A^3 - \int_{2A}^{l(x)} 4\alpha^2 dt. \tag{2.2}$$

Using (2.1) and (2.2) we have

$$\Delta l(x) \leq \frac{m-1}{l(x)} + \frac{8\alpha^2 A^3}{3l(x)^2} + \frac{4\alpha^2}{l(x)} \leq \frac{m-1+2\alpha^2(2+A^2)}{l(x)}.$$

The last inequality follows from $l(x) > 2A$.

If $l(x) \leq 2A$, we get

$$\int_0^{l(x)} t^2 \operatorname{Ric}(\dot\gamma(t))dt \geq -\frac{\alpha^2 l(x)^3}{3}$$

since $\operatorname{Ric} \geq -\alpha^2$. From (2.1) we have

$$\Delta l(x) \leq \frac{m-1}{l(x)} + \frac{\alpha^2 l(x)}{3} \leq \frac{m-1+4\alpha^2A^2/3}{l(x)} \leq \frac{m-1+2\alpha^2A^2}{l(x)}.$$

Thus we have proved (B) for all $x \in M$.

LEMMA 2.2. *Let M be a Cartan-Hadamard manifold of dimension m . Let o be a point of M , and $r(x) = \operatorname{dist}(x, o)$. If the Ricci curvature of M satisfies*

Ric $\geq -\alpha^2(1+r(x)^2)$ *on* M *, then we have the followings :*

(A) $\Delta r(x) \leq \dfrac{m-1}{r(x)} + \dfrac{\alpha^2 r(x)}{3} + \dfrac{\alpha^2 r(x)^3}{5}$ *on* M *.*

(B) Let p *be a point of* M *such that* $p \neq o$ *, and* $l(x) = \mathrm{dist}(x,p)$ *. Then we have*

$$\Delta l(x) \leq \frac{m-1}{l(x)} + \frac{\alpha^2(1+A^2)l(x)}{3} + \frac{\alpha^2 A l(x)^2}{2} + \frac{\alpha^2 l(x)^3}{5}$$

for $x \in M$ *, where* $A = dist(o,p)$ *.*

Proof. From (2.1) we have

$$\Delta r(x) \leq \frac{m-1}{r(x)} + \frac{\alpha^2}{r(x)^2}\int_0^{r(x)} t^2\ (1+t^2)dt. \tag{2.3}$$

Integrating (2.3), we obtain (A) .

Let $\gamma(t)$ be a minimal geodesic joining p and arbitrarily $x \in M$. From $A + l(\gamma(t)) \geq r(\gamma(t))$ and (2.3), we have

$$\Delta l(x) \leq \frac{m-1}{l(x)} + \frac{\alpha^2}{l(x)^2}\int_0^{l(x)} t^2\ (1+(t+A)^2)dt.$$

Integrating the above, we have directly (B) .

2.7 Quadratic decaying condition

In this section we shall give gradient estimetes of a holomorophic map from a Kähler CH manifold whose Ricci curvarure is bounded from below by a quadratic decaying function. Again we recall Yau's general Schwarz lemma [12]

GENERAL SCHWARZ LEMMA. *Let* M *be a Kähler manifold with Ricci curvature bounded below by* $-\alpha^2$ $(\alpha > 0)$ *, and* N *be a Hermitian manifold with holomorphic bisectional curvature bounded from above by* $-\beta^2$ $(\beta > 0)$ *. Then if there is a nonconstant holomorphic map* f *from* M *to* N *, we have* $f^* ds_N^2 \leq \dfrac{\alpha^2}{\beta^2} ds_M^2$.

The following theorem is a partial answer to our problem raised in Section 2.5, and gives more accuracy of gradient estimates adding the curvature conditions of quadratic decay. And Theorem 2.1 shows that every holomorphic map between manifolds satisfying the conditions of the theorem is destance-decreasing up to their Ricci curvatures. The proof is essentially based on Yau [11] and Yau[12]. However in order to simplify the proof, we shall use the argument of Ahlfors [1] and Wu [13].

THEOREM 2.1. *Let* M *be a Kähler CH manifold of complex dimension* $m(m \geq 2)$ *, and* N *be a Hermitian manifold of complex dimension* n *with holomorphic bisectional curvature bounded from above by* $-\beta^2$ $(\beta > 0)$ *. Suppose that the Ricci curvature of* M *satisfies* $\mathrm{Ric} \geq -\alpha^2/(1+r(x)^2)$ *, where* $r(x)$ *is the distance function from a fixed point* o *of* M *. Then if there is a nonconstant holomorphic map* f *from* M *to* N *, we have*

$$f^*ds_N^2 \leq \frac{\alpha^2}{\beta^2(1+r(x)^2)} ds_M^2.$$

Let M and N be a Hermitian manifods of complex dimension m , and n respectively. And let f be a holomorphic map from M to N . According to Lu [7], we define *the general elementary symmetric function* $u(x)$ associated with f on M . By choosing a orthonormal basis $\{X_i\}$ of type (1,0) at $x \in M$, and we define $u(x)$ by

$$u(x) = \Sigma_i |df(X_i)|^2 \quad for \quad x \in M. \tag{2.4}$$

From the definition we can see $f^*ds_N^2 \leq u(x)ds_M^2$. The following result is shown in Lu[7].

PROPOSITION 2.1 *Let* M *and* N *be a Hermitian manifold of dimension* m, n *respectively. And let* $u(x)$ *be the general elementary symmetric function on* M *of a holomorphic map* f *from* M *to* N *defined as above. Suppose that for a point* p *in* M *the Ricci curvature of* M *satisfies* $\mathrm{Ric} > \alpha$ *at* p *, and the holomorphic bisectinal* *curvature of* N *is bounded above by* β *at* $f(p)$ *. Then we have at* p *,*

$$\Delta u(p) \geq 2(\alpha u(p) - \beta u(p)^2) \tag{2.5}$$

$$\Delta \log u(p) \geq 4(\alpha - \beta u(p)). \tag{2.6}$$

Proof of Theorem 2.1. By the definition of the general elementary symmetric function $u(x)$, it suffices to prove that $u(x) \leq \alpha^2/\{\beta^2(1+r(x)^2)\}$. And we have

$$u(x) \leq \frac{\alpha^2}{\beta^2} \tag{2.7}$$

on M from Yau[12], since the Ricci curvature of M is bounded below by $-\alpha^2$. For a sufficiently small positive number ε , we define a C^∞ function $(1+r(x)^{2-\varepsilon})u(x)$ on M . Then either $(1+r(x)^{2-\varepsilon})u(x)$ attains its supremum at some point or, there is a sequence $\{q_k\}$ in M such that

$$\limsup_{k\to\infty}(1+r(q_k)^{2-\varepsilon})u(q_k) = \sup_{x\in M}(1+r(x)^{2-\varepsilon})u(x).$$

First, we suppose that $(1+r(x)^{2-\varepsilon})u(x)$ attains its supremum at $q \in M$. In this case we shall consider for three cases ; $r(q)=0, r(q)<1$, and $r(q)\geq 1$.

If $q=o$, it follows from (2.7) that

$$\begin{aligned}(1+r(x)^{2-\varepsilon})u(x) &\leq (1+r(o)^{2-\varepsilon})u(o)\\ &= u(o)\\ &\leq \frac{\alpha^2}{\beta^2}.\end{aligned}$$

Therefore we have that $u(x) \leq \dfrac{\alpha^2}{\beta^2(1+r(x)^{2-\varepsilon})}$ on M if $q=o$.

Next, we assume $q\neq o$, and $(1+r(q)^{2-\varepsilon})u(q)>0$. Hence $\log\{(1+r(x)^{2-\varepsilon})u(x)\}$ is well defined near q . We see that $\log(1+r(x)^{2-\varepsilon})u(x)$ also achieves its maximum at q . From the maximum principle we have

$$\Delta \log\{(1+r(q)^{2-\varepsilon})u(q)\} \leq 0. \tag{2.8}$$

Direct computation implies

$$\begin{aligned}\Delta \log(1+r(x)^{2-\varepsilon}) &= \frac{2-\varepsilon}{1+r(x)^{2-\varepsilon}}\{r(x)^{1-\varepsilon}\Delta r(x)+(1-\varepsilon)r(x)^{-\varepsilon}-\frac{(2-\varepsilon)r(x)^{2-2\varepsilon}}{1+r(x)^{2-\varepsilon}}\} \\ &\geq \frac{2-\varepsilon}{r(x)^{\varepsilon}+r(x)^2}(r(x)\Delta r(x)-1)\end{aligned}$$

According to Lemma 2.1(A), we see $\Delta r(x) \geq (2m-1)/r(x)$. Then we have

$$\begin{aligned}\Delta \log(1+r(x)^{2-\varepsilon}) &\geq \frac{(2-\varepsilon)(2m-2)}{r(x)^{\varepsilon}+r(x)^2} \\ &\geq \frac{3m-4}{r(x)^{\varepsilon}+r(x)^2}.\end{aligned} \tag{2.9}$$

From (2.8) we see that

$$0 \geq \Delta \log\{(1+r(q)^{2-\varepsilon})u(q)\} = \Delta \log u(q) + \Delta \log(1+r(q)^{2-\varepsilon}).$$

Using (2.6) and (2.9), we get

$$\begin{aligned}0 &\geq 4(\frac{-\alpha^2}{1+r(q)^2}+u(q)\beta^2)+\frac{3m-4}{r(q)^{\varepsilon}+r(q)^2} \\ &\geq 4(\frac{-\alpha^2}{1+r(q)^2}+u(q)\beta^2)\end{aligned}$$

since $m \geq 2$. Therefore we have

$$(1+r(q)^2)u(q) \leq \frac{\alpha^2}{\beta^2}. \tag{2.10}$$

Hence we have for all $x \in M$

$$\begin{aligned}(1+r(x)^{2-\varepsilon})u(x) &\leq (1+r(q)^{2-\varepsilon})u(q) \\ &\leq \frac{\alpha^2(1+r(q)^{2-\varepsilon})}{\beta^2(1+r(q)^2)}.\end{aligned}$$

In particular if we suppose $r(q) \geq 1$, then we have immediately from (2.10)

$$\begin{aligned}(1+r(x)^{2-\varepsilon})u(x) &\leq (1+r(q)^{2-\varepsilon})u(q) \\ &\leq (1+r^2(q))u(q) \\ &\leq \frac{\alpha^2}{\beta^2}.\end{aligned}$$

Then we obtain $u(x) \leq \frac{\alpha^2}{\beta^2(1+r(x)^{2-\varepsilon}}$ on M if $r(q) \geq 1$. Thus we have proved that for any small positive constant ε if $(1+r(x)^{2-\varepsilon})u(x)$ achieves its supremum at some point q_ε , then it follows that

$$u(x) \leq \max\{\frac{\alpha^2}{\beta^2(1+r(x)^{2-\varepsilon})}, \frac{\alpha^2(1+r(q_\varepsilon)^{2-\varepsilon})}{\beta^2(1+r(q_\varepsilon)^2)(1+r(x)^{2-\varepsilon})}\}. \tag{2.11}$$

on M , where q_ε satisfies $0 < r(q_\varepsilon) < 1$. Hence it remains to consider the case that $(1+r(x)^{2-\varepsilon})u(x)$ does not achieves its supremum.

Now let $\{q_k\}$ be a sequence in M such that

$$\limsup_{k\to\infty}(1+r(q_k)^{2-\varepsilon})u(q_k) = \sup_{x\in M}(1+r(x)^{2-\varepsilon})u(x).$$

Choosing a subsequence adequately we may assume that $\lim_{k\to\infty} r(q_k) = \infty$, and

$$\begin{aligned} \sup_{x\in B_o(r(q_k))} \quad (1+r(x)^{2-\varepsilon})u(x) &= (1+r(q_k)^{2-\varepsilon})u(q_k), \\ r(q_k) &< r(q_{k+1}) \end{aligned} \tag{2.12}$$

for all integer k , where $B_o(r(q_k))$ is the ball of radious $r(q_k)$ arround $o \in M$. Let $B_k(R)$ be the ball of radious R arround $q_k \in M$, and let $l(x)$ be the distance from q_k . So we define a real-valued C^∞ function $\Phi(x)$ on $B_k(R)$ by

$$\Phi(x) = (1+r(x)^{2-\varepsilon})u(x)(R^2 - l(x)^2)^2.$$

We see that $\Phi(x) \equiv 0$ on the boundary $\partial B_k(R)$ of $B_k(R)$, and $\Phi(x) > 0$ in $B_k(R)$. Hence $\Phi(x)$ attains its maximum at an interior point p_k of $B_k(R)$. From (2.13), we see

$$r(q_k) \leq r(p_k) \leq r(q_k) + R\ . \tag{2.13}$$

Since $\Phi(p_k) > 0$, $\log\Phi(x)$ is well defined as a C^∞ function near p_k . Then $\log\Phi(x)$ also attains its maximum at p_k , hence $\Delta\log\Phi(p_k) \leq 0$. Therefore, at p_k we have,

$$\begin{aligned} 0 &\geq \Delta\log\{(1+r(p_k)^{2-\varepsilon})u(p_k)(R^2 - l(p_k)^2)^2\} \\ &= \Delta\log(1+r(p_k)^{2-\varepsilon}) + \Delta\log u(p_k) + 2\Delta\log(R^2 - l(p_k)^2)\ . \end{aligned}$$

Since (2.6) and (2.9) , we obtain at p_k

$$0 \geq \frac{3m-4}{r(p_k)^\varepsilon + r(p_k)^2} + 4\{\frac{-\alpha^2}{1+r(p_k)^2} + u(p_k)\beta^2\} + 2\Delta\log(R^2 - l(p_k)^2) \ . \tag{2.14}$$

So that we shall estimate the third term of the right-hand side. Direct computation gives

$$\Delta\log(R^2 - l(p_k)^2) = -\frac{2l(p_k)\Delta l(p_k)}{R^2 - l(p_k)^2} - \frac{2R^2 + 2l(p_k)^2}{(R^2 - l(p_k)^2)^2} \ . \tag{2.15}$$

We may assume $r(q_k) \geq 1$. So from Lemma 2.1(B) we have

$$\begin{aligned}\Delta\log(R^2 - l(p_k)^2) \ &\geq \ -\frac{2C_1}{R^2 - l(p_k)^2} - \frac{2R^2 + 2l(p_k)^2}{(R^2 - l(p_k)^2)^2} \\ &= \ -2\frac{(C_1+1)R^2 - (C_1-1)l(p_k)^2}{(R^2 - l(p_k)^2)^2},\end{aligned}$$

where we set $C_1 = m - 1 + 2\alpha^2(2 + r(q_k)^2)$. We define the real valued function $\varphi(\xi, \eta)$ for $\xi, \eta \in \mathrm{R}$ by

$$\varphi(\xi, \eta) = (C_1 + 1)\xi^2 - (C_1 - 1)\eta^2. \tag{2.16}$$

Using (2.16) and (2.14) we have

$$0 \geq 4(\frac{-\alpha^2}{1+r(p_k)^2} + u(p_k)\beta^2) + \frac{3m-4}{(r(p_k)^\varepsilon + r(p_k)^2)} - \frac{4\varphi(R, l(p_k))}{(R^2 - l(p_k)^2)^2} \ , \tag{2.17}$$

Deviding (2.17) by $4\beta^2$, we get

$$\begin{aligned}u(p_k) \ &\leq \ \frac{1}{\beta^2}\{\frac{\alpha^2}{1+r(p_k)^2} - \frac{(3/4)m-1}{r(p_k)^\varepsilon + r(p_k)^2} + \frac{\varphi(R, l(p_k))}{(R^2 - l(p_k)^2)^2}\} \\ &\leq \ \frac{1}{\beta^2}\{\frac{\alpha^2}{1+r(p_k)^{2-\varepsilon}} + \frac{\varphi(R, l(p_k))}{(R^2 - l(p_k)^2)^2}\} \ .\end{aligned} \tag{2.18}$$

Here we assumed that $r(p_k) > 1$ from (2.13). So the last inequality follows from $m \geq 2$ and $1 + r(p_k)^{2-\varepsilon} < 1 + r(p_k)^2$. Since Φ attains its maximum at p_k , we obtain from (2.18) that

$$\begin{aligned}\Phi(q_k) \leq \Phi(p_k) \ &= \ u(p_k)(1 + r(p_k)^{2-\varepsilon})(R^2 - l(p_k)^2)^2 \\ &\leq \ \frac{\alpha^2(R^2 - l(p_k)^2)^2}{\beta^2} + \frac{(1 + r(p_k)^{2-\varepsilon})\varphi(R, l(p_k))}{\beta^2}.\end{aligned} \tag{2.19}$$

Deviding (2.19) by R^4 , and combining $\Phi(q_k) = u(q_k)(1 + r(q_k)^{2-\varepsilon})R^4$, we have

$$u(q_k)(1 + r(q_k)^{2-\varepsilon}) \leq \frac{\alpha^2}{\beta^2} + \frac{(1 + r(p_k)^{2-\varepsilon})\varphi(R, l(p_k))}{\beta^2 R^4}. \tag{2.20}$$

Next, we shall show that $(1 + r(p_k)^{2-\varepsilon})\varphi(R, l(p_k))/R^4$ tends to zero, when $R \to \infty$. The definition of C_1 implies $C_1 \geq 1$. Then we see that

$$\varphi(R, l) \leq (C_1 + 1)R^2$$

by (2.16). From (2.13) it follows that $r(p_k) \leq 2R$ if R is large enough. Hence we have

$$\frac{(1 + r(p_k)^{2-\varepsilon})\varphi(R, l(p_k))}{R^4} \leq \frac{(1 + (2R)^{2-\varepsilon})(C_1 + 1)R^2}{R^4}$$

for sufficiently large R . Since the right-hand side tends to zero when $R \to \infty$, thus we have proved

$$\lim_{R\to\infty} \frac{(1 + r(p_k)^{2-\varepsilon})\varphi(R, l(p_k))}{R^4} = 0.$$

This implies from (2.20) that

$$(1 + r(q_k)^{2-\varepsilon})u(q_k) \leq \frac{\alpha^2}{\beta^2}.$$

By the definition of q_k , $(1 + r(x)^{2-\varepsilon})u(x) \leq \frac{\alpha^2}{\beta^2}$ follows for all $x \in B_o(r(q_k))$. Letting $k \to \infty$ such that $r(q_k)) \to \infty$, we have

$$u(x) \leq \frac{\alpha^2}{\beta^2(1 + r(x)^{2-\varepsilon})}$$

on M .

Combining (2.11) we have proved that for arbitrarily small constatnt $\varepsilon > 0$

$$u(x) \leq \max\{\frac{\alpha^2}{\beta^2(1 + r(x)^{2-\varepsilon})}, \frac{\alpha^2}{\beta^2(1 + r(x)^2)}, \frac{\alpha^2(1 + r(q_\varepsilon)^{2-\varepsilon})}{\beta^2(1 + r(q_\varepsilon)^2)(1 + r(x)^{2-\varepsilon})}\}.$$

for all x in M , where q_ε is the maximum point of $(1 + r(x)^{2-\varepsilon})u(x)$ while it attains its maximum satisfying $r(q_\varepsilon) < 1$. Since the right-hand side is continuous relative to ε near $\varepsilon = 0$, then we have

$$u(x) \leq \frac{\alpha^2}{\beta^2(1 + r(x)^2)}$$

on M . This completes the proof.

Next, we shall estimate a growth of a holomorphic map from a Kähler CH manifold whose Ricci curvature satisfies the condition of Theorem 2.1. The following corollaries are immediately applications of Theorem 2.1.

COROLLARY 2.1. *Let M be a Kähler CH manifold of complex dimension $m (m \geq 2)$, and N be a Hermitian manifold of complex dimension n with holomorphic bisectional curvature bounded from above by $-\beta^2$ $(\beta > 0)$. Suppose the Ricci curvature of M satisfies $\mathrm{Ric} \geq -\alpha^2/(1 + r(x)^2)$, where $r(x)$ is the distance function from a fixed point o of M . And let f be a holomorphic map from M to N . Then we have*

$$\mathrm{dist}_N(f(x), f(o)) \leq \frac{\alpha}{\beta} \log\{r(x) + (r(x)^2 + 1)^{1/2}\}$$

on M .

Proof. We may assume that f is a nonconstant map. We denote by $|X|_M$ and $|Y|_N$ the length of tangent vectors $X \in TM, Y \in TN$ measured by each metric of M and N respectively. Let $\gamma(t)$ be a unit speed geodesic in M from o to x . So we can define the piecewise-smooth curve $\sigma(t)$ on N by $\sigma(t) = f(\gamma(t))$. Hence we have,

$$\mathrm{dist}_N(f(x), f(o)) \leq \int_0^{r(x)} |\dot{\sigma}(t)|_N \, dt$$

We see that $|\dot{\sigma}(t)|_N^2 = |f_*(\dot{\gamma}(t))|_N^2 = f^* ds_N^2(\dot{\gamma}(t), \dot{\gamma}(t))$. And Theorem 2.1 gives

$$|\dot{\sigma}(t)|_N^2 \leq \frac{\alpha^2}{\beta^2(1 + r(\gamma(t))^2)} ds_M^2(\dot{\gamma}(t), \dot{\gamma}(t)).$$

Since $\gamma(t)$ is parametrized by its arclength, then we have

$$\begin{aligned} \int_0^{r(x)} |\dot{\sigma}(t)|_N \, dt &= \int_0^{r(x)} \frac{\alpha}{\beta(1 + t^2)^{1/2}} dt \\ &= \frac{\alpha}{\beta} \log\{r(x) + (r(x)^2 + 1)^{1/2}\}. \end{aligned}$$

Thus we complete the proof.

COROLLARY 2.2. *Let M be a Kähler CH manifold of complex dimension $m(m \geq 2)$. Suppose the Ricci curvature of M satisfies* $\mathrm{Ric} \geq -\alpha^2/(1+r(x)^2)$ *, where $r(x)$ is the distance function from a fixed point o of M . If there is a bounded holomorphic function f on M such that $f(o)=0$, then we have*

$$|f(x)| \leq \{\sup_{x\in M} |f(x)|\} \frac{(2r(x)+1)^{\alpha/2}-1}{(2r(x)+1)^{\alpha/2}+1}$$

on M .

Proof. We may assume that $\sup|f|=1$ deviding f by $\sup|f|$. Therefore we can regard f as a holomorphic map from M into the unit disc D in $\mathbf{C}$. Let ds_D^2 be the Poincaré-Bergman metric of D defined by

$$ds_D^2 = \frac{1}{(1-|z|^2)^2} dz \wedge d\bar{z},$$

which has the constant holomorphic curvature -4 . Mesuring the distance from o to $z \in D$ by this metric , it follows that

$$\mathrm{dist_D(z,0)} = \frac{1}{2}\log\frac{1+|z|}{1-|z|}.$$

Combining Corollary 2.1, we have

$$\begin{aligned}\log\frac{1+|f(x)|}{1-|f(x)|} &\leq \frac{\alpha}{2}\log\{r(x)+(r(x)^2+1)^{1/2}\} \\ &\leq \frac{\alpha}{2}\log(2r(x)+1).\end{aligned}$$

Hence we obtain

$$\frac{1+|f(x)|}{1-|f(x)|} \leq (2r(x)+1)^{\alpha/2}.$$

Simplifying above equation, we get

$$|f(x)| \leq \frac{(2r(x)+1)^{\alpha/2}-1}{(2r(x)+1)^{\alpha/2}+1}$$

for all $x \in M$. This completes the proof.

2.8 Quadratic growing condition

In this section we shall extend the general Schwarz lemma for Kähler CH manifolds whose Ricci curvature is greater than a negative function with quadratic growth. The point of Theorem 2.2 is that it is the Schwarz-type lemma under unbounded Ricci curvature conditions. And Theorem 2.2 also shows that every holomorphic map between manifolds satisfying the conditions of the theorem is destance-decreasing up to their Ricci curvatures. It is described as follows.

THEOREM 2.2. *Let* M *be a Kähler CH manifold of complex dimension* $m(m \geq 1)$, *and* N *be a Hermitian manifold of complex dimension* n *with holomorphic bisectional curvature bounded from above by* $-\beta^2$ $(\beta > 0)$. *Suppose the Ricci curvature of* M *satisfies* $\mathrm{Ric} \geq -\alpha^2(1+r(x)^2)$, *where* $r(x)$ *is the distance function from a fixed point* o *of* M . *Then if there is a nonconstant holomorphic map* f *from* M *to* N , *we have*

$$f^*ds_N^2 \leq \frac{C_{m,\alpha}(1+r(x)^2)}{\beta^2} ds_M^2,$$

where the constant $C_{m,\alpha}$ *is given by* $C_{m,\alpha} = \max\{3\alpha^2, 4m+2\alpha^2\}$.

Proof. In the first, we have from direct computation that

$$\begin{aligned} \Delta \log(1+r(x)^2) &= \frac{2}{1+r(x)^2}\{r(x)\Delta r(x) + 1 - \frac{2r(x)^2}{1+r(x)^2}\} \\ &\leq \frac{2}{1+r(x)^2}\{r(x)\Delta r(x) + 1\}. \end{aligned}$$

According to Lemma 2.2(A), we have on M

$$\begin{aligned} \Delta \log(1+r(x)^2) &\leq \frac{2\{2m + \alpha^2 r(x)^2/3 + \alpha^2 r(x)^4/5\}}{1+r(x)^2} \\ &\leq \frac{4m}{1+r(x)^2} + \frac{2\alpha^2 r(x)^2}{3}. \end{aligned} \qquad (2.21)$$

Note that Lemma 2.2(A) holds for $q \neq o$, but (2.21) holds globally on M since $\Delta \log(1+r(x)^2)$ is continuous on M . Let $u(x)$ be the general elementary symmetric function associated with f . So it suffices to prove that

$$u(x) \leq \frac{C_{m,\alpha}(1+r(x)^2)}{\beta^2}$$

on M . After the manner of the proof of Theorem 2.1, we define a C^∞ function $u(x)/(1+r(x)^2)$ on M . Then either $u(x)/(1+r(x)^2)$ attains its supremum at some point or, there is a sequence $\{q_k\}$ in M such that

$$\limsup_{k\to\infty} \frac{u(q_k)}{1+r(q_k)^2} = \sup_{x\in M} \frac{u(x)}{1+r(x)^2} .$$

We suppose that $u(x)/(1+r(x)^2)$ attains its supremum at some point $q \in M$. So $\log\{u(x)/(1+r(x)^2)\}$ is well defined near q . We see that $\log\{u(x)/(1+r(x)^2)\}$ also achieves its maximum at q . From the maximum principle we see that

$$\Delta \log \frac{u(q)}{1+r(q)^2} \leq 0.$$

Hence we have

$$0 \geq \Delta \log \frac{u(q)}{1+r(q)^2} = \Delta \log u(q) - \Delta \log(1+r(q)^2).$$

Using (2.6) and (2.21), we get

$$0 \;\geq\; 4\{-\alpha^2(1+r(q)^2) + u(q)\beta^2\} - \frac{4m}{1+r(q)^2} - \frac{2\alpha^2 r^2}{3}$$

at q . Therefore we obtain

$$\begin{aligned} \frac{u(q)}{1+r(q)^2} &\leq \frac{\alpha^2 + 4m/(1+r(q)^2)^2 + \alpha^2 r(q)^2/(1+r(q)^2)}{\beta^2} \\ &\leq \frac{2\alpha^2+4m}{\beta^2}. \end{aligned}$$

Since $\dfrac{u(x)}{1+r(x)^2} \leq \dfrac{u(q)}{1+r(q)^2}$, we have proved that if $\dfrac{u(x)}{1+r(x)^2}$ attains its supremum at some point in M , then for all $x \in M$

$$\frac{u(x)}{1+r(x)^2} \leq \frac{2\alpha^2+4m}{\beta^2}. \tag{2.22}$$

Suppose that $u/(1+r^2)$ does not achieve its supremum in any compact subset in M . So we can find a sequence $\{q_k\}$ in M satisfying $r(q_k) < r(q_{k+1})$, $\lim_{k\to\infty} r(q_k) = \infty$ and

$$\sup_{x\in B_o(r(q_k))} \frac{u(x)}{1+r(x)^2} = \frac{u(q_k)}{1+r(q_k)^2}, \tag{2.23}$$

for all integer k . So it follows that

$$\limsup_{k\to\infty} \frac{u(q_k)}{1+r(q_k)^2} = \sup_{x\in M} \frac{u(x)}{1+r(x)^2}.$$

Let $B_k(R)$ be the R -ball around q_k , and $l(x)$ be the distance from q_k . So we define a real-valued C^∞ function $\Phi(x)$ on $B_k(R)$ by

$$\Phi(x) = \frac{(R^2 - l(x)^2)^2 u(x)}{1+r(x)^2}.$$

We see that $\Phi(x)$ attains its maximum at an interior point p_k of $B_k(R)$ since $\Phi(x) \geq 0$ in $B_k(R)$ and $\Phi(x) = 0$ on $\partial B_k(R)$. From (2.23), we see

$$r(q_k) \leq r(p_k) \leq r(q_k) + R\ . \tag{2.24}$$

Since $\Phi(p_k) > 0$, we get $\Delta \log \Phi(p_k) \leq 0$. Hence we have at p_k ,

$$\begin{aligned} 0 &\geq \Delta \log \frac{(R^2 - l(p_k)^2)^2 u(p_k)}{1+r(p_k)^2} \\ &= 2\Delta \log(R^2 - l(p_k)^2) + \Delta \log u(p_k) - \Delta \log(1+r(p_k)^2). \end{aligned}$$

Since (2.6) and (2.21) , we obtain

$$\begin{aligned} 0 &\geq 4(-\alpha^2(1+r(p_k)^2) + u(p_k)\beta^2) - \frac{4m}{1+r(p_k)^2} \\ &- \frac{2\alpha^2 r(p_k)^2}{3} + 2\Delta \log(R^2 - l(p_k)^2) \end{aligned} \tag{2.25}$$

at p_k . We define a real valued function $\varphi(R,\xi,\eta)$ of R, ξ and $\eta \in \mathbf{R}$ by

$$\begin{aligned} \varphi(R,\xi,\eta) &= R^2 + \xi^2 \\ &+ \alpha^2(R^2 - \xi^2)\{\frac{2m-1}{\alpha^2} + \frac{(1+\eta^2)\xi^2}{3} + \frac{\eta\xi^3}{2} + \frac{\xi^4}{5}\}. \end{aligned} \tag{2.26}$$

So using (2.15) and Lemma 2.2(B), the fourth term of the right-hand side of (2.25) becomes

$$\begin{aligned}\Delta\log(R^2-l(p_k)^2) &= -\frac{2R^2+2l(p_k)^2}{(R^2-l(p_k)^2)^2}-\frac{2\alpha^2}{R^2-l(p_k)^2}\{\frac{2m-1}{\alpha^2}\\ &+ \frac{(1+r(q_k)^2)l(p_k)^2}{3}+\frac{r(q_k)l(p_k)^3}{2}+\frac{l(p_k)^4}{5}\}\\ &= -\frac{2\varphi(R,l(p_k),r(q_k))}{(R^2-l(p_k)^2)^2}, \qquad (2.27)\end{aligned}$$

Substituting (2.27) into (2.25) we have at p_k

$$\begin{aligned}0 &\geq 4\{-\alpha^2(1+r(p_k)^2)+u(p_k)\beta^2\}-\frac{4m}{1+r(p_k)^2}\\ &- \frac{2\alpha^2 r(p_k)^2}{3}-\frac{4\varphi(R,l(p_k),r(q_k))}{(R^2-l(p_k)^2)^2}. \qquad (2.28)\end{aligned}$$

Deviding (2.28) by $4(1+r(p_k)^2)\beta^2$ and simplifying it we have

$$\frac{u(p_k)}{1+r(p_k)^2}\leq\frac{1}{\beta^2}\{\frac{7}{6}\alpha^2+\frac{m}{(1+r(x)^2)^2}+\frac{\varphi(R,l(p_k),r(q_k))}{(R^2-l(p_k)^2)^2(1+r(p_k)^2)}\}. \qquad (2.29)$$

Since Φ attains its maximum at p_k , we obtain from (2.29)

$$\Phi(q_k)\leq\Phi(p_k)\leq\{\frac{7\alpha^2(R^2-l(p_k)^2)^2}{6\beta^2}+\frac{m(R^2-l(p_k)^2)^2}{\beta^2(1+r(p_k)^2)^2}+\frac{\varphi(R,l(p_k),r(q_k))}{\beta^2(1+r(p_k)^2)}\}.$$

Hence we get from $\Phi(q_k)=u(q_k)R^4/(1+r(q_k)^2)$ that

$$\frac{u(q_k)}{1+r(q_k)^2}\leq\frac{7\alpha^2}{6\beta^2}+\frac{m}{\beta^2(1+r(p_k)^2)^2}+\frac{\varphi(R,l(p_k),r(q_k))}{R^4\beta^2(1+r(p_k)^2)}.$$

We may put $R=r(q_k)$, then from $R>l(p_k)$ and (2.26) we see

$$\varphi(R,l(p_k),r(q_k))<(2m+1)r(q_k)^2+\frac{3}{2}\alpha^2 r(q_k)^6.$$

From (2.24) we have

$$\frac{u(q_k)}{1+r(q_k)^2}<\frac{7\alpha^2}{6\beta^2}+\frac{3m+1}{\beta^2 r(q_k)^4}+\frac{3\alpha^2}{2\beta^2}.$$

Therefore we obtain

$$\frac{u(x)}{1+r(x)^2}<\frac{3\alpha^2}{\beta^2}+\frac{3m+1}{\beta^2 r(q_k)^4}$$

for $x \in B_o(r(q_k))$. Letting $k \to \infty$ such that $r(q_k) \to \infty$, then we have proved that

$$\frac{u(x)}{1+r(x)^2} \leq \frac{3\alpha^2}{\beta^2}$$

on M if $u(x)/(1+r(x)^2)$ does not attain its supremum. Combining (2.22) we complete the proof.

Note that we need not the condition $m \geq 2$ in Theorem 2.2. Using the estimate of Theorem 2.2, we can get the following corollaries.

COROLLARY 2.3. *Let M be a Kähler CH manifold of complex dimension $m (m \geq 2)$, and N be a Hermitian manifold of complex dimension n with holomorphic bisectional curvature bounded from above by $-\beta^2$ $(\beta > 0)$. Suppose the Ricci curvature of M satisfies $\mathrm{Ric} \geq -\alpha^2(1+r(x)^2)$, where $r(x)$ is the distance function from a fixed point o of M . And let f be a holomorphic map from M to N . Then we have*

$$\mathrm{dist}_N(f(x), f(o)) \leq \frac{\sqrt{C_{m,\alpha}}}{2\beta}\{2r(x) + r(x)^2\}$$

on M , where $C_{m,\alpha} = \max\{3\alpha^2, 4m + 2\alpha^2\}$.

Proof. We may also assume that f is a nonconstant map. We denote by $|X|_M$ and $|Y|_N$ the length of tangent vectors $X \in TM, Y \in TN$ measured by each metric of M and N respectively. Let $\gamma(t)$ be a unit speed geodesic in M from o to x . So we can define the piecewise-smooth curve $\sigma(t)$ on N by $\sigma(t) = f(\gamma(t))$. Hence we have,

$$\mathrm{dist}_N(f(x), f(o)) \leq \int_0^{r(x)} |\dot{\sigma}(t)|_N \, dt$$

We see that

$$|\dot{\sigma}(t)|_N^2 = |f_*(\dot{\gamma}(t))|_N^2 = f^* ds_N^2(\dot{\gamma}(t), \dot{\gamma}(t)).$$

And Theorem 2.2 gives

$$|\dot\sigma(t)|_N^2 \leq \frac{C_{m,\alpha}(1+r(\gamma(t))^2)}{\beta^2} ds_M^2(\dot\gamma(t),\dot\gamma(t)).$$

Since $\gamma(t)$ is parametrized by its arclength, then we have

$$\begin{aligned}\int_0^{r(x)} |\dot\sigma(t)|_N\,dt &= \int_0^{r(x)} \frac{\sqrt{C_{m,\alpha}(1+t^2)}}{\beta} dt \\ &\leq \frac{\sqrt{C_{m,\alpha}}(2r(x)+r(x)^2)}{2\beta}.\end{aligned}$$

Thus we complete the proof.

COROLLARY 2.4. *Let M be a Kähler CH manifold of complex dimension $m (m \geq 1)$. Suppose the Ricci curvature of M satisfies $\mathrm{Ric} \geq -\alpha^2(1+r(x)^2)$, where $r(x)$ is the distance function from a fixed point o of M. If there is a bounded holomorphic function f on M such that $f(o) = 0$, then we have*

$$|f(x)| \leq \{\sup_{x\in M}|f|\} \tanh\{\frac{\sqrt{C_{m,\alpha}}(2r(x)+r(x)^2)}{8}\}$$

on M, where $C_{m,\alpha} = \max\{3\alpha^2, 4m+2\alpha^2\}$.

Proof. We may assume that $\sup|f| = 1$ deviding f by $\sup|f|$. Therefore we can regard f as a holomorphic map from M into the unit disc D in C. Let ds_D^2 be the Poincaré-Bergman metric of D defined by

$$ds_D^2 = \frac{1}{(1-|z|^2)^2} dz \wedge d\bar z,$$

which has the constant holomorphic curvature -4 . Mesuring the distance from o to $z \in D$ by this metric , it follows that

$$\mathrm{dist_D(z,0)} = \frac{1}{2}\log\frac{1+|\mathrm{z}|}{1-|\mathrm{z}|}.$$

From Corollary 2.3, we have

$$\log\frac{1+|f(x)|}{1-|f(x)|} \leq \frac{\sqrt{C_{m,\alpha}}}{4}\{2r(x)+r(x)^2\}$$

Hence we obtain

$$\frac{1+|f(x)|}{1-|f(x)|} \leq \exp\{\frac{\sqrt{C_{m,\alpha}}}{4}(2r(x)+r(x)^2)\}$$

Simplifying above equation, we get

$$|f(x)| \leq \tanh \frac{\sqrt{C_{m,\alpha}}(2r(x)+r(x)^2)}{8}$$

for all $x \in M$. This completes the proof.

Remark. In the last, the we remark that it is still an open problem whether the exponent 2 of $r(x)$ appeared in those theorems is best possible or not. However we think it is not easy to improve the exponent larger than 2 after the same manner.

Bibliography

[1] L. V. Ahlfors, An extension of Schwarz's lemma, *Trans. Amer. Math. Soc.* **43** (1938),359-364.

[2] S. Bochner and W. T. Martin, Functions of several complex variables, Princeton University Press, Princeton, N.J. 1948.

[3] S. S. Chern, On holomorphic mappings of hermitian manifolds of the same dimension, *Proc. Sym. Pure Math.* **11**,*AMS Prov., R.I.* (1968),157-170.

[4] R. E. Greene and H. Wu, Function theory on manifolds which posses a pole, *Lecture Notes in Math.* **699**, Springer-Verlag, New York,1979.

[5] S. Kobayash, Invariant distances on complex manifolds and holomorphic mappings, *J. Math. Soc. Japan* **19**,4 (1967),460-480. *Proc. Sym. Pure Math.* **11**,*AMS Prov., R.I.* (1968),157-170.

[6] S. Kobayash, Volume elements, holomorphic mappings and Schwarz's lemma, *Proc. Sym. Pure Math.* **11**,*AMS Prov., R.I.* (1968),253-260.

[7] Y.C. Lu, Holomorphic mappings of complex manifolds, *J. Differential Geom.* **2**, (1968),299-312.

[8] J. Milnor, On deciding whether a surface is parabolic or hyperbolic, *Amer. Math. Monthly* **84** (1977),43-46.

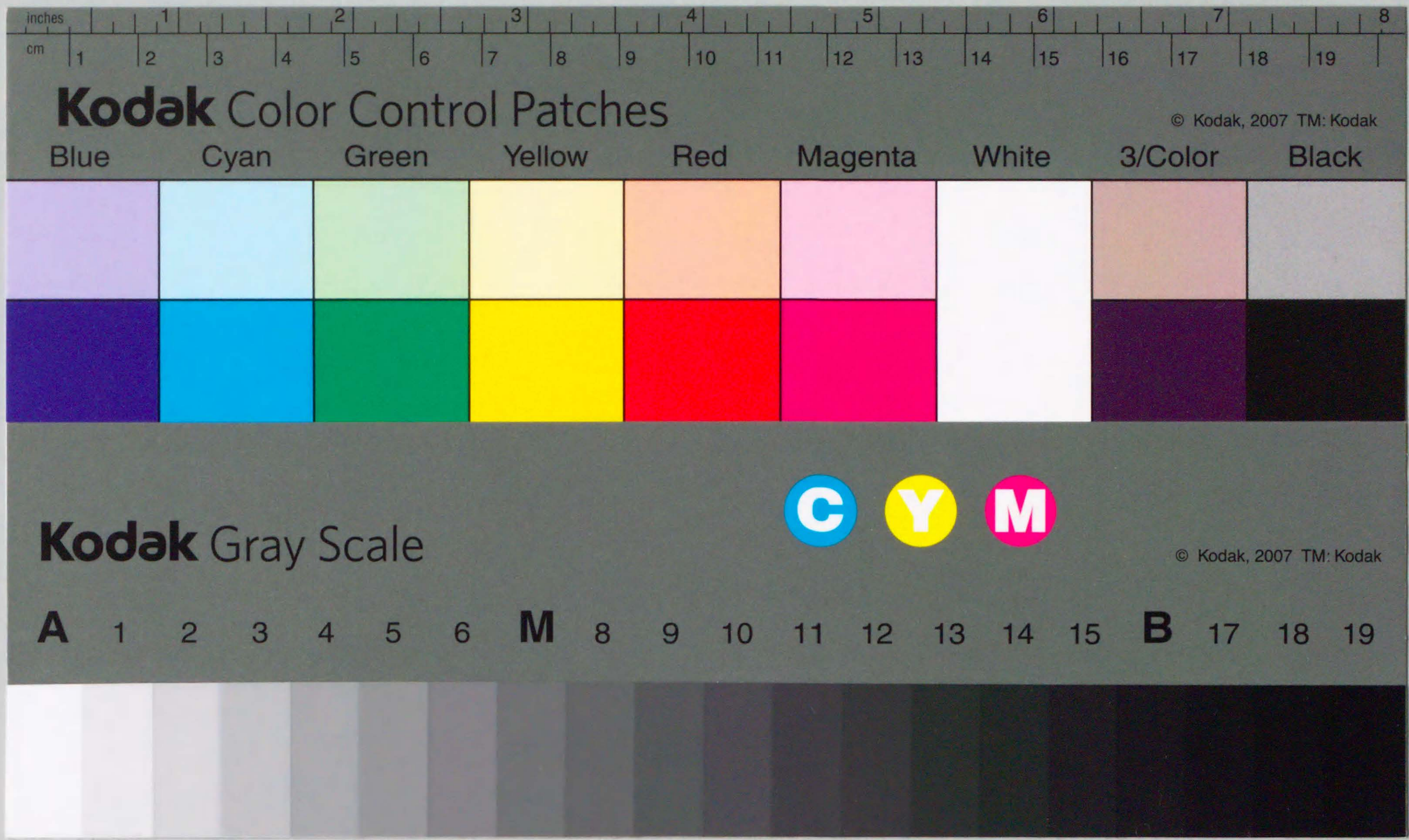
[9] K. Shiga, A geometric characterization of $\mathbf{C}^n$ and open balls, *Nagoya Math. J.* **75** (1979),145-150.

[10] Y. T. Siu and S. T. Yau, Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay, *Ann. of Math.* **105** (1977),225-264.

[11] S. T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure and Appl. Math.* **28** (1975),201-228.

[12] S. T. Yau, A general Schwarz lemma for Kähler manifolds, *Amer. J. Math.* **100** (1978),197-204.

[13] H. Wu, Liouville theorems, Complex Analysis III, *Lecture Notes in Math.* **1277**,331-349, Springer-Verlag, New York,1987.



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