NUMERICAL ANALYSIS OF RADAR CROSS-SECTIONS OF CONDUCTING CYLINDERS EMBEDDED IN STRONG CONTINUOUS-RANDOM MEDIA

https://doi.org/10.11501/3120487
NUMERICAL ANALYSIS OF RADAR CROSS-SECTIONS OF CONDUCTING CYLINDERS EMBEDDED IN STRONG CONTINUOUS-RANDOM MEDIA

ZH. QI MENG
NUMERICAL ANALYSIS OF RADAR CROSS-SECTIONS OF CONDUCTING CYLINDERS EMBEDDED IN STRONG CONTINUOUS-RANDOM MEDIA
Contents

1 Introduction

1.1 Backgrounds and objective .................................................. 1
1.2 Organization on the thesis ..................................................... 3

2 Scattering theory

2.1 Scattering from a conducting body in an inhomogeneous medium ...... 5
2.2 Boundary conditions ............................................................ 7
2.3 A model of scattering and its formulation .................................... 8
2.4 Current generators ............................................................... 12
   2.4.1 Expression under the Dirichlet condition .............................. 12
   2.4.2 Expression under the Neumann condition .............................. 15
2.5 Re-incident waves ............................................................... 15
## Contents

3 Numerical analysis

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Formulation</td>
<td>18</td>
</tr>
<tr>
<td>3.1.1 The second moment of Green's function</td>
<td>22</td>
</tr>
<tr>
<td>3.1.2 The fourth moment of Green's function</td>
<td>24</td>
</tr>
<tr>
<td>3.2 RCS of a conducting circular cylinder</td>
<td>26</td>
</tr>
<tr>
<td>3.2.1 The current generator on a circular cylinder</td>
<td>26</td>
</tr>
<tr>
<td>3.2.2 RCS calculated from coherent scattered waves</td>
<td>27</td>
</tr>
<tr>
<td>3.2.3 Numerical results of RCS</td>
<td>27</td>
</tr>
<tr>
<td>3.3 RCS of a conducting elliptic cylinder</td>
<td>39</td>
</tr>
<tr>
<td>3.3.1 The current generator on an elliptic cylinder</td>
<td>39</td>
</tr>
<tr>
<td>3.3.2 Numerical results of RCS</td>
<td>40</td>
</tr>
</tbody>
</table>

4 Concluding remarks

Acknowledgments

Bibliography

Appendices
Contents

A. Integral equations for random surface currents .................................. 62
B. A construction of the scattering problem by Yasuura's method ............... 65
C. A solution of the second moment equation ........................................ 68
D. A simplified form of the fourth moment ........................................... 70
E. Current generators of a conductor circular cylinder .............................. 73
F. Current generators of a conductor elliptic cylinder ................................ 75
Chapter 1

Introduction

1.1 Backgrounds and objective

When a wave propagates in a medium of which the parameter is randomly varying in time and space, such as the atmosphere, underwater, biological medium and so on, the amplitude and phase of the wave may also fluctuate randomly in time and space. Communication engineers are concerned with the amplitude and phase fluctuations, the coherence time and coherence bandwidth of waves as the waves propagate through atmospheric and ocean turbulence. The detection of clear air turbulence by a radar technique contributes significantly to safe navigation. Geophysicists are interested in the use of wave fluctuations that occur due to propagation through planetary atmospheres in order to remotely determine their turbulence and dynamic characteristics. Bioengineers may use the fluctuation and scattering characteristics of a sound wave as a diagnostic tool. Radar engineers may need to concern themselves with clutter echoes produced by storms, rain, snow, or hail. Therefore the problem of wave propagation and scattering in random media has become increasingly important, particularly in the areas of communication, detection and remote-sensing.

Actually, the problem has been studied for a long time. At first the geometrical optics
method, the usual perturbation method which had been effectively used for analysis of a fluctuated field in an inhomogeneous medium, and the smooth perturbation method were used for analyzing a fluctuated field in a random medium. Although some theoretical results were obtained, it came into notice that the methods were not suitable for many cases, and finding a more accurate and universally applicable method became necessary.

The multiple scattering theory on waves in random media was extensively studied in 1960s. By that time, potentials of the geometrical-optics and perturbation method had been exhausted and the multiple scattering theory invoked from quantum theory was brought into play. Many derivation methods of moment equations of waves in random media were presented from different standpoints after 1970\cite{1-5}. These methods were systematized by M. Tateiba, and he presented a more general method and gave a minute description about applicability limits of the method in 1974\cite{4}. After that, another methods such as the method based on the path integrals have been presented\cite{6, 7}

The multiple scattering theory has been applied to many practical cases (e.g., see references \cite{8-13}). As a special phenomenon of wave scattering in a random medium, backscattering enhancement of waves has been one of the important subject of this area all along and has been investigated from an academic point of view\cite{14-23}. It has thereby been said to be a fundamental phenomenon in a random medium\cite{21, 23} and to be produced by statistical coupling of incident and backscattered waves due to the effect of double passage\cite{15}. When a body is surrounded with a random medium, it may then happen that the radar cross-section (RCS) of the body is remarkably different from that in free space. If the body is regarded as a single point and the backscattering enhancement occurs prominently, RCS of the body has generally been taken to be nearly twice as large as that in free space.
Chapter 1. Introduction

From a practical point of view, a body cannot be regarded as a single point in many cases. For instance, it is important to analyze the RCS of a large body embedded in a random medium for the field of remote sensing or radar engineering. Therefore, the problem of wave scattering by a body in a random medium needs to be treated by taking account of the boundary conditions of incident and scattered waves on the body. The problem has not, however, been analyzed as boundary value problems.

Recently a method for solving the problem and numerical results based on the method have been presented for some cases[24-33]. The method is based on general results of both the independent studies on wave scattering from a conducting body in free space and on wave propagation and scattering in random media. In this paper, we show that the method is effective for solving the present problem as a boundary value problem, and try to conclude general characteristics of RCS of a body embedded in a continuous-random medium by the numerical analysis.

1.2 Organization on the thesis

The thesis consists of four chapters.

After this introduction, chapter 2 describes how the problem is formulated as boundary value problems. Two operators are introduced: Green's function in a random medium and the current generator which transforms incident waves into surface currents on the body surface. Here, a representative form of the Green's function is not required but the moments are done for the analysis of the average quantities concerning observed waves, and the current generator is a non-random operator which depends only on the body surface. Construction of the moments and the current generator is discussed.
In chapter 3, the method described in chapter 2 is first applied to the analysis of wave scattering from a conducting circular cylinder in a continuous-random medium, and the RCS is calculated for the cases of E-wave and H-wave incidence. The numerical results show that the RCS depends not only on the radius of the circular cylinder but also on the coherence of incident waves. After that, in order to make certain the effect of the coherence under different situations of surface curvature, RCS of an elliptic cylinder is analyzed in section 3.3. Many numerical results show the complicated changes of RCS under various situations, and lead to general characteristics of RCS of conducting bodies with convex surfaces embedded in strong continuous-random media.

Chapter 4 is denoted to the summary of this paper and the discussion about forthcoming subjects.

The time factor exp(−iωt) is assumed and suppressed throughout the paper.
Chapter 2

Scattering theory

In this chapter, we describe a method for solving the boundary value problem of wave scattering from a conducting body of arbitrary shape and size in a random medium, by introducing a current generator which transforms incident waves into surface currents on the body.

2.1 Scattering from a conducting body in an inhomogeneous medium

When dealing with a realization of a random medium, the present problem may be regarded as wave scattering from a conducting body in an inhomogeneous medium. Geometry of the problem is shown in figure 2.1 where the coordinate system also is done. Assume for simplicity that the dielectric constant of the medium is a function of location: \( \varepsilon = \varepsilon(\mathbf{r}) \), \( \mathbf{r} = (x, y, z) \), the magnetic permeability \( \mu \) is constant: \( \mu = \mu_0 \) and the electric conductivity \( \sigma = 0 \). In addition, it is assumed that \( \varepsilon(\mathbf{r}) \) is a varying function inside a sphere of radius \( L \) around the body, with the body size \( a \ll L \), and that \( \varepsilon(\mathbf{r}) = \varepsilon_0 \), a constant, elsewhere. Suppose that \( \varepsilon(\mathbf{r}) \) is a piecewise smooth function. Then it may be approximately expressed
in terms of the Fourier series or the wavelets in the three-dimensional region. Even if $\varepsilon(\mathbf{r})$ is expressed in such a form, it is not easy to obtain wave functions in the medium except for the one-dimensional case. This shows that in the case where a conducting body of arbitrary shape and size is surrounded with an inhomogeneous medium, we have no method useful for analyzing generally the wave scattering as boundary value problems. Accordingly, if this is forced to be combined with the fact that an inhomogeneous medium is a realization of a random medium, it may be accepted that when $\varepsilon(\mathbf{r})$ is a random function, it is difficult to find a method for analyzing wave scattering from a body in a random medium as well.

In wave scattering and propagation in random media, we are concerned about not each realization of waves but the moments; and they have been in part obtained for many practical cases. To solve the scattering problem, however, we need to know the moments of surface (electric and magnetic) currents induced by waves or to obtain directly the moments
of scattered waves from known incident waves, by fitting the boundary conditions. How to fulfill this requirement will be described in the following sections.

### 2.2 Boundary conditions

Let $\varepsilon(r)$ defined in the previous section be a random function throughout the paper from now. It can be expressed as

$$\varepsilon(r) = \varepsilon_0[1 + \delta\varepsilon(r)]$$

(2.1)

Here $\delta\varepsilon(r)$ is a continuous random function with the zero mean: $\langle \delta\varepsilon(r) \rangle = 0$ for a continuous medium and $\delta\varepsilon(r) = \sum_{i=1}^{N} \varepsilon_i(r)$ for a discrete random medium, where $\varepsilon_i(r)$ is a random function of position, dielectric constant, shape, size and orientation of the $i$-th scatterer, and $N$ is the number of random scatterers and is very large. In addition, $\delta\varepsilon(r)$ is assumed to be a bounded function:

$$|\delta\varepsilon(r)| < \infty$$

(2.2)

The surface of the body is assumed to be expressed by a smooth function in order to construct operators on the surface in section 2.4. Even on the assumption, the surface changes according as physical situations; for example, it may be regarded as a rough surface or a coated surface with a material, when particles stick partly on the surface. In this paper, we assume that an infinitesimal thin layer of free space exists between the surface and the medium and finally the thickness of the layer tends to zero, as shown in figure 2.2. Accordingly, we can assume a smooth surface and impose two types of boundary condition on wave fields on the body: the Dirichlet condition (DC) and the Neumann condition (NC). The former is used for the electric fields tangential to the body surface and for the magnetic field perpendicular to the body surface, and the latter is used for the magnetic field tangential to the surface of an infinite uniform cylinder. They are expressed for the
field $u$ as

\begin{align}
  u(r) &= 0, \text{ for DC} \tag{2.3} \\
  \frac{\partial}{\partial n} u(r) &= 0, \text{ for NC} \tag{2.4}
\end{align}

where $r$ is on the surface of the body $S$, and $\partial/\partial n$ denotes the outward normal derivative at $r$ on $S$.

Figure 2.2: A model of the boundary between a body and a medium.

### 2.3 A model of scattering and its formulation

According to Appendix A, using Green's function in the random medium, we can obtain integral equations for surface currents on the surface of the body; and then using the
Chapter 2. Scattering theory

solutions of equations, we can express the scattered waves. However, it is also shown that the methods based on integral equations for surface currents on the body are not applicable to the present scattering problem. The reason is that these surface currents are obtained as the solutions of statistically nonlinear equations constructed by the random incident or re-incident waves and the random Green’s function. Instead of obtaining the surface currents directly, we therefore try to express them approximately. Consider the scattering problem qualitatively as follows, referring to figure 2.3. An electromagnetic wave radiated from a source of which the position $r_t$ is beyond the random medium: $r_t > L$, propagates in the random medium, illuminates the body and induces a surface current on the body. A scattered wave from the body is produced by the surface current and propagates in the random medium; then, a part of the scattered wave is scattered by the random medium in the backward direction toward the body and is re-incident on the body. The re-incident wave produces a new surface current and a new scattered-wave. This iteration leads to a general solution of the scattering problem. Of course, observed waves at an observation point are, in general, obtained as the sum of above scattered-waves and the wave scattered only by the random medium.

In above scattering process, the surface current is given as the sum of each surface current produced by the $n$-th re-incident wave where $n = 0, 1, 2, \cdots$ and $n = 0$ means direct incidence. The transform of the $n$-th re-incident wave into the surface wave is performed on the surface of the body. The effects of the random medium are included in the $n$-th re-incident wave and are also done in the surface current only through the transformation. Accordingly, to formulate the scattering process in a solvable form, we introduce a current generator which transforms random incident waves directly into random surface currents on the body and which is a deterministic operator dependent on the body surface. We also introduce the Green’s function which transforms the source distribution
Figure 2.3: A model of scattering.
Chapter 2. Scattering theory

into the incident wave and also transforms the surface current into the scattered wave. According to Appendix A, the Green's function may be approximately obtained under the condition $L \gg a$ as the Green's function in the random medium where the body is replaced with the same random medium.

Using the Green's function and the current generator, let us formulate the scattering problem. The incident wave expressed in terms of the source distribution and the Green's function is transformed into the surface current by the current generator, and the first scattered wave is expressed in terms of the surface current and the Green's function. From the scattered wave, we may express the first re-incident wave, i.e., the second incident wave (see section 2.5). In this way, the $n$-th scattered wave may be expressed and hence the scattered wave may be obtained as the sum of them. Consequently, an approach to the scattering problem can be described schematically as figure 2.4.

![Figure 2.4: Schematic diagram for solving the scattering problem where a conducting body is surrounded with a random medium.](image)

As mentioned in section 2.1, the moments of the Green's function are required and may be approximately obtained in some practical cases by applying the multiple scattering theory of wave propagation in random media. On the other hand, the current generator must be defined and shown to be constructed. This will be done in the following section.
2.4 Current generators

As mentioned in the previous section, a current generator is an operator which transforms incident waves into surface currents on the body. Here, let us designate the incident wave by $u_{in}$, the scattered wave by $u_s$, and the total wave by $u$: $u = u_{in} + u_s$, where $u_{in}$ includes both waves: the incident wave independent of the body and the re-incident wave (see figure 2.3). Surface currents at a point on the body depend on $u_{in}$ on the overall surface. According as the boundary conditions, the current generators, written as $Y$, may be defined on the body as follows:

\[
\frac{\partial u(r)}{\partial n} = \int_S Y_E(r|r')u_{in}(r')dr', \quad \text{for DC} \\
u(r) = \int_S Y_H(r|r')u_{in}(r')dr', \quad \text{for NC}
\]

As mentioned in section 2.3, the $Y$ is a deterministic operator which is dependent on the body surface and independent of the random medium and $u_{in}(r)$.

Above description suggests that $Y$ can be constructed in the case where the body is in free space of $\varepsilon(r) \equiv 0$. Let us try to express $Y$ in an explicit form, which expression can be made in case that the body surface is smooth by applying Yasuura’s method[34-36]. It is a general method for analyzing the scattered wave and the surface current, and is simply described in Appendix B from a viewpoint of operator construction.

2.4.1 Expression under the Dirichlet condition

Let us put $\varepsilon(r) = \varepsilon_0$ in figure 2.1. According to Yasuura’s method, the surface current can be approximated by a truncated modal expansion as follows:

\[
\frac{\partial u(r)}{\partial n} \approx \sum_{m=1}^{M} b_m(M)\phi_m^*(r) = B_M\Phi_M^T
\]

(2.7)
where the basis functions $\phi_m$ are called the modal functions and constitute the complete set of wave functions satisfying the Helmholtz equation in free space and the radiation condition (A.1). Here the asterisk denotes the complex conjugate, $\Phi_M = [\phi_1, \phi_2, \cdots, \phi_M]$ and $\Phi^T_M$ denotes the transposed vector of $\Phi_M$, where $M = 2N + 1$. The coefficient vector $B_M$, defined as $[b_1, b_2, \cdots, b_M]$, can be obtained by the ordinary mode-matching method as shown below.

Let us minimize the mean square error

$$\Omega_E(M) = \int_S \left| \sum_{m=1}^{M} b_m(M)\phi^*_m(r) - \frac{\partial u(r)}{\partial n} \right|^2 \, dr$$  \hspace{1cm} (2.8)

by the method of least squares. That is, we partially differentiate (2.8) with respect to $b^*_m$ and obtain the algebraic equation

$$\sum_{m=1}^{M} b_m(M) \int_S \phi_n(r)\phi^*_m(r) \, dr = \int_S \phi_n(r)\frac{\partial u(r)}{\partial n} \, dr, \quad n = 1 \sim M.$$  \hspace{1cm} (2.9)

Because of $u(r) = u_{in} + u_s = 0$ on $S$, the right-hand side of (2.9) can be written as

$$\int_S \left( \phi_m \frac{\partial u_{in}}{\partial n} - \frac{\partial \phi_m}{\partial n} u_s \right) \, dr = \int_S \left( \phi_m \frac{\partial u_{in}}{\partial n} - \frac{\partial \phi_m}{\partial n} u_{in} \right) \, dr + \int_S \left( \phi_m \frac{\partial u_s}{\partial n} - \frac{\partial \phi_m}{\partial n} u_s \right) \, dr$$  \hspace{1cm} (2.10)

Using Green's theorem for $\phi_m$, $u_s$ in the region surrounded by $S$ and infinity, and using the radiation condition for $\phi_m$, $u_s$, we obtain

$$\int_S \left( \phi_m \frac{\partial u_s}{\partial n} - \frac{\partial \phi_m}{\partial n} u_s \right) \, dr = 0$$  \hspace{1cm} (2.11)

and hence the right-hand side of (2.9) can be given as the reaction of $\phi_m$ and $u_{in}$:

$$\int_S \phi_m \frac{\partial u_{in}}{\partial n} \, dr = \int_S \ll \phi_m(r), u_{in}(r) \gg \, dr$$  \hspace{1cm} (2.12)

where $\ll, \gg$ means

$$\ll \phi_m(r), u_{in}(r) \gg \equiv \phi_m(r) \frac{\partial u_{in}(r)}{\partial n} - \frac{\partial \phi_m(r)}{\partial n} u_{in}(r)$$  \hspace{1cm} (2.13)
Chapter 2. Scattering theory

We can therefore write (2.9) as

\[ A_E B_M^T = \int_S \Phi_M^T(r), u_{in}(r) \, dr \]  

(2.14)

where \( A_E \) is a positive definite Hermitian matrix of \( M \times M \) except for the internal resonance frequencies, and is given by

\[ A_E = \begin{bmatrix} (\phi_1, \phi_1) & \cdots & (\phi_1, \phi_M) \\ \vdots & \ddots & \vdots \\ (\phi_M, \phi_1) & \cdots & (\phi_M, \phi_M) \end{bmatrix} \]  

(2.15)

in which its \( m,n \) elements are the inner products of \( \phi_m \) and \( \phi_n \):

\[ (\phi_m, \phi_n) = \int_S \phi_m(r) \phi_n^*(r) \, dr \]  

(2.16)

From (2.14), the \( B_m^T \) is given by

\[ B_m^T = A_E^{-1} \int_S \Phi_m(r'), u_{in}(r') \, dr' \]  

(2.17)

Substituting (2.17) into (2.7) and comparing it with (2.5), we can approximately express the current generator as follows:

\[ Y_E(r|r') \simeq \Phi_m^*(r) A_E^{-1} \Phi_M^T(r') \]  

(2.18)

where \( \Phi_M^T \), denotes the operation (2.13) of each element of \( \Phi_M^T \) and the function \( u_{in} \) to the right of the \( \Phi_M^T \). Equation (2.7) has been proved to converge in the mean sense as \( M \to \infty \). Therefore (2.18) converges to the true operator in the same sense.

Finally we touch on the set of \( \phi_m, m = 1, 2, 3, \ldots \). In the case of scattering from the body of finite size, it is chosen from the sets of which each set consists of solutions of the Helmholtz equation with the radiation condition, solutions which are obtained by separation of variables. We usually use the spherical Bessel functions-spherical harmonies \( h_n^{(1)}(kr) P_n^m(\cos \theta) \exp(im\phi) \) for three dimensional problems and the Hankel functions \( H_m(kr) \exp(im\theta) \) for two dimensional problems, because they are well known and tractable to computation.
2.4.2 Expression under the Neumann condition

Similarly, the surface current can be approximately expressed as

\[ u(r) \simeq \sum_{m=1}^{M} b_m(M) \frac{\partial \phi^*_m(r)}{\partial n} = B_M \frac{\partial \Phi^T_M}{\partial n} = \frac{\partial \Phi^*_M B^T_m}{\partial n} \tag{2.19} \]

Consider the mean square error

\[ \Omega_H(N) = \int_S \left( \sum_{m=1}^{M} b_m(M) \frac{\partial \phi^*_m(r)}{\partial n} - u(r) \right)^2 \, dr \tag{2.20} \]

and minimize it by the method of least squares. The same procedure as that taken for the

Dirichlet condition yields

\[ A_B B^T_M = \int_S \Phi^T_M(r), u_{in}(r) \gg dr \tag{2.21} \]

where \( A_B \) is \( A_E \) of (2.15) with \( (\phi_m, \phi_n) \) replaced by \( (\partial \phi_m / \partial n, \partial \phi_n / \partial n) \).

From (2.21), the \( B^T_M \) is given by

\[ B^T_M = A^{-1}_B \int_S \Phi^T_M(r'), u_{in}(r') \gg dr' \tag{2.22} \]

Substituting (2.22) into (2.19) and comparing it with (2.6), we can approximately obtain

the current generator \( Y_H \) for the Neumann condition as

\[ Y_H(r|r') \simeq \frac{\partial \phi^*_M(r)}{\partial n} A^{-1}_H \Phi^T_M(r'), \tag{2.23} \]

Here \( Y_H \) also converges in the same sense as \( Y_E \) does, when \( M \to \infty \).

2.5 Re-incident waves

Referring to figure 2.4, we need to show how to describe the re-incident wave explicitly.

Assume that the random medium is in the region of \(-L < z < L\) as shown in figure 2.5.
In order to show shortly an idea of the description, we deal with the scalar wave equation:

\[
[\nabla^2 + k^2(1 + \delta \varepsilon(\mathbf{r}))]u = 0
\]  

(2.24)

Then we can obtain the following equation:

\[
u = u_{\text{in}} + H u
\]  

(2.25)

where \( H \) is the operator in which all effects of the random medium are included; \( H = 0 \) for \( \delta \varepsilon(\mathbf{r}) \equiv 0 \) and \( H \) includes the integral with respect to \( z \) from \(-L\) to \( L\). Let us divide \( H \) into two parts: \( H = H_f + H_b \) where \( H_f \) includes the integral from \(-L\) to \( z \), called the forward scattering operator for convenience, and \( H_b \) does one from \( z \) to \( L \), called the backward scattering operator. Of course, \( H_f \) and \( H_b \) can be given explicitly.

Figure 2.5: Geometry of the propagation problem in a random medium.

Because (2.25) is deformed as \((I - H_f)u = u_{\text{in}} + H_b u\) where \( I \) is the identity operator, we have

\[
u = (I - H_f)^{-1}u_{\text{in}} + (I - H_f)^{-1}H_b u
\]  

(2.26)

where \((I - H_f)^{-1}\) is the inverse operator of \((I - H_f)\) and is expressed in terms of an ordered exponential function[4]. Because (2.26) is a Volterra’s integral equation with respect to \( z \),
we may express its solution formally as follows:

\[ u = u_0 + \sum_{n=1}^{\infty} u_n \]  

(2.27)

where

\[ u_0 = (I - H_f)^{-1} u_{\text{in}} \]  

(2.28)

\[ u_n = [(I - H_f)^{-1} H_b]^n u_0 \]  

(2.29)

Here \( u_n \) represents a wave scattered \( n \) times in the backward direction, and \( u_0 \) may be called a successively forward-scattered wave of which the moments satisfy so-called moment equation\([4, 38]\). By replacing \( u_{\text{in}}, u \) with \( G_0, G \), respectively, we can express the re-incident waves but it is not easy to express the higher order moments in analytic forms because we obtain only a few analytic expressions even for the moments of \( u_0 \).
Chapter 3

Numerical analysis

This chapter shows numerical results of RCS of conducting circular and elliptic cylinders, by applying the theory presented in chapter 2.

3.1 Formulation

Assume that \( \delta \varepsilon(r) \) is a continuous random function with

\[
\langle \delta \varepsilon(r) \rangle = 0, \quad \langle \delta \varepsilon(r_1) \delta \varepsilon(r_2) \rangle = B(r_1, r_2)
\]  
(3.1)

and

\[
B(r, r) \ll 1, \quad kl(r) \gg 1
\]  
(3.2)

where the angular brackets denote the ensemble average, \( B(r, r) \), \( l(r) \) are the local intensity and scale size of random medium, respectively, and \( k \) is the wavenumber in free space:

\[
k = \omega \sqrt{\varepsilon_0 \mu_0}.
\]

Under the condition (3.2), depolarization of electromagnetic waves due to the random medium can be neglected; and the scalar approximation is valid. In addition, the small scattering-angle approximation is also valid[10, 39]; and re-incident waves are negligible at the first stage of analysis. Then the wave equation for an electromagnetic
field component is given as

$$[\nabla^2 + k^2(1 + \delta\varepsilon(r))]v(r) = 0$$  \hspace{1cm} (3.3)$$

in the random medium, where $v$ denotes each component.

Suppose that a conducting cylinder of infinite length is surrounded with above random medium. Geometry of the scattering problem is shown in figure 3.1 where the intensity of random medium is depicted along the $z$ axis. As shown in figure 3.1, when an incident wave propagated along the $z$ axis is scattered and observed at a point close to the $z$ axis, we can approximately express (3.1) under the condition (3.2) as follows:

$$B(r_1, r_2) = B(\rho_1 - \rho_2, z_+, z_-)$$  \hspace{1cm} (3.4)$$

where $r = (\rho, z)$, $\rho = i_x x + i_y y$, $z_+ = (z_1 + z_2)/2$ and $z_- = z_1 - z_2$.

Consider the case where a directly incident wave is produced by a line source $f(r')$ distributed uniformly along the $y$ axis. Then we can deal with this scattering problem two-dimensionally under the condition (3.2) and use $r$ even for this case although $r = (x, z)$.

According as polarizations of incident waves: $E_y$ or $H_y$, where $E_y, H_y$ are the $y$ components of electric and magnetic fields, respectively, the boundary condition becomes (2.3) or (2.4).

From the above mentioned, the incident wave can, in general, be expressed as

$$u_{in}(r) = \int_{V_T} G(r|r')f(r')dr' = G(r|r_1)$$  \hspace{1cm} (3.5)$$

where $G(r|r')$ is the Green's function in the random medium and the dimension coefficient is understood. By referring to figure 2.4, the scattered wave can be given by

\begin{align*}
  u_s(r) &= -\int_{S} G(r|r_1) \frac{\partial}{\partial n_1} u(r_1)dr_1 \\
  &= -\int_{S} G(r|r_1) \int_{S} Y_E(r_1|r_2)u_{in}(r_2)dr_2dr_1  \hspace{1cm} (3.6)
\end{align*}
for the Dirichlet condition and
\[ u_s(r) = \int_S \left[ \frac{\partial}{\partial n_1} G(r|r_1) \right] u(r_1) dr_1 = \int_S \left[ \frac{\partial}{\partial n_1} G(r|r_1) \right] \int_S Y_H(r_1|r_2) u_{in}(r_2) dr_2 dr_1 \]
\[ (3.7) \]
for the Neumann condition, where \( Y_E, Y_H \) are given by (2.18) and (2.23), respectively. From (3.6) and (3.7), the average scattered wave can be expressed as
\[ \langle u_s \rangle = -\int_S dr_1 \int_S dr_2 Y_E(r_1|r_2) \langle G(r|r_1) G(r_2|r_1) \rangle \]
\[ (3.8) \]

\[ \begin{array}{c}
\text{Transmitter} \\
\text{Receiver} \\
\text{Conducting Cylinder} \\
\text{Random Medium} \\
\text{B(r,r') : Local Intensity of Random Medium}
\end{array} \]

Figure 3.1: Geometry of the scattering problem from a conducting body, the coordinate system and the local intensity of random medium.
for the Dirichlet condition and
\[
\langle |u_s|^2 \rangle = \int_\mathcal{S} \, d\mathbf{r}_1 \int_\mathcal{S} \, d\mathbf{r}_2 \int_\mathcal{S} \, d\mathbf{r}_1' \, \int_\mathcal{S} \, d\mathbf{r}_2' \{ Y_H(\mathbf{r}_1|\mathbf{r}_2) Y_H^*(\mathbf{r}_1'|\mathbf{r}_2') \}
\]
for the Neumann condition. The average intensity of scattered waves is given by
\[
\langle |u_s|^2 \rangle = \int_\mathcal{S} \, d\mathbf{r}_1 \int_\mathcal{S} \, d\mathbf{r}_2 \int_\mathcal{S} \, d\mathbf{r}_1' \, \int_\mathcal{S} \, d\mathbf{r}_2' \{ Y_H(\mathbf{r}_1|\mathbf{r}_2) Y_H^*(\mathbf{r}_1'|\mathbf{r}_2') \}
\]
for the Dirichlet condition and
\[
\langle |u_s|^2 \rangle = \int_\mathcal{S} \, d\mathbf{r}_1 \int_\mathcal{S} \, d\mathbf{r}_2 \int_\mathcal{S} \, d\mathbf{r}_1' \, \int_\mathcal{S} \, d\mathbf{r}_2' \{ Y_H(\mathbf{r}_1|\mathbf{r}_2) Y_H^*(\mathbf{r}_1'|\mathbf{r}_2') \}
\]
for the Neumann condition.

The scattered wave can be divided into two parts:
\[
u_s = \langle u_s \rangle + \Delta u_s
\]
where $\langle u_s \rangle$, $\Delta u_s$ are called the coherent and the incoherent scattered waves, respectively.

The coherent scattered waves are given by (3.6) and (3.7). To analyze them, we need to obtain the second moment of the Green's function: $M_{20} = \langle G(\mathbf{r}|\mathbf{r}_1)G(\mathbf{r}_2|\mathbf{r}_1) \rangle$ (see subsection 3.1.1).

For a cylindrical wave incidence, average radar cross-section (RCS) is given by
\[
\sigma \simeq \langle |u_s|^2 \rangle \times 8\pi k z \times 2\pi z
\]
\[
= \sigma_0 + \sigma_{in}
\]
where
\[
\sigma_0 = \langle |u_s|^2 \rangle \times 8\pi k z \times 2\pi z
\]
\[
\sigma_{in} = \langle |\Delta u_s|^2 \rangle \times 8\pi k z \times 2\pi z
\]
Chapter 3. Numerical analysis

The $a_0$ and $a_m$ correspond the coherent and incoherent part of $u$, respectively. The numerical analysis of $a_0$ is given in subsection 3.2.2. When we calculate $\sigma$ by using equations (3.10) and (3.11), the fourth moment of the Green's function

$$M_{22} = \langle G(r|r_1)G(r_2|r_1)G^*(r|r')G^*(r_2|r') \rangle$$

is needed. Although $M_{22}$ have not been expressed in compact forms for a general case, we can use an approximate expression as shown in subsection 3.1.2 for the calculation of RCS in a strong random medium.

### 3.1.1 The second moment of Green's function

It is assumed that $\delta \varepsilon(r)$ is a smooth random function and the order of averaging procedure and differentiation are exchangeable to each other. This moment is approximately expressed as the product of $\langle G(r|r_1) \rangle$ and $\langle G(r_2|r_1) \rangle$ if the angle between $r$ and $r_1$, shown in figure 3.1, is not very small. In this case, $\langle G(r|r') \rangle$ is given in a well known form and hence the second moment also is done. If the angle is quite small, then $G(r|r_1)$ and $G(r_2|r_1)$ are statistically coupled and the double passage effect[15] plays a leading role in analyzing $M_{20}$.

Let us assume that the coherence of waves is kept almost complete in propagation of distance $w$ equal to the $z$-axis width of the cylinder. This assumption is acceptable in practical cases under the condition (3.2). On the assumption, we can replace approximately the random medium effect in propagation from the source to the receiver via the cylinder by that in propagation from the source to the receiver via the plane at $z = w/2$ (see figure 3.1). When the source and the receiver are on the same plane perpendicular to the $z$ axis,
then $M_{20}$ in $z > w/2$ can therefore be given as a solution of the following second moment equation[38].

$$\left[ \frac{\partial}{\partial z} - i \frac{1}{2k} (\nabla^2 + \nabla_z^2) - i 2k \right] M_{20}$$

$$= \left\{ - \frac{k^2}{2} \int_a^z B \left( 0, z - \frac{z'}{2}, z' \right) + B \left( \rho - \rho_1, z - \frac{z'}{2}, z' \right) \right\} M_{20}$$

(3.16)

and

$$M_{20}|_{z=w/2} = G_0(\rho, w/2 | \rho_1, z_1)G_0(\rho_2, w/2 | \rho_2, z_2)$$

(3.17)

where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2, \quad \nabla_z^2 = \partial^2/\partial x_z^2 + \partial^2/\partial y_z^2$$

(3.18)

and $G_0(\mathbf{r} | \mathbf{r}')$ is the Green's function in free space. Although $\rho = i_x x, \, \rho_1 = i_x x_1$ and $\partial/\partial y = \partial/\partial y_1 = 0$ in this case, we use these symbols for convenience.

It is difficult to obtain the solution of (3.16) analytically for a general form of $B(\rho, z_+, z_-)$. Equation (3.17) can be solved, however, on the assumption that $B(\rho, z_+, z_-)$ is approximately expressed in a quadratic form with respect to $\rho$, which assumption leads to the solution valid in the neighborhood of $\rho - \rho_c \approx 0$. That is, let us assume that

$$B(\rho, z_+, z_-) = B(z_+) \left[ 1 - \frac{\rho^2}{l^2(z_+)} \right] \exp \left[ - \frac{z_+^2}{l^2(z_+)} \right]$$

(3.19)

where,

$$B(z_+) = \begin{cases} B_0 & \text{for } a \leq z \leq L \\ B_0 (z/L)^{-m} & \text{for } L \leq z \end{cases}$$

(3.20)

and $l(z_+) = l_0$, a constant, as shown in figure 3.1, and the positive index $m$ denotes the normalized thickness of transition layer from the random medium to free space; $m = 8/3$ is assumed in subsections 3.2.2, 3.2.3 and 3.3.2. Then (3.16) can be solved; i.e., according to the Appendix C, the final form of $M_{20}$ is given by

$$M_{20}(\mathbf{r}, \mathbf{r}_1 : \mathbf{r}_2, \mathbf{r}_t) = G_0(\mathbf{r} | \mathbf{r}_1)G_0(\mathbf{r}_2 | \mathbf{r}_t)$$

$$\exp \left[ - \sqrt{\frac{\pi}{2}} k^2 B_0 l_0 \left( \frac{m}{m-1} L - a \right) \right] M(\rho_4, z);$$

(3.21)
Chapter 3. Numerical analysis

where \( m > 1, m \neq 2, z \gg L, v = 1/(2 - m), \rho_d = \rho - \rho_1, \rho_{dc} = \rho_1 - \rho_2,\)

\[
Q_1(z) = i\sqrt{\pi}k_Bz/l_0
\]

\[
Q_2(z) = i\nu\sqrt{\pi}k_B(z/L)^{-m/2}z/l_0
\]

\[
P_1(z) = J_{3/2}[Q_1(a)]J_{-1/2}[Q_1(z)] + J_{-3/2}[Q_1(a)]J_{1/2}[Q_1(z)]
\]

\[
P_2(z) = J_{3/2}[Q_1(a)]J_{-3/2}[Q_1(z)] + J_{-3/2}[Q_1(a)]J_{3/2}[Q_1(z)]
\]

\[
P_3(z) = P_1(L)[J_{\nu+1}[Q_2(L)]J_{-\nu}[Q_2(z)] + J_{-\nu-1}[Q_2(L)]J_{\nu}[Q_2(z)]]
\]

\[
+P_2(L)[J_{\nu}[Q_2(L)]J_{-\nu}[Q_2(z)] - J_{-\nu}[Q_2(L)]J_{\nu}[Q_2(z)]
\]

\[
P_4(z) = P_1(L)[J_{\nu+1}[Q_2(L)]J_{-\nu-1}[Q_2(z)] + J_{-\nu-1}[Q_2(L)]J_{\nu+1}[Q_2(z)]
\]

\[
+P_2(L)[J_{\nu}[Q_2(L)]J_{-\nu-1}[Q_2(z)] - J_{-\nu}[Q_2(L)]J_{\nu+1}[Q_2(z)]
\]

and \( J_\alpha \) is the Bessel Function of real order \( \alpha \).

3.1.2 The fourth moment of Green’s function

Under a general situation, it is difficult to express the fourth moment in an analytic form.

We concentrate on the state of \( u_s \simeq \Delta u_s \) in (3.12) and the backscattering. In wave propagation through a strong continuous-random medium, therefore we may assume that the Green’s function becomes approximately complex Gaussian random[38], and the fourth
Chapter 3. Numerical analysis

moment in the backward direction is expressed as the product of the second moments[6, 14].

\[
M_{22} = \langle G(r|\mathbf{r}_1)G(r_2|\mathbf{r}_2)G^*(\mathbf{r}|\mathbf{r}_1')G^*(r_2'|\mathbf{r}_1') \rangle \\
\approx \langle G(r|\mathbf{r}_1)G^*(\mathbf{r}|\mathbf{r}_1') \rangle \langle G(r_2|\mathbf{r}_2)G^*(r_2'|\mathbf{r}_1') \rangle \\
+ \langle G(r|\mathbf{r}_1)G^*(r_2'|\mathbf{r}_1') \rangle \langle G(r_2|\mathbf{r}_2)G^*(\mathbf{r}|\mathbf{r}_1') \rangle 
\]

(3.23)

where \( \mathbf{r}_1 = \mathbf{r}_1' = \mathbf{r} \) on the assumptions of backscattering and a single point source.

The second moments in (3.23) have been given[40–42]: for instance,

\[
\langle G(r|\mathbf{r}_1)G^*(\mathbf{r}|\mathbf{r}_1') \rangle = G_0(r|\mathbf{r}_1)G_0^*(r|\mathbf{r}_1') \\
\exp \left\{ -\frac{k^2}{4} \int_0^2 dz_1 \int_0^2 dz_2 D \left[ \frac{z - a - 2z_1}{z - a} (\rho_1 - \rho_1') , z - z_2 - \frac{z_1}{2} , z_1 \right] \right\} 
\]

(3.24)

where

\[
D(\rho , z_+, z_-) = 2[B(0, z_+, z_-) - B(\rho , z_+, z_-)]
\]

(3.25)

which is called the structure function of random medium. Without approximation (3.19), we may calculate (3.23) for a general form of \( D(\rho , z_+, z_-) \); here we assume

\[
D(\rho , z_+, z_-) = 2B(z_+ ) \left\{ 1 - \exp \left[ - \left( \frac{\rho}{l(z_+)} \right)^2 \right] \right\} \exp \left[ - \left( \frac{z}{l(z_+)} \right)^2 \right] 
\]

(3.26)

for computation below, where \( B(z_+) \) is given by (3.20) and \( l(z) = l_0 , \) a constant, as assumed in the previous subsection. According to the Appendix D, the final form of \( M_{22} \) is given by

\[
M_{22} \approx \frac{1}{8 \pi k z} \frac{O(\rho , z|\rho_1 , z_1)O(\rho , z|\rho_2 , z_2)}{O(\rho , z|\rho_1' , z_1')O(\rho , z|\rho_2' , z_2')}
\]

\[
[S(\rho_1 , \rho_1')S(\rho_2 , \rho_2') + S(\rho_1 , \rho_2')S(\rho_2 , \rho_1')] 
\]

(3.27)

\[
O(\rho , z|\rho' , z') = \exp \left\{ ik \left[ - z' + \frac{(\rho - \rho')^2}{2(z - z_0)} \right] \right\} 
\]

(3.28)

\[
S(\rho , \rho') = \exp \left\{ -\frac{k^2}{4} \mu \gamma(z , z_0) (\rho - \rho')^2 \right\} 
\]

(3.29)

\[
\mu = \pi^{1/2} \frac{B_0}{l_0} \frac{L^3}{(z - z_0)^2} 
\]

(3.30)

\[
\gamma(z , z_0) = \frac{2}{(3 - m)(2 - m)(1 - m)} \left( \frac{z}{L} \right)^{3-m} - \left( \frac{m}{1 - m} + \frac{z_0}{L} \right) \left( \frac{z}{L} \right)^2 
\]

(3.31)
Chapter 3. Numerical analysis

\[ m(z_0) - \frac{1}{3} \left( \frac{z}{L} \right) - \frac{1}{3} \left( \frac{z_0}{L} \right)^3 \]  \hspace{1cm} (3.31)

3.2 RCS of a conducting circular cylinder

Let us replace the cylinder in figure 3.1 with a circular cylinder of radius \( a \). The \( \sigma_0 \) and \( \sigma \) are calculated from (3.8), (3.9) and (3.10), (3.11) by using (3.21) and (3.23). For a circular cylinder, the current generators expressed by (2.18), (2.23) can be simplified by using orthogonality of the Bessel functions.

3.2.1 The current generator on a circular cylinder

Let us choose \( H_m(kp) \exp(i m \theta) \), \( m = -N \sim N \), as the modal functions \( \phi_m \), then they form an orthogonal set on the surface of the circular cylinder; that is, \( \langle \phi_m, \phi_n \rangle = 0 \), for \( m \neq n \) and hence \( A_E \) and \( A_H \) become diagonal matrix (see Appendix E). Consequently, as \( N \rightarrow \infty \), we can obtain

\[ Y_E(r|r_0) = \frac{i}{\pi^2 a^2} \sum_{n=-\infty}^{\infty} \exp[in(\theta_0 - \theta)] J_n(ka)H_n^{(1)}(ka) \] \hspace{1cm} (3.32)

\[ Y_H(r|r_0) = \frac{i}{\pi^2 ka^2} \sum_{n=-\infty}^{\infty} \frac{\exp[in(\theta_0 - \theta)]}{J_n(ka) \frac{\partial}{\partial(ka)}H_n^{(1)}(ka)} \] \hspace{1cm} (3.33)

where \( J_n(ka) \neq 0 \); that is, the internal resonance frequencies are excepted.

The solution of the scattering problem in free space is well known for the case of wave incidence on the cylinder[43]. When using the solution, (2.5) and (2.6), we can also obtain (3.32) and (3.33).
3.2.2 RCS calculated from coherent scattered waves

The average radar cross-sections calculated from coherent scattered waves for E-wave and H-wave incidences are shown in figure 3.2, in the case of $ka = 0.1 \sim 5.0$, $z/L = 10/3$ and $kl_0 = 20\pi$. We can see the $\sigma_0$ is smaller than that in free space shown by dotted line in the figure. The attenuation depends on the intensity of the medium.

3.2.3 Numerical results of RCS

When we express the coherent Green's function as
\[
\langle G(\mathbf{r} | \mathbf{r}_1) \rangle = G_0(\mathbf{r} | \mathbf{r}_1) \exp[-\alpha(L)],
\]
then $\alpha(L) > 2$ is required in order that (3.23) holds. In this case,
\[
\alpha(L) = \frac{k^2}{4} \int_a^z dz_1 \int_a^z dz_2 B \left(0, z_1 - \frac{z_2}{2}, z_2\right)
\]
\[
\simeq \frac{\sqrt{\pi}}{5} B_0 \times kl_0 \times kL
\]
and here we let $B_0 kL = 3\pi$ and $kl_0 \gg 1$. Although the incident wave becomes sufficiently incoherent, we should pay attention to spatial coherence of the incident wave because the wave scattering from the cylinder in the random medium is expected to depend largely on the coherence length of the incident wave about the cylinder. The degree of spatial coherence is defined by
\[
\Gamma(\mathbf{r}, z) = \frac{\langle G(\mathbf{r}_1 | \mathbf{r}_1) G^*(\mathbf{r}_2 | \mathbf{r}_1) \rangle}{\langle |G(\mathbf{r}_0 | \mathbf{r}_1)|^2 \rangle}
\]
where $\mathbf{r}_1 = (\mathbf{r}, 0)$, $\mathbf{r}_2 = (-\mathbf{r}, 0)$, $\mathbf{r}_0 = (0, 0)$, $\mathbf{r}_1 = (0, z)$.

Figure 3.3 shows the degree of spatial coherence calculated from (3.36) and that the coherence length of the incident wave is sufficiently larger than the diameter of the cylinder. In this situation, figures 3.4 and 3.5 show the average RCS for the E-wave and H-wave
Chapter 3. Numerical analysis

Figure 3.2: Radar cross-sections vs cylinder sizes, calculated from the coherent scattered waves.
incidences, respectively, compared with those in free space. Their RCS in the random medium become nearly twice as large as those in free space except the internal resonance frequencies: \( J_n(ka) = 0, n = 0, 1, 2, \ldots \). A part of figure 3.5 enlarged about the zero points of \( J_0 \) and \( J_1 \) is shown in figures 3.6 (a) and (b), respectively.

These figures are obtained by substituting (3.23), (3.32) or (3.23), (3.33) into (3.10) or (3.11) according as polarization of incident waves and by carrying out directly the quadruple integrals with respect to \( \theta_i, \theta'_i, i = 1, 2 \). For the E-wave incidence, the RCS computed above is similar in change with \( ka = 0.1 \sim 5.0 \) to that in free space, so that there is not any abnormal change of the RCS in the neighborhood of the internal resonance frequencies. Consequently, it may be concluded that the RCS is nearly twice as large as that in free space in the overall region of \( ka = 0.1 \sim 5.0 \) in the case of E-wave incidence.

Figure 3.3: The degree of spatial coherence of incident waves about the cylinder. \((kl_o = 2000\pi)\)
Figure 3.4: The average radar cross-section in the case of E-wave incidence, where the coherence length of the incident wave is shown in figure 3.3.

Figure 3.5: The average radar cross-section in the case of H-wave incidence, where the coherence length of the incident wave is shown in figure 3.3.
Chapter 3. Numerical analysis

(a) In the neighborhood of the first zero point of $J_0(ka_0)$: $ka_0 = 2.40482 \cdots$.

(b) In the neighborhood of the second zero point of $J_1(ka_0)$: $ka_0 = 3.83171 \cdots$.

Figure 3.6: Enlargement of figure 3.5 about the internal resonance frequencies of the cylinder.
Chapter 3. Numerical analysis

On the other hand, in the case of H-wave incidence, the change of RCS is different from that in free space about the internal resonance frequencies, as shown in figure 3.6. The current induced by H-wave incidence flows circularly along the surface of the cylinder and hence the RCS of a conducting cylinder coated with a thin dielectric layer changes remarkably about the internal resonance frequencies. An illustrative example is shown in figure 3.7. In the present case, however, such a phenomenon does not occur and the abnormal change is considered to be caused by the low accuracy of computation. Although the value of the quadruple integral is expected to take the same order as each value of the Bessel functions near the zero points of the Bessel functions in the dominator of the current generator, it is difficult virtually to carry out the integral with high accuracy so as to do that, which difficulty we do not have for the E-wave incidence.

On the assumption that the average intensity of backscattered waves is finite at the internal resonance frequencies, we carry out the computation. We rewrite (3.33) as

\[
Y_H(r|r_0) = \frac{X_l}{J_l(ka)} + \Delta Y_H
\]

(3.37)

\[
X_l = \frac{i}{\pi^2 ka^2} \exp[i l (\theta_0 - \theta)] \frac{\partial}{\partial (ka)} H^{(1)}_l(ka)
\]

(3.38)

\[
\Delta Y_H = \frac{i}{\pi^2 ka^2} \sum_{n=-\infty}^{\infty} \frac{\exp[i n (\theta_0 - \theta)]}{J_n(ka) \frac{\partial}{\partial (ka)} H^{(1)}_n(ka)} (n \neq l)
\]

(3.39)

Using the above, (3.11) can be expressed as follows:

\[
\langle |u_a|^2 \rangle = \alpha (ka)^{-2} (ka) + \beta (ka)^{-1} (ka) + \gamma (ka)
\]

(3.40)

\[
\alpha = \int_s dr_1 \int_s dr_2 \int_s dr'_1 \int_s dr'_2 X_i X_i^* Z
\]

(3.41)

\[
\beta = \int_s dr_1 \int_s dr_2 \int_s dr'_1 \int_s dr'_2 (X_i \Delta Y_2^* + X_i^* \Delta Y_2) Z
\]

(3.42)

\[
\gamma = \int_s dr_1 \int_s dr_2 \int_s dr'_1 \int_s dr'_2 \Delta Y_2 \Delta Y_2^* Z
\]

(3.43)
where

\[ Z = \left( \frac{\partial}{\partial n_1} G(r_1|\mathbf{r}_1) G(r_2|\mathbf{r}_2) \frac{\partial}{\partial n_1'} G^*(r_1'|\mathbf{r}_1') G^*(r_2'|\mathbf{r}_2') \right) \]  

(3.44)

Let us \( J_l(ka_0) = 0, \ l = 0, 1, 2, \ldots \), and expand \( \alpha \) and \( \beta \) in the Taylor series about \( ka = ka_0 \):

\[ \alpha = \sum_{m=0}^{\infty} \alpha_m (ka - ka_0)^m \]  

(3.45)

\[ \beta = \sum_{m=0}^{\infty} \beta_m (ka - ka_0)^m \]  

(3.46)

Figure 3.7: Radar cross-sections of the cylinder coated with a thin dielectric layer.
Chapter 3. Numerical analysis

Then we have from the above assumption

\[ a_0 = a_0(ka_0) = 0, \quad a_1 = \frac{\partial a_0(ka_0)}{\partial (ka)} = 0 \quad (3.47) \]

\[ \beta_0 = \beta(ka_0) = 0 \quad (3.48) \]

According to the computation of (3.41) and (3.42), the coefficients \( a_0, a_1 \) and \( \beta_0 \) are order of \( 10^{-6}, 10^{-5} \) and \( 10^{-3} \), respectively, and \( \alpha_2, \beta_1 \) are order of 1 at the first zero point of \( J_0(ka_0) = 0 \) and the second zero point of \( J_1(ka_0) = 0 \), when \( J_0(ka_0) \) and \( J_1(ka_0) \) at each zero point are order of \( 10^{-7} \) and \( 10^{-6} \), respectively. This result shows the validity of the assumption from a numerical analysis point of view and also does no good accuracy of direct computation of (3.11) about the resonance frequencies. Using (3.45) and (3.46), we can express (3.40) about \( ka = ka_0 \) as

\[
\langle |u_\text{av}|^2 \rangle = \sum_{m=2}^{\infty} \frac{1}{m!} \frac{\partial^m \alpha(ka_0)}{\partial (ka)^m} (ka - ka_0)^m + \sum_{m=2}^{\infty} \frac{1}{m!} \frac{\partial^m J_0^2(ka_0)}{\partial (ka)^m} (ka - ka_0)^m + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \beta(ka)}{\partial (ka)^m} (ka - ka_0)^m \quad + \gamma(ka) \quad (3.49)
\]

The numerical results of the average RCS calculated from (3.49) are shown by the solid lines in figures 3.8 (a) and (b) where the dotted lines show the RCS calculated directly from (3.11). These figures show clearly that there is not any abnormal change of the RCS about the internal resonance frequencies of the cylinder in both the random medium and free space. Consequently, in the coherence case shown in figure 3.3, the RCS is nearly twice as large as that in the free space in the overall region of \( ka = 0.1 \sim 5.0 \) for the incidence of H-waves as well as E-waves.
Chapter 3. Numerical analysis

(a) In the neighborhood of the first zero point of $J_0(ka_0)$: $ka_0 = 2.40482 \cdots$.

(b) In the neighborhood of the second zero point of $J_1(ka_0)$: $ka_0 = 3.83171 \cdots$.

Figure 3.8: The average radar cross-sections about the internal resonance frequencies in the case of H-wave incidence.
Next, let us see some changes of RCS when the spatial coherence length becomes short. For the situations of coherence shown in figure 3.9, some abnormal changes of RCS appear in figures 3.10 and 3.11: along with the change of \( ka \), for the E-wave incidence, some undulations of the RCS are shown (when \( kl_0 = 2\pi \)); and for the H-wave incidence, the RCS becomes smaller than that in free space somewhere.

The reason of the abnormal changes is considered as an effect of spatial coherence of incident waves. Figure 3.9 shows that small \( kl_0 \) makes the spatial coherence length decrease rapidly with increase of \( ka \). When the coherence length \( l_c \) becomes less than the width of the illuminated surface, as schematically shown in figure 3.12, the spatial coherence of the incident wave at the point crossed by the surface and the incident line is kept only on a part of the illuminated surface \( S_c \) centered at the point, and the scattered wave

![Figure 3.9: The degree of spatial coherence of incident waves about the cylinder. This figure shows that the coherence length of the incident wave becomes short, compared with figure 3.3.](image)
Figure 3.10: The average radar cross-section in the case of E-wave incidence, where the coherence length of the incident wave is shown in figure 3.9.

Figure 3.11: The average radar cross-section in the case of H-wave incidence, where the coherence length of the incident wave is shown in figure 3.9.
from the remainder of the illuminated surface may be roughly assumed to become purely random. Then the RCS thereby depends mainly on the $S_c$. But as well known, RCS of the cylinder in free space depends on the illuminated surface. The difference between $S_c$ and the illuminated surface makes the RCS in random medium change abnormally.

![Figure 3.12: The surface where the spatial coherence of the incident wave is kept.](image)

In fact, RCS of a body in free space depends not only on the width but also on the curvature of the illuminated surface. For example, see figure 3.13. The RCS of a taper is certainly different from RCS of a flat of which size is the same as the bottom of the taper. The curvature of illuminated surface is therefore expected to have an effect on the abnormal changes of RCS in the random medium. However, we can not get any message of the effect of surface curvature from figures 3.10 and 3.11, because the curvature of a circle is constant.

![Figure 3.13: Taper and flat](image)
3.3 RCS of a conducting elliptic cylinder

Because RCS is expected to change largely with the curvature, it becomes necessary to analyze the RCS in a random medium for a body of variable curvature on the surface. In this section, the RCS of a conducting elliptic cylinder in a strong random medium (see figure 3.14) is numerically analyzed and the characteristics of the RCS are made clear.

![Geometry of the scattering problem from a conducting elliptic cylinder surrounded by a random medium.](image)

Figure 3.14: Geometry of the scattering problem from a conducting elliptic cylinder surrounded by a random medium.

3.3.1 The current generator on an elliptic cylinder

For an ellipse, similar to the case of a circle, the current generator can also be obtained in a simple form if Mathieu’s functions are chosen as basis functions of the expansion (see Appendix F): for E-wave incidence,

\[
Y_E(r | r') = -\frac{2j}{\pi^2 c^2} \frac{1}{\sqrt{\cos^2 \xi - \cos^2 \eta \cos^2 \xi' - \cos^2 \eta'}} \times \left\{ \sum_{n=0}^{\infty} \frac{ce_n(\eta)ce_n(\eta')}{Mc_n^{(3)}(\xi)Mc_n^{(1)}(\xi')} + \sum_{n=1}^{\infty} \frac{se_n(\eta)se_n(\eta')}{Ms_n^{(3)}(\xi)Ms_n^{(1)}(\xi')} \right\}
\]  

(3.50)
and for H-wave incidence,

\[
Y_h(r|r') = \frac{2j}{\pi c} \frac{1}{\sqrt{\cosh^2 \zeta' - \cos^2 \eta'}} \left\{ \sum_{n=0}^{\infty} \frac{c_n(\eta)c_n(\eta')}{\partial \text{Mc}_n^{(1)}(\xi)} \text{Mc}_n^{(1)}(\xi') + \sum_{n=1}^{\infty} \frac{c_n(\eta)c_n(\eta')}{\partial \text{Mc}_n^{(3)}(\xi)} \text{Ms}_n^{(1)}(\xi') \right\}
\]

(3.51)

where \(c_n(\eta), s_n(\eta)\) are Mathieu's functions of the first kind, \(\text{Mc}_n^{(1)}(\xi), \text{Ms}_n^{(1)}(\xi)\) and \(\text{Mc}_n^{(3)}(\xi), \text{Ms}_n^{(3)}(\xi)\), respectively, are modified Mathieu's functions of the first and third kind. The elliptic cylindrical coordinates \((\xi, \eta, y)\) used here are related to the rectangular Cartesian coordinates \((z, x, y)\) by the transformation

\[
\begin{align*}
  z &= c \cosh \xi \cos \eta \\
  x &= c \sinh \xi \sin \eta \\
  y &= y
\end{align*}
\]

(3.52)

where \(c\) is the half focal-length of the ellipse.

### 3.3.2 Numerical results of RCS

Let us call the straight-line between the source and the origin the incident line and call the angle between the incident line and the major axis of the ellipse the incident angle.

Consider first the case of \(kl_0 = 200\pi\) in figure 3.9. The RCS is shown in figures 3.15 and 3.16 for the E-wave and H-wave incidences, respectively, compared with that in free space, where \(\theta = 90°\). The RCS in the random medium becomes nearly twice as large as that in free space, independent of the ratio of axis-lengths of the ellipse \(b/a\) and the polarization of the incident wave. The spatial coherence of incident wave has almost no effect on RCS in these figures, because RCS depends mainly on the illuminated surface of the cylinder according to the geometrical optics approximation, and the spatial coherence length in this case is larger enough than the width of the illuminated surface when \(ka \leq 3\), where \(a\) is half the major axis length of the ellipse.
Chapter 3. Numerical analysis

Figure 3.15: RCS for E-wave incidence in the case of \( kl_0 = 200\pi \) shown by the solid line in figure 3.9.

Figure 3.16: RCS for H-wave incidence in the case of \( kl_0 = 200\pi \) shown by the solid line in figure 3.9.
Next consider the cases of $k l_0 = 20\pi$ and $2\pi$ in figure 3.9. As $k a$ increases, $l_c$ becomes less than the width of the illuminated surface, and the RCS in the random medium may become different from that in free space as indicated in subsection 3.2.3.

Figures 3.17, 3.18 for E-wave incidence and 3.19, 3.20 for H-wave incidence show clearly the effects of the spatial coherence on RCS; that is, the RCS for the three cases of $k l_0$ are the same when $k a$ is small, and become different from each other as $k a$ increases. Comparing the RCS with that in free space, we find out that the RCS is not only enhanced but also diminished in some cases. The effects of the surface curvature on RCS are also shown in these figures. For the ellipse of small axis ratio, the curvature of the illuminated surface is large and changes slowly, then the effective width of the illuminated surface for RCS is large and therefore the spatial coherence has a more remarkable effect on RCS.

Figure 3.17: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.9, the incident direction is parallel to the minor axis of the ellipse and the axis ratio of the ellipse $b/a = 0.75$. 
Chapter 3. Numerical analysis

Figure 3.18: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.9, the incident direction is parallel to the minor axis of the ellipse and the axis ratio of the ellipse $b/a = 0.50$.

Figure 3.19: The effect of spatial coherence on RCS for H-wave incidence, where the spatial coherence is shown in figure 3.9, the incident direction is parallel to the minor axis of the ellipse and the axis ratio of the ellipse $b/a = 0.75$. 
Chapter 3. Numerical analysis

Figure 3.20: The effect of spatial coherence on RCS for H-wave incidence, where the spatial coherence is shown in figure 3.9, the incident direction is parallel to the minor axis of the ellipse and the axis ratio of the ellipse $b/a = 0.50$.

The curvature effects can be shown more clearly when $\theta = 0^\circ$. See figures 3.21, 3.22 for E-wave incidence and 3.23, 3.24 for H-wave incidence. In this case, the effective width of the illuminated surface for RCS becomes small because the scattered waves from the surface except the effective surface are canceled out due to the rapid changes of the curvature of the illuminated surface, and hence the effects of the spatial coherence on RCS become small, compared with the case of $\theta = 90^\circ$. However, we should note that figure 3.24 shows the RCS for $k_{l_0} = 2\pi$ can be enhanced to several times as large as that in free space.
Figure 3.21: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.9, the incident direction is parallel to the major axis of the ellipse and the axis ratio of the ellipse $b/a = 0.75$.

Figure 3.22: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.9, the incident direction is parallel to the major axis of the ellipse and the axis ratio of the ellipse $b/a = 0.50$. 
Chapter 3. Numerical analysis

Figure 3.23: The effect of spatial coherence on RCS for H-wave incidence, where the spatial coherence is shown in figure 3.9, the incident direction is parallel to the major axis of the ellipse and the axis ratio of the ellipse $b/a = 0.75$.

Figure 3.24: The effect of spatial coherence on RCS for H-wave incidence, where the spatial coherence is shown in figure 3.9, the incident direction is parallel to the major axis of the ellipse and the axis ratio of the ellipse $b/a = 0.50$. 
The above effects of coherence, curvature and size of the surface on the RCS are further clear when the ellipse becomes more oblate. Before showing it, let us refer to the degrees of spatial coherence over a wider changing range of $k\rho$ in figure 3.25, and note the degree of spatial coherence for $kl_0 = 200\pi$ also becomes very small when $k\rho$ closes to 10. Then see figures 3.26 and 3.27, where $ka$ is changed from zero to ten, $kb$ is a constant: 1.5 or 1.0 and incident angle $\theta = 90^\circ$. For $kl_0 = 200\pi$, the RCS cannot be kept to nearly double that in free space as $ka$ increases, in other words, the ratio between the RCS and that in free space becomes smaller and smaller, because $l_c$ is not large enough. For $kl_0 = 2\pi$ where $l_c$ is the least in the three cases of $kl_0$, because the width of $S_c$ in which the spatial coherence is kept (see figure 3.28(a)) does not become wider with the increase of $ka$, the change of RCS with $ka$ is quite smaller than that of RCS in free space.

![Figure 3.25: The degree of spatial coherence of incident waves about the cylinder.](image)
Figure 3.26: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.25, the incident direction is parallel to the minor axis of the ellipse and $kb = 1.5$.

Figure 3.27: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.25, the incident direction is parallel to the minor axis of the ellipse and $kb = 1.0$. 
On the other hand, for the case of incident angle $\theta = 0^\circ$, see figures 3.29 and 3.30, when $ka$ becomes large, the curvature of the illuminated surface changes rapidly. According to the geometrical optics approximation in free space, the back-scattered wave is mainly determined by a narrow determinative surface centered at the end point of the major axis. Let designate this determinative surface as $S_d$ (see figure 3.28(b)). As a result, RCS in free space decreases as $ka$ increases because the increase of $ka$ makes $S_d$ more narrow. RCS for the random medium of $kl_0 = 200\pi$ is nearly twice as large as that in free space because $S_c$ is wider enough than $S_d$. For the case of $kl_0 = 2\pi$, as $ka$ increases, the RCS changes largely at first where $S_d$ is wider than $S_c$, and is gradually close to nearly double the RCS in free space because $S_d$ is gradually close to $S_c$. 

![Figure 3.28: Oblate ellipse](image)
Figure 3.29: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.25, the incident direction is parallel to the major axis of the ellipse and $k_b = 1.5$.

Figure 3.30: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.25, the incident direction is parallel to the major axis of the ellipse and $k_b = 1.0$. 
The RCS has been analyzed for the case of wave incidence along the minor or major axis of the ellipse. In either case, the axis is coincident with the incident line and the curvature of the illuminated surface is symmetric about the incident line. When the wave is obliquely incident on the ellipse; \( \theta \neq 0^\circ \) and \( \theta \neq 90^\circ \), then the curvature of the illuminated surface does not become symmetric, and as a result, RCS as a function of \( k a \) becomes more complicated especially for the case of \( k l_0 = 2\pi \). As illustrative examples, the RCS are shown in Figs 3.31, 3.32 for E-waves incidence and in Figs 3.33, 3.34 for H-wave incidence, where \( \theta = 45^\circ \).
Figure 3.31: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.9, the incident angle $\theta = 45^\circ$ and $b/a = 0.75$.

Figure 3.32: The effect of spatial coherence on RCS for E-wave incidence, where the spatial coherence is shown in figure 3.9, the incident angle $\theta = 45^\circ$ and $b/a = 0.50$. 
Chapter 3. Numerical analysis

Figure 3.33: The effect of spatial coherence on RCS for H-wave incidence, where the spatial coherence is shown in figure 3.9, the incident angle \( \theta = 45^\circ \) and \( b/a = 0.75 \).

Figure 3.34: The effect of spatial coherence on RCS for H-wave incidence, where the spatial coherence is shown in figure 3.9, the incident angle \( \theta = 45^\circ \) and \( b/a = 0.50 \).
Chapter 4

Concluding remarks

We have presented a method for analyzing wave scattering from a conducting body in a random medium as boundary value problems. In doing that, we have introduced the current generators which transform any incident wave into surface currents on the body, and shown that the generators are constructed by Yasuura's method. The introduction of current generators makes the analysis of the scattering problem separated: that is, the analysis of wave propagation in random media and that of surface currents on the body. The former is based on the multiple scattering theory in random media and is to obtain the moments of Green's functions. The latter is based on the surface integral and the inversion of the matrix of which each element is the inner product of basis functions in the complete set of wave functions.

Applying the method, we have numerically analyzed radar cross-sections (RCS) of a conducting circular and elliptic cylinder embedded in a strong random medium as a boundary value problem. The Numerical analysis gives us the following results.

1. The effect of double passage makes the coherent part of RCS decrease.

2. In the case where the coherent backscattered wave is negligible and the backscattered
Chapter 4. Concluding remarks

wave becomes almost incoherent, the random medium has two kinds of effect on the RCS; the one is that of double passage which results in backscattering enhancement, and the other one is that of the spatial coherence which results in the complicated changes of the RCS. The effect of the spatial coherence depends on the ratio of the spatial coherence to the effective width of the illuminated surface according to the geometrical optics approximation.

2.1. Under the condition that the ratio is sufficiently more than one, the effect of the spatial coherence may be neglected and the RCS in the random medium is nearly twice as large as that in free space. This enhancement of RCS is due to the effect of double passage and independent of the angle of wave incidence and the size of the ellipse.

2.2. If above condition does not hold, the spatial coherence of the incident wave is not kept on the overall illuminated surface and hence the RCS may change complicatedly: it can be enhanced to more than twice as large as that in free space or become smaller than that in free space. These changes of RCS depend on the spatial coherence length, the curvature of the illuminated surface and the size of the cross-section of the cylinder.

Because the effects of the curvature and size of body surface have been taken into account, above results may be considered to be valid for a conducting convex body embedded in a continuous-random medium.

The result that RCS in a strong continuous-random medium depends mainly on wave scattering from a part of surface of the body awakes an interest in close comparison of the RCS and that for wave beam incidence in free space.
Acknowledgments

I wish to express my deepest gratitude to Professor Mitsuo Tateiba of Kyushu University for his proper guidance, support and reading the entire text in its original form.

I would also like to thank Professor Kiyotoshi Yasumoto and Professor Yoshihiko Akaiwa of Kyushu University for valuable advice.

My thanks are also due to the members of Electromagnetic Wave Laboratory in Kyushu University for their cooperation. Among them are Dr. K. Fujisaki, Dr. H. Maeda, Mr. K. Ishida, Mr. H. Koga and others.

This research was supported by a grant from the Ministry of Education, Science, Sports and Culture, Japan.
Bibliography


Bibliography


A. Integral equations for random surface currents

Consider the scattering problem shown in figure 2.1 where $\varepsilon(r)$ is the random function defined by (2.1), (2.2) and the boundary conditions (2.3), (2.4) are valid. For simplicity we deal with scalar waves. To formulate the problem, we introduce the Green's function which satisfies the radiation condition

$$\lim_{r \to \infty} r \left( \frac{\partial G}{\partial r} - i k G \right) = 0; \quad k = \omega \sqrt{\varepsilon_0 \mu_0}$$

(A.1)

and the equation

$$[\nabla^2 + k^2(1 + \delta \varepsilon(r))] G(r|r') = -\delta(r - r'), \quad \text{for any } r$$

(A.2)

where the body is replaced by the random medium with the same property.

When a source distribution is $f(r_t), \ r_t > L$, then the incident wave is defined by

$$u_{in}(r) = \int G(r|r_t) f(r_t) dr_t \tag{A.3}$$

which means the wave propagated in the random medium without the body. In order that (A.3) is the incident wave independent of the body, it is necessary that $G(r|r_t)$ is hardly affected by the random medium replaced instead of the body; that is,

$$L \gg a \tag{A.4}$$
is required. Strictly speaking, it is required that $G(r|r_0)$ satisfies (A.2) where the body is replaced with free space and that the boundary between free space and the random medium is matched; that is, non-reflection is assumed. Some moments of $G(r|r_0)$ in this case, however, are approximately obtainable at present under the condition (A.4): i.e., on the assumption that the effect of free space is neglected. Consequently, under the condition (A.4), we can use the Green's function in the random medium where the body is replaced with the same random medium.

The wave produced by the body under the existence of $u_{in}$ is called here the scattered wave and is designated by $u_s$. Then $u_s$ satisfies the homogeneous equation of (A.2) in the random medium and the radiation condition at infinity. When the total wave is designated by $u$: $u = u_{in} + u_s$, then according to section 2.2, boundary conditions on the body are specified by (2.3) and (2.4).

Using Green's theorem, the radiation condition and the boundary conditions, we can obtain the integral representations of $u_s$:

$$u_s(r) = -\int_s G(r|r_0) \frac{\partial}{\partial n_0} u(r_0) dr_0 \quad \text{for DC}$$

$$u_s(r) = \int_s \left( \frac{\partial}{\partial n_0} G(r|r_0) \right) u(r_0) dr_0 \quad \text{for NC}$$

It can be shown from (2.2) that the singularities of

$$\lim_{r \to r_0} G(r|r_0) \quad \text{and} \quad \lim_{r \to r_0} \frac{\partial}{\partial n_0} G(r|r_0)$$

are the same as these in free space[25]. This fact leads to the Fredholm integral equations of the second kind on the surface:

$$\frac{1}{2} \frac{\partial u(r)}{\partial n} + \int_s \frac{\partial G(r|r_0)}{\partial n} \frac{\partial u(r_0)}{\partial n_0} dr_0 = \frac{\partial u_{in}(r)}{\partial n}, \quad \text{for DC}$$

$$\frac{1}{2} u(r) + \int_s \frac{\partial G(r|r_0)}{\partial n} u(r_0) dr_0 = u_{in}, \quad \text{for NC}$$
For the Dirichlet condition case, substituting the solution $\partial u/\partial n$ of (A.7) into (A.5), we can obtain the scattered wave. However, $G$ and $\partial u/\partial n$ in (A.5) are statistically coupled, and $\partial G/\partial n$ and $\partial u/\partial n_0$ in (A.7) are also done, so that it is difficult to express the moments of $u_s$ in a closed form from (A.5) and (A.7). This holds also for the Neumann condition case. That is, the integral equation method including the boundary element method is not applicable to the problem of wave scattering from a conducting body in the random medium if we want to obtain the moments of $u_s$ directly.
B. A construction of the scattering problem by Yasuura's method

Consider the scattering problem shown in figure 2.1 where \( \varepsilon(r) = \varepsilon_0 \). When Yasuura's method is applied under the Dirichlet condition on the body, the scattered wave can be approximately expressed in terms of the modal functions defined in subsection 2.4.1 as follows:

\[
\begin{align*}
\mathbf{u}_s(r) & \simeq \sum_{m=1}^{M} a_m(M) \phi_m(r) = a_M \phi^T_M = \phi_M a^T_M \\
\end{align*}
\]

(B.1)

where \( a_M = [a_1, a_2, \ldots, a_M] \), given by

\[
a_M = -A_{E}^{-1}(\phi^T_M, u_{in})
\]

(B.2)

in which \( A_{E} \) is given by (2.13) and \( (\phi^T_M, u_{in}) \) denotes the column vector of which each element is the inner product of \( \phi_m \) and \( u_{in} \), defined as (2.14). Equation (B.2) is obtained by minimizing the mean square error of the boundary value

\[
\| u_s + u_{in} \|^2 = \int_S \left| \sum_{m=1}^{M} a_m(M) \phi_m(r) + u_{in}(r) \right|^2 dr
\]

and the deviation procedure is the same as that used to obtain (2.9).

Let us now introduce the scattering operator \( S_E \) defined by

\[
\mathbf{u}_s(r) = \int_S S_E(r|r') u_{in}(r') dr', \ r' \text{ on } S
\]

(B.3)

where \( u_{in} \) is any incident wave satisfying the Helmholtz equation. Substitution of (B.2) into (B.1) and comparison of it with (B.3) lead to

\[
S_E(r|r') \simeq -\phi_M A_{E}^{-1}(\phi^T_M),
\]

(B.4)

where \( (\phi^T_M \phi_M \) means the operation (3.2) of \( \phi^T_M \) and the function \( u_{in} \) to the right of the \( \phi^T_M \).
Under the Neumann condition, $a^T_M$ in (B.1) can be obtained by

$$a^T_M = -A^{-1}_H \left( \frac{\partial \phi_T^M}{\partial n}, \frac{\partial u_{in}}{\partial n} \right)$$

(B.5)

where $A_H$ is given by (2.19), and then the error of the boundary value

$$\left\| \frac{\partial u_s}{\partial n} + \frac{\partial u_{in}}{\partial n} \right\| = \int_S \left\| \sum_{m=1}^M a_m(M) \frac{\partial \phi_n(M)}{\partial n} + \frac{\partial u_{in}(r)}{\partial n} \right\|^2 dr$$

has been minimized in the sense of mean squares. In the same way of (B.3), let us define the scattering operator in this case as follows:

$$u_s(r) = \int_S S_H(r|r')u_{in}(r')dr', \quad r' \text{ on } S$$

(B.6)

Then it can be approximately given by

$$S_H(r|r') \simeq -\phi_M A^{-1}_H \left( \frac{\partial \phi_T^M}{\partial n} \right)$$

(B.7)

Here it should be noted that as $M \to \infty$, the scattering operators $S_E$ and $S_H$ converge uniformly to true operators, respectively, because (B.1) and (B.4) do so[21].

Using the scattering operators defined here and the current generators defined in subsection 2.4, we can schematically describe an approach to the problem of wave scattering from a conducting body in free space as figure B.1. That is, under the Dirichlet condition,

$$u_s = S_E \cdot u_{in} \quad \text{(Direct Type)}$$

(B.8)

$$= -G_0 \cdot \frac{\partial u}{\partial n}; \quad \frac{\partial u}{\partial n} = Y_E \cdot u_{in} \quad \text{(Indirect Type)}$$

(B.9)

and under the Neumann condition,

$$u_s = S_H \cdot u_{in} \quad \text{(Direct Type)}$$

(B.10)

$$= \frac{\partial G_0}{\partial n} \cdot u; \quad u = Y_H \cdot u_{in} \quad \text{(Indirect Type)}$$

(B.11)

where $G_0$ is Green's function in free space and the dot $\cdot$ denotes the integration on the surface of the body. We should pay attention to the fact that $S$ and $Y$ are constructed
by using the same matrix $A$ which depends only on the body surface, although it must be satisfied from a physical point of view.

When dealing with the scattering problem methodically as shown in figure B.1, operators $S$ and $Y$ must be well defined and mathematically constructed. Yasuura's method has definitely constructed them. In general, the scattering problem has been analyzed without getting $S$ or $Y$, even if Yasuura's method is applied. The reason is that the computation is simple and effective, so that it seems that the idea of $S$ and $Y$ has not been specially required. If you try to apply Yasuura's method to some practical cases and to obtain numerical results, you should refer to the article[32]. As obvious from comparison of figure B.1 and figure 2.4, however, an operator construction by Yasuura's method is important for analyzing wave scattering from a conducting body in a random medium as boundary value problems. This appendix puts emphasis on an aspect of Yasuura's method, the aspect which has not usually been paid attention to but may be useful for the development of approaches to some problems.

Figure B.1: Schematic diagram for solving the scattering problem where a conducting body is in free space.
Appendix C

C. A solution of the second moment equation

Assuming that a solution is expressed as

\[ M_{20} = G_0(\mathbf{r}_1|\mathbf{r}_1)G_0(\mathbf{r}_1|\mathbf{r}_2) \exp \left[ -k^2 \int_a^z dz_1 \int_a^{z_1} dz_2 B \left( 0, z_1 - \frac{z_2}{2}, z_2 \right) \right] M(\rho_d, z) \]  

(C.1)

where \( \rho_d = \rho - \rho_t \), and substituting it into (3.16), we have

\[
\left[ \frac{\partial}{\partial z} - i \frac{1}{k} \nabla_d^2 - \frac{1}{z - a} (\rho_d - \rho_{dc}) \cdot \nabla_d - \frac{k^2}{4} \int_a^{z'} D \left( \rho_d, z - \frac{z'}{2}, z' \right) dz' \right] M(\rho_d, z) = 0 \]  

(C.2)

where

\[
M(\rho_d, a) = 1
\]  

(C.3)

Considering \( D(\rho, z_+, z_-) \propto \rho^2 \), we assume that \( M(\rho_d, z) \) is expressed in the following form.

\[
M(\rho_d, z) = \frac{1}{g(z)} \exp \left[ h(z) \rho_d^2 + f(z) \rho_d \cdot \rho_{dc} \right] \]  

(C.4)

where \( \rho_{dc} = \rho_1 - \rho_2 \). The substitution of (C.5) into (C.2) leads to the equations for \( g(z) \), \( h(z) \) and \( f(z) \):

\[
- \frac{1}{g(z)} \frac{dg(z)}{dz} + \frac{4}{k} h(z) + \left[ \frac{1}{z - a} - \frac{f(z)}{z - a} \right] \rho_d^2 = 0
\]  

(C.6)

\[
\frac{df(z)}{dz} + \left[ \frac{1}{z - a} + \frac{4}{k} h(z) \right] f(z) - \frac{2}{z - a} h(z) = 0
\]  

(C.7)

\[
\frac{dh(z)}{dz} + \frac{12}{k} h^2(z) + \frac{2}{z - a} h(z) - \frac{k^2}{4} \rho_d^2 \int_a^{z'} D \left( \rho_d, z - \frac{z'}{2}, z' \right) dz' = 0
\]  

(C.8)

In the case where the random medium intensity is characterized by

\[
\frac{B(z)}{l(z)} = \left( \frac{z}{L} \right)^n \frac{B_0}{l_0}
\]  

(C.9)
we assume
\[ h(z) = -i \frac{k}{4} \frac{1}{p(z)} \frac{dp(z)}{dz} \]  \hspace{1cm} (C.10)
and substitute it into (C.8). Then we obtain the Riccati-type equation for \( p(z) \).
\[ \frac{d^2 p(\xi)}{d\xi^2} - i\sqrt{\pi} \left( k \frac{B_0}{l_0} \right) L^{-n} \xi^{-(n+4)} p(\xi) = 0 \]  \hspace{1cm} (C.11)
where \( \xi = 1/z \). If \( n \neq 1/z \), the solution \( p(z) \) is given by[44]
\[ p(z) = z^{-1/2} \left\{ C_1 J_{-\nu}(Q(z)) + C_2 J_{\nu}(Q(z)) \right\} \]  \hspace{1cm} (C.12)
where \( a \ll z \) is assumed, \( Q(z) = i\nu \sqrt{\pi} k B_0 (z/L)^{n/2} l_0 \), \( \nu = 1/(n+2) \), and \( C_1, C_2 \) are constant. Substituting (C.10) with (C.12) into (C.6) and (C.7), we can readily obtain \( g(z) \) and \( f(z) \) as follows:
\[ g(z) = \frac{p(z)}{p(a)} \exp \left\{ -i \frac{k}{4} \int_a^z d\xi' \frac{1}{(z' - a)^2} \left[ 1 - \left( \frac{p(a)}{p(z)} \right)^2 \right] \right\} \]  \hspace{1cm} (C.13)
\[ f(z) = -i \frac{k}{2 z - a} \left[ 1 - \frac{p(a)}{p(z)} \right] \]  \hspace{1cm} (C.14)
Consequently, the substitution of (C.10), (C.13) and (C.14) into (C.5) yields
\[ M(\rho_d, z) = \frac{p(a)}{p(z)} \exp \left\{ -i \frac{k}{4} \frac{1}{p(z)} \frac{dp(z)}{dz} \rho_d^2 - i \frac{k}{2 z - a} \left[ 1 - \frac{p(a)}{p(z)} \right] \rho_d \cdot \rho_{dc} \right\} + \int_a^z d\xi' \frac{1}{z' - a} \left[ 1 - \left( \frac{p(a)}{p(z)} \right)^2 \right] \rho_{dc}^2 \]  \hspace{1cm} (C.15)
In the case of
\[ \frac{B(z)}{l(z)} = \begin{cases} \frac{B_0}{l_0}, & a < z < L \\ \left( \frac{z}{L} \right)^{-m} \frac{B_0}{l_0}, & L < z \end{cases} \]
which corresponds to the random medium in subsection 3.1.1, we need to connect the waves continuously at the boundary \( z = L \). When we determine the constants \( C_1 \) and \( C_2 \) in (C.12) under the conditions of (C.3) and the continuity of \( M(\rho_d, L) \), then (3.22) can be obtained.
D. A simplified form of the fourth moment

The $\langle |u_4|^2 \rangle$ can be obtained by substituting (2.18), (3.23) into (3.10) or (2.23), (3.23) into (3.11) and using (3.24), (3.26), but the multiple integral in (3.10) and (3.11) will take too much calculation time. It is necessary to simplify the form of the fourth moment.

For a 2-D problem, the Green’s function in free space is given by

$$G_0(\mathbf{r}|\mathbf{r}_1) = G_0(\mathbf{r}, z|\mathbf{\rho}_1, z_1) = \frac{1}{4\pi} H_0^{(1)} \left( k\sqrt{(\mathbf{\rho} - \mathbf{\rho}_1)^2 + (z - z_1)^2} \right)$$  \hspace{1cm} (D.1)

and

$$G_0^*(\mathbf{r}|\mathbf{r}_1') = G_0^*(\mathbf{r}, z|\mathbf{\rho}_1', z_1') = -\frac{1}{4\pi} H_0^{(2)} \left( k\sqrt{(\mathbf{\rho} - \mathbf{\rho}_1')^2 + (z - z_1')^2} \right)$$  \hspace{1cm} (D.2)

where $H_0^{(1)}$ and $H_0^{(2)}$ are the first and second Hankel function, respectively.

Because

$$kz \gg 1$$  \hspace{1cm} (D.3)

let

$$R_1 = k\sqrt{(\mathbf{\rho} - \mathbf{\rho}_1)^2 + (z - z_1)^2} \gg 1$$  \hspace{1cm} (D.4)

$$R_2 = k\sqrt{(\mathbf{\rho} - \mathbf{\rho}_1')^2 + (z - z_1')^2} \gg 1$$  \hspace{1cm} (D.5)

(D.1) and (D.2) can be expressed approximately as

$$G_0(\mathbf{\rho}, z|\mathbf{\rho}_1, z_1) \approx \frac{1}{4\pi i} H_0^{(1)}(kR_1) \simeq \frac{1}{4\pi i} \sqrt{\frac{2}{\pi k R_1}} e^{i(kR_1-\xi)}$$  \hspace{1cm} (D.6)

$$G_0^*(\mathbf{\rho}, z|\mathbf{\rho}_1', z_1') \approx -\frac{1}{4\pi i} H_0^{(2)}(kR_2) \simeq -\frac{1}{4\pi i} \sqrt{\frac{2}{\pi k R_2}} e^{-i(kR_2-\xi)}$$  \hspace{1cm} (D.7)

then we have

$$G_0(\mathbf{\rho}, z|\mathbf{\rho}_1, z_1)G_0^*(\mathbf{\rho}, z|\mathbf{\rho}_1', z_1') = \frac{1}{8\pi k \sqrt{R_1 R_2}} \exp[i(k(R_1 - R_2))]$$  \hspace{1cm} (D.8)
Appendix D

Note that

\[
z \gg z_0, z_1, z_1', |\rho|, |\rho_1|, |\rho_1'| \quad (D.9)
\]

\[
\frac{1}{\sqrt{R_1 R_2}} \approx \frac{1}{z} \quad (D.10)
\]

\[
\begin{align*}
   ik R_1 &= ik \left\{ (z - z_0 + z_0 - z_1)^2 + (\rho - \rho_1)^2 \right\}^{1/2} \\
   &= ik \left\{ (z - z_0)^2 + 2(z_0 - z_1)(z - z_0) + (z_0 - z_1)^2 + (\rho - \rho_1)^2 \right\}^{1/2} \\
   &\approx ik(z - z_0) \left\{ 1 + \frac{z_0 - z_1}{z - z_0} + \frac{1}{2} \frac{(\rho - \rho_1)^2}{(z - z_0)^2} \right\} \quad (D.11)
\end{align*}
\]

\[
\begin{align*}
   ik R_2 &= ik \left\{ (z - z_0 + z_0 - z_1')^2 + (\rho - \rho_1')^2 \right\}^{1/2} \\
   &= ik \left\{ (z - z_0)^2 + 2(z_0 - z_1')(z - z_0) + (z_0 - z_1')^2 + (\rho - \rho_1')^2 \right\}^{1/2} \\
   &\approx ik(z - z_0) \left\{ 1 + \frac{z_0 - z_1'}{z - z_0} + \frac{1}{2} \frac{(\rho - \rho_1')^2}{(z - z_0)^2} \right\} \quad (D.12)
\end{align*}
\]

then we obtain

\[
\begin{align*}
   G_0(\rho, z|\rho_1, z_1)G_0^*(\rho, z|\rho_1', z_1') \\
   \approx \frac{1}{8\pi k z} \exp \left\{ ik(z - z_0) \left[ \frac{z_1 - z_1'}{z - z_0} + \frac{1}{2} \frac{(\rho - \rho_1)^2 - (\rho - \rho_1')^2}{(z - z_0)^2} \right] \right\} \\
   = \frac{1}{8\pi k z} \frac{O(\rho, z|\rho_1, z_1)}{O(\rho, z|\rho_1', z_1')} \quad (D.13)
\end{align*}
\]

\[
O(\rho, z|\rho', z') = \exp \left\{ -ik \left[ -z' + \frac{(\rho - \rho')^2}{2(z - z_0)} \right] \right\} \quad (D.14)
\]

Next consider the exponential part of equation (3.24):

\[
m(\rho, \rho') = \exp \left\{ -\frac{k^2}{4} \int_a^z dz_1 \int_a^{z_1} dz_2 D \left[ z - a - z_2 (\rho - \rho'), z - z_2 - \frac{z_1}{2}, z_1 \right] \right\} \quad (D.15)
\]

Noting (3.19) and (3.20), the following equations can be derived.

\[
m(\rho, \rho') = \exp \left\{ -\frac{k^2}{4} \mu \gamma(z, z_0) (\rho - \rho')^2 \right\} \quad (D.16)
\]

\[
\mu = \pi^{1/2} \frac{B_0}{l_0} \frac{L^3}{(z - z_0)^2} \quad (D.17)
\]

\[
\gamma(z, z_0) = \frac{2}{(3 - m)(2 - m)(1 - m)} \left( \frac{z}{L} \right)^{3-m} - \left[ \frac{m}{1 - m} + \frac{z_0}{L} \right] \left( \frac{z}{L} \right)^2
\]
Finally, we can obtain $M_{22}$ from equation (3.23) as follows:

\[
M_{22} = \langle G(\mathbf{r}_1|\mathbf{r}_1)G(\mathbf{r}_2|\mathbf{r}_2)G^*(\mathbf{r}_1'|\mathbf{r}_1')G^*(\mathbf{r}_2'|\mathbf{r}_2') \rangle
\]

\[
\cong \langle G(\mathbf{r}_1|\mathbf{r}_1)G^*(\mathbf{r}_1'|\mathbf{r}_1') \rangle \langle G(\mathbf{r}_2|\mathbf{r}_2)G^*(\mathbf{r}_2'|\mathbf{r}_2') \rangle
\]

\[
+ \langle G(\mathbf{r}_1|\mathbf{r}_1)G^*(\mathbf{r}_2'|\mathbf{r}_1') \rangle \langle G(\mathbf{r}_2|\mathbf{r}_2)G^*(\mathbf{r}_1'|\mathbf{r}_2') \rangle
\]

\[
= G_0(\mathbf{r}_1|\mathbf{r}_1)G_0^*(\mathbf{r}_1'|\mathbf{r}_1')G_0(\mathbf{r}_2|\mathbf{r}_2)G_0^*(\mathbf{r}_2'|\mathbf{r}_2')
\]

\[
\left[ m(\mathbf{r}_1 - \mathbf{r}_1')m(\mathbf{r}_2 - \mathbf{r}_2') + m(\mathbf{r}_1 - \mathbf{r}_2')m(\mathbf{r}_2 - \mathbf{r}_1') \right]
\]

\[
\cong \frac{1}{8\pi k}\frac{O(\mathbf{r}_1, z_1|\mathbf{r}_1', z_1')}O(\mathbf{r}_2, z_2|\mathbf{r}_2', z_2')
\]

\[
S(\mathbf{r}, \mathbf{r}') = \exp \left[ -\frac{k^2}{4} \mu(\mathbf{r}, \mathbf{r}')^2 \right]
\]

Equation (D.19) shows that the calculation time of multiple integrals in (3.10) and (3.11) can be saved effectively if functions $O(\mathbf{r}, z|\mathbf{r}', z')$ and $S(\mathbf{r}, \mathbf{r}')$ are known beforehand.
Appendix E

E. Current generators of a conductor circular cylinder

In this case, \( H_m^{(1)}(kr) \exp(\text{i}m\theta) \), \( m = -\infty \sim \infty \), is chosen as \( \phi_m \). For E-wave incidence, the inner product in Equation (2.16) is given by

\[
(\phi_m, \phi_n) = \int_S \phi_m(r)\phi_n^*(r)dr
= \int_S H_m^{(1)}(ka)H_n^{(2)}(ka)e^{i(m-n)\theta}d\theta
= 2\pi a H_m^{(1)}(ka)H_n^{(2)}(ka)\delta_{m,n}
\]  

(E.1)

where

\[
\delta_{m,n} = \begin{cases} 
0, & \text{for } m \neq n \\
1, & \text{for } m = n 
\end{cases}
\]  

(E.2)

Then \( A_E \) becomes diagonal matrix, and we have

\[
\Phi_m^*(r)A_E^{-1} = \sum_{n=-\infty}^{\infty} H_n^{(2)}(ka)e^{-\text{i}n\theta} \frac{1}{2\pi a} \frac{1}{H_n^{(1)}(ka)H_n^{(2)}(ka)}
\]

\[
= \frac{1}{2\pi a} \sum_{n=-\infty}^{\infty} \frac{1}{H_n^{(1)}(ka)e^{-\text{i}n\theta}}
\]  

(E.3)

Note the incident wave can be expressed as

\[
u_{in}(r) = \sum_{n=-\infty}^{\infty} a_n J_n(ka)e^{\text{i}n\theta}
\]  

(E.4)

then

\[
\int_S dr \ll \phi_m(r), u_{in}(r) \gg = \int_S dr \left\{ \phi_m(r) \frac{\partial u_{in}(r)}{\partial n} - \frac{\partial \phi_m(r)}{\partial n} u_{in}(r) \right\}
= aH_m^{(1)}(ka) \int_0^{2\pi} \sum_{n=-\infty}^{\infty} a_n k \frac{\partial J_n(ka)}{\partial (ka)} e^{i(m+n)\theta} d\theta
-ak \int_0^{2\pi} \frac{\partial H_m^{(1)}(ka)}{\partial (ka)} e^{i\text{m}\theta} u_{in}(r) d\theta
= akH_m^{(1)}(ka) \frac{J_n(ka)}{J_m(ka)} \frac{1}{\partial (ka)} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} a_n J_n(ka) e^{i(m+n)\theta} d\theta
-ak \int_0^{2\pi} \frac{\partial H_m^{(1)}(ka)}{\partial (ka)} e^{i\text{m}\theta} u_{in}(r) d\theta
\]
Therefore, we obtain
\[
\int_0^{2\pi} e^{im\theta} J_m(ka) u_{in}(r) d\theta
\]
\[
\left[ H_m^{(1)}(ka) \frac{\partial J_m^{(1)}(ka)}{\partial (ka)} - \frac{\partial H_m^{(1)}(ka)}{\partial (ka)} J_m^{(1)}(ka) \right]
\]
\[
= \int_0^{2\pi} e^{im\theta} J_m^{(1)}(ka) u_{in}(r) d\theta
\]
\[
\left[ H_m^{(1)}(ka) \frac{\partial J_m^{(1)}(ka)}{\partial (ka)} - \frac{\partial H_m^{(1)}(ka)}{\partial (ka)} J_m^{(1)}(ka) \right]
\]
\[
= \int \frac{2i}{\pi a} e^{im\theta} J_m^{(1)}(ka) u_{in}(r) dr
\]  
(E.5)

Therefore, we obtain
\[
\frac{\partial u(r)}{\partial n} = \Phi_M(r) A_E^{-1} \int_S \kappa(r) J_m(ka) H_m^{(1)}(ka) u_{in}(r) d\theta
\]
\[
= \int_S dr' \frac{i}{\pi^2 a^2} \sum_{n=-\infty}^{N} e^{-in(\theta-\theta')} J_n(ka) H_m^{(1)}(ka) u_{in}(r')
\]  
(E.6)

Noting the definition of current generator (equation (2.5))
\[
\frac{\partial u(r)}{\partial n} = \int_S Y_E(r|r') u_{in}(r') dr'
\]  
(E.7)

we have the current generator for E-wave incidence
\[
Y_E(r|r_0) = \frac{i}{\pi^2 a^2} \sum_{n=-\infty}^{\infty} \frac{\exp[in(\theta_0 - \theta)]}{J_n(ka) H_m^{(1)}(ka)}
\]
(E.8)

Similarly, we can obtain that for H-wave incidence
\[
Y_H(r|r_0) = \frac{i}{\pi^2 ka^2} \sum_{n=-\infty}^{\infty} \frac{\exp[in(\theta_0 - \theta)]}{J_n(ka) \frac{\partial}{\partial (ka)} H_m^{(1)}(ka)}
\]
(E.9)
F. Current generators of a conductor elliptic cylinder

Consider the E-wave incidence. Let us approximate the wave field with a truncated modal expansion:

$$u(r) \approx \sum_{m=-\infty}^{\infty} c_m \phi_m^*(r)$$  \hspace{1cm} (F.1)

then the surface current is given by

$$\frac{\partial u(r)}{\partial n} \approx \sum_{m=-\infty}^{\infty} c_m \frac{\partial \phi_m^*(r)}{\partial n} = \Phi_M^* C_M$$  \hspace{1cm} (F.2)

where

$$\Phi_M^* = \begin{bmatrix} \frac{\partial \phi_1^*}{\partial n} & \frac{\partial \phi_2^*}{\partial n} & \cdots & \frac{\partial \phi_M^*}{\partial n} \end{bmatrix}$$  \hspace{1cm} (F.3)

$$C_M = [c_1, c_2, \ldots, c_M]$$  \hspace{1cm} (F.4)

For E-wave incidence, referring to subsection 2.4.1, we can obtain

$$C_m^T = A_E^{-1} \int_S \langle \phi_m(r'), u_{in}(r') \rangle dr'$$  \hspace{1cm} (F.5)

$$A_E = \begin{bmatrix} (\phi_1, \frac{\partial \phi_1^*}{\partial n}) & (\phi_1, \frac{\partial \phi_2^*}{\partial n}) & \cdots & (\phi_1, \frac{\partial \phi_M^*}{\partial n}) \\ (\phi_2, \frac{\partial \phi_1^*}{\partial n}) & (\phi_2, \frac{\partial \phi_2^*}{\partial n}) & \cdots & (\phi_2, \frac{\partial \phi_M^*}{\partial n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\phi_M, \frac{\partial \phi_1^*}{\partial n}) & (\phi_M, \frac{\partial \phi_2^*}{\partial n}) & \cdots & (\phi_M, \frac{\partial \phi_M^*}{\partial n}) \end{bmatrix}$$  \hspace{1cm} (F.6)

The basis functions can be chosen as

$$\phi_m(r) = \begin{cases} Mc_m^{(3)}(\xi)c_m(\eta) & m \geq 0 \\ Ms_m^{(3)}(\xi)se_m(\eta) & m < 0 \end{cases}$$  \hspace{1cm} (F.7)

where $c_m(\eta)$, $se_m(\eta)$ are Mathieu's functions of the first kind, $Mc_m^{(0)}(\xi)$, $Ms_m^{(0)}(\xi)$ are modified Mathieu's functions of l-th kind. The elliptic cylindrical coordinates ($\xi$, $\eta$, $\gamma$) used
here are related to the rectangular Cartesian coordinates \((z, x, y)\) by the transformation

\[
\begin{align*}
z &= c \cosh \xi \cos \eta \\
x &= c \sinh \xi \sin \eta \\
y &= y
\end{align*}
\]  

where \(c\) is half the focal-length of the ellipse (figure F.1). Noting

\[
\nabla \phi = \frac{1}{c\sqrt{\cosh^2 \xi - \cos^2 \eta}} \left( a_x \frac{\partial \phi}{\partial \xi} + a_y \frac{\partial \phi}{\partial \eta} \right) + a_y \frac{\partial \phi}{\partial y}
\]  

and

\[
\int_0^{2\pi} c e_m(\eta) s e_n(\eta) d\eta = 0 \quad \text{(F.9)}
\]

\[
\int_0^{2\pi} c e_m(\eta) c e_n(\eta) d\eta = \begin{cases} 
\pi, & \text{for } m = n \\
0, & \text{for } m \neq n
\end{cases} \quad \text{(F.10)}
\]

Figure F.1: The elliptic cylindrical coordinates.
the inner product in (F.6) becomes

\[
(\phi_m, \partial \phi_n^*) \Delta \phi_n = \int_S \phi_m(r) \partial \phi_n^*(r) \, dr
\]

\[
= \int_S \phi_n(r) \alpha \cdot \nabla \phi_n^*(r) \, dr
\]

\[
= \left\{ \begin{array}{l}
\pi \Mc_m^{(3)}(\xi) \frac{\partial \Mc^{(4)}(\xi)}{\partial \xi} \delta_{m,n} & m \geq 0 \\
\pi \Ms_m^{(3)}(\xi) \frac{\partial \Ms^{(4)}(\xi)}{\partial \xi} \delta_{m,n} & m < 0
\end{array} \right.
\]

The incident wave can be expressed as

\[
uin(r) = \sum_{l=0}^{\infty} \frac{1}{2} \left[ \Mc_l^{(3)}(\xi) ce_1(\eta) + \Ms_{-l}^{(3)}(\xi) se_{-1}(\eta) \right]
\]

and we have

\[
\int_{\delta} dr \langle \phi_m(r'), \nuin(r') \rangle = \int_S \left\{ \phi_m(r') \frac{\partial uin(r')}{\partial n} - \frac{\partial \phi_m(r')}{\partial n} uin(r') \right\} \, dr
\]

\[
= \left\{ \begin{array}{l}
\int_S dr \Psi_c & m \geq 0 \\
\int_S dr \Psi_s & m < 0
\end{array} \right.
\]

\[
\Psi_c = \Mc_m^{(3)}(\xi') ce_m(\eta') \sum_{l=0}^{\infty} \frac{1}{2} \frac{\partial \Mc_l^{(4)}(\xi')}{\partial \xi} ce_l(\eta')
\]

\[
- \frac{\partial \Ms_m^{(3)}(\xi')}{\partial \xi} ce_m(\eta') \sum_{l=0}^{\infty} \frac{1}{2} \frac{\partial \Ms_l^{(4)}(\xi')}{\partial \xi} ce_l(\eta')
\]

\[
\Psi_s = \Ms_{-m}^{(3)}(\xi') se_{-m}(\eta') \sum_{l=-\infty}^{-1} \frac{1}{2} \frac{\partial \Ms_{-l}^{(4)}(\xi')}{\partial \xi} se_l(\eta')
\]

\[
- \frac{\partial \Ms_{-m}^{(3)}(\xi')}{\partial \xi} se_{-m}(\eta') \sum_{l=-\infty}^{-1} \frac{1}{2} \frac{\partial \Ms_{-l}^{(4)}(\xi')}{\partial \xi} se_{-l}(\eta')
\]

Because

\[
\frac{\partial \Mc_m^{(4)}(\xi)}{\partial \xi} = \frac{\partial \Mc_m^{(3)}(\xi) - \Mc_m^{(1)}(\xi)}{\partial \xi} = \frac{2i}{\pi} (m \geq 0)
\]

\[
\frac{\partial \Ms_{-m}^{(4)}(\xi)}{\partial \xi} = \frac{\partial \Ms_{-m}^{(3)}(\xi) - \Ms_{-m}^{(1)}(\xi)}{\partial \xi} = \frac{2i}{\pi} (m < 0)
\]
we can rewrite equation (F.15) as
\[
\int_S d\mathbf{r} \ll \phi_m(\mathbf{r}'), u_{in}(\mathbf{r}') \gg
\]
\[
= \left\{ \begin{array}{ll}
\frac{-2i}{\pi} \int_S d\eta' \left[ \frac{c_{em}(\eta')}{M^{(1)}_{m}}(\xi') \sum_{l=0}^{\infty} p_l M^{(1)}_{m} \left( \xi' \right) c_{l}(\eta') \right] & m \geq 0 \\
\frac{-2i}{\pi} \int_S d\eta' \left[ \frac{-s_{-m}(\eta')}{M^{(1)}_{-m}}(\xi') \sum_{l=-\infty}^{-1} p_l M^{(1)}_{-m} \left( \xi' \right) s_{-l}(\eta') \right] & m < 0 \\
\frac{2i}{\pi} \int_S d\eta' \left[ \frac{c_{em}(\eta')}{c_{\sqrt{c} \cosh^2 \xi' - \cos^2 \eta'} M^{(1)}_{m}}(\xi') \right] u_{in}(\mathbf{r}') d\mathbf{r}' & m \geq 0 \\
\frac{2i}{\pi} \int_S d\eta' \left[ \frac{-s_{-m}(\eta')}{c_{\sqrt{c} \cosh^2 \xi' - \cos^2 \eta'} M^{(1)}_{-m}}(\xi') \right] u_{in}(\mathbf{r}') d\mathbf{r}' & m < 0
\end{array} \right.
\]
\[\text{(F.20)}\]

Substituting (F.3), (F.5) into (F.2) and noting (F.6), (F.7), (F.13) and (F.20), we have
\[
\frac{\partial u(\mathbf{r})}{\partial n} \simeq \int_S \frac{2i}{\pi} c^2 \sqrt{(\cosh^2 \xi - \cos^2 \eta)(\cosh^2 \xi' - \cos^2 \eta')} \left\{ \sum_{m=0}^{\infty} \frac{c_{em}(\eta)c_{em}(\eta')}{M^{(3)}_{m}(\xi)M^{(1)}_{m}(\xi')} + \sum_{m=1}^{\infty} \frac{s_{em}(\eta)s_{em}(\eta')}{M^{(3)}_{m}(\xi)M^{(1)}_{m}(\xi')} \right\} u_{in}(\mathbf{r}') d\mathbf{r}'
\]
\[\text{(F.21)}\]
Comparing (F.21) with (2.5), we obtain the current generator of a conducting elliptic cylinder for E-wave incidence:
\[
Y_E(\mathbf{r}|\mathbf{r}') = \frac{2j}{\pi^2 c^2} \sqrt{(\cosh^2 \xi - \cos^2 \eta)(\cosh^2 \xi' - \cos^2 \eta')} \times \left\{ \sum_{m=0}^{\infty} \frac{c_{em}(\eta)c_{em}(\eta')}{M^{(3)}_{m}(\xi)M^{(1)}_{m}(\xi')} + \sum_{m=1}^{\infty} \frac{s_{em}(\eta)s_{em}(\eta')}{M^{(3)}_{m}(\xi)M^{(1)}_{m}(\xi')} \right\}
\]
\[\text{(F.22)}\]
Similarly, for H-wave incidence, we have
\[
Y_H(\mathbf{r}|\mathbf{r}') = \frac{2j}{\pi^2 c^2} \sqrt{\cosh^2 \xi' - \cos^2 \eta'} \frac{1}{\sqrt{\cosh^2 \xi - \cos^2 \eta'}} \times \left\{ \sum_{m=0}^{\infty} \frac{c_{em}(\eta)c_{em}(\eta')}{\partial M^{(3)}_{m}(\xi)\partial M^{(1)}_{m}(\xi')} + \sum_{m=1}^{\infty} \frac{s_{em}(\eta)s_{em}(\eta')}{\partial M^{(3)}_{m}(\xi)\partial M^{(1)}_{m}(\xi')} \right\}
\]
\[\text{(F.23)}\]