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Deriving Five Growth Functions from Bertalanffy Function based on Symmetry and Complexity

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This study was designed to derive five growth functions from Bertalanffy function using symmetry and complexity of them by relating each function to its first and second derivatives. The results obtained were as follows. In equations constructed by relating functions to their derivatives, the left–hand sides were the same in form, but the right–hand sides showed differences from the symmetry essential to exponential function. These differences were related to complexity of functions approaching asymptotes. The asymptotic properties of five functions were shown by phenomena that the right–hand sides tended to zero as t tended to infinity. Six growth functions arranged by the complexity were Bertalanffy > Richards > Mitscherlich = logistic = Gompertz > basic growth. Despite different function forms, Mitscherlich, logistic and Gompertz functions were not distinguished each other from the viewpoint of complexity, a kind of symmetry existing at the base of them. Based on symmetry and complexity, five growth functions were derived from Bertalanffy function, a hierarchic structure of growth functions from Bertalanffy function on down.

Key words: asymptote, Bertalanffy function, complexity, growth function, symmetry.

INTRODUCTION

Richards function includes Mitscherlich, logistic, Gompertz and basic growth functions as special cases (Richards, 1959), a kind of generality that Richards function has. This is also shown by many subsequent studies, for example, Yoshida (1979), Naito and Shiraishi (1983), Osumi and Ishikawa (1983), and Ito and Osumi (1984). This generality of Richards function goes back to the differential equation for Bertalanffy function (1949, 1957), from which Richards function was derived (Richards, 1959). There was a study in which growth was analyzed using Richards function and its first, second and third derivatives (Nath and Moore Ⅲ, 1992). Shimojo *et al.* (2011) made comparisons between the above five functions, except Bertalanffy function, by relating them to their first and second derivatives.

The present study was designed to derive the above five growth functions from Bertalanffy function using symmetry and complexity of them by relating each function to its first and second derivatives.

DERIVING FIVE GROWTH FUNCTIONS FROM BERTALANFFY FUNCTION

Bertalanffy (W_V), Richards (W_R), Mitscherlich (W_M), logistic (W_L), Gompertz (W_G) and basic growth (W_B) functions

The growth rate given by Bertalanffy (1949, 1957) is

$$\frac{dW_{\mathbf{V}}}{dt} = \alpha W_{\mathbf{V}}^{m} - \beta W_{\mathbf{V}} , \tag{1}$$

where $W_{\mathbf{V}}$ = weight, t = time, α = anabolic constant, β = catabolic constant, m = constant. As reported by Bertalanffy (1957), solving (1) gives Bertalanffy function (2),

$$W_{\mathbf{V}} = (\alpha/\beta - (\alpha/\beta - W_{\mathbf{0}}^{1-m})\exp(-\beta(1-m)t))^{1/(1-m)} , \tag{2}$$

where $W_{\mathbf{0}}^{1-m}$= weight at $t = 0$, $m \neq 1$.

The other five growth functions (Richards, 1959; Osumi and Ishikawa, 1983; Ito and Osumi, 1984) are as follows,

$$W_{\mathbf{R}} = A(1-b\exp(-kt))^{1/(1-m)}, \tag{3}$$

$$W_{\mathbf{M}} = A(1-b\exp(-kt)), \tag{4}$$

$$W_{\mathbf{L}} = A/(1+b\exp(-kt)), \tag{5}$$

$$W_{\mathbf{G}} = A\exp(-b\exp(-kt)), \tag{6}$$

$$W_{\mathbf{B}} = W_{\mathbf{0}} \exp(rt), \tag{7}$$

where W = weight, t = time, A, b, k and m are constants, $W_{\mathbf{0}}$ = weight at $t = 0$, r = relative growth rate.

Relating each function to its first and second derivatives

Relating each function to its first and second derivatives gives the following equations,

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$$\frac{(dW_V/dt)^2}{W_V(d^2W_V/dt^2)}$$

$$= \frac{(\alpha/\beta - W_0^{1-m})\exp(-\beta(1-m)t)}{(\alpha/\beta - W_0^{1-m})\exp(-\beta(1-m)t) - (\alpha/\beta)(1-m)} \quad . \tag{2–1}$$

$$\frac{(dW_R/dt)^2}{W_R(d^2W_R/dt^2)} = \frac{b\exp(-kt)}{b\exp(-kt) - (1-m)} \quad , \tag{3–1}$$

$$\frac{(dW_M/dt)^2}{W_M(d^2W_M/dt^2)} = \frac{b\exp(-kt)}{b\exp(-kt) - 1} \quad , \tag{4–1}$$

$$\frac{(dW_L/dt)^2}{W_L(d^2W_L/dt^2)} = \frac{b\exp(-kt)}{b\exp(-kt) - 1} \quad , \tag{5–1}$$

$$\frac{(dW_G/dt)^2}{W_G(d^2W_G/dt^2)} = \frac{b\exp(-kt)}{b\exp(-kt) - 1} \quad , \tag{6–1}$$

$$\frac{(dW_B/dt)^2}{W_B(d^2W_B/dt^2)} = 1. \tag{7–1}$$

Equation (7–1) shows that the form of function (7) is not changed by differential calculus, a kind of symmetry that is essential to exponential function. In equations (2–1)~(6–1), the left–hand sides are constructed to show the same form as that of equation (7–1), and thus, the right–hand sides show differences from the symmetry of exponential function. These differences from the symmetry are related to complexity of functions approaching asymptotes. The asymptotic properties in equations (2–1)~(6–1) are shown by phenomena that the right–hand sides tend to zero as t tends to infinity. Six growth functions arranged by the complexity are as follows: Bertalanffy > Richards > Mitscherlich = logistic = Gompertz > basic growth. Despite different function forms, Mitscherlich, logistic and Gompertz functions are not distinguished each other from the viewpoint of complexity, a kind of symmetry that exists at the base of them.

Deriving five growth functions from Bertalanffy function

If there is a replacement (8) in equation (2–1) for Bertalanffy function,

$$\alpha/\beta - W_0^{1-m} = b,\ \ \alpha/\beta = 1,\ \ \beta(1-m) = k,\ \ m \neq 1 \tag{8}$$

then this gives equation (9–1) whose form is the same as equation (3–1) for Richards function,

$$\frac{(dW_V/dt)^2}{W_V(d^2W_V/dt^2)} = \frac{b\exp(-kt)}{b\exp(-kt) - (1-m)} \quad . \tag{9–1}$$

Equations (4–1)~(6–1) are derived from equation (2–1) by the replacement (9),

$$\alpha/\beta - W_0^{1-m} = b,\ \ \alpha/\beta = 1,\ \ \beta(1-m) = k,\ \ m = 0. \tag{9}$$

Equation (7–1) is derived from equation (2–1) by the replacement (10),

$$\alpha/\beta \neq 1,\ \ m = 1. \tag{10}$$

These replacements, though A and W_0 should be determined to obtain actual growth functions, show a hierarchic structure of growth functions from Bertalanffy function on down. The replacement for deriving basic growth function with an exponential increase is different from those for deriving the other functions approaching asymptotes. Five growth functions are derived from Bertalanffy function using symmetry and complexity of them. Turner *et al.* (1976) reported, using a more generic function, the highly hierarchic structure of growth functions.

Conclusions

The present study suggests that Richards, Mitscherlich, logistic, Gompertz and basic growth functions are derived from Bertalanffy function using symmetry and complexity of them.

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