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A FORMALIZATION OF A FUZZY RELATIONAL DATABASE MODEL USING RELATIONAL CALCULUS

By

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Abstract

We introduce a formalization of a fuzzy relational database model using relational calculus on the category of fuzzy relations. We also introduce general formulas for the notion of database operations such as 'projection', 'selection', 'injection' and 'natural join' which can be used for both traditional and fuzzy database models. We prove several elementary properties of database operations using relational calculus. Next we demonstrate the truck backer-upper control problem using our formulation. Every fuzzy states, operation strategies, and solving procedures are described by database tables and formulas of relational calculus. We also show examples of computations of fuzzy relational databases using our Mathematica implementation.

Key Words and Phrases: Fuzzy Theory, Relational Database, Relational Calculus, Dedekind Category, Fuzzy Database, Database Operations, Control Problem

1. Introduction

The relational database model was first introduced by Codd (1970). One of the advantages of the relational model is soundness for data consistency. A procedure of data processing is sometimes described by dynamic sequences of operations which may have ambiguities in its implementation. Since a procedure in the relational database model is defined by a static formula, we can avoid inconsistencies in their implementations.

Relational databases commonly contain attributes, tuples, database relation, database schema, and database operations. In this paper, we show properties of fuzzy relational databases using relational calculus.

Fuzzy database relation has been introduced by Raju, Majumdar, Arun (1988), and fuzzy operations model was introduced by Umano and Fukami, S. (1994), Nakata (2000). Database operator is one of the important aspects in relational database properties. Fuzzy relational database operators('projection', 'join', and 'selection') has been introduced by Umano and Fukami, S. (1994). In our work, we use relational calculus to compute database operations. We use simpler and clearer formula to compute database operations.

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The first concept of fuzzy relation has been invented by Zadeh (1965). Kawahara, et al. developed an algebraic formalization of fuzzy relations using relational calculus Kawahara (1988), Kawahara and Furusawa (1999), Mori and Kawahara (1995). Okuma, et al. introduced a relational database model using a theory of relational calculus on the Dedekind Category Okuma and Kawahara (2000). Since the Dedekind Category includes the category of fuzzy relations, we can formalize a fuzzy relational database model using relational calculus on the category of fuzzy relations. Further, we formalize the notion of database operations such as 'projection', 'selection', 'injection' and 'natural join' using formulas of relational calculus. We note that the same formulas of relational calculus can be used for a traditional non-fuzzy relational database model. So we can consider our fuzzy relational database theory using intuition of the traditional database operations. We show several elementary properties of database operations using relational calculus.

There exist some trials of fuzzy relational database theory, but they are limited in specific topics such as fuzzy graph problem Mordeson and Nair (2001), Kiss (1991). The advantage of our framework is showing several relational database operations uniformly using the theory of relational calculus.

Next, we apply our formulation to an example of the truck backer-upper control problem using fuzzy logic introduced by Freeman (1994). Every fuzzy states, procedures are described as database tables of the fuzzy relational database theory. Problem-solving procedures are also described by static formulas of the relational calculus on the category of fuzzy relations. Since every property is described statically, the consistency of data can be proved formally. We also implemented operations in the theory of fuzzy relational database using Mathematica[1]. Using our Mathematica library, we demonstrate computations in the truck backer-upper control problem.

Finally, we list functions in our Mathematica library and examples in the Appendix.

2. Fuzzy Relation

In this section, we summarize basic notations for fuzzy relations. We denote the set $\{\mathbf{x} \in \mathbb{R} | 0 \leq \mathbf{x} \leq 1\}$ as [0,1]. The supremum and infimum of a family $\{x_\lambda\}_{\lambda\in\Lambda}$ of elements $x_\lambda \in [0,1]$ is denoted by $\bigvee_{\lambda\in\Lambda} x_\lambda$ and $\bigwedge_{\lambda\in\Lambda} x_\lambda$, respectively. In particular, $x \vee x' = max\{x, x'\}$, $x \wedge x' = min\{x, x'\}$ for $x, x' \in [0,1]$. For two elements $x, x' \in [0,1]$, the relative pseudo-complement $\Rightarrow$ of x relative to x' defined by $x \Rightarrow x' := [x \leq x'] \vee x'$, where

$$[x \leq x'] := \begin{cases} 1 & \text{if } x \leq x', \\ 0 & \text{otherwise .} \end{cases}$$

Lemma 2.1. *Let $x, y, z \in [0,1]$.*

$$x \leq (y \Rightarrow z) \textit{ if and only if } x \wedge y \leq z.$$

[1] `http://www.wolfram.com/mathematica/`

PROOF.
$$\begin{aligned}(\rightarrow)\ x \le (y \Rightarrow z) &\leftrightarrow x \le [y \le z] \vee z \\ &\leftrightarrow (x \le [y \le z]) \vee (x \le z) \\ &\rightarrow (x = 0 \vee (y \le z)) \vee (x \le z) \\ &\rightarrow (x \wedge y \le z). \\ (\leftarrow)\ x \wedge y \le z &\leftrightarrow x \le z \vee y \le z \\ &\rightarrow x \le z \vee x \le [y \le z] \\ &\rightarrow x \le [y \le z] \vee z \\ &\leftrightarrow x \le (y \Rightarrow z).\end{aligned}$$
□

DEFINITION 2.2. A **fuzzy relation** α from a set A to another set B is a fuzzy subset of the Cartesian product $A \times B$. i.e. $\alpha : A \times B \to [0,1]$. It is denoted by $\alpha : A \rightharpoondown B$. The **inverse fuzzy relation** $\alpha^{\#} : B \rightharpoondown A$ of α is a relation defined by $\alpha^{\#}(b,a) := \alpha(a,b)$ for $a \in A, b \in B$. The **composition** $\alpha \cdot \beta : A \rightharpoondown C$ of $\alpha : A \rightharpoondown B$ followed by $\beta : B \rightharpoondown C$ is a fuzzy relation defined by $\alpha \cdot \beta(a,c) := \bigvee_{b \in B} (\alpha(a,b) \wedge \beta(b,c))$. The **identity** relation $id_A : A \rightharpoondown A$ is defined by

$$id_A(a,b) := \begin{cases} 1 & (a = b), \\ 0 & (a \neq b) \ . \end{cases}$$

DEFINITION 2.3. We define a category $FRel$ as follows:
An object X of $FRel$ is a set. For two objects X and Y, a morphism set $FRel(X,Y)$ is a set of fuzzy relations from X to Y. It is easy to check that $FRel$ is a category with a composition and an identity of relation.

DEFINITION 2.4. Let X, Y be objects in $FRel$. α, β morphism in $FRel(X,Y)$, $a \in X$, and $b \in Y$. We define fuzzy operations $\sqcup$(union), $\sqcap$(intersection), $\sqsubseteq$(subset), $\Rightarrow$(the relative pseudo-complement), and constants $\mathbf{0}_{AB}$(least), ∇_{AB}(greatest) in $FRel$, as follows:

1. $(\alpha \sqcup \beta)(a,b) := \alpha(a,b) \vee \beta(a,b)$,
2. $(\alpha \sqcap \beta)(a,b) := \alpha(a,b) \wedge \beta(a,b)$,
3. $\alpha(a,b) \sqsubseteq \beta(a,b)$ iff $\alpha(a,b) \le \beta(a,b)$,
4. $\mathbf{0}_{AB}(a,b) := 0$, and
5. $\nabla_{AB}(a,b) := 1$.

EXAMPLE 2.5. Consider $A = \{a,b,c,d\}$, $B = \{2,3,4\}$, $C = \{x,y,z\}$. Let $\alpha_1 : A \rightharpoondown B$, $\alpha_2 : A \rightharpoondown B$, $\beta : B \rightharpoondown C$ be fuzzy relations such that we defined by $\alpha_1(a,2) = 0.1$, $\alpha_1(a,4) = 0.4$, $\alpha_1(c,2) = 0.3$, $\alpha_1(d,3) = 0.5$, $\alpha_2(a,2) = 0.3$, $\alpha_2(c,2) = 0.2$, $\beta(2,y) = 0.5$, $\beta(4,y) = 0.2$, $\beta(3,z) = 0.6$,and $\beta(3,x) = 0.4$.

The followings are examples of a union, an intersection and a composition relation:

- We have $(\alpha_1 \sqcup \alpha_2)(a,2) = 0.3$, $(\alpha_1 \sqcup \alpha_2)(a,4) = 0.4$, $(\alpha_1 \sqcup \alpha_2)(c,2) = 0.3$ for $(\alpha_1 \sqcup \alpha_2) : A \rightharpoondown B$.

- We have $(\alpha_1 \sqcap \alpha_2)(a, 2) = 0.1$, $(\alpha_1 \sqcap \alpha_2)(c, 2) = 0.2$ for $(\alpha_1 \sqcap \alpha_2) : A \rightharpoondown B$.
- We have $(\alpha_1 \cdot \beta)(a, y) = (\alpha_1(a, 2) \wedge \beta(2, y)) \vee (\alpha_1(a, 3) \wedge \beta(3, y)) \vee (\alpha_1(a, 4) \wedge \beta(4, y)) = 0.1 \vee 0 \vee 0.2 = 0.2$ for $(\alpha_1 \cdot \beta) : A \rightharpoondown C$.

DEFINITION 2.6. Let $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ be fuzzy relations for all $x \in X$, $y \in Y$ and $z \in Z$. The residue composite $\alpha \rhd \beta : X \rightharpoondown Z$ is defined by:

$$(\alpha \rhd \beta)(x, z) = \bigwedge_{y \in Y} (\alpha(x, y) \Rightarrow \beta(y, z))$$

PROPOSITION 2.7. *Let $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ be fuzzy relations.*

1. $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^{\#}\gamma)$,
2. $\gamma \sqsubseteq \alpha \rhd \beta$ *if and only if* $\alpha^{\#} \cdot \gamma \sqsubseteq \beta$, *and*
3. $(0_{XY} \rhd \beta) = \nabla_{XZ}$.

PROOF. 1. *Let fx $\in X$ and $z \in Z$.*

$$\begin{aligned}
(\alpha \cdot \beta \sqcap \gamma)(x, z) &= (\alpha \cdot \beta)(x, y) \wedge \gamma(x, z) \\
&= \bigvee_{y \in Y} (\alpha(x, y) \wedge \beta(y, z) \wedge \gamma(x, z)) \\
&= \bigvee_{y \in Y} (\alpha(x, y) \wedge \beta(y, z) \wedge \alpha^{\#}(y, x) \wedge \gamma(x, z)) \\
&\sqsubseteq \bigvee_{y \in Y} (\alpha(x, y) \wedge \beta(y, z)) \wedge \bigvee_{x' \in X} (\alpha^{\#}(y, x') \wedge \gamma(x', z)) \\
&= \bigvee_{y \in Y} (\alpha(x, y) \wedge \beta(y, z)) \wedge (\alpha^{\#} \cdot \gamma)(y, z) \\
&= \bigvee_{y \in Y} (\alpha(x, y) \wedge (\beta \sqcap \alpha^{\#} \cdot \gamma)(y, z)) \\
&= \alpha \cdot (\beta \sqcap \alpha^{\#} \cdot \gamma)(x, z)
\end{aligned}$$

So we have $\alpha \cdot \beta \sqcap \gamma \sqsubseteq \alpha \cdot (\beta \sqcap \alpha^{\#} \cdot \gamma)$.

2. *Let $y \in Y$ and $z \in Z$*

$$\begin{aligned}
\alpha^{\#} \cdot \gamma(y, z) \sqsubseteq \beta(y, z) &\leftrightarrow \forall_y \forall_z : \alpha^{\#} \cdot \gamma(y, z) \leq \beta(y, z) \\
&\leftrightarrow \forall_y \forall_z : \bigvee_{x \in X} (\alpha^{\#}(y, x) \wedge \gamma(x, z)) \leq \beta(y, z) \\
&\leftrightarrow \forall_y \forall_z \forall_x : \alpha^{\#}(y, x) \wedge \gamma(x, z) \leq \beta(y, z) \\
&\leftrightarrow \forall_y \forall_z \forall_x : \gamma(x, z) \wedge \alpha^{\#}(y, x) \leq \beta(y, z) \\
&\leftrightarrow \forall_y \forall_z \forall_x : \gamma(x, z) \sqsubseteq \alpha^{\#}(y, x) \Rightarrow \beta(y, z) \qquad \textit{[By Lemma 2.1]} \\
&\leftrightarrow \forall_x \forall_z : \gamma(x, z) \sqsubseteq \bigwedge_{y \in Y} \alpha(x, y) \Rightarrow \beta(y, z) \\
&\leftrightarrow \forall_x \forall_z : \gamma(x, z) \sqsubseteq (\alpha \rhd \beta)(x, z)
\end{aligned}$$

So we have $\gamma \sqsubseteq \alpha \rhd \beta \leftrightarrow \gamma \sqsubseteq \alpha \rhd \beta$.

3. Let $x \in X$ and $z \in Z$

$$\begin{aligned}(0_{XY} \rhd \beta)(x,z) &= \bigwedge_{y\in Y} 0_{XY}(x,y) \rhd \beta(y,z)\\ &= \bigwedge_{y\in Y} [0_{XY}(x,y) \leq \beta(y,z)] \vee \beta(y,z)\\ &= \bigwedge_{y\in Y} 1 \vee (\beta(y,z)) = 1 = \nabla_{XZ}(x,z)\end{aligned}$$

So we have $(0_{XY} \rhd \beta) = \nabla_{XZ}$.

Definition 2.8. Furusawa and Kawahara (2014) **Dedekind Category** D is a category satisfying the following axioms D1(Complete Heyting Algebra(CHA)), D2(Converse), D3(Dedekind Formula), and D4(Residual Composition).
We denote a morphism $\alpha \in D(X,Y)$ as $\alpha : X \rightharpoondown Y$.
D1. For all pairs of objects X and Y the hom-set $D(X,Y)$ consisting of all morphisms of X into Y is a complete Heyting algebra (namely, a complete distributive lattice) $D(X,Y) = (D(X,Y), \sqcap, \sqcup, \sqsubseteq, 0_{XY}, \nabla_{XY})$ with the least morphism 0_{XY} and the greatest morphism ∇_{XY}.

That is, if $\alpha, \alpha_i \in D(X,Y)$ for an index set I, we have the followings:

(a) $\sqsubseteq$ is a partial order on $D(X,Y)$,

(b) $0_{XY} \sqsubseteq \alpha \sqsubseteq \nabla_{XY}$,

(c) $\sqcup_{i\in I}\alpha_i \sqsubseteq \alpha$ iff $\alpha_i \sqsubseteq \alpha$ for all $i \in I$,

(d) $\alpha \sqsubseteq \sqcap_{i\in I}\alpha_i$ iff $\alpha_i \sqsubseteq \alpha$ for all $i \in I$, and

(e) $\alpha \sqcap \sqcup_{i\in I}\alpha_i = \sqcup_{i\in I}(\alpha \sqcap \alpha_i)$.

D2. There is given a converse operation$^{\#} : D(X,Y) \to D(Y,X)$. That is, for all morphisms $\alpha, \alpha' \in D(X,Y), \beta \in D(Y,Z)$, the following converse laws hold:

(a) $(\alpha\beta)^{\#} = \beta^{\#}\alpha^{\#}$,

(b) $(\alpha^{\#})^{\#} = \alpha$, and

(c) $\alpha \sqsubseteq \alpha'$, then $\alpha^{\#} \sqsubseteq \alpha'^{\#}$.

D3. For all morphisms $\alpha \in D(X,Y), \beta \in D(Y,Z)$ and $\gamma \in D(X,Z)$, the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^{\#}\gamma)$ holds.
D4. For all morphisms, we denote $\alpha \in D(X,Y)$ as $\alpha : X \rightharpoondown Y$ and $\beta \in D(Y,Z)$ as $\beta : Y \rightharpoondown Z$, the residual composite $\alpha \rhd \beta : X \rightharpoondown Z$ is a morphism such that $\gamma \sqsubseteq \alpha \rhd \beta$ if and only if $\alpha^{\#}.\gamma \sqsubseteq \beta$ for all morphisms $\gamma : X \rightharpoondown Z$.

Example 2.9. Consider a category Rel_o whose objects are all nonempty set and in which a hom-set $Rel_o(X,Y)$ between objects X and Y is the set of all (binary) fuzzy relations on X if $X = Y$, and $\nabla_{XY} = 0_{XY}$, otherwise. That is, a hom-set $Rel_o(X,Y)$ is singleton set when X and Y are distinct. Then it is easy to verify that the category Rel_o is dedekind category. The conditions (D1) and (D2) are trivial, and (D3) and (D4) also hold as follows:

(a). If $X = Y = Z$, then (D3) and (D4) are clear from Proposition 2.7.1 and 2.7.2.

(b). If $X = Y \neq Z$, then $\beta = 0_{YZ}$, and $\gamma = 0_{XZ}$.
So $\alpha \cdot \beta \sqcap \gamma = \alpha \cdot 0_{YZ} \sqcap 0_{XZ} = 0_{XZ}$ holds. Since $0_{XZ} \sqsubseteq \alpha \cdot (\beta \sqcap \alpha^{\#} \cdot \gamma)$, then we have $\alpha \cdot \beta \sqcap \gamma \sqsubseteq \alpha \cdot (\beta \sqcap \alpha^{\#} \cdot \gamma)$ (D3).
Since $\gamma \sqsubseteq \alpha \rhd \beta \leftrightarrow 0_{XZ} \sqsubseteq \alpha \rhd 0_{YZ}$ and $\alpha^{\#} \cdot \gamma \sqsubseteq \beta \leftrightarrow \alpha^{\#} \cdot 0_{XZ} \sqsubseteq 0_{YZ}$, then we have $\gamma \sqsubseteq \alpha \rhd \beta$ and $\alpha^{\#} \cdot \gamma \sqsubseteq \beta$ for any α, β, γ (D4).

(c). If $X \neq Y$ then $\alpha = 0_{XY}$.
So $\alpha \cdot \beta \sqcap \gamma = 0_{XY} \cdot \beta \sqcap \gamma = 0_{XZ}$ holds. Since $0_{XZ} \sqsubseteq \alpha \cdot (\beta \sqcap \alpha^{\#} \cdot \gamma)$, then we have $\alpha \cdot \beta \sqcap \gamma \sqsubseteq \alpha \cdot (\beta \sqcap \alpha^{\#} \cdot \gamma)$ (D3).
Since $\gamma \sqsubseteq \alpha \rhd \beta \leftrightarrow \gamma \sqsubseteq 0_{XY} \rhd \beta \leftrightarrow \gamma \sqsubseteq \nabla_{XZ}$ and $\alpha^{\#} \cdot \gamma \sqsubseteq \beta \leftrightarrow 0_{YX} \cdot \gamma \sqsubseteq \beta \leftrightarrow 0_{YZ} \sqsubseteq \beta$, then we have $\gamma \sqsubseteq \alpha \rhd \beta$ and $\alpha^{\#} \cdot \gamma \sqsubseteq \beta$ for any α, β, γ (D4).

Proposition 2.10. *Furusawa and Kawahara (2014)*
Let $\alpha : A \rightharpoonup B$, $\beta : A \rightharpoonup B$ and $\beta_\lambda : A \rightharpoonup B$ $(\lambda \in \Lambda)$ be fuzzy relations. Then we have Basic properties of dedekind categories:

1. $(\alpha \cdot 0_{YZ} = 0_{XZ}) \wedge (0_{VX} \cdot \alpha = 0_{VY})$.
2. *If* $(\alpha \sqsubseteq \alpha') \wedge (\beta \sqsubseteq \beta')$ *then* $(\alpha \cdot \beta \sqsubseteq \alpha' \cdot \beta')$.
3. *If* $(\alpha \sqsubseteq \alpha') \wedge (\beta \sqsubseteq \beta')$ *then* $(\alpha' \rhd \beta \sqsubseteq \alpha \rhd \beta')$.
4. $\alpha \sqcap (\sqcup_k \alpha_k) = \sqcup_k (\alpha \sqcap \alpha_k)$.
5. $\alpha \cdot (\sqcap_j \beta_j) \cdot \gamma \sqsubseteq \sqcap_j \alpha \cdot \beta_j \cdot \gamma$, $\alpha \cdot (\sqcup_j \beta_j) \cdot \gamma = \sqcup_j \alpha \cdot \beta_j \cdot \gamma$.
6. $id_X^{\#} = id_X$, $0_{XY}^{\#} = 0_{YX}$. $\nabla_{XY}^{\#} = \nabla_{YX}$,
7. $(\sqcap_k \alpha_k)^{\#} = \sqcap_k \alpha_k^{\#}$, $(\sqcup_k \alpha_k)^{\#} = \sqcup_k \alpha_k^{\#}$.
8. $(\sqcup_k \alpha_k) \rhd \beta = \sqcap_k (\alpha_k \rhd \beta)$.
9. $\alpha \sqsubseteq \alpha \cdot \alpha^{\#} \cdot \alpha$.
10. $\alpha \cdot \beta \sqcap \gamma \sqsubseteq (\alpha \sqcap \gamma \cdot \beta^{\#}) \cdot (\beta \sqcap \alpha^{\#} \cdot \gamma)$.
11. $[f \cdot (\beta \sqcap \beta') \cdot g^{\#} = (f \cdot \beta \cdot g^{\#}) \sqcap (f \cdot \beta' \cdot g^{\#})]$, *for all* $f : X \rightharpoonup Y$, *and for all* $g : W \rightharpoonup Z$.
12. $[f \cdot (\beta \Rightarrow \beta') \cdot g^{\#} = (f \cdot \beta \cdot g^{\#}) \Rightarrow (f \cdot \beta' \cdot g^{\#})]$, *for all* $f : X \rightharpoonup Y$, *for all* $g : W \rightharpoonup Z$.
13. *If* $f \sqsubseteq g$ *then* $f = g$, *for all* $f, g : X \rightharpoonup Y$.
14. *If* $(u \sqsubseteq id_X)$ *then* $(u^{\#} = u)$,
15. *If* $(u \sqsubseteq id_X) \wedge (v \sqsubseteq id_X)$ *then* $(u \cdot v = u \sqcap v)$, *and*
16. *If* $(u \sqsubseteq id_X) \wedge (v \sqsubseteq id_X)$ *then* $(u \rhd v) \sqcap id_X = (u \Rightarrow v) \sqcap id_X$.

□

Lemma 2.11. *Let $f : A \rightharpoonup B$, $\alpha, \beta : B \rightharpoonup C$, and $g : C \rightharpoonup D$ be fuzzy relations.*

If $ff^{\#} \sqsubseteq id$ and $g^{\#} g \sqsubseteq id$ then we have $f \cdot (\alpha \sqcap \beta) \cdot g = (f \cdot \alpha \cdot g) \sqcap (f \cdot \beta \cdot g)$

□

3. Fuzzy Relational Database

In this chapter, we review a formalization which is introduced by Okuma and Kawahara (2000) and introduce new formalization of database operations.

Definition 3.1. Let $A = \{a_1, a_2, ..., a_n\}$ be a finite set of attributes. For any attribute $a_i \in A$, we define a set D_{a_i} called an attribute domain. We denote the product set $\prod_{a \in X} D_a$ by D_X for finite subset X of A.

Definition 3.2. A **database scheme** $R = \left(A, \{D_a\}_{a \in A}\right)$ is a pair of an attribute set and a class of attribute domains. Let X, Y be subsets of A, we denote the projection function D_X to D_a by ρ_{Xa} and D_X to D_Y by $\rho_{X,Y}$.

Proposition 3.3. *If $X \sqsubseteq Y \sqsubseteq Z$, then:*

$$\rho^{\#}_{Y,X} \cdot \rho_{Y,X} \sqsubseteq id_X,\ id_Y \sqsubseteq \rho_{Y,X} \cdot \rho^{\#}_{Y,X} \text{ and } \rho_{Z,X} = \rho_{Z,Y} \cdot \rho_{Y,X}.$$

□

Definition 3.4. A **database relation** r is a fuzzy relation $r : D_A \rightharpoonup D_A$ which satisfies $r \sqsubseteq \mathrm{id}_A$ where id_A is an identity function on A. We can consider a fuzzy relation r as a function $r : D_A \times D_A \to [0, 1]$ and for a tuple $t \in D_A$, we call $r(t, t)$ the fuzzy value of t.

Lemma 3.5. *Let $r_1, r_2 : D_A \rightharpoonup D_A$ be database relation, we have*

$$r_1 = r_1^{\#},\ \text{and}\, r_1.r_2 = r_1 \sqcap r_2.$$

□

Definition 3.6. Let f and r be database relations $f, r : D_A \rightharpoonup D_A$. The **selection** $\sigma_f(r)$ of r with f is the database relation defined by

$$\sigma_f(r) = r \cdot f : D_A \rightharpoonup D_A.$$

Proposition 3.7. *Let f, r_1, and r_2 be database relations f, r_1, $r_2 : D_A \rightharpoonup D_A$.*

1. $\sigma_f(r_1 \sqcup r_2) = \sigma_f(r_1) \sqcup \sigma_f(r_2)$, *and*
2. $\sigma_f(r_1 \sqcap r_2) = \sigma_f(r_1) \sqcap \sigma_f(r_2)$.

Proof. 1. $\sigma_f(r_1 \sqcup r_2) = (r_1 \sqcup r_2).f = (r_1.f) \sqcup (r_2.f) = \sigma_f(r_1) \sqcup \sigma_f(r_2)$.

2. *Since $f \sqsubseteq id$ then $f^{\#} f \sqsubseteq id$, so we get* $\sigma_f(r_1 \sqcap r_2) = (r_1 \sqcap r_2).f = (r_1.f) \sqcap (r_2.f) = \sigma_f(r_1) \sqcap \sigma_f(r_2)$ *by Lemma 2.11.*

□

Proposition 3.8. *Let f, r, and g be database relations f, g, $r : D_A \rightharpoonup D_A$.*

1. $\sigma_{f \sqcup g}(r) = \sigma_f(r) \sqcup \sigma_g(r)$, *and*
2. $\sigma_{f \sqcap g}(r) = \sigma_f(r) \sqcap \sigma_g(r)$.

Proof. *1.* $\sigma_{f \sqcup g}(r) = r.(f \sqcup g) = (r.f) \sqcup (r.g) = \sigma_f(r) \sqcup \sigma_g(r)$.

2. Since $r \sqsubseteq id$ *then* $r^{\#} r \sqsubseteq id$*, so we get* $\sigma_{f \sqcap g}(r) = r.(f \sqcap g) = (r.f) \sqcap (r.g) = \sigma_f(r) \sqcap \sigma_g(r)$ *by Lemma 2.11.*

□

Example 3.9. Consider $A = \{Name, Job, Experience, Salary\}$ then we have domain of A $D_A = D_N \times D_J \times D_E \times D_S$, and $r_H : D_A \times D_A \to [0,1]$ (cf. Table 1). We select the relation with salary equal or more than 60000.

Table 1: Table Relation High Experience High Salary

r_H

Name	Job	Experience	Salary	Fuzzy Value
John	Engineer	8	60000	0.67
Ashok	Manager	9	70,000	0.80
Mary	Secretary	8	40,000	0.50
James	Engineer	12	80,000	1.00
Robin	Engineer	9	60,000	0.80

We define $f : D_A \rightharpoonup D_A$, by $f((n,j,e,s),(n,j,e,s)) = \begin{cases} 1 & s > 60000, \\ 0 & \text{otherwise.} \end{cases}$

The **selection** $\sigma_f(r_H) = r \cdot f : D_A \rightharpoonup D_A$ is shown in Table 2. We denote $\sigma_f(r_H)$ by $\sigma_{Salary\ > 60000}(r_H)$.

Table 2: Fuzzy Selection $\sigma_{Salary > 60000}(r_H)$

$\sigma_f(r_H)$

Name	Job	Experience	Salary	Fuzzy Value
Ashok	Manager	9	70000	0.8
James	Engineer	12	80000	1

Definition 3.10. Let $X \sqsubseteq A$ and $r : D_A \rightharpoonup D_A$ a database relation. The **projection** $\pi_{A,X}(r)$ is the database relation defined by

$$\pi_{A,X}(r) = \rho^{\#}_{A,X} \cdot r \cdot \rho_{A,X} : D_X \rightharpoonup D_X.$$

Example 3.11. Let $A = \{Name, Job, Experience, Salary\}$, $r_H : D_A \times D_A \to [0,1]$ be database relation defined by Tabel 1. Table 3.11 is result of the projection to the set of attribute $Y = \{Job, Salary\}$.

Proposition 3.12. *Let* $X \sqsubseteq A$ *and* $r_1 : D_A \rightharpoonup D_A$*,* $r_2 : D_A \rightharpoonup D_A$ *be database relations.*

1. $\pi_{A,X}(r_1 \sqcup r_2) = \pi_{A,X}(r_1) \sqcup \pi_{A,X}(r_2)$*, and*

2. $\pi_{A,X}(r_1 \sqcap r_2) \sqsubseteq \pi_{A,X}(r_1) \sqcap \pi_{A,X}(r_2)$.

Table 3: Fuzzy Projection $\pi_{A,Y}\,(r_H)$

$\pi_{A,Y}(r_H)$

Job	Salary	Fuzzy Value
Engineer	80000	1
Manager	70000	0.8
Secretary	40000	0.5
Engineer	60000	0.8

PROOF. *1.* $\pi_{A,X}\,(r_1 \sqcup r_2) = \rho^{\#}_{A,X} \cdot (r_1 \sqcup r_2) \cdot \rho_{A,X} = (\rho^{\#}_{A,X} \cdot r_1 \cdot \rho_{A,X}) \sqcup (\rho^{\#}_{A,X} \cdot r_2 \cdot \rho_{A,X}) = \pi_{A,X}\,(r_1) \sqcup \pi_{A,X}\,(r_2)$.

2. By Proposition 2.10.5, we have $\pi_{A,X}\,(r_1 \sqcap r_2) = \rho^{\#}_{A,X} \cdot (r_1.r_2) \cdot \rho_{A,X} \sqsubseteq (\rho^{\#}_{A,X} \cdot r_1 \cdot \rho_{A,X}) \sqcap (\rho^{\#}_{A,X} \cdot r_2 \cdot \rho_{A,X}) = \pi_{A,X}\,(r_1) \sqcap \pi_{A,X}\,(r_2)$.

□

EXAMPLE 3.13. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ be domain of attributes. Consider $r_1, r_2 : X \rightharpoondown Y$ database relations defined by Table 4.

Table 4: Relation r_1 and r_2

r_1

X	Y	Fuzzy Value
x_1	y_1	1
x_2	y_2	1

r_2

X	Y	Fuzzy Value
x_1	y_2	1
x_2	y_1	1

From Table 4, then we have, $\pi_{A,Y}(r_1) = \pi_{A,Y}(r_2) = id_Y$. But, $\pi_Y(r_1 \sqcap r_2) = \emptyset \neq id_Y$, so we have $\pi_Y(r_1 \sqcap r_2) \neq \pi_{A,Y}(r_1) \sqcap \pi_{A,Y}(r_2)$.

Table 5: Fuzzy Projection r_1 and r_2

$\pi_{A,Y}$

Y	Fuzzy Value
y_1	1
y_2	1

DEFINITION 3.14. Let $X \sqsubseteq A$ and $r_X : D_X \rightharpoondown D_X$ be a database relation. The **injection** $\eta_{X,A}\,(r_X)$ of r_X to A is the database relation defined by

$$\eta_{X,A}\,(r_X) = (\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X}) \sqcap \mathrm{id}_A : D_A \rightharpoondown D_A.$$

PROPOSITION 3.15. *Let $X \sqsubseteq A \sqsubseteq B$ and $r_X, r'_X : D_X \rightharpoondown D_X$ be a database relation.*

1. $\eta_{X,A}(r_X \sqcup r'_X) = \eta_{X,A}(r_X) \sqcup \eta_{X,A}(r'_X)$,

2. If $r_X \sqsubseteq r'_X$ *then* $\eta_{X,A}(r_X) \sqsubseteq \eta_{X,A}(r'_X)$,

3. $\eta_{X,A}(r_X \cdot r'_X) = \eta_{X,A}(r_X) \cdot \eta_{X,A}(r'_X)$,

4. $\eta_{X,A}(r_X \sqcap r'_X) = \eta_{X,A}(r_X) \sqcap \eta_{X,A}(r'_X)$, *and*

5. $\eta_{A,B}(\eta_{X,A}(r_X)) = \eta_{X,B}(r_X)$.

Proof. *Suppose that* r_X *and* r'_X *be database relation, so* $r_X \sqsubseteq id_X$, *and* $r'_X \sqsubseteq id_X$

1. *Using Proposition 2.10.5,*

$$\begin{aligned}
\eta_{X,A}(r_X \sqcup r'_X) &= \rho_{A,X} \cdot (r_X \sqcup r'_X) \cdot \rho^{\#}_{A,X} \sqcap id_A \\
&= \rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \sqcup \rho_{A,X} \cdot r'_X \cdot \rho^{\#}_{A,X} \sqcap id_A \\
&= \rho_{A,X} \cdot (r_X \sqcup r'_X) \cdot \rho^{\#}_{A,X} \sqcap id_A \\
&= (\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \sqcap id_A) \sqcup (\rho_{A,X} \cdot r'_X \cdot \rho^{\#}_{A,X} \sqcap id_A) \\
&= \eta_{X,A}(r_X) \sqcup \eta_{X,A}(r'_X).
\end{aligned}$$

2. $\eta_{X,A}(r_X) = (\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \sqcap id_A) \sqsubseteq (\rho_{A,X} \cdot r'_X \cdot \rho^{\#}_{A,X} \sqcap id_A) = \eta_{X,A}(r'_X)$.

3. $(\sqsubseteq)$ *Since* $r_X \cdot r'_X \sqsubseteq r_X$ *and* $r_X \cdot r'_X \sqsubseteq r'_X$, *we have* $\eta_{X,A}(r_X \cdot r'_X) \sqsubseteq \eta_{X,A}(r_X)$ *and* $\eta_{X,A}(r_X \cdot r'_X) \sqsubseteq \eta_{X,A}(r'_X)$. *Then we get* $\eta_{X,A}(r_X \cdot r'_X) \sqsubseteq \eta_{X,A}(r_X) \cdot \eta_{X,A}(r'_X)$

   $(\sqsupseteq)$ *Since,* $\rho^{\#}_{A,X} \cdot \rho_{A,X} \sqsubseteq id_X$, *we have,*

$$\begin{aligned}
\eta_{X,A}(r_X) \cdot \eta_{X,A}(r'_X) &= ((\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X}) \sqcap id_A) \cdot ((\rho_{A,X} \cdot r'_X \cdot \rho^{\#}_{A,X}) \sqcap id_A) \\
&\sqsubseteq \rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \cdot \rho_{A,X} \cdot r'_X \cdot \rho^{\#}_{A,X} \\
&\sqsubseteq \rho_{A,X}(r_X \cdot r'_X)\rho^{\#}_{A,X} \\
\eta_{X,A}(r_X) \cdot \eta_{X,A}(r'_X) &\sqsubseteq \rho_{A,X}(r_X \cdot r'_X)\rho^{\#}_{A,X}.
\end{aligned}$$

   and also we have,

$$\begin{aligned}
((\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X}) \sqcap id_A) \cdot ((\rho_{A,X} \cdot r'_X \cdot \rho^{\#}_{A,X}) \sqcap id_A) &\sqsubseteq id_A \\
\eta_{X,A}(r_X) \cdot \eta_{X,A}(r'_X) &\sqsubseteq id_A.
\end{aligned}$$

   Then we have, $\eta_{X,A}(r_X) \cdot \eta_{X,A}(r'_X) \sqsubseteq \rho_{A,X}(r_X \cdot r'_X).\rho^{\#}_{A,X} \sqcap id_A = \eta_{X,A}(r_X \cdot r'_X)$.

4. *Using Lemma 3.5,* $\eta_{X,A}(r_X \sqcap r'_X) = \eta_{X,A}(r_X \cdot r'_X)$. *Using (3),* $\eta_{X,A}(r_X \cdot r'_X) = \eta_{X,A}(r_X) \cdot \eta_{X,A}(r'_X) = \eta_{X,A}(r_X) \sqcap \eta_{X,A}(r'_X)$

5. $(\sqsubseteq)$ *We note that* $\rho_{B,X} = \rho_{B,A} \cdot \rho_{A,X}$.
   We have

$$\begin{aligned}
\eta_{A,B}(\eta_{X,A}(r_X)) &= \rho_{B,A} \cdot (\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \sqcap id_A)\rho^{\#}_{B,A} \sqcap id_B \\
&\sqsubseteq (\rho_{B,A} \cdot \rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \cdot \rho^{\#}_{B,A}) \sqcap (\rho_{B,A} \cdot id_A \cdot \rho^{\#}_{B,A}) \sqcap id_B \\
&\sqsubseteq (\rho_{B,A} \cdot \rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \cdot \rho^{\#}_{B,A}) \sqcap id_B \\
&= \rho_{B,X} \cdot r_X \cdot \rho^{\#}_{B,X} \sqcap id_B, \text{ and} \\
\eta_{A,B}(\eta_{X,A}(r_X)) &\sqsubseteq \eta_{X,B}(r_X)
\end{aligned}$$

($\sqsupseteq$) *Since* $r_X \sqsubseteq id_X$, *We have*

$$\begin{aligned}
\eta_{X,B}(r_X) &= \rho_{B,X} \cdot r_X \cdot \rho^{\#}_{B,X} \sqcap id_B \\
&= \rho_{B,X} \cdot (r_X \sqcap id_X) \cdot \rho^{\#}_{B,X} \sqcap id_B \\
&\sqsubseteq (\rho_{B,X} \cdot r_X \cdot \rho^{\#}_{B,X}) \sqcap (\rho_{B,X} \cdot id_X \cdot \rho^{\#}_{B,X}) \sqcap id_B \\
&= (\rho_{B,X} \cdot r_X \cdot \rho^{\#}_{B,X}) \sqcap (\rho_{B,X} \cdot \rho^{\#}_{B,X}) \sqcap id_B \\
&= (\rho_{B,A} \cdot \rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \cdot \rho^{\#}_{B,A}) \sqcap (\rho_{B,A} \cdot \rho^{\#}_{B,A}) \sqcap id_B \qquad (*) \\
&\sqsubseteq \rho_{B,A} \cdot (\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \cdot \rho^{\#}_{B,A} \sqcap \rho^{\#}_{B,A} \cdot \rho_{B,A} \cdot \rho^{\#}_{B,A}) \sqcap id_B \\
&\sqsubseteq \rho_{B,A} \cdot (\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \cdot \rho^{\#}_{B,A} \sqcap id_A \cdot \rho^{\#}_{B,A}) \sqcap id_B \\
&\sqsubseteq \rho_{B,A} \cdot (\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \sqcap id_A \cdot \rho^{\#}_{B,A} \cdot \rho_{B,A}) \cdot \rho^{\#}_{B,A} \sqcap id_B \\
&\sqsubseteq \rho_{B,A} \cdot (\rho_{A,X} \cdot r_X \cdot \rho^{\#}_{A,X} \sqcap id_A) \cdot \rho^{\#}_{B,A} \sqcap id_B
\end{aligned}$$

So we have $\eta_{X,B}(r_X) \sqsubseteq \eta_{A,B}(\eta_{X,A}(r_X))$.

(*) We note that $id_B \sqsubseteq \rho_{B,X} \cdot \rho^{\#}_{B,X}$ *and* $id_B \sqsubseteq \rho_{B,A} \cdot \rho^{\#}_{B,A}$, *then we get* $\rho_{B,X} \cdot \rho^{\#}_{B,X} \sqcap id_B = \rho_{B,A} \cdot \rho^{\#}_{B,A} \sqcap id_B$.

□

DEFINITION 3.16. Let $X, Y \sqsubseteq A$ and $Z = X \sqcup Y$. For database relations $r_X : D_X \rightharpoonup D_X$ and $r_Y : D_Y \rightharpoonup D_Y$. The **natural join** $r_X \bowtie r_Y$ of r_X and r_Y is the database relation defined by

$$r_X \bowtie r_Y = \eta_{X,Z}(r_X) \cdot \eta_{Y,Z}(r_Y) : D_Z \rightharpoonup D_Z.$$

PROPOSITION 3.17. *Let* $X, Y, Z \sqsubseteq A$ *be a attribute set,* $\alpha : D_X \rightharpoonup D_X$, $\beta : D_Y \rightharpoonup D_Y$, $\gamma : D_Z \rightharpoonup D_Z$ *a fuzzy database relation, then:*

1. $(\alpha \bowtie \beta) \bowtie \gamma = \alpha \bowtie (\beta \bowtie \gamma)$.
2. $(\sqcup_{\lambda\in\Lambda}\alpha_\lambda) \bowtie \beta = \sqcup_{\lambda\in\Lambda}(\alpha_\lambda \bowtie \beta)$.
3. $(\sqcap_{\lambda\in\Lambda}\alpha_\lambda) \bowtie \beta = \sqcap_{\lambda\in\Lambda}(\alpha_\lambda \bowtie \beta)$.
4. *If* $\alpha \sqsubseteq \alpha'$ *and* $\beta \sqsubseteq \beta'$ *then* $\alpha \bowtie \beta \sqsubseteq \alpha' \bowtie \beta'$.

PROOF. *1. We have* $(\alpha \bowtie \beta) \bowtie \gamma = (\eta_{X\cup Y,X\cup Y\cup Z}(\alpha \bowtie \beta) \cdot \eta_{Z,X\cup Y\cup Z}(\gamma)$

$$\begin{aligned}
&= (\eta_{X\cup Y,X\cup Y\cup Z}((\eta_{X,X\cup Y}(\alpha)) \cdot (\eta_{Y,X\cup Y}(\beta)) \cdot \eta_{Z,X\cup Y\cup Z}(\gamma) \\
&= ((\eta_{X\cup Y,X\cup Y\cup Z}(\eta_{X,X\cup Y}(\alpha))) \cdot (\eta_{X\cup Y,X\cup Y\cup Z}((\eta_{Y,X\cup Y}(\beta))) \cdot \eta_{Z,X\cup Y\cup Z}(\gamma) \\
&= (\eta_{X,X\cup Y\cup Z}(\alpha) \cdot \eta_{Y,X\cup Y\cup Z}(\beta)) \cdot \eta_{Z,X\cup Y\cup Z}(\gamma) \\
&= \eta_{X,X\cup Y\cup Z}(\alpha) \cdot (\eta_{Y,X\cup Y\cup Z}(\beta) \cdot \eta_{Z,X\cup Y\cup Z}(\gamma)) \\
&= \eta_{X,X\cup Y\cup Z}(\alpha) \cdot (\eta_{Y\cup Z,X\cup Y\cup Z}(\eta_{Y,Y\cup Z}(\beta)) \cdot (\eta_{Y\cup Z,X\cup Y\cup Z}(\eta_{Z,Y\cup Z}(\gamma))) \\
&= \eta_{X,X\cup Y\cup Z}(\alpha) \cdot (\eta_{Y\cup Z,X\cup Y\cup Z}(\eta_{Y,Y\cup Z}(\beta) \cdot \eta_{Z,Y\cup Z}(\gamma)) \\
&= \eta_{X,X\cup Y\cup Z}(\alpha) \cdot (\eta_{Y\cup Z,X\cup Y\cup Z}(\beta \bowtie \gamma)) \\
&= \alpha \bowtie (\beta \bowtie \gamma).
\end{aligned}$$

Then we get $(\alpha \bowtie \beta) \bowtie \gamma = \alpha \bowtie (\beta \bowtie \gamma)$.

2. *Since* $\eta_{X,X\cup Y}(\sqcup_{\lambda\in\Lambda}\alpha_\lambda) = \sqcup_{\lambda\in\Lambda}(\eta_{X,X\cup Y}\alpha_\lambda)$, *we have*

$$\begin{aligned}((\sqcup_{\lambda\in\Lambda}\alpha_\lambda) \bowtie \beta &= (\sqcup_{\lambda\in\Lambda}\eta_{X,X\cup Y}(\alpha_\lambda)) \cdot \eta_{Y,X\cup Y}(\beta)\\ &= \sqcup_{\lambda\in\Lambda}(\eta_{X,X\cup Y}(\alpha_\lambda) \cdot \eta_{Y,X\cup Y}(\beta))\\ &= \sqcup_{\lambda\in\Lambda}(\alpha_\lambda \bowtie \beta).\end{aligned}$$

3. *Since* $\eta_{X,X\cup Y}(\sqcap_{\lambda\in\Lambda}\alpha_\lambda) = \sqcap_{\lambda\in\Lambda}(\eta_{X,X\cup Y}\alpha_\lambda)$ *we have*
$((\sqcap_{\lambda\in\Lambda}\alpha_\lambda) \bowtie \beta = (\sqcap_{\lambda\in\Lambda}\eta_{X,X\cup Y}(\alpha_\lambda)) \cdot \eta_{Y,X\cup Y}(\beta)$
Since $\eta_{X,X\cup Y}(\alpha_\lambda), \eta_{X,X\cup Y}(\beta) \sqsubseteq id$, *we have*

$$\begin{aligned}(\sqcap_{\lambda\in\Lambda}\eta_{X,X\cup Y}(\alpha_\lambda)) \cdot \eta_{Y,X\cup Y}(\beta) &= \sqcap_{\lambda\in\Lambda}(\eta_{X,X\cup Y}(\alpha_\lambda) \cdot \eta_{Y,X\cup Y}(\beta))\\ &= \sqcap_{\lambda\in\Lambda}(\alpha_\lambda \bowtie \beta).\end{aligned}$$

4. *Prop 2.10.5 and Prop 3.15.2, we have*
$\alpha \bowtie \beta = \eta_{X,X\sqcup Y}(\alpha) \cdot \eta_{Y,X\sqcup Y}(\beta) \sqsubseteq \eta_{X,X\sqcup Y}(\alpha') \cdot \eta_{Y,X\sqcup Y}(\beta') = \alpha' \bowtie \beta'$.

□

Example 3.18. Let $LIKES = \{Student, Course\}$ and $TEACH = \{Teacher, Course\}$. Consider $r_{LIKES} : D_{LIKES} \times D_{LIKES} \to [0,1]$ defined by Table 6 and $r_{TEACH} : D_{TEACH} \times D_{TEACH} \to [0,1]$ defined by Table 7.

Table 6: r_{LIKES}

LIKES

Student	Course	Fuzzy Value
John	DBMS	0.90
Mary	DBMS	0.70
John	AI	0.80
Ashok	AI	0.95

Table 7: r_{TEACH}

TEACH

Teacher	Course	Fuzzy Value
Rao	DBMS	0.80
Rao	AI	0.60
Johnson	DBMS	0.60
Johnson	AI	0.90

We note that:$LIKES \sqcup TEACH = \{Student, Course, Teacher\}$. The natural join of r_{LIKES} and r_{TEACH} is expressed in Table 8

Table 8: Fuzzy Natural Join $r_{LIKES} \bowtie r_{TEACH}$

Student	Teacher	Course	Fuzzy Value
John	Rao	DBMS	0.80
John	Johnson	DBMS	0.60
John	Rao	AI	0.60
John	Johnson	AI	0.80
Marry	Rao	DBMS	0.70
Marry	Johnson	DBMS	0.60
Ashok	Rao	AIS	0.60
Ashok	Johnson	AI	0.90

4. Fuzzy Process on Truck Backer-Upper

In this section, we try to implement fuzzy database table and database operations. We show the advantage of fuzzy database table to solve control problem. In our framework, we show fuzzy membership functions using fuzzy database table and define formalization of the defuzzification using fuzzy database operation. The difficulty of our works is how to define continue function of membership value. The interesting of our implementation is our formalizations of fuzzy database table and operations useful to implement in control problem and show the moving clearly.

One of popular simple control problem is a truck backer-upper. We will work on here is a simple version taken from the work of Kosko (1992). The object of the control system is to back up the truck so that it arrives perpendicular to the target position (x_T, y_T). We consider the target position is (60,100). The point (x, y) is the center of the rear of the truck, ϕ is the angle of the truck axis to the horizontal, and θ is the steering angle measured from the truck axis. The controller takes as input of the position of the truck, specified by the pair (x, ϕ), and outputs the steering angle θ.

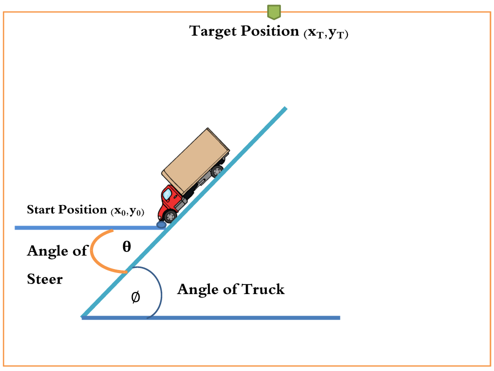

Figure 1: Model of Truck

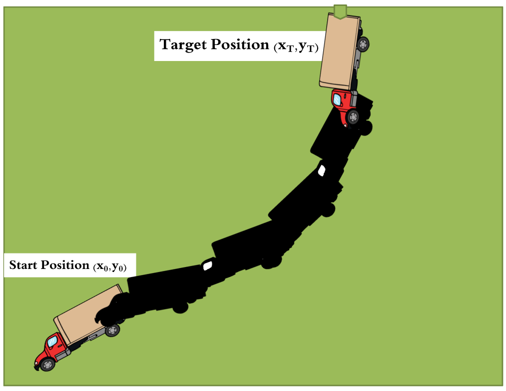

Figure 2: Moving Problem

Freeman (1994) introduced several fuzzy membership functions for positions, truck angle, and steer angles. In our framework work, we define those membership using our fuzzy relational database model. So, we have to design fuzzy relational database for parking problem of truck backer-upper from every start position (x, y, ϕ) to reach target position(x=60, y=100, $\phi = 90$). Using our formalization we try to show automatic moving truck as we show in Figure 2.

4.1. Formulation Using Our Fuzzy Relational Model

In fuzzy truck backer-upper problem, there are input and output elements. The input element are X-axis truck position('XPosition'), and angle of truck('APosition'). Based on information('XPosition' and 'APosition') of truck and target position, we can decide fuzzy rules to compute angle of steer so the truck will move to get target position. For example, if XPosition=50(in the near "Center" from target position), and APosition=100(near "Vertical") so the angle of steer should be small negative or near 0^o. So, we have to define fuzzy relational database table for a classification of XPosition, APosition, SPosition and fuzzy rules.

Let $PT = \{XPosition, PItem\}$. 'PItem' is classification of range X-axis position, we consider D_{PItem} = {Left('LE'), Center('CE'), Right('RI')}, $D_{XPosition} = [0, 100]$, and $r_{PT} : D_{PT} \rightharpoondown D_{PT}$ be a database relation defined by Table 9.

Table 9: Table of classification X Position (r_{PT})

r_{PT}	XPosition	PItem	Fuzzy Value
	$0 \leq x \leq 30$	LE	1
	$30 < x \leq 60$	LE	-1/30 x + 2
	$40 \leq x \leq 60$	CE	1/20 x - 2
	$60 < x \leq 80$	CE	-1/20 x+4
	$60 \leq x \leq 80$	RI	1/20 x-3
	$80 < x \leq 100$	RI	1

Let $AT = \{APosition, AItem\}$. 'AItem' is classification of range angle of truck, we consider D_{AItem} = {Right Below('RB'), Right Upper('RU'), Vertical('VE'), Left Upper ('LU'), Left Below('LB')}, $D_{APosition} = [-180, 180]$, and $r_{AT} : D_{AT} \rightharpoondown D_{AT}$ be a database relation defined by Table 10.

Table 10: Table of Classification Angle(ϕ) Position(r_{AT})

r_{AT}	APosition	AItem	Fuzzy Value
	$-100 \leq a \leq -45$	RB	1/55 a + 100/55
	$-45 < a \leq 10$	RB	-1/55 a + 10/55
	$-10 \leq a \leq 35$	RU	1/45 a - 10/45
	$35 < a \leq 90$	RU	-1/55 a+90/55
	$60 \leq a \leq 90$	VE	1/30 a-2
	$90 < a \leq 120$	VE	/30 a+4
	$100 \leq a \leq 155$	LU	1/55 a-100/55
	$155 < a \leq 190$	LU	-1/55 a+190/55
	$170 \leq a \leq 225$	LB	1/55 a-170/55
	$225 < a \leq 280$	LB	-1/55 a+280/55

Let $ST = \{SPosition, SItem\}$. 'SItem' is classification of range angle of steer, we consider D_{SItem} = {Negative('NE'), Zero('ZE'), Positive('PO')}, $D_{SPosition} = [-30, 30]$, and $r_{ST} : D_{ST} \rightharpoondown D_{ST}$ be a database relation defined by Table 11.

Table 11: Table of Classification angle Steer(r_{ST})

r_{ST}	SPosition	SItem	Fuzzy Value
	$-30 < s \leq -15$	NE	1/15 s + 2
	$-15 < s \leq 0$	NE	-1/15 s
	$-5 < s \leq 0$	ZE	1/5 s + 1
	$0 < s \leq 5$	ZE	-1/5 s + 1
	$0 < s \leq 15$	PO	1/5 s
	$15 < s \leq 30$	PO	1

Example 4.1. In Table 9, the first row $(0 \leq x \leq 30, LE, 1)$ means $r_{PT}(((x, LE), (x, LE)))$ $= 1$ for $0 \leq x \leq 30$.

And the second row $(30 < x \leq 60$, LE, $-1/30x + 2)$ means $r_{PT}(((x, LE), (x, LE)))$ $= -x/30 + 2$ for $30 < x \leq 60$.

4.2. Fuzzy Rules

A fuzzy rules table is a relation between some attributes input table(PItem and AItem) with attribute output table(classification of steer angle 'SItem'). Using a fuzzy rules table, we also can design a defuzzification to compute angle of steer.

Let $FR = \{AItem, PItem, SItem\}$. Consider $r_{FR} : D_{FR} \rightharpoondown D_{FR}$ be a database relation of fuzzy rules defined by Table 12.

Table 12: Table of Fuzzy Rule (r_{FR})

r_{FR}

AItem	PItem	SItem	Fuzzy Value
RB	LE	PO	1
RB	CE	PO	1
RB	RI	PO	1
RU	LE	NE	1
RU	CE	PO	1
RU	RI	PO	1
VE	LE	NE	1
VE	CE	ZE	1
VE	RI	PO	1
LU	LE	NE	1
LU	CE	NE	1
LU	RI	PO	1
LB	LE	NE	1
LB	CE	NE	1
LB	RI	NE	1

4.3. Fuzzy Algorithm

Different from the Freeman method, we use database operations (Selection, Projection, and Natural Join) to construct a fuzzy system.

Let $A = PT \sqcup AT \sqcup ST$, for $a \in A$ and $t_0 \in D_A$, we define a database relation $Eq(t_0) : D_a \rightharpoondown D_a$ by

$$Eq(t_0)(t,t) = \begin{cases} 0 & (t \neq t_0), \\ 1 & (t = t_0). \end{cases}$$

Let $A = PT \sqcup AT \sqcup FR$, $x \in D_{XPosition}$ and $\phi \in D_{APosition}$. A database relation $f : D_X \rightharpoondown D_X$ for the selection σ_f is defined by $f = \eta_{XPosition,X}(Eq(x)) \cdot \eta_{APosition,X}(Eq(\phi))$. The **Output** operations r_{out} can be defined by the equation using database operations as follows:

$$r_{out}(x,\phi) = \pi_{A,SItem}(\sigma_f(r_{PT} \bowtie r_{AT} \bowtie r_{FR})).$$

Example 4.2. Let the start position of truck at $(x, y) = (20, 0)$ and the start angle of truck $\phi = -90$, The selection $\sigma_{XPosition=20,\ APosition=-90}$ of natural join ($r_{PT} \bowtie r_{AT} \bowtie r_{FR}$) is showed in Table 13. The output $r_{Out}(20, -90)$ is showed in Table 14.

Table 13: $\sigma_{XPosition=20,\ APosition=-90}(r_{PT} \bowtie r_{AT} \bowtie r_{FR})$

XPosition	PITem	APosition	AItem	SItem	Fuzzy Value
20	LE	-90	RB	PO	2/11

Table 14: $r_{Out}(20, -90)$

SItem	Fuzzy Value
PO	2/11

Then, we got the result of $r_{out}(20, -90)(PO, PO) = 2/11$. It means the steer position will be in 'Positive' with the fuzzy value 2/11.

The defuzzification process is a process to compute a real value of steer angle using a classification in the output $r_{out}(x, \phi)$. In this case, we use *Mamdani-Centroid* for the defuzzification process.

Let $D_a \sqsubseteq R$, where R is a set of all real numbers. For a database relation $f : D_a \rightharpoonup D_a$, we define the fuzzy sum $S(f)$ and the centroid $E(f)$ as follows:

$$\begin{aligned} S_a(f) &= \sum \{f(t,t) \in [0,1] \mid t \in D_a\}, \\ E_a(f) &= \sum \{t \cdot f(t,t) \in [0,1] \mid t \in D_a\} / S(f). \end{aligned}$$

A database relation r_c is defined by

$$r_c = \pi_{ST,SPosition}(r_{ST} \bowtie r_{out}(x, \phi)) : D_{ST} \rightharpoonup D_{ST}.$$

We get the **defuzzification** θ_c **of** r_c by $\theta_c = E_a(r_c)$.

<u>SimulateTruck</u> (x, y, ϕ);
Input : Start position of truck (x, y, ϕ)
Output : Set of update position (x_t, y_t, ϕ_t)
Variable: r_c, θ_c
while *not TargetPosition*(x_t, y_t) **do**
  $r_{out}(x_t, \phi_t) = \pi_{X,SItem}(\sigma_{X\ Position=x \wedge APosition=\phi_t}(r_{PT} \bowtie r_{AT} \bowtie r_{FR}))$.;
  $r_c = \pi_{ST,SPosition}(r_{ST} \bowtie r_{out}(x, \phi_t))$.;
  $\theta_c = E_{SPosition}(r_c)$;
  $x_{t+1} = x_t + 5cos(\theta_c)$,
  $y_{t+1} = y_t + 5sin(\theta_c)$,
  $\phi_{t+1} = \phi_t + \theta_c$.;
  $t = t + 1$;
end
return collection set $\{(x_0, y_0, \phi_0), (x_1, y_1, \phi_1), \ldots, (x_n, y_n, \phi_n)\}$;

Figure 3: Algorithm Moving of Truck Backer-Upper

In this section, we would like to explain more detail the example of fuzzy truck backer-upper. Using algorithm in Figure 3, we follow the algorithm and fuzzy database tables to make the movement of truck backer-upper automatically.

The followings are equations of moving truckFreeman (1994):

$$\begin{aligned} x_{t+1} &= x_t + 5cos(\theta_c), \\ y_{t+1} &= y_t + 5sin(\theta_c), \text{ and} \\ \phi_{t+1} &= \phi_t + \theta_c. \end{aligned}$$

Controlling the angle of the steer θ_c, the coordinate (x, y) and angle of truck ϕ will be updated, the truck moves to the target position $(60, 100)$.

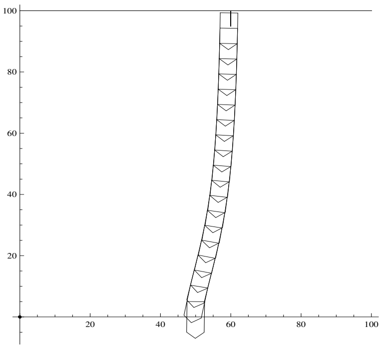

Figure 4: Moving result with start pos. $(x_0 = 50, y_0 = 0, \phi_0 = 100)$

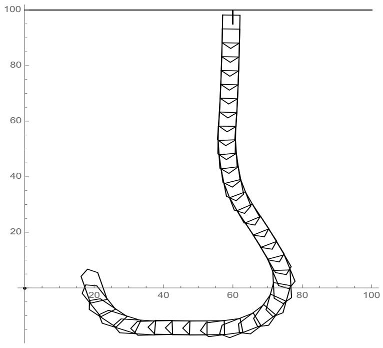

Figure 5: Moving result with start pos. $(x_0 = 20, y_0 = 0, \phi_0 = -90)$

EXAMPLE 4.3. We consider the start position $(x, y) = (20, 0)$ with the angle of truck ϕ=-90. We got the output $r_{Out}(20, -90) = \{(PO, PO), (PO, PO), 2/11\}$ as shown in Example 4.2. Next, we continue to get a real value of the steer angle using the defuzzification process. In the defuzzification process, there are some steps to compute an angle of steer θ_c, we use *Mathematica 10.0.1* to compute the defuzzification process and the algorithm is shown in *Appendix*. In the *Appendix*, we use function '`defuzzy`' to compute the angle of steer θ_c. In the Appendix, the result of `defuzzy[ST, rST, rOut[20,-90]]` is 15.8016, it means angle of steer θ_c is 15.8016 when the position $x = 20$ and $\phi = -90(r_{Out}(20, -90))$. Then, the position of the truck move from $(x_0, y_0, \phi_0) = (20, 0, -90)$ to the new position $(x_1, y_1, \phi_1) = (20 + 5.cos\ 15.8016,\ 0 + 5.sin\ 15.8016,\ -90 + 15.8016)$. The new position causes updating the new output $r_{out}(x_1, \phi_1)$ then we get the new θ_c. This algorithm process of the truck backer-upper will stop if the truck reach the target position $(x_t, y_t, \phi_t) = (60, 100, 90)$.

The result of the truck with the start position $(x, y) = (20, 0)$ and the angle of truck ϕ=-90 showed in the Figure 5. We showed the result of the truck with the start position $(x, y) = (50, 0)$ and the angle of truck ϕ=100 in Figure 4. The results showed an application of our algorithm using fuzzy relational database table to a fuzzy control problem.

5. Conclusion and Future Works

Fuzzy relational database is a combination of relational database and fuzzy theory. We introduce a new method for solving fuzzy problems. In Freeman's method, they introduced several fuzzy membership functions for positions, truck angles, and steer angles. They also defined fuzzy rule tables and introduced several computations for each operations. In our framework, each positions, truck angles, steer angles and a fuzzy rule table are both represented as tables of fuzzy relational database. Further, we do not use problem specific computations. The decision procedure is defined using relational database operations, such as projection, selection and natural join. Using relational calculus, we can extend notions of traditional relational database to fuzzy relational database. Fuzzy relation is useful to make simple institutional notions into the database tables. We can see in the example of truck backer-upper problem, using fuzzy relational database operations the truck can reach the position that we want.

Our example in this paper, we fixed the target position and tables for fuzzy membership properties. So we can change the start position but we can not use the same tables for another target position. One of the next challenges is making fuzzy dynamic relational database. That is we introduce automatic construction of tables for fuzzy membership properties if we change the target position. A system with automatic changing is called a fuzzy dynamic relational database.

Further, we should investigate properties of database using formulas of relational calculus. We are going to formalize the notion of properties of fuzzy relational database such as the traditional notion of functional dependencies.

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References

Codd, E. F. (1970). A relational model of data for large shared data banks, *Communication ACM* Vol. 13(6), 377-387.

Furusawa, H. and Kawahara, Y. (2014). Point axioms and related conditions in Dedekind categories. *Journal of Logical and Algebraic Methods in Programming* Vol. 84(3), 359-376.

James A. Freeman (1994). Fuzzy system for control applications : The Truck Backer Upper, *The Mathematical Journal* Vol.4, 64-69.

J. N. Mordeson and P. S. Nair (2001). Fuzzy Mathematics. *Studies in Fuzziness and Soft Computing.* Physica-Verlag.

Kawahara, Y. (1988). Applications of relational calculus to computer mathematics. *Bulletin of Informatics and Cybernetics* Vol. 23(1-2), 67-78.

Kiss, A. (1991). An application of fuzzy graphs in database theory. *Pure Mathematics and Applications* Vol. 1(3-4), 337-342.

Kosko, Bart (1992). Fuzzy System: Dynamical Systems Approach to Machine Intelegence. Prentice-Hall.

Kawahara, Y., and Furusawa, H. (1999). An algebraic formalization fuzzy relations. *Fuzzy Sets and Systems* Vol. 101, 125-135.

Mori, M.,and Kawahara, Y (1995). Fuzzy graph rewritings. *RIMS KOKYUROKU* Vol. 9(18), 65-71.

Nakata, M (2000). Knowledge management in fuzzy databases Vol. 39(2), 144-156.

Raju, K. V. S. V. N. and Majumdar, Arun K. (1988). . Fuzzy Functional Dependencies and Lossless Join Decomposition of Fuzzy Relational Database Systems. *ACM Trans. Database System* Vol. 13(2), 129-166.

Okuma, H. and Kawahara, Y. (2000). Relational Aspects Of Relational Database Dependencies. *Bulletin of Informatics and Cybernetics* Vol. 32(2), 91-104.

Umano, M.,and Fukami (1994). Fuzzy relational algebra for possibility-distribution-fuzzy-relational model of fuzzy data. *J. Intell.Inf. Syst.* Vol. 3(1), 7-27.

Zadeh, L.A (1965). Fuzzy sets. *Information and Control*. Vol. 8, 338-353.

A Appendix

To compute fuzzy relational database problem, we make Mathematica Library ”Relational Database”. In this section, we show mathematica code in ”Relational Database” Library [2].

A1. Fuzzy Operation

1. `FuzzyProjection[A,r,y]`

	::Projection of fuzzy relation set database rdb with attributes A to attributes y, such that y is subset of A
`A`	Complete set of attribute of a fuzzy relation
`r`	Fuzzy relations set of element every attribute
`y`	Set of attribute that we want to select, subset set of A return pair elements of attribute fuzzy sets y.
`return`	pair elements of fuzzy relation sets.

2. `FuzzyNaturalJoin[A1,A2,r1,r2]`

	::::Join relation between fuzzy relation set database r1 with attributes x and fuzzy relation set database r2 with attributes y
`A1,A2`	Complete set of attribute of a fuzzy relation
`r1,r2`	Fuzzy relations set of element every attribute
`return`	pair elements of union attribute fuzzy sets x with elements r1 and y with elements r2.

3. `FuzzySelection[A,r,Condition]}`

	::Selection of fuzzy relation set database r with attribute A, with some condition in attribute a, for example a¡b, a¿b, b¡a¡c, etc. b and c is value that we want
`A`	Complete set of attribute of a fuzzy relation
`r`	Fuzzy relations set of element every attribute
`Condition`	Condition that is selected, consider a is subset of A. So the condition is mean the condition of a that we want.
`return`	pair elements of fuzzy relation sets.

4. `rOut[{XPosition, X}, {APosition, P}, {a1, r1}, {a2, r2}, {rule, fr, out}]`

[2] `https://github.com/KyushuUniversityMathematics/FuzzyRelationalDatabase`

	::From formalization in section 3.3, we can compute class of steer and fuzzy value
`XPosition`	Attibute "X Position" in r_{PT}
`X`	Actual position of truck in X-Axis
`APosition`	Attribute "A Position" in r_{AT}
`a1,a2`	"a1" Attributes set of r_{PT}, "a2" attributes set of r_{AT}
`r1,r2`	"r1" database relation r_{PT}, "r2" database relation r_{AT}
`rule`	"rule" Attributes set of r_{FR}
`fr`	"fr" database relation r_{FR}
`out`	The attribute that we want to be projected, in this case we want to project in "S Item"
`return`	pair elements of fuzzy relation sets.

5. `Defuzzy[A, r, rOut]`

	::The result of defuzzy is angle of steer ("S Position" attribute)
`A`	Complete set of attribute of r_{ST}
`r`	Fuzzy relations set of element every attribute r_ST
`rOut`	Result of `rOut`.
`return`	pair elements of fuzzy relation sets.

6. Execution Result
   Fuzzy Selection[Example 3.9]

```
In[5]:= Name = 1; Job = 2; Expr = 3; Sal = 4;
A = {Name, Job, Expr, Sal};

In[4]:= DB[Name] = {"John", "Ashok", "Mary", "James", "Robin"};
DB[Job] = {"Engineer", "Manager", "Secretary"};
DB[Expr] = {8, 9, 12};
DB[Sal] = {60000, 70000, 40000, 80000};
rHEHS = {{{"John", "Engineer", 8, 60000}, {"John", "Engineer",
      8, 60000}, 0.67},
      {{"Ashok", "Manager", 9, 70000}, {"Ashok", "Manager", 9,
      70000}, 0.8},
      {{"Mary", "Secretary", 8, 40000}, {"Mary", "Secretary",
      8, 40000}, 0.5},
      {{"James", "Engineer", 12, 80000}, {"James", "Engineer",
      12, 80000}, 1},
      {{"Robin", "Engineer", 9, 60000}, {"Robin", "Engineer",
      9, 60000}, 0.8}};

In[10]:= FuzzySelection[A, rHEHS, {Sal} > 60000]

Out[10]= {{{"Ashok", "Manager", 9, 70000}, {"Ashok", "Manager",9,
   70000}, 0.8}, {{"James", "Engineer", 12, 80000}, {"James",
   "Engineer", 12, 80000}, 1}}
```

Fuzzy Projection[Example 3.11]

```
In[29]:= FuzzyProjection[A, rHEHS, {Job, Sal}]

Out[29]= {{{"Engineer", 60000}, {"Engineer", 60000}, 0.8},
          {{"Engineer", 80000}, {"Engineer", 80000}, 1},
          {{"Manager", 70000}, {"Manager", 70000}, 0.8},
          {{"Secretary", 40000}, {"Secretary", 40000}, 0.5}}
```

Fuzzy Projection[Example 3.13]

```
In[11]:= X = 1; Y = 2; A = {X, Y};

In[12]:= DB[X] = {x1, x2}; DB[Y] = {y1, y2};
r1 = {{{x1, y1}, {x1, y1}, 1}, {{x2, y2}, {x2, y2},
   1}}; r2 = {{{x1, y2}, {x1, y2}, 1}, {{x2, y1}, {x2, y1}, 1}};

In[14]:= FuzzyProjection[A, r1, {Y}]
Out[14]= {{y1, y1, 1}, {y2, y2, 1}}

In[15]:= FuzzyProjection[A, r2, {Y}]
Out[15]= {{y1, y1, 1}, {y2, y2, 1}}

In[16]:= FuzzyProjection[A, FuzzyIntersection[r1, r2], {Y}]
Out[16]= {}
```

Fuzzy Natural Join[Example 3.18]

```
In[17]:= Student = 1; Course = 2; Teacher = 3; Likes = {Student,
  Course}; Teach = {Teacher, Course};
DB[Student] = {"John", "Mary", "Ashok"};
DB[Course] = {"DBMS", "AI"};
DB[Teacher] = {"Rao", "Johnson"};

In[23]:= rLikes = {{{"John", "DBMS"}, {"John", "DBMS"}, .9},
                    {{"Mary","DBMS"}, {"Mary", "DBMS"}, .7},
                    {{"John", "AI"}, {"John", "AI"}, .8},
                    {{"Ashok", "AI"}, {"Ashok", "AI"}, .95}};
        rTeach = {{{"Rao", "DBMS"}, {"Rao", "DBMS"}, 0.8},
                  {{"Rao", "AI"}, {"Rao", "AI"}, 0.6},
                  {{"Johnson", "DBMS"}, {"Johnson", "DBMS"}, 0.6},
                  {{"Johnson", "AI"}, {"Johnson", "AI"}, 0.9}};

In[24]:= FuzzyNaturalJoin[Likes, Teach, rLikes, rTeach]
Out[24]= {{{"Ashok", "AI", "Johnson"}, {"Ashok", "AI", "Johnson"},
  0.9}, {{"Ashok", "AI", "Rao"}, {"Ashok", "AI", "Rao"},
  0.6}, {{"John", "AI", "Johnson"}, {"John", "AI", "Johnson"},
  0.8}, {{"John", "AI", "Rao"}, {"John", "AI", "Rao"},
  0.6}, {{"John", "DBMS", "Johnson"}, {"John", "DBMS", "Johnson"},
  0.6}, {{"John", "DBMS", "Rao"}, {"John", "DBMS", "Rao"},
  0.8}, {{"Mary", "DBMS", "Johnson"}, {"Mary", "DBMS", "Johnson"},
  0.6}, {{"Mary", "DBMS", "Rao"}, {"Mary", "DBMS", "Rao"}, 0.7}}
```

r_{Out} [Example 4.2]

```
In:  rOut[20, -90]
Out: {{"PO", "PO", 2/11}}
```

Deffuzification

```
In: defuzzy[ST, rST, rOut[20,-90]]
Out: 15.8016
```

Simulation

```
simulateTruck[x0_, y0_, phi0_] :=
 Module[{x = x0, y = y0, phi = phi0, newPhi, result = {}},
  While[y <= 95., newPhi = phi + Defuzzy[ST, rST, rOut[x, phi]];
   AppendTo[
    result, {x, y,
      phi} = {x + 5 Cos[newPhi Pi/180], y + 5 Sin[newPhi Pi/180],
       newPhi} // N];];
  result]

In:  v = simulateTruck[20, 0, -90]
Out: {{21.3615, -4.81105, -74.1984}, {24.0561, -9.02287, -57.3906}, \
{27.9092, -12.2093, -39.5897}, {32.5643, -14.0343, -21.4075}, \
{37.5505, -14.4055, -4.2579}, {42.5484, -14.2637,
  1.62512}, {47.5473, -14.3691, -1.20728}, {52.5473, -14.3678,
  0.0141637}, {57.5465, -14.2781, 1.02786}, {62.4818, -13.4764,
  9.22697}, {66.9799, -11.293, 25.8924}, {70.6186, -7.86369,
  43.3035}, {73.0944, -3.51968, 60.3196}, {74.1942, 1.35785,
  77.2928}, {73.9802, 6.35327, 92.4532}, {72.421, 11.1039,
  108.17}, {70.3027, 15.6331, 115.065}, {68.0903, 20.117,
  116.262}, {65.7765, 24.5494, 117.565}, {63.4128, 28.9553,
  118.213}, {61.322, 33.4972, 114.718}, {59.8477, 38.2749,
  107.15}, {59.0678, 43.2137, 98.9734}, {58.5053, 48.182,
  96.4588}, {58.2508, 53.1755, 92.9179}, {58.3342, 58.1748,
  89.0445}, {58.6217, 63.1665, 86.7039}, {58.8276, 68.1623,
  87.6395}, {59.004, 73.1592, 87.9787}, {59.1534, 78.157,
  88.2879}, {59.2795, 83.1554, 88.5538}, {59.3865, 88.1542,
  88.7743}, {59.4772, 93.1534, 88.9601}, {59.5543, 98.1528, 89.1165}}
```

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