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TWO DUALS OF ONE PRIMAL

By

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Abstract

As a primal problem we take a quadratic minimization without constraint. The problem has a Golden terminal function. We associate the primal problem with two dual problems — (1) complementary and (2) identical —. Each dual problem is derived through two dualizations — (i) plus-minus and (ii) dynamic —. Plus-minus dualization is based upon Fenchel duality, while dynamic dualization Lagrange duality. In any derivation, completing the square is performed simultaneously. The primal and both duals are completely solved. The solution is characterized by the Golden number. The optimum points constitute two types of Golden path. It is shown that the primal and the complementary dual have Golden complementary duality and that the primal and the identical dual have Golden identical duality.

Key Words and Phrases: primal, dual, plus-minus dualization, dynamic dualization, completing the square, Golden complementary duality, Golden identical duality

1. Introduction

Recently a dual theory of quadratic optimization without constraint has been developed by Iwamoto (2007), Kira and Iwamoto (2008), Iwamoto (2013), Iwamoto et al. (2013, 2014). The theory is closely related to conjugate function (Fenchel (1953), Rockafeller (1974), Kawasaki (2003)), minimum transform and quasilinearization (Bellman (1981, 1984, 1986), Iwamoto (1987, 2013)). The objective function originates from a linear-quadratic (LQ) model in dynamic optimization (Bellman (1967, 1969, 1971, 1972)). However, until recently any dual approach has never been applied to such LQ model.

In this paper we expand the dual method into a wider class of dualizations and apply it to a new objective function with an additional Golden terminal function. We associate one primal problem with two dual problems — (1) complementary dual and (2) identical dual —. Each dual problem is derived by two dualizations — (i) plus-minus dualization and (ii) dynamic dualization —. While (i) is based upon Fenchel duality, (ii) Lagrange duality. Further each dualization is accompanied by two ones — (a) dualization 1 and (b) dualization 2 —. This paper consists of two parts. Part I discusses a complementary duality. The Golden complementary duality is established.

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Part II discusses an identical duality. The Golden identical duality is established. Due to the Golden premium, the solution and method become very fruitful.

Part I
Complementary Duality

2. Primal problem and dual problem

As an n-variable quadratic optimization, we consider a minimization problem of $x = (x_1, x_2, \ldots, x_n)$:

$$\text{(P)} \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + x_k^2 \right] + \phi^{-1} x_n^2 \\ \text{subject to} & \text{(i)} \ \ x \in R^n \\ & \text{(ii)} \ \ x_0 = c \end{array}$$

where $c \in R$. Hereafter ϕ denotes the *Golden number*

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803.$$

It satisfies

$$1 : \phi = \phi^{-2} : \phi^{-1}, \quad \phi^{-2} + \phi^{-1} = 1.$$

The Golden number ϕ is also defined as a positive solution to a quadratic equation

$$x^2 - x - 1 = 0.$$

Lemma 2.1. (Iwamoto et al.(2014)) *The Golden number ϕ satisfies*

1. $\displaystyle\sum_{k=1}^{n} \phi^{2k-1} = \phi^{2n} - 1$
2. $\displaystyle\sum_{k=1}^{n} \phi^{-2k} = \phi^{-1} - \phi^{-2n-1}$
3. $\phi^n + \phi^{n+1} = \phi^{n+2} \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$
4. $\displaystyle 2\sum_{k=1}^{n} \phi^{-3k-1} + \phi^{-3n-2} = \phi^{-2}.$

Definition 2.2. (Iwamoto(2013)) Let c be any real constant. A finite sequence $\{x_n\}_{n \geq 1}$ with

$$x_n = c\phi^{-2n} \quad \text{or} \quad x_n = c\phi^{-n}$$

is called *Golden path (GP)*. The former is called $1 : \phi$, while the latter $\phi : 1$.

Theorem 2.3. *The primal problem* (P) *has a minimum value* $m = \phi^{-1}c^2$ *at a point*

$$\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}, \hat{x}_n) = c\big(\phi^{-2}, \ \phi^{-4}, \ \ldots, \ \phi^{-(2n-2)}, \ \phi^{-2n}\big).$$

The minimum point $\hat{x}$ is a GP of $1 : \phi$.

PROOF. Theorem 2.3 together with Theorem 2.4 is proved in the following derivation process of dual problem (D) from (P) (see Plus-minus dualization 1 and Lemma 4.1).

The problem (P) has a *dual problem* of n-variable $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$:

$$\text{(D)} \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2 \right] - \phi\mu_n^2 \\ \text{subject to} & \text{(i)} \quad \mu \in R^n. \end{array}$$

THEOREM 2.4. *The dual problem* (D) *has a maximum value* $M = \phi^{-1}c^2$ *at a point*

$$\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_{n-1}^*, \mu_n^*) = c\big(\phi^{-1}, \ \phi^{-3}, \ \ldots, \ \phi^{-(2n-3)}, \ \phi^{-(2n-1)}\big).$$

The maximum point μ^* is also a GP of $1 : \phi$.

A triplet between the minimum solution of (P) and the maximum solution of (D) holds as follows.

1. (**Duality**) The minimum value is equal to the maximum value : $m = M$. The common value is a quadratic function of initial value c, whose coefficient is the inverse ϕ^{-1} to Golden number.
2. (**Golden**) Both the minimum point $(\hat{x}_1, \ \hat{x}_2, \ \ldots, \ \hat{x}_n)$ and the maximum point $(\mu_1^*, \ \mu_2^*, \ \ldots, \ \mu_n^*)$ are Golden paths of $1 : \phi$.
3. (**Complementarity**) An alternate sequence of the minimum point and the maximum point constitutes a Golden path of $\phi : 1$:

$$\begin{aligned} & (\, x_0, \ \mu_1^*, \ \hat{x}_1, \ \mu_2^*, \ \hat{x}_2, \ \ldots, \ \mu_n^*, \ \hat{x}_n \,) \\ = \ & c(\, 1, \ \phi^{-1}, \ \phi^{-2}, \ \phi^{-3}, \ \ldots, \ \phi^{-(2n-1)}, \ \phi^{-2n} \,). \end{aligned}$$

The triplet is called *Golden complementary duality (GCD)*.

3. Duality theorem

We consider a function $f : R^n \to (-\infty, \infty]$. An *effective domain* is defined by

$$\mathrm{dom}(f) = \{x \in R^n \ : \ f(x) < \infty\}.$$

A convex function $f : R^n \to (-\infty, \infty]$ and a concave function $g : R^n \to (-\infty, \infty]$ define its conjugate functions $f^*, \ g_*$ as follows, respectively:

$$\begin{aligned} f^*(\lambda) &= \operatorname*{Sup}_{x \in R^n} \left[(\lambda, x) - f(x) \right], \quad \lambda \in R^n \\ g_*(\lambda) &= \inf_{x \in R^n} \left[(\lambda, x) - g(x) \right], \quad \lambda \in R^n. \end{aligned}$$

Theorem 3.1. Fenchel duality theorem (e.g. see Fenchel (1953), Rockafeller (1974), Borwein and Lewis (2000), Kawasaki (2003)) *Let a function f be convex, and g be concave. If two effective domains* $\mathrm{dom}(f)$, $\mathrm{dom}(-g)$ *are not separated, then it holds that*

$$\inf_{x\in R^n}\,[\,f(x)-g(x)\,] \;=\; \mathop{\mathrm{Sup}}_{\lambda\in R^n}\,[\,g_*(\lambda)-f^*(\lambda)\,]\,.$$

In the following we consider a function $h : R^n \to (-\infty,\infty)$. A convex function $h : R^n \to (-\infty,\infty)$ defines its minimum transform (e.g. see Bellman (1957, 1981, 1984, 1986), Iwamoto (1987, 2013)) $h_\star : R^n \to (-\infty,\infty)$ as follows:

$$h_\star(\lambda) \;=\; \min_{x\in R^n}\,[\,h(x)-(\lambda,x)\,]\,.$$

Hereafter we assume that both the minimum and maximum exist.

Corollary 3.2. *Let two functions f, g : $R^n \to (-\infty,\infty)$ be differentiable and convex. Then it holds that*

$$\min_{x\in R^n}\,[\,f(x)+g(x)\,] \;=\; \mathop{\mathrm{Max}}_{\lambda\in R^n}\,[\,f_\star(\lambda)+g_\star(-\lambda)\,]\,. \tag{1}$$

Corollary 3.2 holds true for a more general setting, as Theorem 3.1 does. Here we choose to not give the setting. Instead, we give a simple proof under the existence of minimum in (1). This proof suggests plus-minus dualization. The proof is outlined as follows. First it holds that

$$\begin{aligned} f(x)+g(x) &= f(x)-(\lambda,x)+g(x)+(\lambda,x) \\ &\ge \min_{x\in R^n}\,[\,f(x)-(\lambda,x)+g(x)+(\lambda,x)\,] \\ &\ge \min_{x\in R^n}\,[\,f(x)-(\lambda,x)\,]+\min_{x\in R^n}\,[\,g(x)-(-\lambda,x)\,] \\ &= f_\star(\lambda)+g_\star(-\lambda) \qquad (x,\lambda)\in R^n{\times}R^n. \end{aligned}$$

This implies that

$$\min_{x\in R^n}\,[\,f(x)+g(x)\,] \;\ge\; \mathop{\mathrm{Max}}_{\lambda\in R^n}\,[\,f_\star(\lambda)+g_\star(-\lambda)\,]\,.$$

On the other hand, let $\overline{x}$ be a minimizer. Then we get $f'(\overline{x})+g'(\overline{x}) = 0$. Setting $\overline{\lambda} := f'(\overline{x}) = -g'(\overline{x})$, we have

$$\begin{aligned} f_\star(\overline{\lambda}) &= f(\overline{x})-(\overline{\lambda},\overline{x}) \\ g_\star(-\overline{\lambda}) &= g(\overline{x})-(-\overline{\lambda},\overline{x}). \end{aligned}$$

Thus we have

$$\begin{aligned} \min_{x\in R^n}\,[\,f(x)+g(x)\,] &= f(\overline{x})+g(\overline{x}) \\ &= f(\overline{x})-(\overline{\lambda},\overline{x})+g(\overline{x})+(\overline{\lambda},\overline{x}) \\ &= f_\star(\overline{\lambda})+g_\star(-\overline{\lambda}) \\ &\le \mathop{\mathrm{Max}}_{\lambda\in R^n}\,[\,f_\star(\lambda)+g_\star(-\lambda)\,]\,. \end{aligned}$$

Hence

$$\min_{x \in R^n} [\, f(x) + g(x) \,] \; = \; \operatorname*{Max}_{\lambda \in R^n} [\, f_\star(\lambda) + g_\star(-\lambda) \,] \, .$$

Our plus-minus dualization is based upon (1). In particular, two equalities

$$\begin{aligned} f(\overline{x}) + g(\overline{x}) \; &= \; f(\overline{x}) - (\overline{\lambda}, \overline{x}) + g(\overline{x}) + (\overline{\lambda}, \overline{x}) \\ f'(\overline{x}) + g'(\overline{x}) \; &= \; 0 \end{aligned}$$

are crucial under differentiable convexity. This hints plus-minus dualization, which is applied in the following.

4. Plus-minus dualization

Now we show that

$$\text{(D)} \qquad \begin{aligned} &\text{Maximize} \quad 2c\mu_1 - \sum_{k=1}^{n-1} \left[\, \mu_k^2 + (\mu_k - \mu_{k+1})^2 \,\right] - \phi \mu_n^2 \\ &\text{subject to} \quad \text{(i)} \quad \mu \in R^n \end{aligned}$$

is derived from (P) through two plus-minus dualizations.

We consider the primal problem (P). Let $I(x)$ be the objective value for $x = (x_1, x_2, \ldots, x_n)$ satisfying (i), (ii):

$$I(x) \; = \; \sum_{k=1}^{n} \left[\, (x_{k-1} - x_k)^2 + x_k^2 \,\right] + \phi^{-1} x_n^2, \qquad x_0 = c.$$

4.1. Plus-minus dualization 1

Then take any $\mu = (\mu_1, \ldots, \mu_n) \in R^n$. Subtracting $2\mu_k(x_{k-1} - x_k)$ from $(x_{k-1} - x_k)^2$ and adding it to x_k^2, we have [1]

$$I(x) \; = \; \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 - 2\mu_k(x_{k-1} - x_k) + x_k^2 + 2\mu_k(x_{n-1} - x_n) \right] + \phi^{-1} x_n^2.$$

Plus-minus dualization 1 completes the square of $(x_{k-1} - x_k)$ first, and completes the square of x_k second. Separating the summation into two, we get

$$\begin{aligned} I(x) \; = \; & \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 - 2\mu_k(x_{k-1} - x_k) \right] \\ & \qquad + \sum_{k=1}^{n-1} \left[x_k^2 + 2\mu_k(x_{k-1} - x_k) \right] + \phi x_n^2 + 2\mu_n(x_{n-1} - x_n). \end{aligned}$$

The first completion yields

$$\sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 - 2\mu_k(x_{k-1} - x_k) \right] \; = \; \sum_{k=1}^{n} \left[(x_{k-1} - x_k - \mu_k)^2 - \mu_k^2 \right].$$

[1] The subtraction/addition leads to Fenchel duality theorem. Thus the dualization is called *plus-minus*.

The second yields

$$\begin{aligned}
&\sum_{k=1}^{n-1}\left[x_k^2+2\mu_k(x_{k-1}-x_k)\right]+\phi x_n^2+2\mu_n(x_{n-1}-x_n)\\
&= 2c\mu_1+\sum_{k=1}^{n-1}\left[x_k^2-2(\mu_k-\mu_{k+1})x_k\right]+\phi x_n^2-2x_n\mu_n\\
&= 2c\mu_1+\sum_{k=1}^{n-1}\left[\left\{x_k-(\mu_k-\mu_{k+1})\right\}^2-(\mu_k-\mu_{k+1})^2\right]+\phi(x_n-\phi^{-1}\mu_n)^2-\phi^{-1}\mu_n^2.
\end{aligned}$$

Summing up the two completions, we obtain

$$\begin{aligned}
I(x) \;=\; &\sum_{k=1}^{n}\left[(x_{k-1}-x_k-\mu_k)^2-\mu_k^2\right]+2c\mu_1\\
&+\sum_{k=1}^{n-1}\left[\left\{x_k-(\mu_k-\mu_{k+1})\right\}^2-(\mu_k-\mu_{k+1})^2\right]+\phi(x_n-\phi^{-1}\mu_n)^2-\phi^{-1}\mu_n^2.
\end{aligned}\tag{2}$$

Let us define

$$J(\mu):=2c\mu_1-\sum_{k=1}^{n-1}\left[\,\mu_k^2+(\mu_k-\mu_{k+1})^2\,\right]-\phi\mu_n^2.$$

Then it holds that

$$J(\mu)\;\le\;I(x)\quad\text{ on }\;R^n\times R^n. \tag{3}$$

The sign of equality holds iff

$$\begin{aligned}
&x_{k-1}-x_k\;=\;\mu_k\,,\quad x_k\;=\;\mu_k-\mu_{k+1}\quad 1\le k\le n-1\\
&x_{n-1}-x_n\;=\;\mu_n\,,\quad x_n\;=\;\phi^{-1}\mu_n
\end{aligned}\tag{4}$$

holds. The equality condition (4) constitutes a system of $2n$ linear equations in $2n$ variables $(x,\,\mu)$.

Lemma 4.1. *The system (4) has a unique solution* $(\hat{x};\mu^*)$:

$$\begin{aligned}
(\hat{x}_1,\,\hat{x}_2,\,\ldots,\,\hat{x}_{n-1},\,\hat{x}_n)\;&=\;c\big(\phi^{-2},\;\phi^{-4},\;\ldots,\;\phi^{-(2n-2)},\;\phi^{-2n}\big),\\
(\mu_1^*,\,\mu_2^*,\,\ldots,\,\mu_{n-1}^*,\,\mu_n^*)\;&=\;c\big(\phi^{-1},\;\phi^{-3},\;\ldots,\;\phi^{-(2n-3)},\;\phi^{-(2n-1)}\big).
\end{aligned}$$

Then both sides in (3) *are equal to* $\phi^{-1}c^2$.

The solution $(\hat{x};\mu^*)$ is called *Golden complementary.*

Therefore, as a dual to minimization of $I(x)$, we get maximization of $J(\mu)$. Conversely, minimization of $I(x)$ leads maximization of $J(\mu)$.

Hence we have

Theorem 4.2. *Both* (P) *and* (D) *are dual to each other.*

This duality is called *complementary*, which comes from the equality condition (4).

4.2. Plus-minus dualization 2

The objective value $I(x)$ is written as

$$I(x) \;=\; \sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + x_k^2\right] + (x_{n-1} - x_n)^2 + \phi x_n^2, \qquad x_0 = c.$$

Then take any $\mu = (\mu_1, \ldots, \mu_n) \in R^n$. Subtracting $2(\mu_k - \mu_{k+1})x_k$ from x_k^2 and adding it to $(x_{k-1} - x_k)^2$, we have

$$\begin{aligned} I(x) \;=\; & \sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + 2(\mu_k - \mu_{k+1})x_k + x_k^2 - 2(\mu_k - \mu_{k+1})x_k\right] \\ & \qquad\qquad + (x_{n-1} - x_n)^2 + 2\mu_n x_n + \phi x_n^2 - 2\mu_n x_n. \end{aligned}$$

Plus-minus dualization 2 completes the square of x_k first, and then does the square of $(x_{k-1} - x_k)$. Separating the summation into two, we get

$$\begin{aligned} I(x) \;=\; & \sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + 2(\mu_k - \mu_{k+1})x_k\right] + (x_{n-1} - x_n)^2 + 2\mu_n x_n \\ & + \sum_{k=1}^{n-1} \left[x_k^2 - 2(\mu_k - \mu_{k+1})x_k\right] + \phi x_n^2 - 2\mu_n x_n. \end{aligned}$$

The former is completed as

$$\begin{aligned} & \sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + 2(\mu_k - \mu_{k+1})x_k\right] + (x_{n-1} - x_n)^2 + 2\mu_n x_n \\ = \;& 2c\mu_1 + \sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 - 2\mu_k(x_{k-1} - x_k)\right] - 2\mu_n x_{n-1} \\ & \qquad\qquad + (x_{n-1} - x_n)^2 + 2\mu_n x_n \\ = \;& 2c\mu_1 + \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 - 2\mu_k(x_{k-1} - x_k)\right] \\ = \;& 2c\mu_1 + \sum_{k=1}^{n} \left[(x_{k-1} - x_k - \mu_k)^2 - \mu_k^2\right]. \end{aligned}$$

The latter is as follows:

$$\begin{aligned} & \sum_{k=1}^{n-1} \left[x_k^2 - 2(\mu_k - \mu_{k+1})x_k\right] + \phi x_n^2 - 2\mu_n x_n \\ = \;& \sum_{k=1}^{n-1} \left[\left\{x_k - (\mu_k - \mu_{k+1})\right\}^2 - (\mu_k - \mu_{k+1})^2\right] + \phi(x_n - \phi^{-1}\mu_n)^2 - \phi^{-1}\mu_n^2. \end{aligned}$$

The two completions are summed up to

$$\begin{aligned} I(x) \;=\; & 2c\mu_1 + \sum_{k=1}^{n} \left[(x_{k-1} - x_k - \mu_k)^2 - \mu_k^2\right] \\ & + \sum_{k=1}^{n-1} \left[\left\{x_k - (\mu_k - \mu_{k+1})\right\}^2 - (\mu_k - \mu_{k+1})^2\right] + \phi(x_n - \phi^{-1}\mu_n)^2 - \phi^{-1}\mu_n^2. \end{aligned}$$

This identity is the same as (2). Thus the same procedure as in Plus-minus duality 1 shows that both the problems are dual to each other.

5. Dynamic dualization

This section shows that (D) is derived from (P) through two dynamic dualizations.

5.1. Dynamic dualization 1

The problem (P) has the objective function

$$I(x) \;=\; \sum_{k=1}^{n} \left[\, (x_{k-1} - x_k)^2 + x_k^2 \,\right] + \phi^{-1} x_n^2, \qquad x_0 = c.$$

In dynamic dualization 1, let us define $u = (u_1, u_2, \ldots, u_n)$ by

$$u_k \;=\; x_{k-1} - x_k \qquad 1 \le k \le n. \tag{5}$$

Then $I(x)$ is written as [2]

$$I(x) \;=\; \sum_{k=1}^{n} (u_k^2 + x_k^2) + \phi^{-1} x_n^2.$$

The equality (5) implies that $\sum_{k=1}^{n} c_k (x_{k-1} - x_k - u_k) = 0$ for any constants $\{c_k\}$. Hence

$$I(x) \;=\; \sum_{k=1}^{n} \left[\, u_k^2 + c_k (x_{k-1} - x_k - u_k) + x_k^2 \,\right] + \phi^{-1} x_n^2.$$

Now let us take any $\mu = (\mu_1, \ldots, \mu_n) \in R^n$ and set c_k as

$$c_k \;=\; 2\mu_k \qquad 1 \le k \le n.$$

Then it holds that

$$I(x) \;=\; \sum_{k=1}^{n} \left[\, u_k^2 + 2\mu_k (x_{k-1} - x_k - u_k) + x_k^2 \,\right] + \phi^{-1} x_n^2.$$

Separating the summation into two, we get

$$I(x) \;=\; \sum_{k=1}^{n} \left(u_k^2 - 2\mu_k u_k \right) + \sum_{k=1}^{n} \left[\, x_k^2 + 2\mu_k (x_{k-1} - x_k) \,\right] + \phi^{-1} x_n^2.$$

[2] The given unconditional minimization (primal) problem is extended to an equivalent conditional problem by introduction of new variables defined as a linear form. Then the Lagrange dual approach together with the completion of the square yields an identity with respect to the expanded variables. Finally reverting to an equality between the original (primal) variables and the Lagrange multipliers (dual variables), we obtain an inequality together with equality condition. Thus *dynamic dualization* consists of three steps (i) expansion to conditional problem, (ii) Lagrange dualization, and (iii) reversion to only original variables.

From

$$\sum_{k=1}^{n} \mu_k(x_{k-1} - x_k) \;=\; c\mu_1 - \sum_{k=1}^{n-1}(\mu_k - \mu_{k+1})x_k - \mu_n x_n$$

and $1 + \phi^{-1} \;=\; \phi$, we have

$$\begin{aligned}
&\sum_{k=1}^{n} \left[\, x_k^2 + 2\mu_k(x_{k-1} - x_k) \,\right] + \phi^{-1}x_n^2 \\
&= \; 2c\mu_1 + \sum_{k=1}^{n-1} \left[\, x_k^2 - 2(\mu_k - \mu_{k+1})x_k \,\right] + \phi x_n^2 - \mu_n x_n.
\end{aligned}$$

Completing the two squares, we get

$$\begin{aligned}
I(x) \;&=\; \sum_{k=1}^{n} \left(u_k^2 - 2\mu_k u_k\right) + 2c\mu_1 + \sum_{k=1}^{n-1} \left[\, x_k^2 - 2(\mu_k - \mu_{k+1})x_k \,\right] + \phi x_n^2 - \mu_n x_n \\
&=\; \sum_{k=1}^{n} \left\{(u_k - \mu_k)^2 - \mu_k^2\right\} + 2c\mu_1 + \sum_{k=1}^{n-1} \left[\, \{x_k - (\mu_k - \mu_{k+1})\}^2 - (\mu_k - \mu_{k+1})^2 \,\right] \\
&\qquad + \phi(x_n - \phi^{-1}\mu_n)^2 - \phi^{-1}\mu_n^2.
\end{aligned}$$

Reverting to (5), we obtain an identity:

$$\begin{aligned}
I(x) \;=\; &\sum_{k=1}^{n} \left\{(x_{k-1} - x_k - \mu_k)^2 - \mu_k^2\right\} + 2c\mu_1 + \sum_{k=1}^{n-1} \left[\, \{x_k - (\mu_k - \mu_{k+1})\}^2 - (\mu_k - \mu_{k+1})^2 \,\right] \\
&+ \phi(x_n - \phi^{-1}\mu_n)^2 - \phi^{-1}\mu_n^2.
\end{aligned}$$

We note that the objective function of (D) is

$$J(\mu) = 2c\mu_1 - \sum_{k=1}^{n-1} \left[\, \mu_k^2 + (\mu_k - \mu_{k+1})^2 \,\right] - \phi\mu_n^2.$$

Therefore like as in Plus-minus dualization 1 we see that both problems are dual to each other.

5.2. Dynamic dualization 2

The problem (P) has the objective function

$$I(x) \;=\; \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + x_k^2 \,\right] + (x_{n-1} - x_n)^2 + \phi x_n^2.$$

In dynamic dualization 2, we define $u \;=\; (u_1, u_2, \ldots, u_n)$ by

$$u_k \;=\; x_k \qquad 1 \le k \le n. \tag{6}$$

Then we have

$$I(x) \ = \ \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + u_k^2 \,\right] + (x_{n-1} - x_n)^2 + \phi u_n^2.$$

It holds that for any constants $\{c_k\}$

$$I(x) \ = \ \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + u_k^2 + c_k(x_k - u_k) \,\right] + (x_{n-1} - x_n)^2 + \phi u_n^2 + c_n(x_n - u_n).$$

Let us take any $\mu = (\mu_1, \ldots, \mu_n)$ and set c_k as

$$c_k \ = \ 2(\mu_k - \mu_{k+1}) \quad 1 \le k \le n-1, \qquad c_n \ = \ 2\mu_n.$$

Then we have

$$\begin{aligned} I(x) \ &= \ \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + u_k^2 + 2(\mu_k - \mu_{k+1})(x_k - u_k) \,\right] \\ &\qquad + (x_{n-1} - x_n)^2 + \phi u_n^2 + 2\mu_n(x_n - u_n). \end{aligned} \tag{7}$$

One completion in (7) yields

$$\begin{aligned} I(x) \ = \ &\sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + u_k^2 + 2(\mu_k - \mu_{k+1})(x_k - u_k) \,\right] + (x_{n-1} - x_n)^2 + 2\mu_n x_n \\ &+ \phi(u_n - \phi^{-1}\mu_n)^2 - \phi^{-1}\mu_n^2. \end{aligned}$$

The remaining term is

$$\begin{aligned} &\sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + u_k^2 + 2(\mu_k - \mu_{k+1})(x_k - u_k) \,\right] + (x_{n-1} - x_n)^2 + 2\mu_n x_n \\ = \ &\sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + 2(\mu_k - \mu_{k+1})x_k + u_k^2 - 2(\mu_k - \mu_{k+1})u_k \,\right] + (x_{n-1} - x_n)^2 + 2\mu_n x_n. \end{aligned}$$

From

$$\sum_{k=1}^{n-1} (\mu_k - \mu_{k+1})x_k \ = \ c\mu_1 - \sum_{k=1}^{n-1} \mu_k(x_{k-1} - x_k) + \mu_n x_{n-1}$$

we complete the term as follows:

$$\begin{aligned} &\sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + 2(\mu_k - \mu_{k+1})x_k + u_k^2 - 2(\mu_k - \mu_{k+1})u_k \,\right] + (x_{n-1} - x_n)^2 + 2\mu_n x_n \\ = \ &2c\mu_1 + \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 - 2\mu_k(x_{k-1} - x_k) + u_k^2 - 2(\mu_k - \mu_{k+1})u_k \,\right] \\ &\qquad + (x_{n-1} - x_n)^2 - 2\mu_n(x_{n-1} - x_n) \\ = \ &2c\mu_1 + \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k - \mu_k)^2 - \mu_k^2 + \{u_k - (\mu_k - \mu_{k+1})\}^2 - (\mu_k - \mu_{k+1})^2 \,\right] \\ &\qquad + (x_{n-1} - x_n - \mu_n)^2 - \mu_n^2. \end{aligned}$$

Hence we have an equality

$$\begin{aligned} I(x) \;&=\; 2c\mu_1 + \sum_{k=1}^{n-1}\left[\,(x_{k-1}-x_k-\mu_k)^2 - \mu_k^2 + \{u_k-(\mu_k-\mu_{k+1})\}^2 - (\mu_k-\mu_{k+1})^2\,\right] \\ &\quad + (x_{n-1}-x_n-\mu_n)^2 - \mu_n^2 + \phi(u_n-\phi^{-1}\mu_n)^2 - \phi^{-1}\mu_n^2. \end{aligned}$$

Reverting to (6), we have an identity:

$$\begin{aligned} I(x) \;&=\; 2c\mu_1 + \sum_{k=1}^{n-1}\left[\,(x_{k-1}-x_k-\mu_k)^2 - \mu_k^2 + \{x_k-(\mu_k-\mu_{k+1})\}^2 - (\mu_k-\mu_{k+1})^2\,\right] \\ &\quad + (x_{n-1}-x_n-\mu_n)^2 - \mu_n^2 + \phi(x_n-\phi^{-1}\mu_n)^2 - \phi^{-1}\mu_n^2. \end{aligned}$$

Hence the same discussion as in the preceding three dualizations claims that both problems are dual to each other.

Part II
Identical Duality

6. Another dual problem

We have shown that the problem (P) has a dual problem (D). It is shown that both problems have the Golden complementary duality (GCD). Now we show that (P) has another dual problem (D^i) with a different kind of duality. The problem (D^i) has an n-variable $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, which is defined as follows.

$$(\mathrm{D}^i) \qquad \begin{array}{ll} \text{Maximize} & 2c\Big(\sum_{k=1}^{n-1}\lambda_k + \phi\lambda_n\Big) - \Big[\sum_{k=1}^{n}\Big(\sum_{l=k}^{n-1}\lambda_l + \phi\lambda_n\Big)^2 + \sum_{k=1}^{n-1}\lambda_k^2 + \phi\lambda_n^2\Big] \\ \text{subject to} & \text{(i)} \;\; \lambda \in R^n \end{array}$$

where $\sum_{l=n}^{n-1}\lambda_l = 0$. From $1+\phi^{-1} = \phi$, (D^i) is also expressed as

$$\begin{array}{ll} \text{Maximize} & 2c\Big(\sum_{k=1}^{n}\lambda_k + \phi^{-1}\lambda_n\Big) - \Big[\sum_{k=1}^{n}\Big(\sum_{l=k}^{n}\lambda_l + \phi^{-1}\lambda_n\Big)^2 + \sum_{k=1}^{n}\lambda_k^2 + \phi^{-1}\lambda_n^2\Big] \\ \text{subject to} & \text{(i)} \;\; \lambda \in R^n. \end{array}$$

Theorem 6.1. *The problem* (D^i) *has a maximum value* $M = \phi^{-1}c^2$ *at a point*

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_{n-1}^*, \lambda_n^*) = c\big(\phi^{-2},\ \phi^{-4},\ \ldots,\ \phi^{-(2n-2)},\ \phi^{-2n}\big).$$

Proof. This is shown in the following derivation process of dual problem (D^i) from (P) (see Plus-minus dualization 1 and Lemma 7.1).

Thus we have the following triplet between the minimum solution of (P) and the maximum solution of (D^i):

1. (**Duality**) The minimum value is equal to the maximum value : $m = M$. The common value is a quadratic function of initial value c, whose coefficient is the inverse ϕ^{-1} to Golden number.
2. (**Golden**) Both the minimum point $(\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_n)$ and the maximum point $(\lambda_1^*,\ \lambda_2^*,\ \ldots,\ \lambda_n^*)$ are Golden paths of $1 : \phi$.
3. (**Identical**) Further both the optimum points are identical :

$$(\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_n) = (\lambda_1^*,\ \lambda_2^*,\ \ldots,\ \lambda_n^*).$$

This triplet is called *Golden identical duality (GID)*.

7. Plus-minus dualization

This section shows that (D^i) is derived from (P) through two plus-minus dualizations. We note that the primal problem (P) has the objective value

$$I(x) = \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + x_k^2 \,\right] + (x_{n-1} - x_n)^2 + \phi x_n^2, \qquad x_0 = c.$$

7.1. Plus-minus dualization 1

Take any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n$. First subtract $2\lambda_k x_k$ from x_k^2 and add it to $(x_{k-1} - x_k)^2$. Second subtract $2\phi\lambda_n x_n$ from ϕx_n^2 and add it to $(x_{n-1} - x_n)^2$. Then we have

$$\begin{aligned} I(x) \;=\; & \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + 2\lambda_k x_k + x_k^2 - 2\lambda_k x_k \,\right] \\ & \qquad + (x_{n-1} - x_n)^2 + 2\phi\lambda_n x_n + \phi x_n^2 - 2\phi\lambda_n x_n. \end{aligned}$$

Plus-minus dualization 1 completes the square of x_k first, and completes the square of $(x_{k-1} - x_k)$ second. The first completion is

$$\begin{aligned} I(x) \;=\; & \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + 2\lambda_k x_k + (x_k - \lambda_k)^2 - \lambda_k^2 \,\right] \\ & \quad + (x_{n-1} - x_n)^2 + 2\phi\lambda_n x_n + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2 \\ =\; & \sum_{k=1}^{n-1} \left[\, (x_{k-1} - x_k)^2 + 2\lambda_k x_k \,\right] + (x_{n-1} - x_n)^2 + 2\phi\lambda_n x_n \\ & \quad + \sum_{k=1}^{n-1} \left[\, (x_k - \lambda_k)^2 - \lambda_k^2 \,\right] + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2. \end{aligned}$$

As for the second, let us transform x to y by

$$y_k := x_{k-1} - x_k \quad 1 \le k \le n$$

namely

$$x_k = c - y_1 - y_2 - \cdots - y_k \quad 1 \le k \le n.$$

Then we have

$$\begin{aligned}
&\sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + 2\lambda_k x_k \right] + (x_{n-1} - x_n)^2 + 2\phi\lambda_n x_n \\
=\ & \sum_{k=1}^{n-1} \Big[y_k^2 + 2\lambda_k \big(c - \sum_{l=1}^{k} y_l\big) \Big] + y_n^2 + 2\phi\lambda_n \big(c - \sum_{l=1}^{n} y_l\big) \\
=\ & 2c\big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\big) + \sum_{k=1}^{n-1} \Big[y_k^2 - 2\big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big) y_k \Big] + y_n^2 - 2\phi\lambda_n y_n \\
=\ & 2c\big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\big) + \sum_{k=1}^{n-1} \Big[\big\{ y_k - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big) \big\}^2 - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big)^2 \Big] \\
& \qquad + (y_n - \phi\lambda_n)^2 - (\phi\lambda_n)^2.
\end{aligned}$$

Summing up the two completions, we have

$$\begin{aligned}
I(x) \ =\ & 2c\big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\big) + \sum_{k=1}^{n-1} \Big[\Big\{ y_k - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big) \Big\}^2 - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big)^2 \Big] \\
& + (y_n - \phi\lambda_n)^2 - (\phi\lambda_n)^2 + \sum_{k=1}^{n-1} \left[(x_k - \lambda_k)^2 - \lambda_k^2 \right] + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2.
\end{aligned}$$

From $y_k = x_{k-1} - x_k$, we have an identity:

$$\begin{aligned}
&\sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + x_k^2 \right] + \phi^{-1} x_n^2 \\
=\ & 2c\big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\big) + \sum_{k=1}^{n-1} \Big[\Big\{ x_{k-1} - x_k - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big) \Big\}^2 - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big)^2 \Big] \\
& + (x_{n-1} - x_n - \phi\lambda_n)^2 - (\phi\lambda_n)^2 + \sum_{k=1}^{n-1} \left[(x_k - \lambda_k)^2 - \lambda_k^2 \right] + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2. \qquad (8)
\end{aligned}$$

Now let us define

$$J(\lambda) := 2c\big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\big) - \Big[\sum_{k=1}^{n} \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big)^2 + \sum_{k=1}^{n-1} \lambda_k^2 + \phi\lambda_n^2 \Big].$$

Then we have an inequality

$$J(\lambda) \ \le\ I(x) \quad \text{on } \ R^n \times R^n. \qquad (9)$$

The sign of equality holds iff

$$
\begin{aligned}
x_{k-1} - x_k &= \sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\,, \quad x_k = \lambda_k \quad 1 \le k \le n-1 \\
x_{n-1} - x_n &= \phi\lambda_n\,, \quad x_n = \lambda_n
\end{aligned}
\tag{10}
$$

holds.

The equality condition (10) constitutes a system of $2n$ linear equations in $2n$-variables (x, λ).

Lemma 7.1. *The system (10) has a unique solution* $(\hat{x}; \lambda^*)$:

$$
\begin{aligned}
(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}, \hat{x}_n) &= c\big(\phi^{-2}, \phi^{-4}, \ldots, \phi^{-(2n-2)}, \phi^{-2n}\big), \\
(\lambda_1^*, \lambda_2^*, \ldots, \lambda_{n-1}^*, \lambda_n^*) &= c\big(\phi^{-2}, \phi^{-4}, \ldots, \phi^{-(2n-2)}, \phi^{-2n}\big).
\end{aligned}
$$

Then both the sides in (9) are equal to $\phi^{-1}c^2$.

The solution $(\hat{x}; \lambda^*)$ is called *Golden identical*.

Therefore, as a dual to minimization of $I(x)$, we have maximization of $J(\lambda)$. Conversely, a dual of maximization of $J(\lambda)$ is minimization of $I(x)$.

Thus we have

Theorem 7.2. *Both problems* (P) *and* (D^i) *are dual to each other.*

This duality is called *identical*, which comes from the equality condition (10).

7.2. Plus-minus dualization 2

For brevity let us take

$$
x_{k-1} - x_k = u_k \qquad 1 \le k \le n.
$$

The objective value of (P) becomes

$$
I(x) = \sum_{k=1}^{n-1} (u_k^2 + x_k^2) + u_n^2 + \phi x_n^2.
$$

Then we take any $\lambda = (\lambda_1, \ldots, \lambda_n) \in R^n$ and set

$$
c_k = \sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n \quad 1 \le k \le n-1, \qquad c_n = \phi\lambda_n.
$$

Subtracting $2c_k u_k$ from u_k^2 and adding it to x_k^2, we have

$$
I(x) = \sum_{k=1}^{n-1} (u_k^2 - 2c_k u_k + x_k^2 + 2c_k u_k) + u_n^2 - 2c_n u_n + \phi x_n^2 + 2c_n u_n.
$$

Plus-minus dualization 2 completes the square of u_k first, and then does the square of x_k. Completing the first square and separating the summation into two, we get

$$\begin{aligned} I(x) &= \sum_{k=1}^{n-1} \left[\, (u_k - c_k)^2 - c_k^2 + x_k^2 + 2c_k(x_{k-1} - x_k) \,\right] \\ &\qquad + (u_n - c_n)^2 - c_n^2 + \phi x_n^2 + 2c_n(x_{n-1} - x_n) \\ &= \sum_{k=1}^{n} \left[\, (u_k - c_k)^2 - c_k^2 \,\right] \\ &\qquad + \sum_{k=1}^{n-1} \left[\, x_k^2 + 2c_k(x_{k-1} - x_k) \,\right] + \phi x_n^2 + 2c_n(x_{n-1} - x_n). \end{aligned}$$

From

$$\sum_{k=1}^{n-1} c_k(x_{k-1} - x_k) \;=\; x_0 c_1 - \sum_{k=1}^{n-1} (c_k - c_{k+1}) x_k - c_n x_{n-1}$$

we have the second completion as follows:

$$\begin{aligned} &\sum_{k=1}^{n-1} \left[\, x_k^2 + 2c_k(x_{k-1} - x_k) \,\right] + \phi x_n^2 + 2c_n(x_{n-1} - x_n) \\ &= 2x_0c_1 + \sum_{k=1}^{n-1} \left[\, x_k^2 - 2(c_k - c_{k+1})x_k \,\right] + \phi x_n^2 - 2c_n x_n \\ &= 2x_0c_1 + \sum_{k=1}^{n-1} \left(\, x_k^2 - 2\lambda_k x_k \,\right) + \phi(x_n^2 - 2\lambda_n x_n) \\ &= 2x_0c_1 + \sum_{k=1}^{n-1} \left[\, (x_k - \lambda_k)^2 - \lambda_k^2 \,\right] + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2. \end{aligned}$$

Summing up both completions, we obtain

$$\begin{aligned} I(x) &= \sum_{k=1}^{n} \left[\, (u_k - c_k)^2 - c_k^2 \,\right] \\ &\qquad + 2x_0c_1 + \sum_{k=1}^{n-1} \left[\, (x_k - \lambda_k)^2 - \lambda_k^2 \,\right] + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2. \end{aligned}$$

This is nothing but (8). Thus the remaining discussion is same as in Plus-minus dualization 1. Hence we see that both problems (P) and (D^i) are dual to each other.

Remark 1

The primal problem

$$(\mathrm{P}) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n} \left[\, (x_{k-1} - x_k)^2 + x_k^2 \,\right] + \phi^{-1} x_n^2 \\ \text{subject to} & \text{(i)} \;\; x \in R^n \\ & \text{(ii)} \;\; x_0 = c \end{array}$$

has the identical dual problem

$$(\mathrm{D}^i) \quad \begin{array}{ll} \text{Maximize} & 2c\Big(\sum_{k=1}^{n-1}\lambda_k + \phi\lambda_n\Big) - \Big[\sum_{k=1}^{n}\Big(\sum_{l=k}^{n-1}\lambda_l + \phi\lambda_n\Big)^2 + \sum_{k=1}^{n-1}\lambda_k^2 + \phi\lambda_n^2\Big] \\ \text{subject to} & \text{(i)} \ \ \lambda \in R^n. \end{array}$$

The (D^i) is transformed into the complementary dual problem

$$(\mathrm{D}^c) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \Big\{\sum_{k=1}^{n-1}\big[\,\mu_k^2 + (\mu_k - \mu_{k+1})^2\,\big] + \phi\mu_n^2\Big\} \\ \text{subject to} & \text{(i)} \quad \mu \in R^n \end{array}$$

through a transformation $\lambda \to \mu$:

$$\mu_k \;=\; \sum_{l=k}^{n-1}\lambda_l + \phi\lambda_n \quad 1 \le k \le n-1, \qquad \mu_n \;=\; \phi\lambda_n. \tag{11}$$

The (D^c) is nothing but (D), which is discussed in Part 1: Complementary Duality. The inverse transformation is $\mu \to \lambda$:

$$\lambda_k \;=\; \mu_k - \mu_{k+1} \quad 1 \le k \le n-1, \qquad \lambda_n \;=\; \phi^{-1}\mu_n. \tag{12}$$

That is, both duals (D^c), (D^i) are transformed into each other through transformations (11), (12).

8. Dynamic dualization

This section shows that (D^i) is derived through two dynamic dualizations. We remark that (P) has the objective function

$$I(x) \;=\; \sum_{k=1}^{n-1}\big[\,(x_{k-1}-x_k)^2 + x_k^2\,\big] + (x_{n-1}-x_n)^2 + \phi x_n^2, \qquad x_0 = c.$$

8.1. Dynamic dualization 1

In dynamic dualization 1, let us define $u = (u_1, u_2, \ldots, u_n)$ by

$$u_k \;=\; x_k \qquad 1 \le k \le n. \tag{13}$$

Then the objective value of (P) becomes

$$I(x) \;=\; \sum_{k=1}^{n-1}\big[\,(x_{k-1}-x_k)^2 + u_k^2\,\big] + (x_{n-1}-x_n)^2 + \phi u_n^2.$$

The equality (13) implies that $\sum_{k=1}^{n} c_k(x_k - u_k) = 0$ for any constants $\{c_k\}$. Hence

$$I(x) \;=\; \sum_{k=1}^{n-1}\big[\,(x_{k-1}-x_k)^2 + u_k^2 + c_k(x_k-u_k)\,\big] + (x_{n-1}-x_n)^2 + \phi u_n^2 + c_n(x_n-u_n).$$

Now take any $\lambda = (\lambda_1, \ldots, \lambda_n) \in R^n$ and set

$$c_k \;=\; 2\lambda_k \quad 1 \le k \le n-1, \qquad c_n \;=\; 2\phi\lambda_n.$$

Then we have

$$\begin{aligned} I(x) \;&=\; \sum_{k=1}^{n-1} \big[\, (x_{k-1} - x_k)^2 + u_k^2 + 2\lambda_k (x_k - u_k) \,\big] \\ &\qquad + (x_{n-1} - x_n)^2 + \phi u_n^2 + 2\phi\lambda_n (x_n - u_n). \end{aligned}$$

First completing the square of u_k and then separating the summation into two, we get

$$\begin{aligned} I(x) \;&=\; \sum_{k=1}^{n-1} \big[\, (x_{k-1} - x_k)^2 + 2\lambda_k x_k + (u_k - \lambda_k)^2 - \lambda_k^2 \,\big] \\ &\qquad + (x_{n-1} - x_n)^2 + 2\phi\lambda_n x_n + \phi(u_n - \lambda_n)^2 - \phi\lambda_n^2 \\ &=\; \sum_{k=1}^{n-1} \big[\, (x_{k-1} - x_k)^2 + 2\lambda_k x_k \,\big] + (x_{n-1} - x_n)^2 + 2\phi\lambda_n x_n \\ &\qquad + \sum_{k=1}^{n-1} \big[\, (u_k - \lambda_k)^2 - \lambda_k^2 \,\big] + \phi(u_n - \lambda_n)^2 - \phi\lambda_n^2. \end{aligned}$$

In order to complete the square of $(x_{k-1} - x_k)$, we introduce a transformation $x \to y$:

$$y_k := x_{k-1} - x_k \quad 1 \le k \le n.$$

This yields

$$x_k = c - y_1 - y_2 - \cdots - y_k \quad 1 \le k \le n.$$

Then the completion is

$$\begin{aligned} &\sum_{k=1}^{n-1} \big[\, (x_{k-1} - x_k)^2 + 2\lambda_k x_k \,\big] + (x_{n-1} - x_n)^2 + 2\phi\lambda_n x_n \\ &= \sum_{k=1}^{n-1} \Big[\, y_k^2 + 2\lambda_k \big(c - \sum_{l=1}^{k} y_l\big) \,\Big] + y_n^2 + 2\phi\lambda_n \big(c - \sum_{l=1}^{n} y_l\big) \\ &= 2c\big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\big) + \sum_{k=1}^{n-1} \Big[\, y_k^2 - 2\big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big) y_k \,\Big] + y_n^2 - 2\phi\lambda_n y_n \\ &= 2c\big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\big) + \sum_{k=1}^{n-1} \Big[\, \Big\{ y_k - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big) \Big\}^2 - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big)^2 \,\Big] \\ &\qquad + (y_n - \phi\lambda_n)^2 - (\phi\lambda_n)^2. \end{aligned}$$

Summing up the two completions, we have

$$\begin{aligned} I(x) \;&=\; 2c\big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\big) + \sum_{k=1}^{n-1} \Big[\, \Big\{ y_k - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big) \Big\}^2 - \big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\big)^2 \,\Big] \\ &\quad + (y_n - \phi\lambda_n)^2 - (\phi\lambda_n)^2 + \sum_{k=1}^{n-1} \big[\, (u_k - \lambda_k)^2 - \lambda_k^2 \,\big] + \phi(u_n - \lambda_n)^2 - \phi\lambda_n^2. \end{aligned}$$

From $y_k = x_{k-1} - x_k,\ u_k = x_k$, we obtain an identity:

$$\sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + x_k^2 \right] + \phi^{-1} x_n^2$$
$$= 2c\Big(\sum_{k=1}^{n-1} \lambda_k + \phi\lambda_n\Big) + \sum_{k=1}^{n-1} \Big[\Big\{ x_{k-1} - x_k - \Big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\Big) \Big\}^2 - \Big(\sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n\Big)^2 \Big]$$
$$+ (x_{n-1} - x_n - \phi\lambda_n)^2 - (\phi\lambda_n)^2 + \sum_{k=1}^{n-1} \left[(x_k - \lambda_k)^2 - \lambda_k^2 \right] + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2.$$

This is the same as (8). Thus both problems (P) and (D^i) are dual to each other.

8.2. Dynamic dualization 2

In dynamic dualization 2, we define $u = (u_1, u_2, \ldots, u_n)$ by

$$u_k = x_{k-1} - x_k \qquad 1 \le k \le n. \tag{14}$$

The objective value of (P) is

$$I(x) = \sum_{k=1}^{n-1} (u_k^2 + x_k^2) + u_n^2 + \phi x_n^2.$$

It holds that for any constants $\{c_k\}$

$$I(x) = \sum_{k=1}^{n-1} \left[u_k^2 + 2c_k(x_{k-1} - x_k - u_k) + x_k^2 \right] + u_n^2 + \phi x_n^2 + 2c_n(x_{n-1} - x_n - u_n).$$

Now take any $\lambda = (\lambda_1, \ldots, \lambda_n) \in R^n$ and set

$$c_k = \sum_{l=k}^{n-1} \lambda_l + \phi\lambda_n \quad 1 \le k \le n-1, \qquad c_n = \phi\lambda_n. \tag{15}$$

First completing the square of u_k and separating the summation into two, we have

$$\begin{aligned} I(x) &= \sum_{k=1}^{n-1} \left[(u_k - c_k)^2 - c_k^2 + x_k^2 + 2c_k(x_{k-1} - x_k) \right] \\ &\qquad + (u_n - c_n)^2 - c_n^2 + \phi x_n^2 + 2c_n(x_{n-1} - x_n) \\ &= \sum_{k=1}^{n} \left[(u_k - c_k)^2 - c_k^2 \right] \\ &\qquad + \sum_{k=1}^{n-1} \left[x_k^2 + 2c_k(x_{k-1} - x_k) \right] + \phi x_n^2 + 2c_n(x_{n-1} - x_n). \end{aligned}$$

From

$$\sum_{k=1}^{n-1} c_k(x_{k-1} - x_k) = x_0 c_1 - \sum_{k=1}^{n-1} (c_k - c_{k+1}) x_k - c_n x_{n-1}$$

we have the second completion

$$
\begin{aligned}
& \sum_{k=1}^{n-1} \left[x_k^2 + 2c_k(x_{k-1} - x_k) \right] + \phi x_n^2 + 2c_n(x_{n-1} - x_n) \\
= \;& 2x_0c_1 + \sum_{k=1}^{n-1} \left[x_k^2 - 2(c_k - c_{k+1})x_k \right] + \phi x_n^2 - 2c_n x_n \\
= \;& 2x_0c_1 + \sum_{k=1}^{n-1} \left(x_k^2 - 2\lambda_k x_k \right) + \phi(x_n^2 - 2\lambda_n x_n) \\
= \;& 2x_0c_1 + \sum_{k=1}^{n-1} \left[(x_k - \lambda_k)^2 - \lambda_k^2 \right] + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2.
\end{aligned}
$$

Summing up the two completions, we obtain an identity:

$$
\begin{aligned}
I(x) \;=\; & \sum_{k=1}^{n} \left[(u_k - c_k)^2 - c_k^2 \right] \\
& + 2x_0c_1 + \sum_{k=1}^{n-1} \left[(x_k - \lambda_k)^2 - \lambda_k^2 \right] + \phi(x_n - \lambda_n)^2 - \phi\lambda_n^2.
\end{aligned}
$$

From (14) and (15), it turns out that the identity is the same as (8). Thus both the problems are dual to each other.

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