

## Asymptotic tail dependence of the normal copula

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# Asymptotic tail dependence of the normal copula

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**Abstract.** Copulas have lately attracted much attention as a tool in finance and insurance for dealing with multiple risks that cannot be considered independent. The normal copula, widely used in practice, is known to have the same tail dependence parameter as the product copula. The present paper brings into question the common interpretation of this fact as evidence that the normal copula lacks tail dependence, both by providing numerical examples and by mathematically determining the asymptotic behaviour of the tail dependence.

*Keywords.* copula, normal copula, tail dependence

## 1. INTRODUCTION

### 1.1. COPULAS

Copulas have gained increasing popularity in risk management as a tool for investigating dependent risks. We begin by reviewing rudimentary definitions and facts on copulas. See Nelsen [2] for further reference.

**Definition 1.** A *copula* is  $C: [0, 1]^2 \rightarrow [0, 1]$  with the following properties:

- (1)  $C(u, 0) = C(0, v) = 0$ ,  $C(u, 1) = u$ , and  $C(1, v) = v$  for all  $u, v \in [0, 1]$ ;
- (2) if  $0 \leq u_1 \leq u_2 \leq 1$  and  $0 \leq v_1 \leq v_2 \leq 1$ , then  $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0$ .

**Example 1.** The function  $C(u, v) = uv$  is a copula and called the *product copula*.

For a bivariate random variable  $(X, Y)$ , let  $F_X$  and  $F_Y$  denote the marginal distribution functions and let  $F_{X,Y}$  denote the joint distribution function:  $F_X(x) = P(X \leq x)$ ,  $F_Y(y) = P(Y \leq y)$ , and  $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$  for  $x, y \in \mathbb{R}$ . We say that  $(X, Y)$  is *continuous* if  $F_X$  and  $F_Y$  are both continuous.

**Theorem 1** (Sklar). *If  $(X, Y)$  is a continuous bivariate random variable, then there exists a unique copula  $C_{X,Y}$  such that*

$$F_{X,Y}(x, y) = C_{X,Y}(F_X(x), F_Y(y))$$

for all  $x, y \in \mathbb{R}$ .

**Example 2.** The independence of  $X$  and  $Y$  is equivalent to  $C_{X,Y}$  being the product copula.

**Remark 1.** If we write

$$F^{-1}(u) = \inf\{x \in \mathbb{R} \mid F(x) \geq u\}$$

for univariate distribution functions  $F$ , we have

$$C_{X,Y}(u, v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)).$$

In this paper, the focus will be on the *normal copula*:

**Definition 2.** Let  $-1 < \rho < 1$ . If  $(X, Y)$  is a normally distributed bivariate random variable such that  $E[X] = E[Y] = 0$ ,  $V(X) = V(Y) = 1$ , and  $\text{Cov}(X, Y) = \rho$ , then  $C_{X,Y}$  is called the *normal copula* (or *Gaussian copula*) with correlation  $\rho$  and denoted by  $C_\rho$ .

### 1.2. TAIL DEPENDENCE OF COPULAS

**Definition 3.** Let  $C$  be a copula. We define  $\lambda_C: (0, 1) \rightarrow [0, 1]$  by

$$\lambda_C(t) = \frac{1 - 2t + C(t, t)}{1 - t}.$$

We call  $\lim_{t \nearrow 1} \lambda_C(t)$  the *upper tail dependence parameter* of  $C$ , if it exists.

**Remark 2.** If  $(X, Y)$  is a continuous bivariate random variable, then

$$\begin{aligned} \lambda_{C_{X,Y}}(t) &= \frac{\left(1 - P(X \leq F_X^{-1}(t)) - P(Y \leq F_Y^{-1}(t))\right)}{1 - P(X \leq F_X^{-1}(t))} \\ &= \frac{P(X > F_X^{-1}(t), Y > F_Y^{-1}(t))}{P(X > F_X^{-1}(t))} \\ &= P(Y > F_Y^{-1}(t) \mid X > F_X^{-1}(t)). \end{aligned}$$

**Example 3.** If  $C$  is the product copula, then  $\lambda_C(t) = 1 - t \rightarrow 0$  as  $t \nearrow 1$ .

The normal copula is known to have upper tail dependence parameter 0:

**Proposition 1.** *The normal copula with arbitrary correlation  $\rho \in (-1, 1)$  has upper tail dependence parameter 0.*

	Product copula	Normal copula $C_\rho$ with $\rho = 0.5$
$t = 0.8$	0.2000	0.4358
$t = 0.9$	0.1000	0.3240
$t = 0.95$	0.0500	0.2438
$t = 0.99$	0.0100	0.1294
$t = 0.995$	0.0050	0.0993
$t = 0.999$	0.0010	0.0543

Table 1: Upper tail dependence  $\lambda_C(t)$  of the product and normal copulas

This proposition, with Example 3 in mind, is often interpreted to mean that the normal copula exhibits no tail dependence. However, Table 1 suggests that the product and normal copulas have different rates at which  $\lambda_C(t)$  converges to 0. The purpose of this paper is to completely describe how  $\lambda_{C_\rho}(t)$  converges to 0.

Now we state a particular case of our main theorem, of which the complete statement will be given in Section 2 (Theorem 3).

**Theorem 2.** *We have*

$$\lambda_{C_\rho}(t) = \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}} e^{-\frac{1-\rho}{2(1+\rho)}s^2} \times \left( s^{-1} - \frac{1+2\rho-\rho^2}{1-\rho}s^{-3} + O(s^{-5}) \right)$$

as  $t \nearrow 1$ , where  $s = \Phi^{-1}(t) \nearrow \infty$ , with  $\Phi$  denoting the distribution function of the standard normal distribution:  $t = \Phi(s) = (2\pi)^{-1/2} \int_{-\infty}^s \exp(-x^2/2) dx$ .

Theorem 2 gives the leading behaviour of  $\lambda_{C_\rho}(t)$ :

**Corollary 1.** *We have*

$$\lambda_{C_\rho}(t) \sim (4\pi)^{-\frac{\rho}{1+\rho}} \sqrt{\frac{(1+\rho)^3}{1-\rho}} (1-t)^{\frac{1-\rho}{1+\rho}} (-\log(1-t))^{-\frac{\rho}{1+\rho}}$$

as  $t \nearrow 1$ .

Since the proof of Corollary 1 requires Proposition 3, we defer it to the end of Subsection 3.1.

**Remark 3.** Heffernan [1] mentions the asymptotic order in a different language. Let  $(X, Y)$  be a continuous bivariate random variable such that  $C_{X,Y} = C_\rho$  and the marginals are both unit Fréchet, i.e.  $F_X(x) = F_Y(x) = \exp(-1/x)$  for  $x > 0$ . Then [1] states that

$$P(X, Y > x) \sim c_\rho (\log x)^{-\frac{\rho}{1+\rho}} P(X > x)^{\frac{2}{1+\rho}}$$

as  $x \nearrow \infty$ , where  $c_\rho$  is a positive constant depending on  $\rho$ . By Remark 2, this implies that

$$\begin{aligned} \lambda_{C_\rho}(t) &= \frac{P(X, Y > -1/\log t)}{1-t} \\ &\sim c_\rho (1-t)^{\frac{2}{1+\rho}-1} (\log(-1/\log t))^{-\frac{\rho}{1+\rho}} \\ &\sim c_\rho (1-t)^{\frac{1-\rho}{1+\rho}} (-\log(1-t))^{-\frac{\rho}{1+\rho}} \end{aligned}$$

as  $t \nearrow 1$ .

## 2. PRECISE STATEMENT OF THE MAIN THEOREM

This section is devoted to giving the precise statement of our main theorem. Henceforth we fix a real number  $\rho$  with  $-1 < \rho < 1$  and denote  $\lambda_{C_\rho}(t)$  simply by  $\lambda(t)$ .

We define sequences  $(a_n)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$  of real numbers by

$$\begin{aligned} a_n &= (-1)^n n! (1+\rho)^n \sum_{l=0}^n \frac{(2l-1)!!}{l!} (1-\rho)^{-l}, \\ b_n &= (-1)^n (2n-1)!! \end{aligned}$$

where  $(-1)!! = 1$  by definition. We further define a sequence  $(c_n)_{n \geq 0}$  of real numbers by the following equation between formal power series in  $X$ :

$$\sum_{n=0}^{\infty} c_n X^n = \frac{\sum_{n=0}^{\infty} a_n X^n}{\sum_{n=0}^{\infty} b_n X^n} \in \mathbb{R}[[X]].$$

In other words, we define  $(c_n)_{n \geq 0}$  recursively by setting  $c_0 = a_0/b_0$  and

$$c_n = \frac{1}{b_0} \left( a_n - \sum_{k=0}^{n-1} b_{n-k} c_k \right)$$

for  $n \geq 1$ .

The first three terms of the sequences are as follows:

$$\begin{aligned} a_0 &= 1, & a_1 &= -(1+\rho) \left( 1 + \frac{1}{1-\rho} \right), \\ a_2 &= (1+\rho)^2 \left( 2 + \frac{2}{1-\rho} + \frac{3}{(1-\rho)^2} \right), \\ b_0 &= 1, & b_1 &= -1, & b_2 &= 3, \\ c_0 &= 1, & c_1 &= -\frac{1+2\rho-\rho^2}{1-\rho}, \\ c_2 &= \frac{3+13\rho-3\rho^2-3\rho^3+2\rho^4}{(1-\rho)^2}. \end{aligned}$$

Now our main theorem goes as follows:

**Theorem 3 (Main Theorem).** *For every positive integer  $N$ , we have*

$$\lambda(t) = \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}} e^{-\frac{1-\rho}{2(1+\rho)}s^2} \left( \sum_{n=0}^{N-1} c_n s^{-2n-1} + O(s^{-2N-1}) \right)$$

as  $t \nearrow 1$ , where  $s = \Phi^{-1}(t) \nearrow \infty$ .

## 3. PROOF OF THE MAIN THEOREM

Let  $1/2 < t < 1$  and put  $s = \Phi^{-1}(t) > 0$ . If we set

$$\begin{aligned} A &= \int_s^\infty \int_s^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right) dx dy, \\ B &= \int_s^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx, \end{aligned}$$

then  $\lambda(t) = A/B$  by Remark 2. We shall estimate  $A$  and  $B$  separately.

Let  $\mathbb{R}_+$ ,  $\mathbb{N}_0$ , and  $\mathbb{N}$  denote the sets of positive real numbers, nonnegative integers, and positive integers, respectively.

### 3.1. ESTIMATE OF $B$

**Proposition 2.** *If  $\theta \in \mathbb{R}_+$  and  $N \in \mathbb{N}$ , then*

$$(-1)^N \int_{\theta}^{\infty} e^{-x^2/2} dx > (-1)^N e^{-\theta^2/2} \sum_{n=0}^{N-1} b_n \theta^{-2n-1}.$$

*Proof.* For  $n \in \mathbb{N}_0$ , set

$$I_n = \int_{\theta}^{\infty} x^{-n} e^{-x^2/2} dx.$$

Then the left-hand side of the required inequality is  $(-1)^N I_0$ .

Since integration by parts gives

$$\begin{aligned} I_n &= - \int_{\theta}^{\infty} x^{-n-1} (e^{-x^2/2})' dx \\ &= -[x^{-n-1} e^{-x^2/2}]_{\theta}^{\infty} + \int_{\theta}^{\infty} (-n-1)x^{-n-2} e^{-x^2/2} dx \\ &= \theta^{-n-1} e^{-\theta^2/2} - (n+1)I_{n+2}, \end{aligned}$$

we have

$$\begin{aligned} &(-1)^N e^{-\theta^2/2} \sum_{n=0}^{N-1} b_n \theta^{-2n-1} \\ &= \sum_{n=0}^{N-1} (-1)^{N+n} (2n-1)!! \theta^{-2n-1} e^{-\theta^2/2} \\ &= \sum_{n=0}^{N-1} (-1)^{N+n} (2n-1)!! (I_{2n} + (2n+1)I_{2n+2}) \\ &= \sum_{n=0}^{N-1} ((-1)^{N+n} (2n-1)!! I_{2n} \\ &\quad - (-1)^{N+n+1} (2n+1)!! I_{2n+2}) \\ &= (-1)^N I_0 - (2N-1)!! I_{2N} \\ &< (-1)^N I_0. \end{aligned}$$

**Proposition 3.** *For every  $N \in \mathbb{N}$ , we have*

$$B = \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \left( \sum_{n=0}^{N-1} b_n s^{-2n-1} + O(s^{-2N-1}) \right)$$

as  $s \nearrow \infty$ .

*Proof.* If  $N'$  is an even integer with  $N' \geq N$ , then Proposition 2 shows that

$$\begin{aligned} B &> \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \sum_{n=0}^{N'-1} b_n s^{-2n-1} \\ &= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \left( \sum_{n=0}^{N-1} b_n s^{-2n-1} + O(s^{-2N-1}) \right). \end{aligned}$$

By taking  $N'$  to be an odd integer with  $N' \geq N$ , we may similarly obtain

$$B < \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \left( \sum_{n=0}^{N-1} b_n s^{-2n-1} + O(s^{-2N-1}) \right).$$

The proposition follows from these estimates.  $\square$

*Proof of Corollary 1, assuming Theorem 2.* Proposition 3 shows that

$$1-t = B \sim \frac{e^{-s^2/2} s^{-1}}{\sqrt{2\pi}},$$

which in turn implies that

$$-\log(1-t) \sim \frac{s^2}{2}.$$

It follows from Theorem 2 that

$$\begin{aligned} \lambda(t) &\sim \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}} e^{-\frac{1-\rho}{2(1+\rho)} s^2} s^{-1} \\ &= (4\pi)^{-\frac{\rho}{1+\rho}} \sqrt{\frac{(1+\rho)^3}{1-\rho}} \left( \frac{e^{-s^2/2} s^{-1}}{\sqrt{2\pi}} \right)^{\frac{1-\rho}{1+\rho}} \left( \frac{s^2}{2} \right)^{-\frac{\rho}{1+\rho}} \\ &\sim (4\pi)^{-\frac{\rho}{1+\rho}} \sqrt{\frac{(1+\rho)^3}{1-\rho}} (1-t)^{\frac{1-\rho}{1+\rho}} (-\log(1-t))^{-\frac{\rho}{1+\rho}}. \square \end{aligned}$$

### 3.2. ESTIMATE OF $A$

We set  $\alpha = \sqrt{(1-\rho)/2}$  and  $\beta = \sqrt{(1+\rho)/2}$ , so that  $\alpha$  and  $\beta$  are positive real numbers with  $\alpha^2 + \beta^2 = 1$ .

**Lemma 1.** *We have*

$$A = \frac{\beta}{\pi} e^{-s^2/2} \int_{\alpha s/\beta}^{\infty} \left( \int_{\alpha w + \beta s}^{\infty} e^{-z^2/2} dz \right) \times e^{(\alpha w + \beta s)^2/2} e^{-w^2/2} dw.$$

*Proof.* Symmetry gives

$$\begin{aligned} \square \quad A &= 2 \iint_{x \geq y \geq s} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) dx dy \\ &= \frac{1}{2\pi\alpha\beta} \iint_{x \geq y \geq s} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) dx dy. \end{aligned}$$

We use the change of variables

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \beta z + \alpha\beta w - \alpha^2 s \\ \beta z - \alpha\beta w + \alpha^2 s \end{pmatrix} \\ \iff \begin{pmatrix} z \\ w \end{pmatrix} &= \begin{pmatrix} (x+y)/2\beta \\ (x-y)/2\alpha\beta + \alpha s/\beta \end{pmatrix}. \end{aligned}$$

Since

$$\begin{aligned} x \geq y \geq s &\iff \beta z + \alpha\beta w - \alpha^2 s \geq \beta z - \alpha\beta w + \alpha^2 s \geq s \\ &\iff w \geq \alpha s/\beta, z \geq \alpha w + \beta s \end{aligned}$$

and

$$\begin{aligned} \frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} &= \frac{(x + y)^2}{4(1 + \rho)} + \frac{(x - y)^2}{4(1 - \rho)} \\ &= \frac{(2\beta z)^2}{8\beta^2} + \frac{(2\alpha\beta w - 2\alpha^2 s)^2}{8\alpha^2} \\ &= \frac{z^2}{2} + \frac{(\beta w - \alpha s)^2}{2} \\ &= \frac{z^2}{2} - \frac{(\alpha w + \beta s)^2}{2} + \frac{w^2 + s^2}{2}, \end{aligned}$$

we have

$$\begin{aligned} A &= \frac{1}{2\pi\alpha\beta} \int_{\alpha s/\beta}^{\infty} \int_{\alpha w + \beta s}^{\infty} \exp\left(-\frac{z^2}{2} + \frac{(\alpha w + \beta s)^2}{2} - \frac{w^2 + s^2}{2}\right) \\ &\quad \times \left| \det \begin{pmatrix} \beta & \alpha\beta \\ \beta & -\alpha\beta \end{pmatrix} \right| dz dw \\ &= \frac{\beta}{\pi} e^{-s^2/2} \int_{\alpha s/\beta}^{\infty} \left( \int_{\alpha w + \beta s}^{\infty} e^{-z^2/2} dz \right) \\ &\quad \times e^{(\alpha w + \beta s)^2/2} e^{-w^2/2} dw. \quad \square \end{aligned}$$

**Lemma 2.** For every  $K \in \mathbb{N}$ , we have

$$\begin{aligned} &(-1)^K A \\ &> (-1)^K \frac{\beta}{\pi} e^{-s^2/2} \sum_{n=0}^{K-1} b_n \int_{\alpha s/\beta}^{\infty} (\alpha w + \beta s)^{-2n-1} e^{-w^2/2} dw. \end{aligned}$$

*Proof.* Lemma 1 and Proposition 2 show that

$$\begin{aligned} &(-1)^K A \\ &= \frac{\beta}{\pi} e^{-s^2/2} \int_{\alpha s/\beta}^{\infty} \left( (-1)^K \int_{\alpha w + \beta s}^{\infty} e^{-z^2/2} dz \right) \\ &\quad \times e^{(\alpha w + \beta s)^2/2} e^{-w^2/2} dw \\ &> \frac{\beta}{\pi} e^{-s^2/2} \int_{\alpha s/\beta}^{\infty} (-1)^K \left( \sum_{n=0}^{K-1} b_n (\alpha w + \beta s)^{-2n-1} \right) \\ &\quad \times e^{-w^2/2} dw \\ &= (-1)^K \frac{\beta}{\pi} e^{-s^2/2} \\ &\quad \times \sum_{n=0}^{K-1} b_n \int_{\alpha s/\beta}^{\infty} (\alpha w + \beta s)^{-2n-1} e^{-w^2/2} dw. \quad \square \end{aligned}$$

For  $n \in \mathbb{N}$  and  $j, k \in \mathbb{N}_0$  with  $j \leq k$ , we define

$$r_{j,k,n} = \frac{(2k - j)!(n + j - 1)!}{(2k - 2j)!!j!(n - 1)!}.$$

**Lemma 3.** If  $n \in \mathbb{N}$  and  $k \in \mathbb{N}_0$ , then we have the following:

- (1)  $r_{0,k+1,n} = r_{0,k,n}(2k + 1)$ .
- (2)  $r_{k+1,k+1,n} = r_{k,k,n}(n + k)$ .
- (3)  $r_{j,k+1,n} = r_{j,k,n}(2k - j + 1) + r_{j-1,k,n}(n + j - 1)$  for  $j = 1, \dots, k$ .

*Proof.* Straightforward. □

**Lemma 4.** If  $n, K \in \mathbb{N}$ , then

$$\begin{aligned} &(-1)^K \int_{\alpha s/\beta}^{\infty} (\alpha w + \beta s)^{-n} e^{-w^2/2} dw \\ &> (-1)^K \sum_{0 \leq j \leq k \leq K-1} \left( (-1)^k r_{j,k,n} \alpha^{-2k+2j-1} \beta^{n+2k+1} \right. \\ &\quad \left. \times s^{-n-2k-1} e^{-\alpha^2 s^2/2\beta^2} \right). \end{aligned}$$

*Proof.* Put  $u = s/\beta$  for simplicity. For  $m \in \mathbb{N}_0$  and  $n \in \mathbb{N}$ , set

$$\begin{aligned} I_{m,n} &= \int_{\alpha s/\beta}^{\infty} w^{-m} (\alpha w + \beta s)^{-n} e^{-w^2/2} dw \\ &= \int_{\alpha u}^{\infty} w^{-m} (\alpha w + \beta^2 u)^{-n} e^{-w^2/2} dw. \end{aligned}$$

Then what we need to show is that

$$\begin{aligned} &(-1)^K I_{0,n} \\ &> (-1)^K \sum_{k=0}^{K-1} (-1)^k \sum_{j=0}^k r_{j,k,n} \alpha^{-2k+2j-1} u^{-n-2k-1} e^{-\alpha^2 u^2/2}. \end{aligned}$$

Since integration by parts gives

$$\begin{aligned} I_{m,n} &= - \int_{\alpha u}^{\infty} w^{-m-1} (\alpha w + \beta^2 u)^{-n} (e^{-w^2/2})' dw \\ &= - [w^{-m-1} (\alpha w + \beta^2 u)^{-n} e^{-w^2/2}]_{\alpha u}^{\infty} \\ &\quad + \int_{\alpha u}^{\infty} ((-m - 1)w^{-m-2} (\alpha w + \beta^2 u)^{-n} \\ &\quad \quad + w^{-m-1} (-\alpha n) (\alpha w + \beta^2 u)^{-n-1}) e^{-w^2/2} dw \\ &= \alpha^{-m-1} u^{-m-n-1} e^{-\alpha^2 u^2/2} \\ &\quad - (m + 1) I_{m+2,n} - \alpha n I_{m+1,n+1}, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{j=0}^k r_{j,k,n} \alpha^j (\alpha^{-2k+j-1} u^{-n-2k-1} e^{-\alpha^2 u^2/2} - I_{2k-j,n+j}) \\ &= \sum_{j=0}^k r_{j,k,n} \alpha^j ((2k - j + 1) I_{2k-j+2,n+j} \\ &\quad \quad + \alpha(n + j) I_{2k-j+1,n+j+1}) \\ &= \sum_{j=0}^k r_{j,k,n} \alpha^j (2k - j + 1) I_{2k-j+2,n+j} \\ &\quad + \sum_{j=1}^{k+1} r_{j-1,k,n} \alpha^j (n + j - 1) I_{2k-j+2,n+j} \\ &= \sum_{j=0}^{k+1} r_{j,k+1,n} \alpha^j I_{2k-j+2,n+j} \end{aligned}$$

by Lemma 3. It follows that

$$\begin{aligned}
& (-1)^K \sum_{k=0}^{K-1} (-1)^k \sum_{j=0}^k r_{j,k,n} \alpha^{-2k+2j-1} u^{-n-2k-1} e^{-\alpha^2 u^2/2} \\
&= (-1)^K \sum_{k=0}^{K-1} (-1)^k \left( \sum_{j=0}^k r_{j,k,n} \alpha^j I_{2k-j,n+j} \right. \\
&\quad \left. + \sum_{j=0}^{k+1} r_{j,k+1,n} \alpha^j I_{2k-j+2,n+j} \right) \\
&= (-1)^K \sum_{k=0}^{K-1} \left( (-1)^k \sum_{j=0}^k r_{j,k,n} \alpha^j I_{2k-j,n+j} \right. \\
&\quad \left. - (-1)^{k+1} \sum_{j=0}^{k+1} r_{j,k+1,n} \alpha^j I_{2k-j+2,n+j} \right) \\
&= (-1)^K \left( r_{0,0,n} I_{0,n} - (-1)^K \sum_{j=0}^K r_{j,K,n} \alpha^j I_{2K-j,n+j} \right) \\
&= (-1)^K I_{0,n} - \sum_{j=0}^K r_{j,K,n} \alpha^j I_{2K-j,n+j} \\
&< (-1)^K I_{0,n}. \quad \square
\end{aligned}$$

**Lemma 5.** *If  $l$  and  $m$  are integers with  $0 \leq l \leq m$ , then*

$$\sum_{n=0}^{m-l} \frac{(m+l-n)!(m-l+n)!}{(m-l-n)!(2n)!!} = (2l-1)!!(2m)!!.$$

*Proof.* For each  $l \in \mathbb{N}_0$ , let  $P_l$  be the statement that the lemma is true for all  $m \geq l$ . We shall prove  $P_l$  by induction on  $l$ .

To establish  $P_0$ , we need to prove that

$$\sum_{n=0}^m \frac{(m+n)!}{(2n)!!} = (2m)!!$$

for all  $m \geq 0$ . If  $m = 0$ , then both sides are 1. Suppose that equality holds for  $m$ . Then

$$\begin{aligned}
& \sum_{n=0}^{m+1} \frac{(m+n+1)!}{(2n)!!} \\
&= \sum_{n=0}^{m+1} \frac{(m+n)!}{(2n)!!} (m+n+1) \\
&= (m+1) \sum_{n=0}^{m+1} \frac{(m+n)!}{(2n)!!} + \frac{1}{2} \sum_{n=1}^{m+1} \frac{(m+n)!}{(2n-2)!!} \\
&= (m+1) \left( (2m)!! + \frac{(2m+1)!}{(2m+2)!!} \right) + \frac{1}{2} \sum_{n=0}^m \frac{(m+n+1)!}{(2n)!!} \\
&= \frac{(2m+2)!!}{2} + \frac{1}{2} \sum_{n=0}^{m+1} \frac{(m+n+1)!}{(2n)!!},
\end{aligned}$$

from which it follows that

$$\sum_{n=0}^{m+1} \frac{(m+n+1)!}{(2n)!!} = (2m+2)!!.$$

Therefore equality holds for  $m+1$  as well. Hence  $P_0$  has been verified.

Now suppose that  $P_l$  is true. Let  $m \geq l+1$ . Since

$$\begin{aligned}
& \frac{(m-l+n+1)!}{(m-l-n+1)!} - \frac{(m-l+n-1)!}{(m-l-n-1)!} \\
&= \frac{(m-l+n-1)!}{(m-l-n+1)!} \left( (m-l+n)(m-l+n+1) \right. \\
&\quad \left. - (m-l-n)(m-l-n+1) \right) \\
&= \frac{(m-l+n-1)!}{(m-l-n+1)!} \cdot 2n(2m-2l+1)
\end{aligned}$$

for  $0 \leq n \leq m-l-1$ , we have

$$\begin{aligned}
& \sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n-1)!(2n)!!} \\
&= \sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n+1)!}{(m-l-n+1)!(2n)!!} \\
&\quad - \sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n+1)!(2n)!!} 2n(2m-2l+1).
\end{aligned}$$

The inductive hypothesis shows that

$$\begin{aligned}
& \sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n+1)!}{(m-l-n+1)!(2n)!!} \\
&= \sum_{n=0}^{m-l+1} \frac{(m+l-n+1)!(m-l+n+1)!}{(m-l-n+1)!(2n)!!} \\
&\quad - \frac{(2l+1)!(2m-2l+1)!}{1!(2m-2l)!!} - \frac{(2l)!(2m-2l+2)!}{0!(2m-2l+2)!!} \\
&= (2l-1)!!(2m+2)!! - \frac{(2l+1)!(2m-2l+1)!}{(2m-2l)!!} \\
&\quad - \frac{(2l)!(2m-2l+1)!}{(2m-2l)!!}
\end{aligned}$$

and that

$$\begin{aligned}
& \sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n+1)!(2n)!!} 2n(2m-2l+1) \\
&= (2m-2l+1) \sum_{n=1}^{m-l-1} \frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n+1)!(2n-2)!!} \\
&= (2m-2l+1) \sum_{n=0}^{m-l-2} \frac{(m+l-n)!(m-l+n)!}{(m-l-n)!(2n)!!} \\
&= (2m-2l+1) \left( (2l-1)!!(2m)!! - \frac{(2l+1)!(2m-2l-1)!}{1!(2m-2l-2)!!} \right. \\
&\quad \left. - \frac{(2l)!(2m-2l)!}{0!(2m-2l)!!} \right) \\
&= (2l-1)!!(2m+2)!! - (2l+1)!!(2m)!! \\
&\quad - \frac{(2l+1)!(2m-2l+1)!}{(2m-2l)!!} - \frac{(2l)!(2m-2l+1)!}{(2m-2l)!!}.
\end{aligned}$$

Therefore we have

$$\sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n-1)!(2n)!!} = (2l+1)!!(2m)!!,$$

as required.

**Proposition 4.** For every  $N \in \mathbb{N}$ , we have

$$A = \frac{1}{2\pi} \sqrt{\frac{(1+\rho)^3}{1-\rho}} e^{-s^2/2} e^{-\frac{1-\rho}{2(1+\rho)}s^2} \times \left( \sum_{n=0}^{N-1} a_n s^{-2n-2} + O(s^{-2N-2}) \right).$$

*Proof.* Lemmas 2 and 4 show that

$$\begin{aligned} & (-1)^K A \\ & > (-1)^K \frac{\beta}{\pi} e^{-s^2/2} \sum_{n=0}^{K-1} b_n \int_{\alpha s/\beta}^{\infty} (\alpha w + \beta s)^{-2n-1} e^{-w^2/2} dw \\ & > (-1)^K \frac{\beta}{\pi} e^{-s^2/2} e^{-\alpha^2 s^2/2\beta^2} \\ & \quad \times \sum_{\substack{0 \leq n \leq K-1 \\ 0 \leq j \leq k \leq K-1}} ((-1)^{n+k} (2n-1)!! r_{j,k,2n+1} \\ & \quad \times \alpha^{-2k+2j-1} \beta^{2n+2k+2} s^{-2n-2k-2}) \end{aligned}$$

for every  $K \in \mathbb{N}$ .

Now let  $N \in \mathbb{N}$ . If  $K \geq N$ , then

$$\begin{aligned} & \sum_{\substack{0 \leq n \leq K-1 \\ 0 \leq j \leq k \leq K-1}} ((-1)^{n+k} (2n-1)!! r_{j,k,2n+1} \\ & \quad \times \alpha^{-2k+2j-1} \beta^{2n+2k+2} s^{-2n-2k-2}) \\ & = \sum_{m=0}^{N-1} \left( (-1)^m \beta^{2m+2} s^{-2m-2} \right. \\ & \quad \times \sum_{\substack{n \geq 0, 0 \leq j \leq k \\ n+k=m}} (2n-1)!! r_{j,k,2n+1} \alpha^{-2k+2j-1} \left. \right) \\ & \quad + O(s^{-2N-2}) \\ & = \sum_{m=0}^{N-1} \left( (-1)^m \beta^{2m+2} s^{-2m-2} \right. \\ & \quad \times \sum_{l=0}^m \alpha^{-2l-1} \sum_{n=0}^{m-l} (2n-1)!! r_{m-l-n, m-n, 2n+1} \left. \right) \\ & \quad + O(s^{-2N-2}) \\ & = \sum_{m=0}^{N-1} \left( (-1)^m \beta^{2m+2} s^{-2m-2} \right. \\ & \quad \times \sum_{l=0}^m \frac{1}{(2l)!!} \alpha^{-2l-1} \sum_{n=0}^{m-l} \frac{(m+l-n)!(m-l+n)!}{(m-l-n)!(2n)!!} \left. \right) \\ & \quad + O(s^{-2N-2}) \\ & = \sum_{m=0}^{N-1} (-1)^m (2m)!! \left( \sum_{l=0}^m \frac{(2l-1)!!}{(2l)!!} \alpha^{-2l-1} \right) \beta^{2m+2} s^{-2m-2} \\ & \quad + O(s^{-2N-2}) \end{aligned}$$

□ by Lemma 5, and so

$$\begin{aligned} & (-1)^K A \\ & > (-1)^K \frac{\beta}{\pi} e^{-s^2/2} e^{-\alpha^2 s^2/2\beta^2} \\ & \quad \times \left( \sum_{m=0}^{N-1} (-1)^m (2m)!! \left( \sum_{l=0}^m \frac{(2l-1)!!}{(2l)!!} \alpha^{-2l-1} \right) \beta^{2m+2} s^{-2m-2} \right. \\ & \quad \left. + O(s^{-2N-2}) \right) \\ & = (-1)^K \frac{1}{2\pi} \sqrt{\frac{(1+\rho)^3}{1-\rho}} e^{-s^2/2} e^{-\frac{1-\rho}{2(1+\rho)}s^2} \\ & \quad \times \left( \sum_{m=0}^{N-1} \left( (-1)^m \frac{(2m)!!}{2^m} (1+\rho)^m \right. \right. \\ & \quad \times \left. \left( \sum_{l=0}^m \frac{(2l-1)!!}{(2l)!!/2^l} (1-\rho)^{-l} \right) s^{-2m-2} \right. \\ & \quad \left. + O(s^{-2N-2}) \right) \\ & = (-1)^K \frac{1}{2\pi} \sqrt{\frac{(1+\rho)^3}{1-\rho}} e^{-s^2/2} e^{-\frac{1-\rho}{2(1+\rho)}s^2} \\ & \quad \times \left( \sum_{m=0}^{N-1} a_m s^{-2m-2} + O(s^{-2N-2}) \right). \end{aligned}$$

By taking an odd  $K$  and an even  $K$ , we may obtain the proposition. □

Propositions 3 and 4 complete the proof of Theorem 3.

## REFERENCES

- [1] Heffernan, J. E.: A directory of coefficients of tail dependence, *Extremes* **3** (2000) 279–290.
- [2] Nelsen, R. B.: *An introduction to copulas*, second edition, Springer Series in Statistics, Springer, New York, 2006.

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