Piecewise truncated conical minimal surfaces and the Gauss hypergeometric functions

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**Piecewise truncated conical minimal surfaces and the Gauss hypergeometric functions**

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**Abstract.** The catenary is the curve which a hanging chain forms, that is, mathematically, the graph of the function $t \mapsto c \cosh \frac{t}{c}$ for a constant $c > 0$. The study of catenaries is applied to the design of arches and suspension bridges. The surface of revolution generated by a catenary is called a catenoid. It is well-known that a catenoid is a minimal surface and the shape which a soap film between two parallel circles forms. In this article, we consider the approximation of a catenoid by combinations of some truncated cones keeping the minimality in a certain sense. In investigating the minimal combinations, the theory of the Gauss hypergeometric functions plays an important role.

**Keywords.** hypergeometric function, truncated cone, catenoid

1. **INTRODUCTION**

It is interesting to approximate a surface by good surfaces from an industrial point of view. In this article, we consider the approximation of a catenoid bounded by two circles of the same radii by a sequence of piecewise truncated conical minimal surfaces.

Throughout this article, a truncated cone means a right circular cone with its apex cut off by a plane parallel to the cone base.

For $x_0, x_1 > 0$ and $\ell > 0$, let $D_{1,\ell}(x_0, x_1)$ be the truncated cone such that the radii of two circles of it are $x_0$ and $x_1$, and its height is $\ell$. Here, we do not consider the interior of the two circles of radii $x_0$ and $x_1$ of $D_{1,\ell}(x_0, x_1)$. Putting

$$S_{1,\ell}(x_0, x_1) := \pi (x_0 + x_1) \sqrt{(x_1 - x_0)^2 + \ell^2},$$

the area of $D_{1,\ell}(x_0, x_1)$ is equal to $\pi \cdot S_{1,\ell}(x_0, x_1)$.

For $x_0, x_1, x_2 > 0$ and $\ell > 0$, let $D_{2,\ell}(x_0, x_1, x_2)$ be the figure consisting of the union of $D_{1,\ell}(x_0, x_1)$ and $D_{1,\ell}(x_1, x_2)$ attached along the circle of radius $x_1$. Similarly, for $n \geq 3$, we define $D_{n,\ell}(x_0, x_1, \ldots, x_n)$ inductively as the union of $D_{n-1,\ell}(x_0, x_1, \ldots, x_{n-1}, x_n)$ and $D_{1,\ell}(x_{n-1}, x_n)$ attached along the circle of radius $x_{n-1}$.

$D_{n,\ell}(x_0, x_1, \ldots, x_n)$ consists of $n$ truncated cones and is called a piecewise truncated conical surface with length $(n; \ell)$ or simply a PTC surface with $L(n; \ell)$ by definition. $D_{n,\ell}(x_0, x_1, \ldots, x_n)$ has the boundary consisting of two circles of radii $x_0$ and $x_n$, and its area is equal to

$$\pi \sum_{i=1}^{n} S_{1,\ell}(x_{i-1}, x_i).$$

We put

$$S_{n,\ell}(x_0, x_1, \ldots, x_n) := \sum_{i=1}^{n} S_{1,\ell}(x_{i-1}, x_i).$$

For arbitrary fixed $a, b > 0$ and $n \in \mathbb{N}$,

$$D_{n+2,\ell}(a, x_0, x_1, \ldots, x_n, b)$$

is a PTC surface with boundary condition $(a, b)$ and length $(n + 2; \ell)$ or simply BCL-$(a, b; n + 2; \ell)$. A PTC surface $D_{n+2,\ell}(a, x_0, x_1, \ldots, x_n, b)$ with BCL-$(a, b; n + 2; \ell)$ is said to be minimal by definition if

$$(x_0, x_1, \ldots, x_n) \mapsto S_{n+2,\ell}(a, x_0, x_1, \ldots, x_n, b).$$

Moreover a PTC minimal surface

$$D_{n+2,\ell}(a, x_0, x_1, \ldots, x_n, b)$$

with BCL-$(a, b; n + 2; \ell)$ is said to be stable if and only if the Hessian matrix of the above function is positive definite at

$$\left( x_0^{(0)}, x_1^{(0)}, \ldots, x_n^{(0)} \right).$$

Putting

$$2D_{n,\ell}(x_0, x_1, \ldots, x_n)$$

:= $D_{2n,\ell}(x_0, x_{n-1}, \ldots, x_0, x_1, \ldots, x_{n-1}, x_n)$,

$$2D_{n,\ell}(x_0, x_1, \ldots, x_{n-1}, a)$$

is a PTC surface with BCL-$(a, a; 2n; \ell)$ for arbitrary fixed $a > 0$. We put

$$\tilde{S}_{n,\ell}(x_0, x_1, \ldots, x_{n-1}) := S_{n,\ell}(x_0, x_1, \ldots, x_{n-1}, a).$$
If \( (x_0^{(0)}, x_1^{(0)}, \ldots, x_{n-1}^{(0)}) \) is a critical point of \( \tilde{S}_{a,n,\ell} \), then \( 2D_{a,n,\ell}(x_0^{(0)}, x_1^{(0)}, \ldots, x_{n-1}^{(0)}, a) \) is minimal with BCL-(a, a; n + 2; \ell). Moreover, if the Hessian matrix of \( \tilde{S}_{a,n,\ell} \) is positive definite there, then \( 2D_{a,n,\ell}(x_0^{(0)}, x_1^{(0)}, \ldots, x_{n-1}^{(0)}, a) \) is stable.

Now, we introduce the main results.

**Theorem 1.** For \( n \in \mathbb{N} \) and \( \ell > 0 \), there are an explicit function \( g_n,\ell(x) \) and a positive number \( \eta_{n,\ell} > 0 \) satisfying the following:

1. If \( a > \eta_{n,\ell} \), then the equation \( g_n,\ell(x) - a = 0 \) has two positive solutions \( x_{a,n,\ell}^+ < x_{a,n,\ell}^- \).
2. We see that
   \[
   2D_{a,n,\ell}(x_{a,n,\ell}^\pm, g_n,\ell(x_{a,n,\ell}^\pm), \ldots, g_{n-1,\ell}(x_{a,n,\ell}^\pm), a)
   \]
   are PTC minimal surfaces with BCL-(a, a; 2n; \ell).

Moreover,
   \[
   g_n,\ell(x) = xT_n \left( 1 + \frac{\ell^2}{2x^2} \right),
   \]
where \( T_n \) is the (first kind) Chebyshev polynomial.

**Theorem 2.** Under the same situation as Theorem 1,
   \[
   2D_{a,n,\ell}(x_{a,n,\ell}^+, g_n,\ell(x_{a,n,\ell}^+), \ldots, g_{n-1,\ell}(x_{a,n,\ell}^+), a)
   \]
is stable.

2. **The cases \( n = 1, 2, 3 \)**

In this section we see Theorems 1 and 2 in the cases where \( n = 1, 2, 3 \).

2.1. **The case \( n = 1 \)**

For \( a > 0 \) and \( \ell > 0 \), we consider the critical points of the function
   \[
   \tilde{S}_{1,1,\ell}(x_0) := S_{1,\ell}(x_0, a) = (x_0 + a)\sqrt{(a - x_0)^2 + \ell^2}.
   \]
Since
   \[
   \frac{d\tilde{S}_{1,1,\ell}}{dx_0} = \frac{2x_0^2 - 2ax_0 + \ell^2}{\sqrt{(a - x_0)^2 + \ell^2}},
   \]
if \( a > \sqrt{2}\ell := \eta_{1,\ell} \), then there are two critical points
   \[
   a \pm \sqrt{a^2 - 2\ell^2}/2
   \]
of \( \tilde{S}_{1,1,\ell}(x_0) \). We put
   \[
   x_{1,1,\ell}^+ := \frac{a + \sqrt{a^2 - 2\ell^2}}{2}, \quad x_{1,1,\ell}^- := a - \frac{\sqrt{a^2 - 2\ell^2}}{2}.
   \]
Then, since
   \[
   a = x_{1,1,\ell}^+ + \frac{\ell^2}{2x_{1,1,\ell}^+} = x_{1,1,\ell}^- + \frac{\ell^2}{2x_{1,1,\ell}^-},
   \]
if we put
   \[
   g_{1,\ell}(x) := x + \frac{\ell^2}{2x},
   \]
for \( x > 0 \), then
   \[
   \{x_{a,1,\ell}^\pm\} = g_{1,\ell}^{-1}(a)
   \]
for \( a > \eta_{1,\ell} \). In other words, a positive number \( x \) is a critical point of \( \tilde{S}_{g_1,\ell}(x_0, x_{1,\ell}(x_0)) \).

We remark that \( g_1(x) \) takes the minimum \( \eta_{1,\ell} \) at \( x = \ell/\sqrt{2} \), that is, \( g_1'((\ell/\sqrt{2})) = 0 \) and \( g_1((\ell/\sqrt{2})) = \eta_{1,\ell} \). Thus, putting \( \xi_{1,\ell} := \ell/\sqrt{2} \),
   \[
   x_{a,1,\ell}^+ > \xi_{1,\ell} > x_{a,1,\ell}^-
   \]
for \( a > \eta_{1,\ell} \).

Moreover, if we put for \( x > 0 \), \( a := g_{1,\ell}(x) \),
   \[
   H_{1,\ell}(x) := \frac{d^2\tilde{S}_{1,1,\ell}}{dx_0^2}(x),
   \]
and
   \[
   \det H_{1,\ell}(x) := H_{1,\ell}(x)
   \]
itself, then
   \[
   \det H_1(x) = \frac{\ell^2(3x - a) + 2(x - a)^3}{((a - x)^2 + \ell^2)^{3/2}} = \frac{\ell^2(3x - (x + \frac{\ell^2}{2})) + 2(x - (x + \frac{\ell^2}{2}))^3}{((a - x)^2 + \ell^2)^{3/2}}
   \]
   \[
   = \frac{\ell^2(2x^2 - \ell^2)(4x^2 + 1)}{4x^3((a - x)^2 + \ell^2)^{3/2}} = \frac{\ell^2(4x^2 + 1)g_{1,\ell}'(x)}{2x((a - x)^2 + \ell^2)^{3/2}} = \frac{4x^2g_{1,\ell}'(x)}{(4x^2 + \ell^2)^{1/2}}.
   \]
Together with the behavior of \( g_{1,\ell}'(x) \), this formula means that for \( a > \eta_{1,\ell} \), \( \tilde{S}_{1,1,\ell}(x_0) \) takes the local minimum at \( x_{a,1,\ell}^+ \) because \( g_{1,\ell}(x_{a,1,\ell}^+) > 0 \).

2.2. **The case \( n = 2 \)**

For \( a > 0 \) and \( \ell > 0 \), we consider the critical points of the function \( \tilde{S}_{2,\ell}(x_0, x_1) := S_{1,\ell}(x_0, x_1) + S_{1,\ell}(x_1, a) \), that is, we consider a point \( (x_0, x_1) \) satisfying
   \[
   \frac{\partial\tilde{S}_{2,\ell}}{\partial x_0}(x_0, x_1) = \frac{\partial\tilde{S}_{2,\ell}}{\partial x_1}(x_0, x_1) = 0.
   \]
By the case where \( n = 1 \) and the formula
   \[
   \frac{\partial \tilde{S}_{2,\ell}}{\partial x_0}(x_0, x_1) = 0,
   \]
we see that \( x_1 = g_{1,\ell}(x_0) \). Moreover,
   \[
   0 = \frac{\partial \tilde{S}_{2,\ell}}{\partial x_1}(x_0, x_1)
   \]
   \[
   = \frac{2x_1^2 - 2ax_1 + \ell^2}{(x_1 - x_0)^2 + \ell^2} + \frac{2x_1^2 - 2ax_1 + \ell^2}{(a - x_1)^2 + \ell^2}.
   \] (2.1)
implies that
\[
0 = (2x_1^2 - 2x_0x_1 + \ell^2)^2((a - x_1)^2 + \ell^2)
- (2x_1^2 - 2ax_1 + \ell^2)^2((x_1 - x_0)^2 + \ell^2)
= \ell^2(a - x_0)(4x_1^2 - 4ax_0x_1 + 2\ell^2x_1 + \ell^2x_0 + a\ell^2).
\]
Here, if \(a = x_0\), then Formula (2.1) does not hold and so we have
\[
(\ell^2 - 4x_0x_1)a + 4x_1^4 + 2\ell^2x_1 + \ell^2x_0 = 0,
\]
that is,
\[
a = \frac{4x_1^4 + 2\ell^2x_1 + \ell^2x_0}{4x_0x_1 - \ell^2}
= \frac{4(g_{1,\ell}(x_0))^3 + 2\ell^2g_{1,\ell}(x_0) + \ell^2x_0}{4g_{1,\ell}(x_0) - \ell^2}
= x_0 + \frac{\ell^2}{2x_0} + \frac{\ell^4}{2x_0^2},
\]
where we can check that in this case, Formula (2.1) holds.

If we put
\[
g_{2,\ell}(x) := x + \frac{\ell^2}{x} + \frac{\ell^4}{2x^3},
\]
then \(g_{2,\ell}(x)\) is positive, convex in \((0, \infty)\), and
\[
\lim_{x \to 0} g_{2,\ell}(x) = \lim_{x \to \infty} g_{2,\ell}(x) = \infty
\]
and thus, it takes the unique minimal value \(\eta_{2,\ell} > 0\) at a point \(\xi_{2,\ell} > 0\). Hence, if \(a > \eta_{2,\ell}\), then there are two solutions \(x_{a,2,\ell}^\pm\) of \(g_{2,\ell}(x) = a\), where \(x_{a,2,\ell}^- < \xi_{2,\ell} < x_{a,2,\ell}^+\). Consequently, if \(a > \eta_{2,\ell}\), then there are two critical points \((x_{a,2,\ell}^\pm, g_{1,\ell}(x_{a,2,\ell}^\pm))\) of \(\tilde{S}_{a,2,\ell}(x_0, x_1)\); if \(a = \eta_{2,\ell}\), only one critical point \((\xi_{2,\ell}, g_{1,\ell}(\xi_{2,\ell}))\); and if \(a < \eta_{2,\ell}\), there is no critical point.

We should remark that since \(g_{1,\ell}^\prime(x) > g_{2,\ell}^\prime(x)\) for \(x > 0\), \(\xi_{1,\ell} < \xi_{2,\ell}\). By numerical calculations we see that \(\xi_{2,\ell} \approx 1.66066\) and \(\eta_{2,\ell} \approx 2.9720\).

Seeing the above argument in terms of \(x_0\), we have that for \(x > 0\), if we put \(a := g_{2,\ell}(x)\), then \((x, g_{1,\ell}(x))\) is a critical point of \(\tilde{S}_{a,\ell}(x_0, x_1)\).

Next, for \(x > 0\), putting \(a = g_{2,\ell}(x)\), we investigate the Hessian matrix
\[
H_{2,\ell}(x) := \begin{pmatrix}
\frac{\partial^2 \tilde{S}_{a,2,\ell}}{\partial x_0^2}(x, g_{1,\ell}(x)) & \frac{\partial^2 \tilde{S}_{a,2,\ell}}{\partial x_0 \partial x_1}(x, g_{1,\ell}(x)) \\
\frac{\partial^2 \tilde{S}_{a,2,\ell}}{\partial x_1^2}(x, g_{1,\ell}(x)) & \frac{\partial^2 \tilde{S}_{a,2,\ell}}{\partial x_1}(x, g_{1,\ell}(x))
\end{pmatrix}
\]
of \(\tilde{S}_{a,2,\ell}(x_0, x_1)\) at \((x, g_{1,\ell}(x))\).

From this point on, we put, for \(\ell > 0\) and \(s, t > 0\),
\[
S_{\ell}(s, t) := (s + t)\sqrt{(t - s)^2 + \ell^2} = (S_{1,\ell}(s, t)).
\]
Then, we see
\[
\frac{\partial^2 \tilde{S}_{a,2,\ell}}{\partial x_0^2}(x, g_{1,\ell}(x)) = \frac{\partial^2 S_{1,\ell}}{\partial s^2}(x, g_{1,\ell}(x)),
\]
\[
\frac{\partial^2 \tilde{S}_{a,2,\ell}}{\partial x_0 \partial x_1}(x, g_{1,\ell}(x)) = \frac{\partial^2 S_{1,\ell}}{\partial s \partial t}(x, g_{1,\ell}(x)),
\]
and
\[
\frac{\partial^2 \tilde{S}_{a,2,\ell}}{\partial x_1^2}(x, g_{1,\ell}(x)) = \frac{\partial^2 S_{1,\ell}}{\partial t^2}(x, g_{1,\ell}(x))
+ \frac{\partial^2 S_{1,\ell}}{\partial s \partial t}(x, g_{1,\ell}(x), g_{2,\ell}(x)).
\]
Then by a direct but long calculation using
\[
(g_{2,\ell}(x) - g_{1,\ell}(x))^2 + \ell^2 = \left(\frac{x^2 + \ell^2}{x^4}\right)^2 \left(g_{1,\ell}(x) - x\right)^2 + \ell^2,
\]
we see that the determinant \(\det H_{2,\ell}(x)\) of \(H_{2,\ell}(x)\) satisfies
\[
\det H_{2,\ell}(x) = \frac{16x^3}{\ell^2}(1 - \frac{2x^2}{x^2} - \frac{3x^4}{2x^2})
= \frac{16x^3}{\ell^2}(1 - \frac{2x^2}{x^2} - \frac{3x^4}{2x^2}).
\]
Thus, if \(x > \xi_{2,\ell}\), then \(\det H_{2,\ell}(x) > 0\). Moreover \(\xi_{2,\ell} > \xi_{1,\ell}\) implies that if \(x > \xi_{2,\ell}\), then \(x > \xi_{1,\ell}\) and
\[
\frac{\partial^2 \tilde{S}_{a,2,\ell}}{\partial x_0^2}(x, g_{1,\ell}(x)) = \frac{\partial^2 S_{1,\ell}}{\partial s^2}(x, g_{1,\ell}(x)) = H_{1,\ell}(x) > 0
\]
from the case where \(n = 1\). This implies that \(H_{2,\ell}(x)\) is positive definite at \((x_{a,2,\ell}^+, g_{1,\ell}(x_{a,2,\ell}^+))\) if \(a > \eta_{2,\ell}\) and
\[
2D_{2,\ell}(x_{a,2,\ell}^+, g_{1,\ell}(x_{a,2,\ell}^+), a)
\]
is a stable PTC minimal surface with BCL-(a, a; 4; \ell).

2.3. THE CASE \(n = 3\)

We consider the critical points of
\[
\tilde{S}_{a,3,\ell}(x_0, x_1, x_2) := S_{\ell}(x_0, x_1) + S_{\ell}(x_1, x_2) + S_{\ell}(x_2, a)
\]
for \(a > 0\). If \((x_0, x_1, x_2)\) is a critical point of \(\tilde{S}_{a,3,\ell}\), then as in the case where \(n = 2\), we have
\[
x_1 = g_{1,\ell}(x_0),
\]
\[
x_2 = g_{2,\ell}(x_0),
\]
and
\[
a = \frac{4x_0^3 + 2\ell^2x_2 + \ell^2x_1}{4x_1x_2 - \ell^2}
= \frac{4(g_{2,\ell}(x_0))^3 + 2\ell^2g_{2,\ell}(x_0) + \ell^2g_{1,\ell}(x_0)}{4g_{1,\ell}(x_0)g_{2,\ell}(x_0) - \ell^2}
= x_0 + \frac{9\ell^2}{2x_0} + \frac{3\ell^4}{x_0^2} + \frac{\ell^6}{2x_0^3}.
\]
Putting
\[
g_{3,\ell}(x) := x + \frac{9\ell^2}{2x} + \frac{3\ell^4}{x^2} + \frac{\ell^6}{2x^3},
\]
similarly as in the case \(n = 2\), we see that there is \(\xi_{3,\ell} > 0\) with \(g_{3,\ell}(\xi_{3,\ell}) = 0\) such that if \(a > \eta_{3,\ell} :=\)
Lemma 3 described in Section 6.) Consequently, det $H_{x_g^\ell,\ell}$ is the equation $g_{x_g^\ell}(x) = a$ has two solutions $x_{a,3,\ell}^{\pm}$ with $x_{a,3,\ell}^+ > \xi_{3,\ell} > x_{a,3,\ell}^-$. Moreover, $(x_{a,3,\ell}^+, g_{1,\ell}(x_{a,3,\ell}^+), g_{2,\ell}(x_{a,3,\ell}^+))$ are the critical points of $\bar{S}_{a,3,\ell}(x_0, x_1, x_2)$. The same argument as in the case $n = 2$ implies $\xi_{a,3,\ell} > \xi_{2,\ell}$.

We define $H_{3,\ell}(x)$ for $x > 0$ as the Hessian matrix of $\bar{S}_{a,3,\ell}$ at $(x, g_{1,\ell}(x), g_{2,\ell}(x))$, where $a := g_{3,\ell}(x)$. Then

$$\frac{\partial^2 \bar{S}_{a,3,\ell}}{\partial x_0 \partial x_2} = 0$$

implies

$$\det H_{3,\ell}(x) = \left( \frac{\partial^2 \bar{S}_{a,3,\ell}}{\partial x_1 \partial x_2} \right)^2 \times \det H_{1,\ell}(x).$$

Making a large calculation (with the help of a computer), we see that

$$\det H_{3,\ell}(x) = \frac{64 x^{18} g_{3,\ell}'(x)}{\ell^3 (x^2 + \ell^2)^2 (4x^2 + 3\ell^2)^3 (4x^2 + \ell^2)^3/2}.$$ 

Thus, by $\xi_{3,\ell} > \xi_{2,\ell} > \xi_{1,\ell}$, if $x > \xi_{3,\ell}$, then det $H_{3,\ell}(x) > 0$, det $H_{2,\ell}(x) > 0$, and det $H_{1,\ell}(x) > 0$ and as is well-known in linear algebra, this implies $H_{3,\ell}$ is positive definite. (See Lemma 3 described in Section 6.) Consequently,

$$2D_{3,\ell}(x_{a,3,\ell}^+, g_{1,\ell}(x_{a,3,\ell}^+), g_{2,\ell}(x_{a,3,\ell}^+), a)$$

is stable.

The calculation of the determinant of $H_n(x)$ is mentioned later.

Repeating the above argument, we see that $g_{4,\ell}(x)$ and $g_{5,\ell}(x)$ should be defined as

$$g_{4,\ell}(x) := \frac{4 \left( g_{3,\ell}(x) \right)^3 + 2\ell^2 g_{3,\ell}(x)}{4 g_{2,\ell}(x) g_{3,\ell}(x) - \ell^2}$$

$$= x + \frac{8\ell^2}{x} + \frac{10\ell^4}{x^3} + \frac{4\ell^6}{x^5} + \frac{\ell^8}{2x^7},$$

$$g_{5,\ell}(x) := \frac{4 \left( g_{4,\ell}(x) \right)^3 + 2\ell^2 g_{4,\ell}(x)}{4 g_{3,\ell}(x) g_{4,\ell}(x) - \ell^2}$$

$$= x + \frac{25\ell^2}{2x} + \frac{25\ell^4}{x^3} + \frac{35\ell^6}{2x^5} + \frac{5\ell^8}{x^7} + \frac{10}{2x^9},$$

and in general,

$$g_{n,\ell}(x) := \frac{4 \left( g_{n-1,\ell}(x) \right)^3 + 2\ell^2 g_{n-1,\ell}(x)}{4 g_{n-2,\ell}(x) g_{n-1,\ell}(x) - \ell^2}$$

for $n \geq 2$, here $g_{0,\ell}(x) := x$.

3. CATENOIDS AND APPROXIMATIONS OF THEM

We put for $c > 0$,

$$C_c(t) := c \cosh \left( \frac{t}{c} \right).$$

The curve $(t, C_c(t))$ is called a catenary. The function $c \mapsto c \cosh \left( \frac{1}{c} \right)$ is positive, convex and takes the unique minimum $\eta_\infty := 1.5088 \ldots$ at $c = 0.8355 \ldots =: C_{\infty}$. Thus, if $a > \eta_\infty$, there are two positive numbers $c_{a}^+ < \eta_\infty < c_{a}^-$ such that $c_{a}^\pm \cosh \left( \frac{1}{c_{a}^\pm} \right) = a$.

The surface $R(C_c) := (t, C_c(t) \cos \theta, C_c(t) \sin \theta)$ is called a catenoid, which is known as a minimal surface of revolution, where “minimal” means “of mean curvature 0”. Let $C_{c,1}$ be $C_c|_{(-1,1)}$. For $a > \eta_\infty$, $R \left( C_{c_{a}^{-},1} \right)$ have the same boundary. The area of $R \left( C_{c_{a}^{-},1} \right)$ is minimal in the set of surfaces having the same boundary and that of $R \left( C_{c_{a}^{+},1} \right)$ is not.

In the view of the previous section, if $a > \eta_\infty$, the sequence

$$2D_{1,1} \left( x_{a_{1,1},1}^+, a \right),$$

$$2D_{3,1} \left( x_{a_{2,1},3}^+, g_{1,3} \left( x_{a_{2,1},3}^+ \right), a \right),$$

$$2D_{4,1} \left( x_{a_{3,1},4}^+, g_{1,4} \left( x_{a_{3,1},4}^+ \right), g_{2,4} \left( x_{a_{3,1},4}^+ \right), a \right),$$

$$\vdots$$

might give an approximation of $R \left( C_{c_{a}^{-},1} \right)$ as PTC minimal surfaces, where the formula $\eta_\infty > \eta_{n,2}$ is proved later.

For example, if $a = 2$, then

$$x_{2,1,1}^+ = 1.707 \ldots,$$

$$x_{2,2,3}^+ = 1.699 \ldots, g_{1,1} \left( x_{2,2,3}^+ \right) = 1.772 \ldots,$$

$$x_{2,3,3}^+ = 1.697 \ldots, g_{1,1} \left( x_{2,3,3}^+ \right) = 1.730 \ldots,$$

$$g_{2,4} \left( x_{2,3,3}^+ \right) = 1.830 \ldots, (3.1)$$

and thus,

$$2D_1 \left( 1.707 \ldots, 2 \right),$$

$$2D_3 \left( 1.699 \ldots, 1.772 \ldots, 2 \right),$$

$$2D_4 \left( 1.697 \ldots, 1.730 \ldots, 1.830 \ldots, 2 \right),$$

$$\vdots$$

might give an approximate of $R \left( C_{c_{2}^{-},1} \right)$ as PTC minimal surfaces, here

$$c_{2}^+ = 1.696 \ldots,$$

$$c_{2}^+ = 1.696 \ldots, C_{c_{2}^{-}} \left( 1/2 \right) = 1.770 \ldots,$$

$$c_{2}^+ = 1.696 \ldots, C_{c_{2}^{-}} \left( 1/3 \right) = 1.729 \ldots,$$

$$C_{c_{2}^{-}} \left( 2/3 \right) = 1.829 \ldots.$$
In fact, we prove this formula in the next section. Assuming 
imath \xi
imath \eta
we change
\[ g_{2,3}(x) = x + \frac{1}{2x} + \frac{1}{32x^3} \]
\[ g_{3,4}(x) = x + \frac{1}{2x} + \frac{1}{32x^3} + \frac{1}{4x^3} \]
\[ g_{4,5}(x) = x + \frac{1}{2x} + \frac{1}{32x^3} + \frac{1}{4x^3} + \frac{1}{6x^3} \]
Thus, it is indicated that
\[ g_{n,\frac{1}{n}}(x) = \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-k)! \cdot (2k)! \cdot n^{2k-1} \cdot x^{2k-1}}. \]  
(3.2)

In fact, we prove this formula in the next section. Assuming this, we see the following remark.

**Remark 1.** We put
\[ g_{\infty}(x) := x \cosh\left(\frac{x}{x}\right). \]
Then, the coefficient of \( \frac{1}{2x-x} \) of \( g_{n,\frac{1}{n}}(x) \) is larger than that of \( g_{n-1,\frac{1}{n}}(x) \) and smaller than that of \( g_{\infty}(x) \) for \( n \geq 2 \) and \( 2 \leq n \leq 4 \). Thus, we see that \( g_{\infty}(x) > g_{n,\frac{1}{n}}(x) \), \( g_{n-1,\frac{1}{n}}(x) \) and \( g_{\infty}(x) \) are homothetic to each other. Consequently, we have
\[ g_{n,\ell}(x) \equiv \frac{\ell}{\ell} g_{n,\ell} \left( \frac{\ell}{x} \right), \]
and if \( \ell = \frac{1}{n} \), then
\[ g_{n,\ell}(x) = n\ell \cdot g_{n,\frac{1}{n}} \left( \frac{x}{n} \right). \]

Substituting \( \frac{1}{n} \) instead of \( x \) in Formula (4.1), we propose that
\[ g_{n,\ell}(x) = \sum_{k=0}^{n} \frac{(n)^{(k)} \cdot (-n)^{(k)}}{(1/2)k \cdot k!} \left( -\left( \frac{\ell}{2x} \right) \right)^{k}. \]
For \( \alpha, \beta, \gamma \in \mathbb{R} \), where \( \gamma \neq 0, -1, -2, \ldots \), the series
\[ F(\alpha, \beta; \gamma; z) := \sum_{k=0}^{\infty} \frac{(\alpha)_{k} \cdot (\beta)_{k}}{(\gamma)_{k} \cdot k!} z^{k} \]
is called a Gauss hypergeometric function. Since \( (-n)_{k} = 0 \) for \( k \geq n+1 \), we see
\[ \sum_{k=0}^{n} \frac{(n)_{k} \cdot (-n)_{k}}{(1/2)k \cdot k!} \left( -\left( \frac{\ell}{2x} \right) \right)^{k} \]
= \( F \left( n, -n, \frac{1}{2} \right) - \left( \frac{\ell}{2x} \right)^{2} \).
for \( x > 0 \).

Let \( T_n \) for \( n \in \mathbb{N} \cup \{0\} \) be the Chebyshev polynomial, that is,
\[
T_0(z) := 1, \quad T_1(z) := z
\]
and for \( n \geq 2 \),
\[
T_n(z) := 2zT_{n-1}(z) - T_{n-2}(z).
\]
Then, it is well-known that \( F(n, -n, \frac{1}{2}; z) = T_n(1 - 2z) \) (See 15.4.3 in [1]). Moreover, it is also well-known that
\[
F(n, -n, \frac{1}{2}; z^2) = \frac{1}{2} \left( [(1 + z^2)^{\frac{1}{2}} + z]^{2n} + [(1 + z^2)^{\frac{1}{2}} - z]^{2n} \right).
\]
(See 15.1.11 in [1].)

**Lemma 1.** For \( n \geq 2 \), we see that
\[
T_{n-1}^2(x) - T_n(x)T_{n-2}(x) = 1 - x^2.
\]

**Proof.** In the case of \( n = 2 \), we obtain this by direct calculation. For \( n \geq 3 \), by the recursion of the Chebyshev polynomials,
\[
T_{n-1}^2(x) - T_n(x)T_{n-2}(x) = T_{n-1}^2(x) - (2xT_{n-1}(x) - T_{n-2}(x))T_{n-2}(x) = T_{n-2}^2(x) + T_{n-1}(x)(T_{n-1}(x) - 2xT_{n-2}(x)) = T_{n-2}^2(x) - T_{n-1}T_{n-3}(x)
\]
\[
\vdots
\]
\[
= T_2^2(x) - T_0(x)T_2(x) = 1 - x^2.
\]

**Proof of Theorem 1.** Recall that the recursion formula which \( g_{n,\ell}(x) \) should satisfy is
\[
g_{n,\ell}(x) := \frac{4g_{n-1,\ell}(x)^3 + 2\ell^2g_{n-1,\ell}(x) + \ell^2g_{n-2,\ell}(x)}{4g_{n-2,\ell}(x)g_{n-1,\ell}(x) - \ell^2}
\]
for \( n \geq 2 \). (See the last paragraph of Section 2.) Since
\[
g_{0,\ell}(x) = xT_0 \left( 1 + \frac{\ell^2}{2x^2} \right)
\]
and
\[
g_{1,\ell}(x) = xT_1 \left( 1 + \frac{\ell^2}{2x^2} \right),
\]
it suffices to prove that \( xT_n \left( 1 + \frac{\ell^2}{2x^2} \right) \) satisfies the same recursion for \( n \geq 2 \). Rearranging the formula we should prove is
\[
4x^2T_{n-1}^2(X) \left( T_{n-1}^2(X) - T_n(X)T_{n-2}(X) \right) + \ell^2 \left( T_n(X) + 2T_{n-1}(X) + T_{n-2}(X) \right) = 0,
\]
where \( X = 1 + \frac{\ell^2}{2x^2} \). Lemma 1 implies
\[
T_{n-1}^2(X) - T_n(X)T_{n-2}(X) = 1 - X^2
\]
\[
= -\left( \frac{\ell^2}{x^2} + \frac{\ell^4}{4x^4} \right),
\]
and the left side of Formula (4.2) is equal to
\[
\ell^2 \left( T_n(X) - 2XT_{n-1}(X) + T_{n-2}(X) \right) = 0.
\]
Given these facts, we obtain
\[
g_{n,\ell}(x) = x \sum_{k=0}^{n} \frac{(n)! \cdot (-n)!}{(1/2)_k \cdot k!} \cdot \left( -\frac{\ell^2}{2x^2} \right)^k
\]
or
\[
g_{n,\ell}(x) = \sum_{k=0}^{n} \frac{n \cdot (n + k - 1)! \cdot \ell^{2k}}{(n-k)! \cdot (2k)!} \cdot x^{2k-1}.
\]
Since this function is positive and convex for \( x > 0 \), and
\[
\lim_{x \to 0} g_{n,\ell}(x) = \lim_{x \to \infty} g_{n,\ell}(x) = \infty,
\]
there is a unique zero point \( \xi_{n,\ell} \) of \( g_{n,\ell}(x) \). Moreover, if we put \( \eta_{n,\ell} := g_{n,\ell}(\xi_{n,\ell}) \), then \( \eta_{n,\ell} \) is the minimum of \( g_{n,\ell} \).

The role of \( \eta_{n,\ell} \) and the minimal property of
\[
2D_{n,\ell}(x_{a,n,\ell}, z, g_{n,\ell}(x_{a,n,\ell}), \ldots, g_{n-1,\ell}(x_{a,n,\ell}), a)
\]
are obtained similarly as in the case of \( n = 1, 2, 3 \). \( \square \)

**Remark 2.** The coefficient of \( \frac{1}{2x^2} \) of \( g_{n,\ell}(x) \) is larger than that of \( g_{n-1,\ell}(x) \) for \( 2 \leq k \leq n \). Thus, \( g_{n,\ell}(x) > g_{n-1,\ell}(x) \) and \( g'_{n,\ell}(x) < g'_{n-1,\ell}(x) \) for \( x > 0 \). This implies that
\[
\eta_{1,\ell} < \eta_{2,\ell} < \cdots < \eta_{n,\ell} \cdots
\]
and
\[
\xi_{1,\ell} < \xi_{2,\ell} < \cdots < \xi_{n,\ell} \cdots.
\]
As is seen in Remark 1, we have
\[
\lim_{n \to \infty} \xi_{n, \frac{1}{n}} = 0.83355 \cdots \quad \text{and} \quad \lim_{n \to \infty} \eta_{n, \frac{1}{n}} = 1.5088 \cdots.
\]
Thus, by using the fact that
\[
D_{\ell,n}(x_0, x_1, \ldots, x_n)
\]
is homothetic to
\[
D'_{\ell,n}(\ell x_0, \ell x_1, \ldots, \ell x_n)
\]
for \( \ell, \ell' > 0 \), we see that \( \xi_{n,\ell} = (\ell/\ell')\xi_{n,\ell'} \) and \( \eta_{n,\ell} = (\ell/\ell')\eta_{n,\ell'} \) and that
\[
\lim_{n \to \infty} \frac{\xi_{n,\ell}}{n} = 0.83355 \cdots \cdot \ell, \quad \lim_{n \to \infty} \frac{\eta_{n,\ell}}{n} = 1.5088 \cdots \cdot \ell.
\]
5. The Hessian Matrices

The purpose of this section is to investigate the Hessian matrix of the function

\[ S_{a,n,t}(x_0, x_1, x_2, \ldots, x_{n-1}) \]

at

\[ \left( g_0(x^+_{a,n,t}), g_1(x^+_{a,n,t}), g_2(x^+_{a,n,t}), \ldots, g_{n-1}(x^+_{a,n,t}) \right), \]

where we should remark that \( g_a(x^+_{a,n,t}) = a \). For investigating the positive definiteness of the matrix, we may assume that \( \ell = 1 \) without loss of generality. Thus, we put \( S(s,t) := S_1(s,t) = (s+t)\sqrt{(t-s)^2+1} \), \( g_k(x) := g_{k,1}(x) \), and \( x^+_{a,n} := x^+_{a,n,1} \). Then, we have

\[
\frac{\partial^2 S}{\partial s^2}(s,t) = \frac{(3s-t) - 2(t-s)^3}{((t-s)^2 + 1)^{3/2}}
\]

\[
= \frac{(s+t) - 2(t-s)((t-s)^2 + 1)}{((t-s)^2 + 1)^{3/2}},
\]

and

\[
\frac{\partial^2 S}{\partial t^2}(s,t) = \frac{(3t-s) + 2(t-s)^3}{((t-s)^2 + 1)^{3/2}}
\]

\[
= \frac{(s+t) + 2(t-s)((t-s)^2 + 1)}{((t-s)^2 + 1)^{3/2}},
\]

and

\[
\frac{\partial^2 S}{\partial s \partial t}(s,t) = \frac{s + t}{(t-s)^2 + 1}^{3/2}.
\]

By using a theorem in the hypergeometric function theory, we see

\[
g_k(x) = (x/2) \times \left( \left( \sqrt{1 + \left( \frac{1}{2x} \right)^2} + \frac{1}{2x} \right)^{2k} + \left( \sqrt{1 + \left( \frac{1}{2x} \right)^2} - \frac{1}{2x} \right)^{2k} \right).
\]

If we put

\( A = A(x) := \sqrt{1 + \left( \frac{1}{2x} \right)^2} + \frac{1}{2x} \),

then

\( \sqrt{1 + \left( \frac{1}{2x} \right)^2} - \frac{1}{2x} = \frac{1}{A} \)

and

\( x = \frac{1}{A - \frac{1}{A}} \).

Now, we put for \( i \in \mathbb{Z} \),

\[
\alpha_i = \alpha_i(x) := A^i + \frac{1}{A^i}, \quad \beta_i = \beta_i(x) := A^i - \frac{1}{A^i}.
\]

Then, we easily check that

\[
\alpha_1 = \alpha_{-1}, \quad \beta_1 = -\beta_{-1}, \quad \alpha_0 = 2, \quad \beta_0 = 0,
\]

\[
\alpha_i \alpha_j = \alpha_{i+j} + \alpha_{i-j}, \quad \beta_i \beta_j = \alpha_{i+j} - \alpha_{i-j},
\]

\[
\beta_i^2 + 4 = \alpha_i^2, \quad \alpha_i \beta_i = \beta_2i,
\]

and

\[
g_k(x) = \frac{\alpha_{2k}}{2\beta_1}.
\]

Moreover we have

\[
(g_k(x) - g_{k-1}(x))^2 + 1 = \frac{(\alpha_{2k} - 2\alpha_{2k-2})^2}{2\beta_1} + 1
\]

\[
= \frac{(\beta_{2k-1}\beta_1)^2}{2\beta_1} + 1
\]

\[
= \frac{(\alpha_{2k-1})^2}{2}.
\]

From the above, we can write simply

\[
\frac{\partial^2 S}{\partial s^2}(g_{k-1}(x), g_k(x)) = \frac{2(2\alpha_1 - \alpha_{4k-1} + \alpha_{4k-3})}{\beta_1 \alpha_{2k-1}},
\]

and

\[
\frac{\partial^2 S}{\partial t^2}(g_{k-1}(x), g_k(x)) = \frac{2(2\alpha_1 + \alpha_{4k-1} - \alpha_{4k-3})}{\beta_1 \alpha_{2k-1}},
\]

and

\[
\frac{\partial^2 S}{\partial s \partial t}(g_{k-1}(x), g_k(x)) = \frac{-4\alpha_1}{\beta_1 \alpha_{2k-1}}.
\]

Next, we consider \( g'_k(x) \). Since

\[
A' = \frac{1}{\sqrt{1 + \left( \frac{1}{2x} \right)^2}} - \frac{1}{2x^2}
\]

\[
= -\frac{1}{2x^2} \sqrt{1 + \left( \frac{1}{2x} \right)^2} + \frac{1}{2x}
\]

\[
= \frac{-\beta_1^2 A}{A + \frac{1}{A}}
\]

\[
= \frac{-\beta_1^2 A}{\alpha_1},
\]

we see that for \( i \in \mathbb{N} \),

\[
\alpha'_i = A'(iA^{i-1} - iA^{-i-1})
\]

\[
= \frac{i\beta_1^2}{\alpha_1} \left( A' - \frac{1}{A} \right)
\]

\[
= \frac{i\beta_1^2 \beta_i}{\alpha_1}.
\]

Thus, together with \( 1/\beta_1 = x \), we see that

\[
g'_k(x) = \frac{1}{2} \left( \frac{\alpha_{2k} + \alpha_{2k}}{\beta_1} \right)
\]

\[
= \frac{\alpha_1 \alpha_{2k} - 2\beta_k \beta_{2k}}{2\alpha_1}
\]

\[
= \frac{(1 - 2k)\alpha_{2k+1} + (1 + 2k)\alpha_{2k-1}}{2\alpha_1}.
\]
Now we consider, for \( n \in \mathbb{N} \) and \( x > 0 \), the Hessian matrix \( H_n(x) \) of
\[
\hat{S}_{g_n(x),n,1}(x_0,x_1,x_2,\ldots,x_{n-1})
\]
at \((x,g_1(x),g_2(x),\ldots,g_{n-1}(x))\).

**Lemma 2.** We have
\[
\det H_n(x) = 4^n \left(\frac{\alpha_1}{\beta_1}\right)^n \frac{g'_n(x)}{\alpha_1^2 \alpha_2^3 \cdots \alpha_{2n-1}^2}.
\]

**Proof.** We prove this by induction. In the cases where \( n = 1, 2 \), we obtain the lemma by direct calculation. We assume that the lemma holds for \( 1, 2, \ldots, n-1 \), here \( n \geq 3 \).

Recalling that
\[
\hat{S}_{g_n(x),n,1}(x_0,x_1,x_2,\ldots,x_{n-1}) = S(x_0,x_1) + S(x_1,x_2) + \cdots + S(x_{n-2},x_{n-1}) + S(x_{n-1},g_n(x)),
\]
\( H_n(x) = (h_{i,j})_{i,j=1,2,\ldots,n} \) is expressed as
\[
h_{1,1} = \frac{\partial^2 S}{\partial x^2}(g_1(x),g_1(x)),
\]
\[
h_{i,i} = \frac{\partial^2 S}{\partial x^2}(g_i(x),g_i(x)) + \frac{\partial^2 S}{\partial x^2}(g_{i-1}(x),g_{i}(x))
\]
for \( i = 2, 3, \ldots, n \),
\[
h_{i,i+1} = h_{i+1,i} = \frac{\partial^2 S}{\partial x \partial t}(g_{i-1}(x),g_i(x))
\]
for \( i = 1, 2, \ldots, n \), and
\[
h_{i,j} = 0
\]
if \( |i-j| \geq 2 \). Consequently, we have
\[
\det H_n(x) = \det H_{n-1}(x) \times \left(\frac{\partial^2 S}{\partial x^2}(g_{n-2}(x),g_{n-1}(x)) + \frac{\partial^2 S}{\partial x^2}(g_{n-1}(x),g_n(x))\right) - \det H_{n-2}(x) \times \left(\frac{\partial^2 S}{\partial x \partial t}(g_{n-2}(x),g_{n-1}(x))\right)^2.
\]

Omitting the middle formulas, we see
\[
\frac{\partial^2 S}{\partial x^2}(g_{n-2}(x),g_{n-1}(x)) + \frac{\partial^2 S}{\partial x^2}(g_{n-1}(x),g_n(x)) = 2(2\alpha_1 + \alpha_{4n-5} + \alpha_{4n-7}) + 2(2\alpha_1 - \alpha_{4n-1} + \alpha_{4n-3})
\]
\[
= 4(2\alpha_{4n-3} + 2\alpha_{4n-5} - \alpha_5 + \alpha_3 + 4\alpha_1) \beta_1 \alpha_{2n-3} \alpha_{2n-1}^2
\]
and from the induction hypothesis,
\[
\det H_{n-1}(x) = \left(\frac{\alpha_1}{\beta_1}\right)^{n-1} \frac{4^{n-1} \times 2}{\alpha_1^2 \alpha_2^3 \cdots \alpha_{2n-3}^2 \alpha_{2n-1}^2} \times ((-2n+1)\alpha_{2n-3} + 2\alpha_{6n-5} + 2\alpha_{6n-7}) - ((-2n+1)\alpha_{2n-3} + 2\alpha_{6n-5} + 2\alpha_{6n-7})
\]
\[
\times (2\alpha_{4n-3} + 2\alpha_{4n-5} - \alpha_5 + \alpha_3 + 4\alpha_1).
\]

Moreover,
\[
(3 - 2n)\alpha_{2n-1} + (2n - 1)\alpha_{2n-3}
\]
\[
\times (2\alpha_{4n-3} + 2\alpha_{4n-5} - \alpha_5 + \alpha_3 + 4\alpha_1)
\]
\[
= 2(3 - 2n)\alpha_{6n-4} + 4\alpha_{6n-6} + 2(2n - 1)\alpha_{6n-8}
\]
\[
+ (2n - 3)\alpha_{2n-1} + ((-2n + 1)\alpha_{2n-1} + (2n - 1)\alpha_{2n-3} + 9 - 2n)\alpha_{2n-3} + (2n + 5)\alpha_{2n-1} + 4(2n - 1)\alpha_{2n-6} - (2n - 1)\alpha_{2n-8}.
\]

Similarly,
\[
\det H_{n-2}(x) = \left(\frac{\partial^2 S}{\partial x \partial t}(g_{n-2}(x),g_{n-1}(x))\right)^2
\]
\[
= \left(\frac{\alpha_1}{\beta_1}\right)^{n-1} \frac{4^{n-1} \times 2}{\alpha_1^2 \alpha_2^3 \cdots \alpha_{2n-3}^2 \alpha_{2n-1}^2} \times ((-2n+1)\alpha_{6n-4} + 2\alpha_{6n-6} + 2(2n + 1)\alpha_{6n-8}
\]
\[
+ (-4n + 2)\alpha_{2n-2} + 4\alpha_{2n-4} + (4n + 2)\alpha_{2n-6} + (2n + 1)\alpha_{2n-4} + 2\alpha_{2n-6} + (-2n + 1)\alpha_{2n-8}.
\]

On the other hand,
\[
g'_n(x) = \frac{(1 - 2n)\alpha_{2n+1} + (2n + 1)\alpha_{2n-1}}{2\alpha_1}
\]
and by direct calculation, we obtain that
\[
\alpha_1 \alpha_{2n-3} ((1 - 2n)\alpha_{2n+1} + (2n + 1)\alpha_{2n-1})
\]
\[
= (-2n + 1)\alpha_{6n-4} + 2\alpha_{6n-6} + (2n + 1)\alpha_{6n-8}
\]
\[
+ (-4n + 2)\alpha_{2n+2} + 4\alpha_{2n-4} + (4n + 2)\alpha_{2n-6} + (2n + 1)\alpha_{2n-4} + 2\alpha_{2n-6} + (-2n + 1)\alpha_{2n-8}.
\]

This completes the proof. \(\square\)

### 6. Proof of Theorem 2

The following lemma is well-known.

**Lemma 3.** A symmetric \( n \times n \) matrix \( A = (a_{ij})_{i,j=1,\ldots,n} \) is positive definite if and only if \( \det A_k > 0 \) for any \( k = 1, 2, \ldots, n \), where \( A_k := (a_{ij})_{i,j=1,\ldots,k} \).

**Proof of Theorem 2.** Lemma 2 implies that if \( x > \xi_{n,1} \), then \( H_n(x) > 0 \). Moreover, as is seen in Remark 2, \( \xi_{n,1} > \xi_{n-1,1} > \cdots > \xi_{1,1} \) and thus if \( x > \xi_{1,1} \), then \( H_k(x) > 0 \) for \( k = 1, 2, \ldots, n \). Together with Lemma 3, we see that \( H_n(x_{a,n,1}) \) is positive definite and
\[
2D_{n,1}(x_{a,n,1}^+,g_1(x_{a,n,1}^+),\ldots,g_{n-1}(x_{a,n,1}^+),a)
\]
is stable for \( a > \eta_{n,1} \). \(\square\)
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