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On some properties of a discrete hungry Lotka-Volterra system of multiplicative type

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Abstract. Two kinds of discrete hungry Lotka-Volterra systems (dhLV) are known as discretizations of the additive type hungry Lotka-Volterra system and the multiplicative one. By associating the dhLV of additive type (dhLV_I) and the discrete hungry Toda equation (dhToda) with LR transformations, some of the authors give a Bäcklund transformation between these two systems. In this paper, from the dhLV of multiplicative type (dhLV_{II}), we first derive the qd-type dhLV_{II}. Through finding the positivity of the qd-type dhLV_{II} and the LR transformation associated with the dhLV_{II}, we present Bäcklund transformations among the dhLV_I, the dhLV_{II} and the dhToda. Moreover, by using one of the Bäcklund transformations, we show asymptotic convergence of the qd-type dhLV_{II}.

Keywords. Bäcklund transformation, LR transformation, asymptotic convergence, discrete hungry Toda equation, discrete hungry Lotka-Volterra system

1. INTRODUCTION

The integrable Lotka-Volterra system (LV) is known as one of the ordinary differential equations that describe predator-prey dynamics in mathematical biology. In [1, 2, 3], one of extended LV is presented as

$$\begin{cases} \frac{du_k(t)}{dt} = u_k(t) \left(\sum_{p=1}^M u_{k+p}(t) - \sum_{p=1}^M u_{k-p}(t) \right), \\ k = 1, 2, \dots, M_m, \quad t \geq 0, \\ u_{1-M}(t) \equiv 0, \dots, u_0(t) \equiv 0, \\ u_{M_m+1}(t) \equiv 0, \dots, u_{M_m+M}(t) \equiv 0, \end{cases} \quad (1)$$

and another extended LV is given in [2, 3] as

$$\begin{cases} \frac{dv_k(t)}{dt} = v_k(t) \left(\prod_{p=1}^M v_{k+p}(t) - \prod_{p=1}^M v_{k-p}(t) \right), \\ k = 1, 2, \dots, M_m + M - 1, \quad t \geq 0, \\ v_{1-M}(t) \equiv 0, \dots, v_0(t) \equiv 0, \\ v_{M_m+M}(t) \equiv 0, \dots, v_{M_m+M+(M-1)}(t) \equiv 0, \end{cases} \quad (2)$$

where M is a positive integer, $M_k := (M+1)k - M$, and $u_k(t)$ and $v_k(t)$ denote the populations of the k th species at the continuous time t . Eqs. (1) and (2) describe the competition such that the k th species is predator of the $(k+1)$ th, the $(k+2)$ th, \dots , the $(k+M)$ th species and is prey of the $(k-1)$ th, the $(k-2)$ th, \dots , the $(k-M)$ th species. In the case of $M = 1$, both (1) and (2) become the original LV. As M grows larger, for the k th species, the number of species of both the preys and the predators increase. So,

(1) and (2) are called the hungry LV (hLV) of additive type and multiplicative type, respectively. Sometimes, (1) and (2) are referred to as the Bogoyavlensky lattices. The hLV (1) and (2) are also derived from a spatial discretization of the Korteweg-de Vries equation [4].

The discretized version of (1) is presented in [5, 6] as

$$\begin{cases} u_k^{(n+1)} \prod_{p=1}^M \left(1 + \delta^{(n+1)} u_{k-p}^{(n+1)} \right) = u_k^{(n)} \prod_{p=1}^M \left(1 + \delta^{(n)} u_{k+p}^{(n)} \right), \\ k = 1, 2, \dots, M_m, \quad n = 0, 1, \dots, \\ u_{1-M}^{(n)} \equiv 0, \dots, u_0^{(n)} \equiv 0, \quad u_{M_m+1}^{(n)} \equiv 0, \dots, u_{M_m+M}^{(n)} \equiv 0, \end{cases} \quad (3)$$

and that of (2) is given in [6] as

$$\begin{cases} v_k^{(n+1)} \left(1 + \delta^{(n+1)} \prod_{p=1}^M v_{k-p}^{(n+1)} \right) = v_k^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^M v_{k+p}^{(n)} \right), \\ k = 1, 2, \dots, M_m + M - 1, \quad n = 0, 1, \dots, \\ v_{1-M}^{(n)} \equiv 0, \dots, v_0^{(n)} \equiv 0, \\ v_{M_m+M}^{(n)} \equiv 0, \dots, v_{M_m+M+(M-1)}^{(n)} \equiv 0, \end{cases} \quad (4)$$

respectively. Both (3) and (4) are called the discrete hLV (dhLV). In this paper, in order to distinguish two kinds of the dhLVs, we simply refer to (3) and (4) as the dhLV_I associated with the continuous hLV of additive type (1) and the dhLV_{II} associated with the continuous hLV of multiplicative one (2), respectively. In (3) and (4), $\delta^{(n)}$ represents the step size at the discrete time n . The variables $u_k^{(n)}$ and $v_k^{(n)}$ denote the population of the k th species at the

discrete time n . The dhLV_I (3) is shown in [7] to have an application for computing complex eigenvalues of a certain band matrix.

The discrete Toda equation

$$\begin{cases} q_i^{(n+1)} + e_{i-1}^{(n+1)} = q_i^{(n)} + e_i^{(n)}, & i = 1, 2, \dots, m, \\ q_i^{(n+1)} e_i^{(n+1)} = q_{i+1}^{(n)} e_i^{(n)}, & i = 1, 2, \dots, m-1, \\ e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, & n = 0, 1, \dots, \end{cases} \quad (5)$$

is also a famous integrable system. Here, the superscript n is the time variable, as in (3) and (4), and the subscript i denotes the spatial variable. A study on box and ball system in [8] leads to an extended version of the discrete Toda equation (5),

$$\begin{cases} Q_i^{(n+M)} + E_{i-1}^{(n+1)} = Q_i^{(n)} + E_i^{(n)}, & i = 1, 2, \dots, m, \\ Q_i^{(n+M)} E_i^{(n+1)} = Q_{i+1}^{(n)} E_i^{(n)}, & i = 1, 2, \dots, m-1, \\ E_0^{(n)} \equiv 0, \quad E_m^{(n)} \equiv 0, & n = 0, 1, \dots, \end{cases} \quad (6)$$

with positive integer M , which is named *the discrete hungry Toda equation*. In this paper, for the simplicity, we call (6) *the dhToda*. In [9], a new algorithm for computing matrix eigenvalues is designed based on the dhToda (6).

Some of the authors in [10] found a relationship of dependent variables, namely, a Bäcklund transformation, between the dhLV_I (3) and the dhToda (6) through associating these integrable systems with a sequence of LR transformations of matrices. Bäcklund transformation is originally derived from the study of differential geometry. Explicit form of the Bäcklund transformation helps us to understand intrinsic features of an integrable system such as the solutions and symmetry and its relationship with another integrable system [11].

Here, for a nonsingular matrix A , the LR transformation [12] is defined as

$$A = LR, \quad \hat{A} = RL. \quad (7)$$

The 1st equation of (7) represents the LR decomposition of A where L is a lower triangular and R is a unit upper triangular matrix. It is to be noted that the LR decomposition where R has unit diagonal entries is uniquely given. The 2nd equation generates \hat{A} as the matrix product RL . Let $\hat{A} = \hat{L}\hat{R}$ be the LR decomposition of \hat{A} . From (7), we get $\hat{L}\hat{R} = RL$. This type of equation appears in the matrix representation of some discrete integrable systems, and is called the Lax representation of them. The eigenvalues of \hat{A} coincide with those of A . So, the LR transformation (7) yields a similarity transformation from A to \hat{A} , namely, $\hat{A} = RAR^{-1}$. For example, in order to compute the eigenvalues of a symmetric tridiagonal matrix, the quotient difference (qd) algorithm employs a sequence of LR transformations. It is interesting that the recursion formula of the qd algorithm is just equal to the discrete Toda equation (5).

However, there is no observation that the dhLV_{II} (4) is associated with a sequence of LR transformations. In this paper, we first associate the dhLV_{II} (4) with a sequence of LR transformations. Based on this result, we present a

Bäcklund transformation between the dhLV_{II} (4) and the dhToda (6). Additionally, a Bäcklund transformation between the dhLV_I (3) and the dhLV_{II} (4) is also presented for the case of $\delta^{(n)} \rightarrow \infty$.

With the help of the relationship among the dhLV_I (3), the dhLV_{II} (4) and the dhToda (6), we next show the asymptotic behavior of the dhLV_{II} (4) as $n \rightarrow \infty$, by using the convergence property of the dhToda (6) given in [9]. The dhLV_{II} (4) is also shown to be applicable for matrix eigenvalue computation.

This paper is organized as follows. In Section 2, we derive a system called *the qd-type dhLV_{II}* from the original dhLV_{II} (4) through variable transformation. We also show the positivity of the qd-type dhLV_{II} under suitable conditions. In Section 3, we give a Lax representation for the dhLV_{II} (4), and then relate it to the LR transformation of a band matrix. We also review the Lax representation for the dhToda (6) and the LR transformation associated with it. In Section 4, by comparing two LR transformations in Section 2, we derive a Bäcklund transformation between the dhLV_{II} (4) and the dhToda (6). By taking account of the Bäcklund transformation between the dhLV_I (3) and the dhToda (6) given in [10], we also derive a Bäcklund transformation between the dhLV_I (3) and the dhLV_{II} (4) for the case of $\delta^{(n)} \rightarrow \infty$. We investigate the asymptotic behaviour of the dhLV_{II} variables through the Bäcklund transformation between the dhLV_{II} (4) and the dhToda (6). The asymptotic behaviour of the dhToda variables is already shown in [9]. In Section 5, we give numerical examples in order to demonstrate some theorems in the previous sections. Finally, in Section 6, conclusion is presented.

2. THE QD-TYPE DHLV_{II} AND POSITIVITY OF ITS VARIABLES

In this section, we introduce the qd-type dhLV_{II} which is derived from the dhLV_{II} (4) and show the positivity of its variables. For the simplicity, we employ the notations $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ and Φ_5 defined as

$$\begin{aligned} \Phi_1 &:= \{1, 2, \dots, M_m + M - 1\}, \\ \Phi_2 &:= \{1, 2, \dots, M_{m-1} + M\}, \\ \Phi_3 &:= \{1, 2, \dots, m - 1\}, \\ \Phi_4 &:= \{1, 2, \dots, m\}, \\ \Phi_5 &:= \{0, 1, \dots, M - 1\}. \end{aligned}$$

These index sets appear throughout this paper frequently.

2.1. THE QD-TYPE DHLV_{II}

Let us introduce new variables

$$\omega_k^{(n)} := v_k^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^M v_{k-p}^{(n)} \right), \quad \forall k \in \Phi_1, \quad (8)$$

$$\gamma_k^{(n)} := \delta^{(n)} \prod_{p=0}^M v_{k+p}^{(n)}, \quad \forall k \in \Phi_2, \quad (9)$$

from the dhLV_{II} variable $v_k^{(n)}$ and the discrete step size $\delta^{(n)}$. From the boundary condition of $v_k^{(n)}$, we have

$$\omega_k^{(n)} = v_k^{(n)}, \quad \forall k \in \Phi_5 \cup \{M\} \setminus \{0\}, \quad (10)$$

$$\gamma_{-k}^{(n)} = 0, \quad \forall k \in \Phi_5, \quad (11)$$

$$\gamma_{M_m+j}^{(n)} = 0, \quad \forall j \in \Phi_5. \quad (12)$$

Then these variables satisfy the recursion formula

$$\begin{cases} \omega_k^{(n+1)} + \gamma_{k-M}^{(n)} = \omega_k^{(n)} + \gamma_k^{(n)}, & \forall k \in \Phi_1, \\ \omega_k^{(n+1)} \gamma_{k+1}^{(n)} = \omega_{k+M+1}^{(n)} \gamma_k^{(n)}, & \forall k \in \Phi_2 \setminus \{M_{m-1} + M\}. \end{cases} \quad (13)$$

This is easily checked as follows.

$$\begin{aligned} & \omega_k^{(n+1)} + \gamma_{k-M}^{(n)} \\ &= v_k^{(n+1)} \left(1 + \delta^{(n+1)} \prod_{p=1}^M v_{k-p}^{(n+1)} \right) + \delta^{(n)} \prod_{p=0}^M v_{k-M+p}^{(n)} \\ &= v_k^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^M v_{k+p}^{(n)} \right) + \delta^{(n)} \prod_{p=0}^M v_{k-p}^{(n)} \\ &= v_k^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^M v_{k-p}^{(n)} \right) + \delta^{(n)} \prod_{p=0}^M v_{k+p}^{(n)} \\ &= \omega_k^{(n)} + \gamma_k^{(n)}, \end{aligned}$$

$$\begin{aligned} & \omega_k^{(n+1)} \gamma_{k+1}^{(n)} \\ &= \left[v_k^{(n+1)} \left(1 + \delta^{(n+1)} \prod_{p=1}^M v_{k-p}^{(n+1)} \right) \right] \left[\delta^{(n)} \prod_{p=0}^M v_{k+p+1}^{(n)} \right] \\ &= \left[v_k^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^M v_{k+p}^{(n)} \right) \right] \left[\delta^{(n)} \prod_{p=0}^M v_{k+p+1}^{(n)} \right] \\ &= v_{k+M+1}^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^M v_{k+p}^{(n)} \right) \left(\delta^{(n)} \prod_{p=0}^M v_{k+p}^{(n)} \right) \\ &= v_{k+M+1}^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^M v_{k+M+1-p}^{(n)} \right) \left(\delta^{(n)} \prod_{p=0}^M v_{k+p}^{(n)} \right) \\ &= \omega_{k+M+1}^{(n)} \gamma_k^{(n)}. \end{aligned}$$

Eq. (13) has the form similar to the recursion formula of the qd algorithm (5). In order to distinguish (13) from the dhLV_{II} (4), we hereinafter call (13) *the qd-type dhLV_{II}*. Also, we can rewrite the qd-type dhLV_{II} as

$$\begin{cases} \omega_k^{(n+1)} = \omega_k^{(n)} + \gamma_k^{(n)} - \gamma_{k-M}^{(n)}, \\ \gamma_{k+1}^{(n)} = \frac{\omega_{k+M+1}^{(n)} \gamma_k^{(n)}}{\omega_k^{(n+1)}}. \end{cases} \quad (14)$$

If $\omega_k^{(n)}$ for $\forall k \in \Phi_1$ and $\gamma_1^{(n)}$ are given, we can obtain $\omega_k^{(n+1)}$ for $\forall k \in \Phi_1$ and $\gamma_k^{(n)}$ for $\forall k \in \Phi_2 \setminus \{1\}$ by using (14). Let us assume that $\omega_k^{(n)} > 0$ for $\forall k \in \Phi_1$ and $\gamma_1^{(n)} > 0$. Then

we can relate $\gamma_1^{(n)}$ to $\delta^{(n)}$ as follows. From (9) and (10), we derive

$$\begin{aligned} \gamma_1^{(n)} &= \delta^{(n)} \prod_{p=1}^{M+1} v_p^{(n)} \\ &= \delta^{(n)} v_{M+1}^{(n)} \prod_{p=1}^M \omega_p^{(n)} \\ &= \frac{\delta^{(n)} \omega_{M+1}^{(n)} \prod_{p=1}^M \omega_p^{(n)}}{1 + \delta^{(n)} \prod_{p=1}^M \omega_p^{(n)}} \\ &= \frac{\prod_{p=1}^{M+1} \omega_p^{(n)}}{\frac{1}{\delta^{(n)}} + \prod_{p=1}^M \omega_p^{(n)}}. \end{aligned} \quad (15)$$

From (15), it holds that

$$\delta^{(n)} = \frac{\gamma_1^{(n)}}{(\omega_{M+1}^{(n)} - \gamma_1^{(n)}) \prod_{p=1}^M \omega_p^{(n)}}.$$

Hence, the condition $\delta^{(n)} > 0$ is equivalent to

$$0 < \gamma_1^{(n)} < \omega_{M+1}^{(n)}.$$

2.2. POSITIVITY OF THE QD-TYPE dhLV_{II} VARIABLES

We give a theorem concerning the positivity of the qd-type dhLV_{II} variables $\omega_k^{(n)}$ and $\gamma_k^{(n)}$.

Theorem 1. *Let us assume that $\omega_k^{(n)} > 0$, $\forall k \in \Phi_1$ and $0 < \gamma_1^{(n)} < \omega_{M+1}^{(n)}$, then it holds that*

$$\begin{aligned} \omega_k^{(n+1)} &> 0, \quad \forall k \in \Phi_1, \\ \gamma_k^{(n)} &> 0, \quad \forall k \in \Phi_2. \end{aligned}$$

Proof. In the discussion for the positivity of the qd-type dhLV_{II} variables, it is useful to introduce an auxiliary variable $d_k^{(n)}$ defined by

$$d_k^{(n)} = \omega_k^{(n)} - \gamma_{k-M}^{(n)}, \quad \forall k \in \Phi_1. \quad (16)$$

From (11) and (16), it follows that

$$\begin{cases} d_k^{(n)} = \omega_k^{(n)}, & \forall k \in \Phi_5 \cup \{M\} \setminus \{0\}, \\ d_{M+1}^{(n)} = \omega_{M+1}^{(n)} - \gamma_1^{(n)}. \end{cases} \quad (17)$$

By combining (17) with the assumption, we have

$$d_k^{(n)} > 0, \quad \forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}.$$

with $\underline{d}_k = \bar{d}_k = \omega_k^{(n)}$. Similarly, from (17), we get

$$d_{M+1}^{(n)} \leq \bar{d}_{M+1}, \quad (27)$$

with $\bar{d}_{M+1} = \omega_{M+1}^{(n)}$. Obviously, the assumption leads to

$$\underline{\gamma}_1 \leq \gamma_1^{(n)} \leq \bar{\gamma}_1, \quad (28)$$

with $\underline{\gamma}_1 = c$ and $\bar{\gamma}_1 = \omega_{M+1}^{(n)}$. By combining (26)–(28) with (20), we can prove the following inequalities by induction.

$$\begin{cases} \underline{\gamma}_k \leq \gamma_k^{(n)} \leq \bar{\gamma}_k, & \forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0, 1\}, \\ \underline{\omega}_k \leq \omega_k^{(n+1)} \leq \bar{\omega}_k, & \forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}. \end{cases} \quad (29)$$

In the case where $\forall k \in \{i+M+1\}, \forall i \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}$, from the 3rd equation of (20), (26), (27) and the 2nd equation of (29), it holds that

$$\underline{d}_k \leq d_k^{(n)} \leq \bar{d}_k, \quad \forall k \in \{i+M+1\}, \quad \forall i \in \Phi_5 \cup \{M\} \setminus \{0\}, \quad (30)$$

$$d_{2(M+1)}^{(n)} \leq \bar{d}_{2(M+1)}. \quad (31)$$

Moreover, from (29), it follows that

$$\underline{\gamma}_{M+2} \leq \gamma_{M+2}^{(n)} \leq \bar{\gamma}_{M+2}. \quad (32)$$

Eqs. (30)–(32) lead to

$$\begin{cases} \underline{\gamma}_k \leq \gamma_k^{(n)} \leq \bar{\gamma}_k, \\ \forall k \in \{i+M+1\}, \quad \forall i \in \Phi_5 \cup \{M, M+1\} \setminus \{0, 1\}, \\ \underline{\omega}_k \leq \omega_k^{(n+1)} \leq \bar{\omega}_k, \\ \forall k \in \{i+M+1\}, \quad \forall i \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}. \end{cases}$$

Similarly, for $\forall k \in \{(i-1)(M+1) + j + 1\}, \forall i \in \Phi_3 \setminus \{1, 2\}, \forall j \in \Phi_5$, it follows that

$$\begin{cases} \underline{d}_k \leq d_k^{(n)} \leq \bar{d}_k, \\ \underline{\gamma}_k \leq \gamma_k^{(n)} \leq \bar{\gamma}_k, \\ \underline{\omega}_k \leq \omega_k^{(n+1)} \leq \bar{\omega}_k, \end{cases} \quad (33)$$

and for $\forall k \in \{i(M+1)\}, \forall i \in \Phi_3 \setminus \{1, 2\}$, we have

$$d_k^{(n)} \leq \bar{d}_k.$$

We next consider the case where $\forall k \in \{(m-1)(M+1) + i + 1\}, \forall i \in \Phi_5$. By combining the 3rd equation of (20), (33) with (12), we have

$$\underline{d}_{k+i} \leq d_{k+i}^{(n)} \leq \bar{d}_{k+i}, \quad \forall i \in \Phi_5. \quad (34)$$

From (34) and the 1st equation of (20), we have

$$\underline{\omega}_{k+i} \leq \omega_{k+i}^{(n+1)} \leq \bar{\omega}_{k+i}, \quad \forall i \in \Phi_5.$$

To sum up, we obtain (25).

By using (25), we discuss the behavior of variables in (13) as $\delta^{(n)} \rightarrow \infty$. We first consider the case of $k = M_i + M, \forall i \in \Phi_3$. By using (19) repeatedly, we derive

$$d_{M_i+M}^{(n)} = \frac{\prod_{p=2}^i \omega_{M_p+M}^{(n)}}{\prod_{p=1}^{i-1} \omega_{M_p+M}^{(n+1)}} d_{M+1}^{(n)}, \quad \forall i \in \Phi_3 \setminus \{1\}. \quad (35)$$

Eqs. (15) and (17) lead to

$$\begin{aligned} d_{M+1}^{(n)} &= \omega_{M+1}^{(n)} - \gamma_1^{(n)} \\ &= \omega_{M+1}^{(n)} - \frac{\prod_{p=1}^{M+1} \omega_p^{(n)}}{\frac{1}{\delta^{(n)}} + \prod_{p=1}^M \omega_p^{(n)}} \\ &= \frac{\omega_{M+1}^{(n)}}{\delta^{(n)} \prod_{p=1}^M \omega_p^{(n)} + 1}. \end{aligned} \quad (36)$$

As $\delta^{(n)} \rightarrow \infty$, from (25), it follows that $d_{M+1}^{(n)} \rightarrow 0$ in (36). From (35), we derive $d_{M_i+M}^{(n)} \rightarrow 0$ for $\forall i \in \Phi_3 \setminus \{1\}$. By combining them with (16), we get

$$\lim_{\delta^{(n)} \rightarrow \infty} (\omega_{M_i+M}^{(n)} - \gamma_{M_i}^{(n)}) = 0, \quad \forall i \in \Phi_3. \quad (37)$$

Moreover, from (18) and $d_{M_i+M}^{(n)} \rightarrow 0$ for $\forall i \in \Phi_3$, we have

$$\lim_{\delta^{(n)} \rightarrow \infty} (\omega_{M_i+M}^{(n+1)} - \gamma_{M_i+M}^{(n)}) = 0, \quad \forall i \in \Phi_3. \quad (38)$$

Thus, as $\delta^{(n)} \rightarrow \infty$, (13) becomes the trivial equalities $\omega_{M_i+M}^{(n+1)} + \gamma_{M_i}^{(n)} = \omega_{M_i+M}^{(n)} + \gamma_{M_i+M}^{(n)}, \forall i \in \Phi_3$ and $\omega_{M_i+M}^{(n+1)} \gamma_{M_i+1}^{(n)} = \omega_{M_i+1+M}^{(n)} \gamma_{M_i+M}^{(n)}, \forall i \in \Phi_3 \setminus \{m-1\}$.

Next, we consider the cases except for $k = M_i + M, \forall i \in \Phi_3$ in the 1st and 2nd equations of (13). We here focus on the product of $\mathcal{L}_{j+1}^{(n+1)}$ and $\mathcal{R}_{j+1}^{(n)}$. The (i, i) and $(i, i+1)$ entries of $\mathcal{L}_{j+1}^{(n+1)} \mathcal{R}_{j+1}^{(n)}$ are given as, respectively,

$$(\mathcal{L}_{j+1}^{(n+1)} \mathcal{R}_{j+1}^{(n)})_{i,i} = \omega_{M_i+j}^{(n+1)} + \gamma_{M_i-M+j}^{(n)}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \quad (39)$$

$$(\mathcal{L}_{j+1}^{(n+1)} \mathcal{R}_{j+1}^{(n)})_{i,i+1} = \omega_{M_i+j}^{(n+1)} \gamma_{M_i+j+1}^{(n)}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5. \quad (40)$$

Similarly, it follows that

$$(\mathcal{R}_j^{(n)} \mathcal{L}_{j+1}^{(n)})_{i,i} = \omega_{M_i+j}^{(n)} + \gamma_{M_i+j}^{(n)}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \quad (41)$$

$$(\mathcal{R}_j^{(n)} \mathcal{L}_{j+1}^{(n)})_{i,i+1} = \omega_{M_i+M+j+1}^{(n)} \gamma_{M_i+j}^{(n)}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5. \quad (42)$$

Eqs. (22), (39) and (41) bring to the 1st equation of (13). Also, (22), (40) and (42) lead to the 2nd one. This indicates that (22) is a Lax representation for the qd-type dhLV_{II} (13). \square

Moreover, we give a lemma concerning the relationship of the Lax matrices $\mathcal{R}_0^{(n+1)}$ and $\mathcal{R}_M^{(n)}$ as $\delta^{(n)} \rightarrow \infty$.

Lemma 1. *As $\delta^{(n)} \rightarrow \infty$, it holds that*

$$\mathcal{R}_0^{(n+1)} = \mathcal{R}_M^{(n)}. \tag{43}$$

Proof. Obviously, from (37) and (38), $\omega_{M_i+M}^{(n+1)} \rightarrow \gamma_{M_i}^{(n+1)}$ and $\omega_{M_i+M}^{(n+1)} \rightarrow \gamma_{M_i+M}^{(n)}$ as $\delta^{(n)} \rightarrow \infty$. So, it holds that $\gamma_{M_i}^{(n+1)} \rightarrow \gamma_{M_i+M}^{(n)}$ as $\delta^{(n)} \rightarrow \infty$. This leads to (43). \square

Let us introduce the matrix, given by the product of the Lax matrices $\mathcal{L}_1^{(n)}, \mathcal{L}_2^{(n)}, \dots, \mathcal{L}_M^{(n)}$ in (23) and $\mathcal{R}_0^{(n)}$ in (24),

$$\mathcal{A}^{(n)} = \mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \dots \mathcal{L}_M^{(n)} \mathcal{R}_0^{(n)}. \tag{44}$$

Let us consider $\mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1}$ as a similarity transformation of $\mathcal{A}^{(n)}$ by $\mathcal{R}_0^{(n)}$. Then, with the help of Theorem 2, we derive

$$\begin{aligned} \mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1} &= \mathcal{R}_0^{(n)} \mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \dots \mathcal{L}_M^{(n)} \\ &= \mathcal{L}_1^{(n+1)} \mathcal{R}_1^{(n)} \mathcal{L}_2^{(n)} \mathcal{L}_3^{(n)} \dots \mathcal{L}_M^{(n)} \\ &= \mathcal{L}_1^{(n+1)} \mathcal{L}_2^{(n+1)} \mathcal{R}_2^{(n)} \mathcal{L}_3^{(n)} \dots \mathcal{L}_M^{(n)} \\ &\quad \vdots \\ &= \mathcal{L}_1^{(n+1)} \mathcal{L}_2^{(n+1)} \dots \mathcal{L}_{M-1}^{(n+1)} \mathcal{R}_{M-1}^{(n)} \mathcal{L}_M^{(n)} \\ &= \mathcal{L}_1^{(n+1)} \mathcal{L}_2^{(n+1)} \dots \mathcal{L}_{M-1}^{(n+1)} \mathcal{L}_M^{(n+1)} \mathcal{R}_M^{(n)}. \end{aligned} \tag{45}$$

By combining it with Lemma 1, we see that

$$\mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1} = \mathcal{A}^{(n+1)}. \tag{46}$$

This means that the eigenvalues of $\mathcal{A}^{(n)}$ are invariant under the time evolution from n to $n+1$. Eqs. (45) and (46) also lead to

$$\begin{cases} \mathcal{A}^{(n)} = (\mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \dots \mathcal{L}_M^{(n)}) \mathcal{R}_0^{(n)}, \\ \mathcal{A}^{(n+1)} = \mathcal{R}_0^{(n)} (\mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \dots \mathcal{L}_M^{(n)}). \end{cases} \tag{47}$$

Hence, we know that $\mathcal{A}^{(n+1)}$ is given through the LR transformation of $\mathcal{A}^{(n)}$. According to (45), the LR transformation in (47) coincides with M times LR transformations in (22). Let us recall here that (22) is a Lax representation for the dhLV_{II} (4) with $\delta^{(n)} \rightarrow \infty$. We therefore have the following theorem.

Theorem 3. *The dhLV_{II} (4) with $\delta^{(n)} \rightarrow \infty$ generates the LR transformation from $\mathcal{A}^{(n)}$ to $\mathcal{A}^{(n+1)}$ as in (47).*

According to [8], the dhToda (6) satisfies the Lax representation,

$$L^{(n+M)} R^{(n+1)} = R^{(n)} L^{(n)}, \tag{48}$$

$$L^{(n)} = \begin{pmatrix} Q_1^{(n)} & & & & \\ 1 & Q_2^{(n)} & & & \\ & \ddots & \ddots & & \\ & & & 1 & Q_m^{(n)} \end{pmatrix}, \tag{49}$$

$$R^{(n)} = \begin{pmatrix} 1 & E_1^{(n)} & & & \\ & 1 & \ddots & & \\ & & \ddots & E_{m-1}^{(n)} & \\ & & & & 1 \end{pmatrix}, \tag{50}$$

where $Q_i^{(n)} > 0, \forall i \in \Phi_4$ and $E_i^{(n)} > 0, \forall i \in \Phi_3$. The Lax representation (48) may look different from that in [8]. Actually, we can easily get the same Lax representation as in [8] through matrix transposition on both sides of (48).

Let $A^{(n)}$ be the product of the Lax matrices $L^{(n)}, L^{(n+1)}, \dots, L^{(n+M-1)}$ in (49) and $R^{(n)}$ in (50), namely,

$$A^{(n)} = L^{(n)} L^{(n+1)} \dots L^{(n+M-1)} R^{(n)}. \tag{51}$$

Then, from (48), it follows that

$$\begin{aligned} R^{(n)} A^{(n)} (R^{(n)})^{-1} &= R^{(n)} L^{(n)} L^{(n+1)} \dots L^{(n+M-1)} \\ &= L^{(n+M)} R^{(n+1)} L^{(n+1)} L^{(n+2)} \dots L^{(n+M-1)} \\ &= L^{(n+M)} L^{(n+M+1)} R^{(n+2)} L^{(n+2)} \dots L^{(n+M-1)} \\ &\quad \vdots \\ &= L^{(n+M)} L^{(n+M+1)} \dots L^{(n+2M-1)} R^{(n+M)} \\ &= A^{(n+M)}. \end{aligned} \tag{52}$$

Obviously, from (52), the dhToda (6) gives the similarity transformation from $A^{(n)}$ to $A^{(n+M)}$. Eq. (52) is also rewritten as

$$\begin{cases} A^{(n)} = (L^{(n)} L^{(n+1)} \dots L^{(n+M-1)}) R^{(n)}, \\ A^{(n+M)} = R^{(n)} (L^{(n)} L^{(n+1)} \dots L^{(n+M-1)}). \end{cases} \tag{53}$$

Thus, the dhToda (6) has a relationship with the LR transformation as follows.

Theorem 4 ([10]). *The dhToda (6) generates the LR transformation from $A^{(n)}$ to $A^{(n+M)}$ as in (53).*

4. BÄCKLUND TRANSFORMATIONS AMONG THE DISCRETE HUNGRY SYSTEMS

In this section, by considering the relationship between the two LR transformations associated with the dhLV_{II} (4) and the dhToda (6), we give a Bäcklund transformation between the dhLV_{II} (4) and the dhToda (6). By referring to [10], we establish a Bäcklund transformation between the dhLV_I (3) and the dhLV_{II} (4). We also investigate the asymptotic behavior of the qd-type dhLV_{II} (13) with the help of the obtained Bäcklund transformation.

4.1. THE BÄCKLUND TRANSFORMATION BETWEEN THE dhLV_{II} AND THE dhTODA

We first show the relationship of the matrices in two LR transformations associated with the dhLV_{II} (4) and the dhToda (6).

Lemma 2. *For some fixed n , let $\mathcal{R}_0^{(n)} = R^{(n)}$ and $\mathcal{L}_{j+1}^{(n)} = L^{(n+j)}$, $\forall j \in \Phi_5$. Then, it holds that*

$$\mathcal{L}_{j+1}^{(n+1)} = L^{(n+M+j)}, \quad \mathcal{R}_{j+1}^{(n)} = R^{(n+j+1)}, \quad \forall j \in \Phi_5.$$

Proof. The assumption leads to $\mathcal{R}_0^{(n)} \mathcal{L}_1^{(n)} = R^{(n)} L^{(n)}$. Let us recall that $\mathcal{R}_0^{(n)} \mathcal{L}_1^{(n)} = \mathcal{L}_1^{(n+1)} \mathcal{R}_1^{(n)}$ in (22) and $R^{(n)} L^{(n)} = L^{(n+M)} R^{(n+1)}$ in (48). So, it follows that

$$\mathcal{L}_1^{(n+1)} \mathcal{R}_1^{(n)} = L^{(n+M)} R^{(n+1)}.$$

Recall that the upper bidiagonal matrices $\mathcal{R}_1^{(n)}$ and $R^{(n+1)}$ have 1 in every diagonal entry. Hence, by taking account of the uniqueness of LR decomposition, we get

$$\mathcal{L}_1^{(n+1)} = L^{(n+M)}, \quad \mathcal{R}_1^{(n)} = R^{(n+1)}.$$

Similarly, it is easily proved by induction for $j = 1, 2, \dots, M-1$ that $\mathcal{L}_{j+1}^{(n+1)} = L^{(n+M+j)}$ and $\mathcal{R}_{j+1}^{(n)} = R^{(n+j+1)}$. \square

From (44) and (51), it is obvious that $\mathcal{A}^{(n)} = A^{(n)}$ if $\mathcal{L}_{j+1}^{(n)} = L^{(n+j)}$, $\forall j \in \Phi_5$ and $\mathcal{R}_0^{(n)} = R^{(n)}$. So, by using Lemma 2, we see that $\mathcal{A}^{(n+1)} = A^{(n+M)}$ since $\mathcal{R}_{j+1}^{(n)} = R^{(n+j+1)}$, $\forall j \in \Phi_5$ and $\mathcal{L}_{j+1}^{(n+1)} = L^{(n+M+j)}$, $\forall j \in \Phi_5$. In other words, the evolution from n to $n+1$ of the dhLV_{II} (4) can generate the LR transformation given by the evolution from n to $n+M$ of the dhToda (6).

Let us replace n with $\ell M + j$ in the superscripts of the dhLV_{II} and the dhToda variables. Hereinafter, we consider the evolution from ℓ to $\ell+1$ by the dhLV_{II} (4) and the dhToda (6). Let us assume that, for some fixed ℓ ,

$$\begin{cases} \mathcal{L}_{j+1}^{(\ell)} = L^{(\ell M+j)}, & \forall j \in \Phi_5, \\ \mathcal{R}_0^{(\ell)} = R^{(\ell M)}. \end{cases} \quad (54)$$

Then, from Lemma 2, it follows that,

$$\mathcal{L}_{j+1}^{(\ell+1)} = L^{((\ell+1)M+j)}, \quad \mathcal{R}_{j+1}^{(\ell)} = R^{(\ell M+j+1)}, \quad \forall j \in \Phi_5. \quad (55)$$

From the 2nd equation of (55) and Lemma 1, it holds that $\mathcal{R}_0^{(\ell+1)} = R^{((\ell+1)M)}$. By focusing on the entries of matrices in (54), we derive

$$\begin{cases} E_i^{(\ell M)} = \gamma_{M_i}^{(\ell)}, & \forall i \in \Phi_3, \\ Q_i^{(\ell M+j)} = \omega_{M_i+j}^{(\ell)}, & \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \end{cases}$$

for $\ell = 0, 1, \dots$. Moreover, from (55), we obtain

$$E_i^{(\ell M+j+1)} = \gamma_{M_i+j+1}^{(\ell)}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5,$$

for $\ell = 0, 1, \dots$. To sum up, we derive a theorem on the relationship of the variables, namely, the Bäcklund transformation, between the qd-type dhLV_{II} (13) and the dhToda (6).

Theorem 5. *A Bäcklund transformation between the qd-type dhLV_{II} (13) with $\delta^{(\ell)} \rightarrow \infty$ and the dhToda (6) is given by*

$$\begin{cases} E_i^{(\ell M+j)} = \gamma_{M_i+j}^{(\ell)}, & \forall i \in \Phi_3, \quad \forall j \in \Phi_5, \\ Q_i^{(\ell M+j)} = \omega_{M_i+j}^{(\ell)}, & \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \end{cases}$$

for $\ell = 0, 1, \dots$.

It is observed that (8) and (9) are the Bäcklund transformation between the qd-type dhLV_{II} (13) and the original dhLV_{II} (4). So, by combining it with Theorem 5, we have a main theorem in this paper.

Theorem 6. *A Bäcklund transformation between the dhLV_{II} (4) with $\delta^{(\ell)} \rightarrow \infty$ and the dhToda (6) is given by*

$$\begin{cases} E_i^{(\ell M+j)} = \delta^{(\ell)} \prod_{p=0}^M v_{M_i+j+p}^{(\ell)}, & \forall i \in \Phi_3, \quad \forall j \in \Phi_5, \\ Q_i^{(\ell M+j)} = v_{M_i+j}^{(\ell)} \left(1 + \delta^{(\ell)} \prod_{p=1}^M v_{M_i+j-p}^{(\ell)} \right), \\ \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \end{cases}$$

for $\ell = 0, 1, \dots$.

4.2. THE BÄCKLUND TRANSFORMATION BETWEEN THE dhLV_I AND THE dhLV_{II}

Let us introduce the new variables

$$\begin{cases} U_k^{(n)} = u_k^{(n)} \prod_{p=1}^M \left(1 + \delta^{(n)} u_{k-p}^{(n)} \right), & \forall k \in \Phi_2 \cup \{M_m\}, \\ V_k^{(n)} = \frac{1}{\delta^{(n)}} \prod_{p=0}^M \left(1 + \delta^{(n)} u_{k-p}^{(n)} \right), & \forall k \in \Phi_1 \cup \{M_m + M\}, \end{cases} \quad (56)$$

in terms of the dhLV_I variable $u_k^{(n)}$. Then the dhLV_I (3) can be rewritten as

$$\begin{cases} U_k^{(n+1)} + V_{M+k+1}^{(n)} = U_{M+k+1}^{(n)} + V_{M+k}^{(n)}, & \forall k \in \Phi_2, \\ U_k^{(n+1)} V_k^{(n)} = U_k^{(n)} V_{M+k}^{(n)}, & \forall k \in \Phi_2 \cup \{M_m\}, \\ U_{M_m+j+1}^{(n)} := 0, \quad V_{M_m+M+j+1}^{(n)} := \frac{1}{\delta^{(n)}}, & \forall j \in \Phi_5. \end{cases} \quad (57)$$

Eq. (57) is named the qd-type dhLV_I in [13]. Eq. (56) is a Bäcklund transformation between the original dhLV_I (3) and the qd-type dhLV_I (57). Some of the authors, in [10], give a Bäcklund transformation between the qd-type

dhLV_I (57) with $\delta^{(n)} \rightarrow \infty$ and the dhToda (6) as follows.

$$\begin{cases} U_{M_i}^{(\ell)} = \prod_{p=0}^{M-1} Q_i^{(\ell M+p)}, & \forall i \in \Phi_4, \\ U_{M_i+j+1}^{(\ell)} = \left(\prod_{p=0}^{M-2} Q_i^{(\ell M+p+j+1)} \right) E_i^{(\ell M+j)}, \\ \forall i \in \Phi_3, \quad \forall j \in \Phi_5. \end{cases} \quad (58)$$

By combining the 1st equation of (58) with Theorem 5, for $\ell = 0, 1, \dots$, we get

$$U_{M_i}^{(\ell)} = \prod_{p=0}^{M-1} \omega_{M_i+p}^{(\ell)}, \quad \forall i \in \Phi_4. \quad (59)$$

Similarly, from Theorem 5 and the 2nd of (58), we derive

$$\begin{aligned} & U_{M_i+j+1}^{(\ell)} \\ &= Q_i^{(\ell M+j+1)} Q_i^{(\ell M+j+2)} \dots Q_i^{(\ell M+M-1)} \\ &\quad \times Q_i^{((\ell+1)M)} Q_i^{((\ell+1)M+1)} Q_i^{((\ell+1)M+2)} \dots Q_i^{((\ell+1)M+j-2)} \\ &\quad \times \left(Q_i^{((\ell+1)M+j-1)} E_i^{(\ell M+j)} \right) \\ &= \omega_{M_i+j+1}^{(\ell)} \omega_{M_i+j+2}^{(\ell)} \dots \omega_{M_i+M-1}^{(\ell)} \\ &\quad \times \omega_{M_i}^{(\ell+1)} \omega_{M_i+1}^{(\ell+1)} \omega_{M_i+2}^{(\ell+1)} \dots \omega_{M_i+j-2}^{(\ell+1)} \\ &\quad \times \left(\omega_{M_i+j-1}^{(\ell+1)} \gamma_{M_i+j}^{(\ell)} \right). \end{aligned}$$

By taking account of the 2nd equation of (13), we successively rewrite $U_{M_i+j+1}^{(\ell)}$ as

$$\begin{aligned} U_{M_i+j+1}^{(\ell)} &= \omega_{M_i+j+1}^{(\ell)} \omega_{M_i+j+2}^{(\ell)} \dots \omega_{M_i+M-1}^{(\ell)} \\ &\quad \times \omega_{M_i}^{(\ell+1)} \omega_{M_i+1}^{(\ell+1)} \omega_{M_i+2}^{(\ell+1)} \dots \omega_{M_i+j-2}^{(\ell+1)} \\ &\quad \times \left(\gamma_{M_i+j-1}^{(\ell)} \omega_{M_i+M+j}^{(\ell)} \right) \\ &\quad \vdots \\ &= \omega_{M_i+j+1}^{(\ell)} \omega_{M_i+j+2}^{(\ell)} \dots \omega_{M_i+M-1}^{(\ell)} \\ &\quad \times \left(\omega_{M_i}^{(\ell+1)} \gamma_{M_i+1}^{(\ell)} \right) \omega_{M_i+M+2}^{(\ell)} \dots \omega_{M_i+M+j-2}^{(\ell)} \\ &\quad \times \omega_{M_i+M+j-1}^{(\ell)} \omega_{M_i+M+j}^{(\ell)} \\ &= \omega_{M_i+j+1}^{(\ell)} \omega_{M_i+j+2}^{(\ell)} \dots \omega_{M_i+M-1}^{(\ell)} \\ &\quad \times \left(\gamma_{M_i}^{(\ell)} \omega_{M_i+M+1}^{(\ell)} \right) \omega_{M_i+M+2}^{(\ell)} \dots \omega_{M_i+M+j-2}^{(\ell)} \\ &\quad \times \omega_{M_i+M+j-1}^{(\ell)} \omega_{M_i+M+j}^{(\ell)}. \end{aligned}$$

Note here that $\gamma_{M_i}^{(\ell)} \rightarrow \omega_{M_i+M}^{(\ell)}$ as $\delta^{(n)} \rightarrow \infty$. So, it follows that

$$U_{M_i+j+1}^{(\ell)} = \prod_{p=0}^{M-1} \omega_{M_i+p+j+1}^{(\ell)}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5. \quad (60)$$

From (59) and (60), we have

$$U_k^{(n)} = \prod_{p=0}^{M-1} \omega_{k+p}^{(n)}, \quad \forall k \in \Phi_2 \cup \{M_m\}. \quad (61)$$

By combining (61) with (8) and (56), it follows that

$$\begin{aligned} & u_k^{(n)} \prod_{p=1}^M (1 + \delta^{(n)} u_{k-p}^{(n)}) \\ &= \prod_{p=0}^{M-1} \left(v_{k+p}^{(n)} \left(1 + \delta^{(n)} \prod_{r=1}^M v_{k+p-r}^{(n)} \right) \right) \\ &= \left(\prod_{p=0}^{M-1} v_{k+p}^{(n)} \right) \prod_{p=0}^{M-1} \left(1 + \delta^{(n)} \prod_{r=1}^M v_{k+p-r}^{(n)} \right) \\ &= \left(\prod_{p=0}^{M-1} v_{k+p}^{(n)} \right) \prod_{p=1}^M \left(1 + \delta^{(n)} \prod_{r=0}^{M-1} v_{k+r-p}^{(n)} \right). \end{aligned} \quad (62)$$

From the boundary condition of $u_k^{(n)}$ and $v_k^{(n)}$, the case where $k = 1$ in (62) leads to

$$u_1^{(n)} = \prod_{p=0}^{M-1} v_{1+p}^{(n)}.$$

Similarly, by considering the cases where $k = 2, 3, \dots, M_m$, we have $u_k^{(n)} = \prod_{p=0}^{M-1} v_{k+p}^{(n)}$.

The above discussion leads to the following theorem.

Theorem 7. As $\delta^{(n)} \rightarrow \infty$, a Bäcklund transformation between the dhLV_I (3) and the dhLV_{II} (4) is given by

$$u_k^{(n)} = \prod_{p=0}^{M-1} v_{k+p}^{(n)}, \quad \forall k \in \Phi_2 \cup \{M_m\},$$

for $n = 0, 1, \dots$.

4.3. THE ASYMPTOTIC BEHAVIOR OF THE QD-TYPE DHLV_{II} VARIABLES

We next clarify the asymptotic behavior of the qd-type dhLV_{II} variables by combining Theorem 5 with the asymptotic behavior of the dhToda variables given in [9]. Let us again replace n with $\ell M + j$, $\forall j \in \Phi_5$ in the superscript of the dhToda variable. Of course, the limit of $\ell \rightarrow \infty$ is equivalent to that of $n \rightarrow \infty$. A minor change of the limit in [9] brings to the following theorem with respect to the convergence of the dhToda variables.

Theorem 8 (cf.[9]). Let $Q_i^{(0)} > 0, Q_i^{(1)} > 0, \dots, Q_i^{(M-1)} > 0, \forall i \in \Phi_4$ and $E_i^{(0)} > 0, \forall i \in \Phi_3$. As $\ell \rightarrow \infty$, the limits of $Q_i^{(\ell M+j)}$ and $E_i^{(\ell M+j)}$ are given by

$$\begin{aligned} & \lim_{\ell \rightarrow \infty} \prod_{p=0}^{M-1} Q_i^{(\ell M+p)} = C_i, \quad \forall i \in \Phi_4, \\ & \lim_{\ell \rightarrow \infty} E_i^{(\ell M+j)} = 0, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5, \end{aligned} \quad (63)$$

where C_i is some constant and $C_1 \geq C_2 \geq \dots \geq C_m > 0$.

In [9], it is also shown that $\{Q_i^{(\ell M+j)}\}_{\ell=0,1,\dots}$ is a Cauchy sequence. This implies that $Q_i^{(\ell M+j)}$ converges to some constant $C_{i,j} > 0$ as $\ell \rightarrow \infty$, namely,

$$\lim_{\ell \rightarrow \infty} Q_i^{(\ell M+j)} = C_{i,j}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5. \quad (64)$$

Since it is shown in Theorem 5 that $Q_i^{(\ell M+j)} = \omega_{M_i+j}^{(\ell)}$, by using it in (64), we get

$$\lim_{\ell \rightarrow \infty} \omega_{M_i+j}^{(\ell)} = C_{i,j}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5. \quad (65)$$

By taking account that $E_i^{(\ell M+j)} = \gamma_{M_i+j}^{(\ell)}$ shown in Theorem 5, from (63), we derive

$$\lim_{\ell \rightarrow \infty} \gamma_{M_i+j}^{(\ell)} = 0, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5. \quad (66)$$

It is remarkable that (65) and (66) show the convergence of $\omega_k^{(n)}$ and $\gamma_k^{(n)}$ except for $k = M_i + M$, $\forall i \in \Phi_3$ as $\ell \rightarrow \infty$. We next study the convergence of $\omega_{M_i+M}^{(\ell)}$ and $\gamma_{M_i+M}^{(\ell)}$. Eqs. (37) and (66) with $j = 0$ leads to

$$\lim_{\ell \rightarrow \infty} \omega_{M_i+M}^{(\ell)} = 0, \quad \forall i \in \Phi_3. \quad (67)$$

From (38) and (67), it follows that

$$\lim_{\ell \rightarrow \infty} \gamma_{M_i+M}^{(\ell)} = 0, \quad \forall i \in \Phi_3.$$

We summarize the asymptotic behavior of the qd-type dhLV_{II} variables as follows.

Theorem 9. *Let us assume that $\omega_k^{(0)} > 0$, $\forall k \in \Phi_1$. Then, the limits of the qd-type dhLV_{II} variables as $\delta^{(n)} \rightarrow \infty$ are*

$$\begin{aligned} \lim_{n \rightarrow \infty} \omega_{M_i+j}^{(n)} &= C_{i,j}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \\ \lim_{n \rightarrow \infty} \omega_{M_i+M}^{(n)} &= 0, \quad \forall i \in \Phi_3, \\ \lim_{n \rightarrow \infty} \gamma_k^{(n)} &= 0, \quad \forall k \in \Phi_2. \end{aligned}$$

5. NUMERICAL EXAMPLES

In this section, we numerically observe some properties of the qd-type dhLV_{II} (13) shown in the previous sections. We easily realize numerical properties of the original dhLV_{II} (4) through those of the qd-type dhLV_{II}.

We first demonstrate the asymptotic behavior of the qd-type dhLV_{II} (13) shown in Theorem 9 numerically. Let $\omega_1^{(0)} = \omega_2^{(0)} = 5$, $\omega_3^{(0)} = \omega_4^{(0)} = \omega_5^{(0)} = 2$, $\omega_6^{(0)} = \omega_7^{(0)} = \omega_8^{(0)} = 1$ and $M = 2, m = 3, \delta^{(n)} = 10^{12}$, respectively, in the qd-type dhLV_{II} (13). It is emphasized here that the qd-type dhLV_{II} variables, except for $\omega_k^{(0)}$, depend on the value of $\delta^{(n)}$ through $\gamma_1^{(n)}$, as shown in (15). Figures 1 and 2 show the behavior of $\omega_k^{(n)}$, $k = 1, 2, 4, 5, 7, 8$ and $\omega_k^{(n)}$, $k = 3, 6$, $\gamma_k^{(n)}$, $k = 1, 2, 3, 4, 5, 6$, for $n = 0, 1, \dots, 19$, respectively. We see from Figures 1 and 2 that, as n becomes larger, $\omega_k^{(n)}$, $k = 1, 2, 4, 5, 7, 8$ and $\omega_k^{(n)}$, $k = 3, 6$ approach some positive constants and zero, respectively. This numerical result agrees with Theorem 9.

We next give a numerical example in order to confirm the Bäcklund transformation, shown in Theorem 5, between the dhLV_{II} (4) and the dhToda (6), as $\delta^{(n)} \rightarrow \infty$. Let $Q_i^{(0)} = 5$, $i = 1, 2, \dots, 12$ and $E_i^{(0)} = 2$, $i = 1, 2, 3$ with $M = 3$ and $m = 4$ in the dhToda (6). Moreover, let

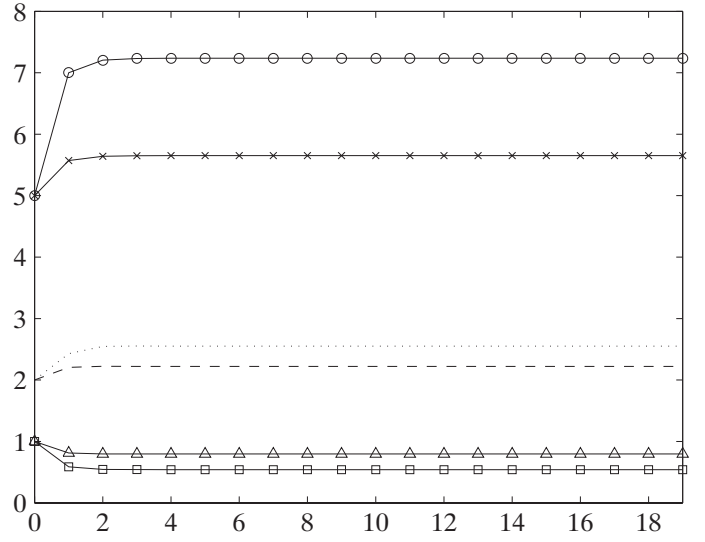


Figure 1: A graph of the iteration number n (x-axis) and the values of $\omega_1^{(n)}, \omega_2^{(n)}, \omega_4^{(n)}, \omega_5^{(n)}, \omega_7^{(n)}$ and $\omega_8^{(n)}$ (y-axis). \circ : $\omega_1^{(n)}$, \times : $\omega_2^{(n)}$, dotted line: $\omega_4^{(n)}$, dashed line: $\omega_5^{(n)}$, \square : $\omega_7^{(n)}$ and \triangle : $\omega_8^{(n)}$.

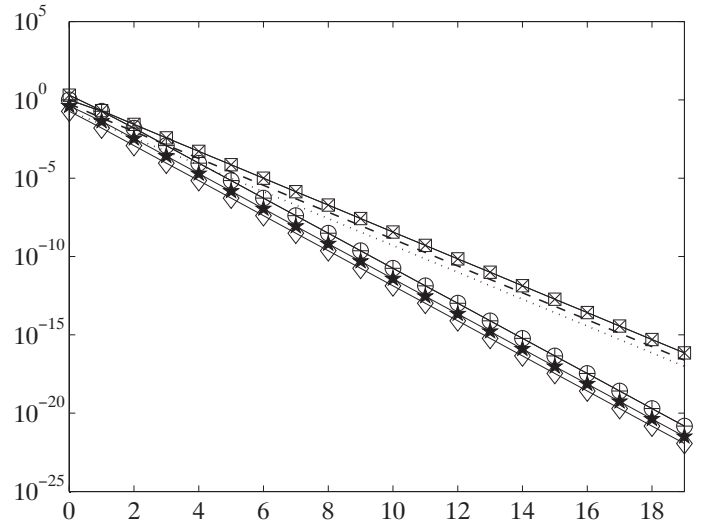


Figure 2: A graph of the iteration number n (x-axis) and the values of $\omega_3^{(n)}, \omega_6^{(n)}$ and $\gamma_k^{(n)}$ for $k = 1, 2, 3, 4, 5, 6$ (y-axis). \times : $\omega_3^{(n)}$, \circ : $\omega_6^{(n)}$, \square : $\gamma_1^{(n)}$, dashed line: $\gamma_2^{(n)}$, dotted line: $\gamma_3^{(n)}$, $+$: $\gamma_4^{(n)}$, \star : $\gamma_5^{(n)}$ and \diamond : $\gamma_6^{(n)}$.

$\omega_{M_i+j}^{(0)} = 5$, $i = 1, 2, 3, 4$, $j = 0, 1, 2$ and $\omega_{M_i+M}^{(0)} = 2$, $i = 1, 2, 3$ with $M = 3$ and $m = 4$ in the qd-type dhLV_{II} (13). We consider two cases where $\delta^{(\ell)} = 0.5$ and $\delta^{(\ell)} = 10^{12}$ for $\ell = 0, 1, \dots$. In Tables 1 and 2, the 1st, the 2nd and the 3rd columns denote the iteration number ℓ , the dhToda variables and the qd-type dhLV_{II} ones, respectively. Table 1 illustrates that, in the case where $\delta^{(\ell)} = 0.5$, $\ell = 0, 1, 2, 3, 4, 5$, the dhToda and dhLV_{II} variables coincide with each other in only a few digits. On the other hand, Table 2 shows that there are little difference

Table 1: Values of $Q_1^{(\ell M)}, \omega_{M_1}^{(\ell)}$ and $E_1^{(\ell M)}, \gamma_{M_1}^{(\ell)}$ in the case where $\delta^{(\ell)} = 0.5$ for $\ell = 0, 1, \dots, 5$ and 50.

ℓ	$Q_1^{(\ell M)}$	$\omega_{M_1}^{(\ell)}$
0	5.00000000000	5.00000000000
1	7.00000000000	6.96850393700
2	7.90909090909	7.89572522147
3	8.34969325153	8.34251316996
4	8.60502692998	8.60022866410
5	8.76967867989	8.76612205167
\vdots	\vdots	\vdots
50	9.04510897048	9.04510897048

ℓ	$E_1^{(\ell M)}$	$\gamma_{M_1}^{(\ell)}$
0	2.00000000000	1.96850393700
1	0.90909090909	0.927221284472
2	0.440602342442	0.446787948489
3	0.255333678448	0.257715494134
4	0.164651749916	0.165893387566
5	0.107582241979	0.108492550373
\vdots	\vdots	\vdots
50	1.225191026031E-12	1.407359466438E-12

Table 2: Values of $Q_1^{(\ell M)}, \omega_{M_1}^{(\ell)}$ and $E_1^{(\ell M)}, \gamma_{M_1}^{(\ell)}$ in the case where $\delta^{(\ell)} = 10^{12}$ for $\ell = 0, 1, \dots, 5$ and 50.

ℓ	$Q_1^{(\ell M)}$	$\omega_{M_1}^{(\ell)}$
0	5.00000000000	5.00000000000
1	7.00000000000	6.99999999999
2	7.90909090909	7.90909090909
3	8.34969325153	8.34969325153
4	8.60502692998	8.60502692998
5	8.76967867989	8.76967867989
\vdots	\vdots	\vdots
50	9.04510897048	9.04510897048

ℓ	$E_1^{(\ell M)}$	$\gamma_{M_1}^{(\ell)}$
0	2.00000000000	1.99999999999
1	0.90909090909	0.90909090909
2	0.440602342442	0.440602342442
3	0.255333678448	0.255333678448
4	0.164651749916	0.164651749916
5	0.107582241979	0.107582241979
\vdots	\vdots	\vdots
50	1.225191026031E-12	1.225191026031E-12

between $Q_1^{(\ell M)}, E_1^{(\ell M)}$ and $\omega_{M_1}^{(\ell)}, \gamma_{M_1}^{(\ell)}$, respectively. Tables 1 and 2 show that Theorem 5 approximately holds for a sufficiently large $\delta^{(\ell)}$.

In [9], some of the authors have proposed an algorithm for computing eigenvalues of totally nonnegative matrices, for which all the minors are nonnegative. So, from the Bäcklund transformation among the dhLV_{II} (4), the qd-type dhLV_{II} (13) and the dhToda (6) shown in Section 4, it is easily expected that the dhLV_{II} (4) and the qd-type dhLV_{II} (13) are applicable for computing the eigenvalues of a totally nonnegative matrix. In particular, $\lim_{n \rightarrow \infty} \prod_{p=0}^{M-1} \omega_{M_1+p}^{(n)}, \forall i \in \Phi_4$ give the eigenvalues of the totally nonnegative matrix $A^{(0)}$.

6. CONCLUSION

In this paper, we first introduce the qd-type dhLV_{II} and show the positivity of its variables. We also give a new Lax representation for the dhLV_{II}. As $\delta^{(n)} \rightarrow \infty$, it is observed that the Lax representation for the dhLV_{II} is related to the LR transformation for a band matrix. In other words, the time evolution of the dhLV_{II} with $\delta^{(n)} \rightarrow \infty$ corresponds to the LR transformation. We next explain how to associate the dhToda with the LR transformation. Remarkably, the dhLV_{II} with $\delta^{(n)} \rightarrow \infty$ is associated with the same form of LR transformation associated with the dhToda. By identifying two these LR transformations, we finally obtain a Bäcklund transformation between the dhLV_{II} and the dhToda. Additionally, through considering a Bäcklund transformation between the dhLV_I and the dhToda in [10],

we establish a Bäcklund transformation between the dhLV_I and the dhLV_{II} for the case of $\delta^{(n)} \rightarrow \infty$. We therefore have Bäcklund transformations among the dhLV_I, the dhLV_{II} and the dhToda. With the help of the Bäcklund transformation between the dhLV_{II} and the dhToda, we investigate the asymptotic convergence of the qd-type dhLV_{II} variables as $n \rightarrow \infty$ through that of the dhToda. We finally give some numerical examples which demonstrate our theoretical results.

We give a comment that the dhLV_{II} and the qd-type dhLV_{II} are applicable for computing eigenvalues of a totally nonnegative matrix. A future work is to derive the Bäcklund transformations among the dhLV_I, the dhLV_{II} and the dhToda in the case where $\delta^{(n)}$ is finite.

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