# Kernel perturbations for convolution first kind Volterra integral equations

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## Frank R. de Hoog and Robert S. Anderssen

**Abstract.** Because of their causal structure, (convolution) Volterra integral equations arise as models in a variety of real-world situations including rheological stress-strain analysis, population dynamics and insurance risk prediction. In such practical situations, often only an approximation for the kernel is available. Consequently, a key aspect in the analysis of such equations is estimating the effect of kernel perturbations on the solutions. In this paper, it is shown how kernel perturbation results derived for the interconversion equation of rheology can be extended to the analysis of kernel perturbations for first kind convolutional integral equations with positive kernels, solutions and forcing terms.

*Keywords.* linear viscoelasticity, interconversion, kernel perturbations, convolution, first kind Volterra integral equations

## 1. Introduction

Much practical modelling of real-world processes is performed using ordinary and partial differential equations, because the underlying model corresponds to a properly posed problem with known parameter values. A different situation arises in parameter estimation, when it is necessary to estimate, from measurements, the value of the parameters which define the specific structure of the properly posed problem to be solved; or in the recovery of information from indirect measurements such as arise in tomography, geophysical exploration and medical diagnosis. Now the models which must be solved are improperly posed [9, 14].

An important subclass of improperly posed problems are first kind Volterra integral equations [6, 7, 13, 17]

$$\int_0^t K\,(t,s)\,u\,(s)\,ds = f,$$

where $K$, $u$ and $f$ denote respectively the kernel, solution and forcing term. In many industrial situations, including material science [15, 12], population dynamics and insurance risk assessement [10, 11, 8, 20], the underlying mathematical model is a first kind convolution Volterra integral equation

$$(k*u)(t) = \int_0^t k\,(t-s)\,u\,(s)\,ds = f, \qquad (1)$$

where $k$, $u$ and $f$ denote respectively the convolution kernel, solution and forcing term. An important subclass of such equations are the equations with positive kernels and solutions. Such a situation arises in the recovery of information about the relaxation behaviour of a polymer from stress-strain measurements [1, 18, 19], where $k$ corresponds to the linear viscoelastic relaxation modulus $G(t)$, $u$ the applied strain-rate $\dot{\gamma} = d\gamma/dt$ and $f$ the measured stress response $\sigma$. In this situation, the focus is on the construction, from measurements of the stress response $\sigma$ and the strain-rate $\dot{\gamma}$, of an approximation $\tilde{G}$ to $G$, for which it is necessary to solve some appropriate form of the Boltzmann equation of linear viscoelasticity

$$\sigma = \int_{-\infty}^t G(t-\tau)\dot{\gamma}(\tau)d\tau.$$

This approximation is often used to determine the corresponding approximation $\tilde{J}$ to the creep (retardation) modulus $J$ by solving the *"interconversion equations"*

$$(J*G)(t) = \int_0^t J\,(t-s)\,G\,(s)\,ds$$
$$= (G*J)(t) = \int_0^t G\,(t-s)\,J\,(s)\,ds = t, \qquad (2)$$

where, in terms of the first kind convolution Volterra equation (1), the kernel $k$ corresponds to the linear viscoelastic creep modulus $J(t)$, $u$ the applied stress-rate $\dot{\sigma} = d\sigma/dt$ and $f$ the measured strain response $\gamma$.

The motivation for solving the interconversion equation is both practical and theoretical. The cost of instruments to measure either the stress response $\sigma$ to a given applied strain-rate $\dot{\gamma}$ or the strain response $\gamma$ to a given

applied stress-rate $\dot{\sigma}$ is very expensive, and considerable time is required to perform either one of these experiments [1, 18, 19]. If it is assumed that the stress-strain response of the material being studies is linear viscoelastic, then the interconversion equation (2) is an immediate consequence. Consequently, for a given approximation $\tilde{G}$, the corresponding approximation $\tilde{J}$ must satisfy the interconversion equation (2), in order to guarantee consistency with the linear viscoelasticity assumption. This leads naturally to the need to examine the effect of the error $\delta G = G - \tilde{G}$ on the resulting error $\delta J = J - \tilde{J}$. This has already been examined in some detail in [1, 2, 3]. The historical importance of these practical considerations is reflected in the fact that the interconversion equation was one of the first problems to be examined computationally on the early electronic computers [16].

An alternative rheological use of equation (1) arises when, for a given relaxation modulus $G$ and stress $\sigma$, the corresponding form for the strain rate $\dot{\gamma}$ is required, which corresponds to solving equation (1) with $k = G$, $u = \dot{\gamma}$ and $f = \sigma$. The need to perform such simulations for a variety of choices for $G$ and $\sigma$ arise in the design of rheometers. Now, in equation (1), the effect on the solution $u$ of perturbations $\tilde{k}$ in the kernel $k$ is required. Here, the analysis for kernel perturbations associated with the interconversion equation (2) [2, 3, 4] is modified to derive the corresponding bounds for this situation.

Consequently, the corresponding approximation $\tilde{u}$ to the solution of equation (1) will satisfy (for fixed forcing term $f$)

$$(\tilde{k} * \tilde{u})(t) = f. \tag{3}$$

The behaviour of the error $\delta u = (u - \tilde{u})$ will depend on the effect of kernel perturbations $\delta k = (k - \tilde{k})$.

## 2.  Interconversion

The interconversion equation (2) plays a fundamental role in determining estimates of the relaxation $G(t)$ and creep (retardation) $J(t)$ modulii. Once an estimate $\tilde{G}$ or $\tilde{J}$ for either $G$ or $J$, respectively, has been determined experimentally, the corresponding estimate for $\tilde{J}$ or $\tilde{G}$ is obtained by solving the interconversion equation (2) without the need to perform an additional independent experiment. In addition, it ensures the consistency of the resulting $\tilde{J}$ or $\tilde{G}$ by ensuring that they satisfy, at least approximately, the interconversion equation (2).

In a series of papers [1, 2, 3, 4], it has been established and analysed theoretically that the recovery of $\tilde{J}$ from $\tilde{G}$ is stable whereas the recovery of $\tilde{G}$ from $\tilde{J}$ cannot be guaranteed, in a given situation, to be stable. Consequently, from a kernel perturbation analysis of the stability of the solution of the interconversion equation (2), attention must be limited to the recovery of $\tilde{J}$ from $\tilde{G}$. Recently, it has

been shown [4] that

$$|\delta J(t)| \leq J(t)\,\tilde{J}(t) \max_{0 \leq s \leq t} |\delta G(s)|, \tag{4}$$

$$\left| \frac{\delta J(t)}{J(t)} \right| \leq \max_{0 \leq s \leq t} \left| \frac{\delta G(s)}{\tilde{G}(s)} \right|, \tag{5}$$

$$\left| \frac{\delta J(t)}{\tilde{J}(t)} \right| \leq \max_{0 \leq s \leq t} \left| \frac{\delta G(s)}{G(s)} \right|. \tag{6}$$

The assumed complete monotonicity [5] of $G$, $\tilde{G}$, $dJ/dt$ and $d\tilde{J}/dt$, in order to guarantee consistency with the conservation of energy, is much stronger than the regularity invoked in [4] to prove the basic lemmas from which these bounds are derived.

The goal of this paper is to derive the counterparts of the above results for the first kind convolution Volterra integral equation (1), under the minimum regularity needed to guarantee their validity.

## 3.  The Kernel Perturbation Estimates

Corresponding to the situation for the interconversion equation (2) and the associated rheological application attention will focus on the practical situation where the kernel $k$, the solution $u$ and the forcing term $f$ are nonnegative. For simplicity, attention will focus on regularity conditions defined in terms of the continuity of appropriate derivatives.

**Lemma 1.** *Let $k \in C^1[0, \infty)$, $f \in C^2[0, \infty)$, $k(0) > 0$, $\dot{k} < 0$, $f(0) = 0$, $\dot{f}(0) > 0$, and $\ddot{f} \geq 0$. Then, the convolution first kind Volterra equation (1) has a unique solution $u \in C^1[0, \infty)$ which satisfies $u > 0$, $\dot{u} > 0$ and $ku \leq \dot{f}$.*

*Proof.* Differentiation of equation (1) with respect to $t$ yields

$$k(0)u(t) + (\dot{k} * u)(t) = \dot{f}, \tag{7}$$

which is a second kind Volterra integral equation. Standard theory for second kind Volterra integral equations [7] ensures the existence of a unique solution $u \in C^1[0, \infty)$ of (7) and hence (1). In addition,

$$u(0) = \frac{\dot{f}(0)}{k(0)} > 0.$$

On assuming that $u(t) > 0$ for $0 < t < t^{\#} < \infty$ with $u(t^{\#}) = 0$, it follows that, since the conditions of the Lemma imply that $\dot{f} > 0$,

$$0 = k(0)u(t^{\#}) = \dot{f}(t^{\#}) - (\dot{k} * u)(t^{\#}) > 0,$$

which contradicts the assumption that $u(t^{\#}) = 0$ and thereby yields $u(t) > 0$, $t \in [0, \infty)$. Differentiation of equation (7) with respect to $t$ then yields

$$k(0)\dot{u}(t) + (\dot{k} * \dot{u})(t) = \ddot{f}(t) - u(0)\dot{k}(t). \tag{8}$$

Standard second kind Volterra integral equation theory now guarantees the existence of $\dot{u} \in C^0[0, \infty)$. From the conditions of the Lemma, it follows that

$$\dot{u}(0) = \frac{1}{k(0)}\left( \ddot{f}(0) - u(0)\dot{k}(0) \right) > 0.$$

On assuming that $\dot{u}(t) > 0$ for $0 < t < t^{\#} < \infty$ with $\dot{u}(t^{\#}) = 0$, it follows that, since $\ddot{f} \geq 0$,

$$0 = k(0)\dot{u}(t^{\#}) = \ddot{f}(t^{\#}) - (\dot{k} * \dot{u})(t^{\#}) - u(0)\dot{k}(t^{\#}) > 0,$$

which yields the contradiction which proves that $\dot{u}(t) > 0$ for $t \in [0, \infty)$. Finally, from (7), it follows, on using the fact that $u > 0$, $\dot{u} > 0$, that

$$\begin{aligned} \dot{f} &= k(0)u(t) + (\dot{k} * u)(t), \\ &\geq k(0)u(t) + (\dot{k} * 1)(t)u(t), \\ &= k(t)u(t). \qquad \square \end{aligned}$$

We now take advantage of the analysis of the interconversion equation given in [2, 3] and generalized in [4].

**Lemma 2.** *For a given $k \in C^1[0, \infty)$ with $k(0) > 0$, there exists a unique $h \in C[0, \infty)$ such that*

$$(k * h)(t) = t. \qquad (9)$$

*Proof.* As in Lemma 1, differentiation of equation (9) yields the second kind equation

$$k(0)h(t) + \dot{k} * h = 1,$$

to which standard Volterra integral equation theory [7] can be applied to yield the stated result. $\qquad \square$

It is such relationships that played the pivotal role in allowing the types of estimates given in equations (4), (5) and (6) to be derived. For the perturbed kernel $\tilde{k}$ of equation (3), the counterpart of this lemma, under the same regularity, also holds; namely, there exist a unique $\tilde{h}$ such that

$$\tilde{k} * \tilde{h} = t.$$

The properties of $h$ (and the corresponding properties of $\tilde{h}$) required in the sequel are:

**Corollary 1.** *For $k \in C^1[0, \infty)$, $k(0) > 0$ and $\dot{k} < 0$, the solution $h$ of equation (9) satisfies $h \in C^1[0, \infty)$, $h > 0$, $\dot{h} > 0$, and $h(0)k(0) = 1$.*

*Proof.* This is an immediate consequence of Lemma 1 on setting $f = t$. $\qquad \square$

**Lemma 3.** *Let $k \in C^1[0, \infty)$ with $k(0) > 0$ and $f \in C^1[0, \infty)$. Then, the solution of equation (1) is given by*

$$u(t) = \frac{d^2(h * f)}{dt^2}. \qquad (10)$$

*Proof.* From equation (9), it follows that

$$t * u = h * f.$$

On the basis of Lemma 2, the differentiation of $h * f$ twice with respect to $t$ yields the relationship

$$\frac{d^2(h * f)}{dt^2} = h(0)\dot{f}(t) + f(0)\dot{h}(t) + \dot{h} * \dot{f},$$

which, in conjunction with the assumptions of this Lemma and Corollary 1, establishes the twice differentiability of $h * f$ and hence of $t * u$. The stated result follows on actually performing the twice differentiation of $t * u$. $\qquad \square$

For the perturbed kernel $\tilde{k}$ of equation (3), the counterpart of this lemma, under the same regularity, also holds; namely,

$$\tilde{u} = \frac{d^2(\tilde{h} * f)}{dt^2}.$$

**Theorem 1.** *Let $k \in C^1[0, \infty)$, $f \in C^2[0, \infty)$, $k(0) > 0$, $\dot{k} < 0$, $f(0) = 0$, $\dot{f}(0) > 0$ and $\ddot{f} \geq 0$. In addition, let $\tilde{k} \in C^1[0, \infty)$, $\tilde{k}(0) > 0$, and $\dot{\tilde{k}} < 0$. Then*

$$\left| \frac{\delta u(t)}{u(t)} \right| \leq \max_{0 \leq s \leq t} \left| \frac{\delta k(s)}{\tilde{k}(s)} \right|, \qquad (11)$$

$$\left| \frac{\delta u(t)}{\tilde{u}(t)} \right| \leq \max_{0 \leq s \leq t} \left| \frac{\delta k(s)}{k(s)} \right|. \qquad (12)$$

*Proof.* On assuming that, for the kernel and solution of the perturbed equation (3), $\tilde{k} \in C^1[0, \infty)$, $\tilde{k}(0) > 0$, $\dot{\tilde{k}} < 0$, it follows from Lemma 1 that $\tilde{u} > 0$, $\dot{\tilde{u}} > 0$, $\tilde{k}\tilde{u} \leq \dot{f}$. In addition, since the right hand side term in equations (1) and (3) is the same, it follows that $\tilde{k} * \tilde{u} = k * u$, and, hence, that

$$\tilde{k} * \delta u = -u * \delta k, \qquad (13)$$

and

$$k * \delta u = -\tilde{u} * \delta k. \qquad (14)$$

The application of Lemma 3 to equation (13) yields

$$\begin{aligned} \delta u &= -\frac{d^2(\tilde{h} * u * \delta k)}{dt^2}, \\ &= -\tilde{h}(0)u(0)\delta k - (\tilde{h}(0)\dot{u} + u(0)\dot{\tilde{h}} + \dot{\tilde{h}} * \dot{u}) * \delta k. \end{aligned}$$

On the basis of the assumptions made about $k$, $\tilde{k}$, $f$ and $\tilde{f}$ along with the results of Lemma 1 and Corollary 1, it follows that

$$a = \tilde{h}(0)u(0) > 0 \text{ and } b(t) = (h(0)\dot{u} + u(0)\dot{\tilde{h}} + \dot{\tilde{h}} * \tilde{u})(t) > 0.$$

Using these facts, in conjunction with the inequalities

$$a\delta k \leq a\tilde{k} \max_{0 \leq s \leq t} \left| \frac{\delta k(s)}{\tilde{k}(s)} \right|, \quad |a * \delta k(t)| \leq a * \tilde{k}(t) \max_{0 \leq s \leq t} \left| \frac{\delta k(s)}{\tilde{k}(s)} \right|,$$

yields

$$\begin{aligned} |\delta u(t)| &\leq (\tilde{h}(0)u(0)\tilde{k} + (\tilde{h}(0)\dot{u} + u(0)\dot{\tilde{h}} + \dot{\tilde{h}} * \dot{u}) * \tilde{k})(t) \\ &\qquad\qquad\qquad\qquad \max_{0 \leq s \leq t} \left| \frac{\delta k(s)}{\tilde{k}(s)} \right|, \\ &= \frac{d^2(\tilde{h} * \tilde{k} * u)}{dt^2} \max_{0 \leq s \leq t} \left| \frac{\delta k(s)}{\tilde{k}(s)} \right|, \\ &= u(t) \max_{0 \leq s \leq t} \left| \frac{\delta k(s)}{\tilde{k}(s)} \right|. \end{aligned}$$

The first estimate (11) in Theorem 1 is an immediate consequence of this last result, since, from Lemma 1, $u(t) > 0$. The second estimate (12) in Theorem 1 follows on appling an analogous argument to equation (14). $\qquad \square$

## 4. Conclusions

The current paper builds on the early work of Anderssen, Davies and de Hoog [1, 2, 3] and establishes that the interconversion relationship (9) has wider applicability than the original interconversion kernel stability analysis for which it was initially invoked.

## Acknowledgements

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