Hamiltonian bifurcation theory for a rotating flow subject to elliptic straining field

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Abstract. A weakly nonlinear stability theory is developed for a rotating flow confined in a cylinder of elliptic cross-section. The straining field associated with elliptic deformation of the cross-section breaks the SO(2)-symmetry of the basic flow and amplifies a pair of Kelvin waves whose azimuthal wavenumbers are separated by 2, being referred to as the Moore-Saffman-Tsai-Widnall (MSTW) instability. The Eulerian approach is unable to fully determine the mean flow induced by nonlinear interaction of the Kelvin waves. We establish a general framework for deriving the mean flow by a restriction to isovortical disturbances with use of the Lagrangian variables and put it on the ground of the generalized Lagrangian-mean theory. The resulting formula reveals enhancement of mass transport in regions dominated by the vorticity of the basic flow. With the mean flow at hand, we derive unambiguously the weakly nonlinear amplitude equations to third order for a nonstationary mode. By an appropriate normalization of the amplitude, the resulting equations are made Hamiltonian systems of four degrees of freedom, possibly with three first integrals identifiable as the wave energy and the mean flow.

1. Introduction

An axisymmetric rotating flow, uniform along the rotating axis, supports a family of three-dimensional neutrally stable oscillations called the Kelvin waves. When rotational and/or translational symmetry is broken, a pair of Kelvin waves are amplified via the symmetry breaking perturbations. A typical example is the Moore-Saffman-Tsai-Widnall (MSTW) instability [26, 29, 5, 18, 7], for which the rotational or SO(2)-symmetry of streamlines of the basic flow is reduced to $\mathbb{Z}_2$-symmetry by a straining field causing the elliptical deformation of the cross-section. The MSTW instability is detected not only for an antiparallel vortex pair in an open space [20] but also for a rotating flow confined in a cylinder [23, 6].

This pertains to the inviscid instability, and the linear stability follows the scenario of Krein's theory of Hamiltonian spectra, though the latter is valid only to finite-dimensional systems. As an extension to nonlinear stage, Hamiltonian bifurcation theories were established in general settings [22, 12, 19], but Calculation of the coefficients of the weakly nonlinear amplitude equations calls for an individual work depending on a specific flow configuration. In parallel with the development of the general framework, weakly nonlinear analyses of the MSTW instability were addressed over two decades [30, 28, 24, 27]. We pointed out [25, 10] that the traditional treatment made within the Eulerian framework has difficulty in handling the mean-flow induced by nonlinear interaction of a Kelvin wave with itself. A resolution is brought by restricting disturbances to so called the isovortical or kinematically accessible ones, for which the wave-induced mean flow and hence the amplitude equations are unambiguously determined. The Lagrangian description is instrumental in treating the isovortical disturbances. The wave-induced mean flow is closely tied with the energy of a wave, a key ingredient of Krein's Hamiltonian spectral theory. In a series of our recent papers [14, 15, 9, 10], we developed a foundation of the Lagrangian description of the spectra, both point and continuous, of the hydrodynamic and the hydromagnetic waves, and gave the recipe to calculate the energy and the wave-induced mean flow. Our explicit formula indicates vigorous generation of mean flow of wave origin and thus enhancement of mass transport in regions dominated by the vorticity of the basic flow.

A commonly observed mode of the MSTW instability is the resonance between the left- ($m = 1$) and right-handed ($m = -1$) helical waves [20]. The resonance between the axisymmetric ($m = 0$) and the elliptic deformation mode ($m = 2$) was also detected in a confined geometry [6, 18]. Here $m(\in \mathbb{Z})$ is the azimuthal wavenumber. The objective of this paper is first to give an interpretation of the mean flow in the language of the Lagrangian description, and is second to give a sketch of the weakly nonlinear evolution of the disturbance amplitude for the resonance pair $m = 0$ and 2 of the Kelvin waves going through the MSTW instability.

In §2, we collect the known result of evolution of the Lagrangian displacement field, and in §3, we give a general description of the mean flow induced by waves on a steady flow, and put it on the ground of the generalized Lagrangian-mean (GLM) theory [4, 1].

Thereafter we inquire into the weakly nonlinear evolution of a nonstationary resonance pair of $(m, m+2)$ waves for which, unlike the stationary case of $(m, m+2) = (-1, 1)$ [25, 10], the wave induced mean-flow of second order in amplitude takes a non-trivial distribution over the domain. After recollecting the MSTW instability in §5, we derive the weakly nonlinear amplitude equation in §6. It is shown that the Hamiltonian property of the Euler equations is lost for the resulting amplitude equations, but is recovered for properly normalized amplitude. The resulting equations exhibit chaotic behavior, and a discussion is made of the origin of chaos. We close with concluding remarks (§7).

2. Lagrangian displacement field

The presence of the basic flow makes it difficult to calculate the wave energy, as the wave field is required to second order in amplitude. This difficulty is circumvented by resorting to Kelvin-Arnold's theorem [3] that a steady state of the Euler flows is an extremal of the kinetic energy with respect to isovortical disturbance. This statement is constructively implemented in the framework of the Lagrangian description (see [14, 15] for the context of parallel shear flows.). In our previous papers, we made asymptotic expansions of temporal evolution of the Lagrangian displacement field to second order [9] and to arbitrary order in wave amplitude [10]. The explicit form of the second-order field manifests the existence of the mean-field induced by self-interaction of a wave [9]. The advantage of restricting to isovortical disturbances, the second-order mean field is expressible solely in terms of the first-order Lagrangian displacement field, being referred to as the wave property [1, 4]. Here we collect the result on the relation of the Lagrangian displacement field with the disturbance velocity field and on the evolution equations of the Lagrangian displacement field to second order.

We assume that the fluid is incompressible, with uniform mass density, as well as inviscid. The motion of an inviscid incompressible fluid is regarded as an orbit on SDiff$(\mathcal{D})$, the group of the volume-preserving diffeomorphisms of the domain $\mathcal{D} \subset \mathbb{R}^3$ [2]. Its Lie algebra $\mathfrak{g}$ is the velocity field of the fluid. A one-parameter subgroup of $\varphi_t \in$ SDiff$(\mathcal{D})$ and its generator $u(t) \in \mathfrak{g}$ are linked by the definition

$$u(t_0) = \left.\frac{\partial}{\partial t}\right|_{t_0} \left(\varphi_t \circ \varphi_{t_0}^{-1}\right). \tag{1}$$

Suppose that a disturbed orbit from φ_t is written at each instant t as $\varphi_{\alpha,t} \circ \varphi_t$ by means of a near-identity map $\varphi_{\alpha,t}$ labeled with a small parameter $\alpha (\in \mathbb{R})$. There exists a generator $\xi_\alpha(t) \in \mathfrak{g}$ for it, defined by $\varphi_{\alpha,t} = \exp \xi_\alpha(t)$. The disturbance velocity field $u_\alpha(t)$ is calculated from

$$u_\alpha(t_0) = \left.\frac{\partial}{\partial t}\right|_{t_0} \left(\varphi_{\alpha,t} \circ \varphi_t \circ \varphi_{t_0}^{-1} \circ \varphi_{\alpha,t_0}^{-1}\right). \tag{2}$$

Use of a geometric setting, combined with symbolic manipulation of the Lie algebra, facilitates perturbation expansions, in powers of α, of the Lagrangian field to a higher order, and a series representation u_α to arbitrary order in α was manipulated in the previous investigation [13, 10].

Translation into the language of the vector calculus is straightforward. Given a steady basic flow $\boldsymbol{U}_0(\boldsymbol{x})$, an orbit $\boldsymbol{x}(t)$ of a fluid particle constituting this basic flow is defined by $\mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t = \boldsymbol{U}_0\left(\boldsymbol{x}(t)\right)$. Suppose that the particle position $\boldsymbol{x}$ is disturbed to $\boldsymbol{x}+\boldsymbol{\xi}_\alpha(\boldsymbol{x},t)$. We expand the Lagrangian displacement field $\boldsymbol{\xi}_\alpha$ in a power series in α as

$$\boldsymbol{\xi}_\alpha(\boldsymbol{x},t) = \alpha\boldsymbol{\xi}_1(\boldsymbol{x},t) + \frac{\alpha^2}{2}\boldsymbol{\xi}_2(\boldsymbol{x},t) + \cdots, \tag{3}$$

and correspondingly $\boldsymbol{u}_\alpha(\boldsymbol{x},t)$ as

$$\boldsymbol{u}_\alpha(\boldsymbol{x},t) = \alpha\boldsymbol{u}_1(\boldsymbol{x},t) + \frac{\alpha^2}{2}\boldsymbol{u}_2(\boldsymbol{x},t) + \cdots. \tag{4}$$

The series development of (2) reads, to $O(\alpha^2)$,

$$\frac{\partial\boldsymbol{\xi}_1}{\partial t} + (\boldsymbol{U}_0\cdot\nabla)\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{U}_0 = \boldsymbol{u}_1, \tag{5}$$

$$\frac{\partial\boldsymbol{\xi}_2}{\partial t} + (\boldsymbol{U}_0\cdot\nabla)\boldsymbol{\xi}_2 - (\boldsymbol{\xi}_2\cdot\nabla)\boldsymbol{U}_0 + (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{u}_1 = \boldsymbol{u}_2. \tag{6}$$

The second-order equation (6) was derived in our recent papers [9, 10]. It is worth noting that the geometric approach is crucial to derive (6); a naive definition

$$\frac{D}{Dt}(\boldsymbol{x}+\boldsymbol{\xi}_\alpha) = \boldsymbol{U}_0(\boldsymbol{x}+\boldsymbol{\xi}_\alpha) + \boldsymbol{u}_\alpha(\boldsymbol{x}+\boldsymbol{\xi}_\alpha,t), \tag{7}$$

supplemented by $D/Dt = \partial/\partial t + \boldsymbol{U}_0\cdot\nabla$ would lead to a distinct representation at $O(\alpha^2)$, though correct to $O(\alpha)$.

The requirement that the disturbance be *isovortical* or *kinematically accessible* is dictated by preservation of the vorticity flux across an arbitrary infinitesimal material surface as represented, in the local Cartesian coordinates, by $\omega_x\mathrm{d}y\wedge\mathrm{d}z + \omega_y\mathrm{d}z\wedge\mathrm{d}x + \omega_z\mathrm{d}x\wedge\mathrm{d}y$ or equivalently by preservation of the circulation $\oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}$ with respect to an arbitrary material loop C contained in the domain [2, 3, 17]. This requirement brings in a series representation of the disturbance vorticity field $\boldsymbol{\omega}_\alpha$ [9]. Define the vector potential $\boldsymbol{v}_\alpha$ for $\boldsymbol{\omega}_\alpha$ by $\boldsymbol{\omega}_\alpha = \nabla\times\boldsymbol{v}_\alpha$, allowing for the freedom of the gauge transformation. The vector field $\boldsymbol{v}_\alpha$ belongs to $\mathfrak{g}^*$, dual of $\mathfrak{g}$. If $\boldsymbol{v}_\alpha(\boldsymbol{x},t)$ is expanded in a power series in α as

$$\boldsymbol{v}_\alpha(\boldsymbol{x},t) = \alpha\boldsymbol{v}_1(\boldsymbol{x},t) + \frac{\alpha^2}{2}\boldsymbol{v}_2(\boldsymbol{x},t) + \cdots, \tag{8}$$

the isovortical disturbance is represented, order by order, with use of the Lagrangian displacement field, as

$$\boldsymbol{v}_1 = \mathcal{P}\left[\boldsymbol{\xi}_1\times\boldsymbol{\omega}_0\right], \tag{9}$$

$$\boldsymbol{v}_2 = \mathcal{P}\left[\boldsymbol{\xi}_1\times(\nabla\times(\boldsymbol{\xi}_1\times\boldsymbol{\omega}_0)) + \boldsymbol{\xi}_2\times\boldsymbol{\omega}_0\right], \tag{10}$$

where $\boldsymbol{\omega}_0 = \nabla\times\boldsymbol{U}_0$ is the vorticity of the basic field and $\mathcal{P}$ is an operator projecting to solenoidal vector field complying with the boundary condition.

The above formulas, along with the mass conservation, embody the essence of the kinematics of the vorticity. The kinematics of vorticity disregards the concept of the energy, the momentum and the force. The conservation laws of the energy and the momentum, in a manner compatible with the isovortical property, are retrieved by the identification [16]

$$\boldsymbol{v}_\alpha(\boldsymbol{x},t) = \boldsymbol{u}_\alpha(\boldsymbol{x},t). \tag{11}$$

In what follows, we think of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as the disturbance velocity $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, respectively, and, with this identification, (5) and (6) are made closed for the Lagrangian variables $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.

3. Wave-induced mean flow

The local mean flow of $O(\alpha^2)$ induced by the self-interaction of a wave is obtained by taking an ensemble average of (10). In the context of the Kelvin wave, an ensemble average may be replaced by the temporal average over a period. Combined with (9), it takes the form [9]:

$$\overline{\boldsymbol{v}_2} = \overline{\mathcal{P}\left[\boldsymbol{\xi}_1 \times (\nabla \times \boldsymbol{v}_1) + \boldsymbol{\xi}_2 \times \boldsymbol{\omega}_0\right]}. \tag{12}$$

In view of (12) or more directly of (10), $\overline{\boldsymbol{v}_2}$ needs the vorticity $\boldsymbol{\omega}_0$ of the basic flow. This has an implication that transport and mixing of materials are promoted in regions with significant values of $|\boldsymbol{\omega}_0|$. In the previous paper [10], we found for a circular cylindrical vortex that the azimuthal and the axial components of (12) coincide with the pseudomomenta. Subsequently, we place this relation on the firm basis of the generalized Lagrangian-mean theory (GLM) [1, 4].

Given the trajectory $\boldsymbol{x}(t)$ of a fluid particle driven by the basic flow $\boldsymbol{U}_0(\boldsymbol{x})$. Suppose that, subject to a disturbance, this basic trajectory is displaced by the Lagrangian displacement $\boldsymbol{\xi}_\alpha$. In GLM theory, the Lagrangian displacement $\boldsymbol{\xi}_\alpha$ is considered only to $O(\alpha)$ and the position of the fluid particle is displaced as $\boldsymbol{x} \to \boldsymbol{x}^\xi = \boldsymbol{x} + \alpha\boldsymbol{\xi}_1(\boldsymbol{x},t)$. By definition, the Eulerian mean of the disturbance field satisfies $\overline{\boldsymbol{\xi}_1(\boldsymbol{x},t)} = \boldsymbol{0}$. This is called the lifting map. We lift the velocity vector $\boldsymbol{v} = \boldsymbol{U}_0 + \boldsymbol{v}_\alpha$ as

$$\boldsymbol{v}^\xi(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{x} + \alpha\boldsymbol{\xi}_1(\boldsymbol{x},t),t). \tag{13}$$

Given the series form $\boldsymbol{v}_\alpha = \alpha\boldsymbol{v}_1 + \alpha^2\boldsymbol{v}_2/2 + (\alpha^3)$, we perform expansion of (13) in powers of α to obtain, to $O(\alpha^2)$,

$$\boldsymbol{v}^\xi = \boldsymbol{U}_0 + \alpha\left[\boldsymbol{v}_1 + (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{U}_0\right] + \frac{\alpha^2}{2}\left[\boldsymbol{v}_2 + 2(\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{v}_1 + \xi_{1k}\xi_{1l}\partial_k\partial_l\boldsymbol{U}_0\right] + O(\alpha^3), \tag{14}$$

as being evaluated at the undisturbed position $\boldsymbol{x}$ and the time t. The Lagrangian-mean velocity is defined by the Eulerian mean of the lifted velocity

$$\begin{aligned}\bar{\boldsymbol{v}}^{\mathrm{L}} &= \overline{\boldsymbol{v}^\xi} = \overline{\boldsymbol{v}(\boldsymbol{x} + \alpha\boldsymbol{\xi}_1, t)} \\ &= \boldsymbol{U}_0(\boldsymbol{x}) + \frac{\alpha^2}{2}\overline{\boldsymbol{v}_2} + \alpha^2\left[\overline{(\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{v}_1} + \frac{1}{2}\overline{\xi_{1k}\xi_{1l}}\partial_k\partial_l\boldsymbol{U}_0\right].\end{aligned} \tag{15}$$

In general, the Lagrangian average of a field is partitioned into the Eulerian average and the Stokes correction. The first two terms on the right-hand side of (15) correspond to the Eulerian-mean velocity $\bar{\boldsymbol{v}}^{\mathrm{E}}$ and the last two terms to the Stokes drift $\bar{\boldsymbol{v}}^{\mathrm{S}}$, namely,

$$\bar{\boldsymbol{v}}^{\mathrm{E}} = \boldsymbol{U}_0(\boldsymbol{x}) + \frac{\alpha^2}{2}\overline{\boldsymbol{v}_2}, \tag{16}$$

$$\bar{\boldsymbol{v}}^{\mathrm{S}} = \alpha^2\left[\overline{(\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{v}_1} + \frac{1}{2}\overline{\xi_{1k}\xi_{1l}}\partial_k\partial_l\boldsymbol{U}_0\right]. \tag{17}$$

We are now prepared to introduce the pseudomomentum [1, 4]. Define the Lagrangian disturbance velocity

$$\boldsymbol{v}^{\ell}(\boldsymbol{x},t) = \boldsymbol{v}^{\xi}(\boldsymbol{x},t) - \bar{\boldsymbol{v}}^{\mathrm{L}}(\boldsymbol{x},t) = \alpha\left[\boldsymbol{v}_1 + (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{U}_0\right] + O(\alpha^2). \tag{18}$$

Built on (18), the i-th component of the pseudomomentum is defined by

$$\mathsf{p}_i = -\alpha\overline{\boldsymbol{v}^{\ell}\cdot\partial_i\boldsymbol{\xi}_1} \quad (i=1,2,3). \tag{19}$$

Qualification as the pseudomomentum manifests itself by rewriting (19), with the help of (5) substituted from the identification (11), into

$$\begin{aligned}\mathsf{p}_i =& -\alpha^2\overline{\frac{\partial\boldsymbol{\xi}_1}{\partial x_i}\cdot\left[\boldsymbol{v}_1 + (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{U}_0\right]} \\ =& -\alpha^2\overline{\frac{\partial\boldsymbol{\xi}_1}{\partial x_i}\cdot\left[\frac{\partial\boldsymbol{\xi}_1}{\partial t} + (\boldsymbol{U}_0\cdot\nabla)\boldsymbol{\xi}_1\right]},\end{aligned} \tag{20}$$

to $O(\alpha^2)$. The total kinetic energy of disturbed flow $\boldsymbol{v} = \boldsymbol{U}_0 + \boldsymbol{v}_\alpha$ is expanded in powers of α as $H(\boldsymbol{v}) = H(\boldsymbol{U}_0) + \alpha^2 H_2/2 + O(\alpha^3)$. The term of $\mathrm{O}(\alpha)$ vanishes identically for isovortical disturbances because restriction of disturbances to this class renders the steady flow $\boldsymbol{U}_0$ to be the extremum of the energy functional [3]. By the wave energy, we mean the energy $\alpha^2 H_2/2$ increased by excitation of waves. It takes several alternative forms [11], among which is

$$H_2 = 2\int\frac{\partial\boldsymbol{\xi}_1}{\partial t}\cdot\left(\frac{\partial\boldsymbol{\xi}_1}{\partial t} + (\boldsymbol{U}_0\cdot\nabla)\boldsymbol{\xi}_1\right)\mathrm{d}V. \tag{21}$$

Here we have assumed the fluid density to be unity. The close bearing of the volume flux of (20) with (21) is apparent.

A precise relation between the wave-induced mean flow and the pseudomomentum is elaborated by returning to the definition of the isovortical disturbances. Suppose that, by the map $\boldsymbol{x}\to\boldsymbol{x}^{\xi} = \boldsymbol{x} + \alpha\boldsymbol{\xi}_1$ a closed material loop C in the basic flow is deformed into C_ξ. The circulation with respect to C_ξ is

$$\begin{aligned}\Gamma =& \oint_{C_\xi}\boldsymbol{v}(\boldsymbol{x},t)\cdot\mathrm{d}\boldsymbol{x} = \oint_C\boldsymbol{v}(\boldsymbol{x}+\alpha\boldsymbol{\xi}_1,t)\cdot\mathrm{d}\left(\boldsymbol{x}+\alpha\boldsymbol{\xi}_1\right) \\ =& \oint_C\left[v_i^{\xi} + \alpha\boldsymbol{v}^{\xi}\cdot(\partial_i\boldsymbol{\xi}_1)\right]\mathrm{d}x_i,\end{aligned} \tag{22}$$

where the summation is taken with respect to the repeated index i over $i=1,2,3$. When the average is taken, we are left with

$$\bar{\Gamma} = \oint_C\left(\bar{v}_i^{\mathrm{L}} - \mathsf{p}_i\right)\mathrm{d}x_i = \oint_C\left(\boldsymbol{U}_0 + \frac{\alpha^2}{2}\overline{\boldsymbol{v}_2} + \bar{\boldsymbol{v}}^{\mathrm{S}} - \mathsf{p}\right)\cdot\mathrm{d}\boldsymbol{x}, \tag{23}$$

where use has been made of (15)–(17) and (19) [4]. The postulation of isovorticity on the disturbance requires $\Gamma = \bar{\Gamma} = \oint_C \boldsymbol{U}_0 \cdot \mathrm{d}\boldsymbol{x}$ for an arbitrary closed material loop C, and thus furnishes

$$\mathsf{p} = \mathcal{P}\left[\frac{\alpha^2}{2}\overline{v_2} + \bar{\boldsymbol{v}}^{\mathrm{S}}\right]. \tag{24}$$

Notably, this is a pointwise relation, unlike the integral relation found in ref [10].

In the following sections, we exemplify these relations by a rotating flow confined in a cylinder. It will be shown that the Stokes-drift term $\bar{\boldsymbol{v}}^{\mathrm{S}}$ gives no contribution to the axial-flow flux when integrated over the cross-section. As a consequence, a simple integral relation holds true between the mean-flow flux and the wave energy as suggested by (20) and (21).

4. Kelvin wave

Kelvin waves are a family of neutrally stable oscillations, thus linearized disturbances of $O(\alpha)$, on the core of a circular cylindrical vortex. We briefly recall the Kelvin waves in a confined geometry [25]. We take, as the basic flow, the rigid-body rotation of an inviscid incompressible fluid confined in a cylinder of circular cross-section of unit radius. The basic flow has both rotation symmetry about the cylinder axis and translation symmetry along it, featured by $\mathrm{SO}(2) \times \mathrm{O}(2)$.

Let us introduce cylindrical coordinates (r, θ, z) with the z-axis along the centerline. Let the r- and the θ-components of the two-dimensional basic velocity field $\boldsymbol{U}_0$ be U_0 and V_0, and the pressure be P_0. The suffix 0 signifies that these quantities pertain to the case of circular cross-section. The basic flow is confined in $r \leq 1$, with the velocity field given by

$$U_0 = 0, \quad V_0 = r, \quad P_0 = r^2/2 - 1. \tag{25}$$

We may take, as the disturbance field $\tilde{\boldsymbol{u}} = \alpha \boldsymbol{u}_{01}$, a normal mode

$$\boldsymbol{u}_{01} = A_m(t)\boldsymbol{u}_{01}^{(m)}(r)\mathrm{e}^{im\theta}\mathrm{e}^{ik_0 z}, \quad A_m(t) \propto \mathrm{e}^{-i\omega_0 t}, \tag{26}$$

where A_m is a complex function of time t and ω_0 is the frequency. This velocity field represents a Kelvin wave with azimuthal wavenumber $m(\in \mathbb{Z})$ and axial wavenumber k_0. The radial functions $\boldsymbol{u}_{01}^{(m)} = (u_{01}^{(m)}, v_{01}^{(m)}, w_{01}^{(m)})$ of the disturbance velocity and $p_{01}^{(m)}$ of the disturbance pressure are found from the equation of continuity and the linearized Euler equations. Here we write down the resulting functions only.

$$\begin{aligned}
p_{01}^{(m)} &= J_m(\eta_m r), \\
u_{01}^{(m)} &= \frac{i}{\omega_0 - m + 2}\left\{-\frac{m}{r}J_m(\eta_m r) + \frac{\omega_0 - m}{\omega_0 - m - 2}\eta_m J_{m+1}(\eta_m r)\right\}, \\
v_{01}^{(m)} &= \frac{1}{\omega_0 - m + 2}\left\{\frac{m}{r}J_m(\eta_m r) + \frac{2\eta_m}{\omega_0 - m - 2}J_{m+1}(\eta_m r)\right\}, \\
w_{01}^{(m)} &= \frac{k_0}{\omega_0 - m}J_m(\eta_m r),
\end{aligned} \tag{27}$$

where η_m is the radial wavenumber

$$\eta_m^2 = \left[\frac{4}{(\omega_0 - m)^2} - 1\right] k_0^2, \tag{28}$$

and J_m is the m-th Bessel function of the first kind. The boundary condition on the cylinder surface, $\boldsymbol{u}_{01} \cdot \boldsymbol{n} = u_{01}^{(m)} = 0$ at $r = 1$, provides the dispersion relation [25]

$$J_{m+1}(\eta_m) = \frac{(\omega_0 - m - 2)m}{(\omega_0 - m)\eta_m} J_m(\eta_m). \tag{29}$$

The Lagrangian displacement field is found from (5) with identification of $\boldsymbol{u}_1 = \boldsymbol{u}_{01}$, and, for the rigid-body rotation (25), is simply

$$\boldsymbol{\xi}_1 = \frac{i}{\omega_0 - m} \boldsymbol{u}_{01}. \tag{30}$$

When calculating the nonlinear quantity, the real part of the linear field $\boldsymbol{u}_{01}$ and $\boldsymbol{\xi}_1$ is to be taken, for example, as $\boldsymbol{u}_{01} = \mathrm{Re}[A_m(t)\boldsymbol{u}_{01}^{(m)}(r)\mathrm{e}^{i(m\theta + k_0 z)}]$. The axial mean flow of $O(\alpha^2)$ induced by a Kelvin wave is of our concern. Because of the absence of the axial component of the basic flow, $U_{0z} = 0$, the Stokes drift (17) reduces to $\bar{\boldsymbol{v}}^{\mathrm{S}} = \alpha^2 \overline{(\boldsymbol{\xi}_1 \cdot \nabla)\boldsymbol{u}_{01}}$. This is integrated over the circular cross-section of the cylinder to produce

$$\int \bar{v}_z^{\mathrm{S}} \mathrm{d}A = \int \alpha^2 \overline{\nabla \cdot (\boldsymbol{\xi}_1 w_{02})} \mathrm{d}A = 2\pi\alpha^2 \int_0^1 \frac{\partial}{\partial r}\left(\overline{\xi_{1r} w_{02}}\right) \mathrm{d}r = 0, \tag{31}$$

by virtue of the boundary condition $\xi_{1r} = 0$ at $r = 1$. The net axial flux of the Stokes drift is zero. Then, in view of (20) and (21), the total axial-flow flux, the integral of z-component of (24) over the cross-section, is connected with the wave energy $\alpha^2 H_2/2$

$$\frac{\alpha^2}{2} \int \overline{v_{2z}} \mathrm{d}A = \int \mathsf{p}_z \mathrm{d}A = k_0 \mu_0, \tag{32}$$

via the action μ defined by

$$\mu_0 = \frac{\alpha^2}{2} H_2 \Big/ \omega_0. \tag{33}$$

The qualification of p_z as the pseudomomentum is evident from this integral relation. It can be shown that the net flux of the Stokes drift survives when vorticity is created at a part of flow domain, for example, in the cases of a surface gravity wave and an internal gravity wave.

Many of the existing weakly nonlinear theories of rotating flows tried to intentionally ignore the mean flow of $O(\alpha^2)$ induced as a result of nonlinear interaction of Kelvin waves (see, for example, [27, 24]). In the following sections, we illustrate that the wave-induced mean flow is requisite for preserving the Hamiltonian structure of the weakly nonlinear amplitude equations, a desired property inherited from the Euler equations.

5. Linear stability of rotating flow in an elliptic cylinder

A rotating flow, endowed with SO(2) × O(2) symmetries, supports neutrally stable oscillations, but is destabilized, when either of these symmetries is lost by some perturbation. Breaking the SO(2) symmetry triggers the Moore-Saffman-Tsai-Widnall (MSTW) instability [26, 29]. An imposed pure shear, deforming the shape of circular streamlines into ellipse, couples a pair of Kelvin waves whose azimuthal wavenumbers are separated by 2, amplifying them together. The typically observable instability mode is caused by a pair of left- and right-handed helical waves ($m = \pm 1$) degenerated at zero frequency ($\omega_0 = 0$). The axial flows induced by $m = \pm 1$ waves cancel each other at every point of the domain at the exact resonance frequency $\omega_0 = 0$. The resonance of $(m, m + 2) = (0, 2)$ pair is also detected for the confined flow [18]. The mean axial velocity of $O(\alpha)$, induced by the combination of this Kelvin-wave pair, takes a nontrivial distribution over the cross-section, though the contributions from the $m = 0$ and the $m = 2$ waves have tendency to cancel together.

To write down the detail of the weakly nonlinear stability analysis, starting from the linear stability, makes this paper rather lengthy. Instead, we here give only a brief outline of the derivation of the amplitude equations to $O(\alpha^3)$, with an emphasis put on the influence of the mean flow of $O(\alpha^2)$. The detail will be reported elsewhere.

We express the elliptic shape of the cylinder cross-section, in terms of the small-parameter ϵ ($|\epsilon| \ll 1$), as

$$\frac{x^2}{1+\epsilon} + \frac{y^2}{1-\epsilon} = 1, \tag{34}$$

and consider only to $O(\epsilon)$. In conjunction with this distortion, the basic flow is perturbed as

$$\begin{aligned} &\boldsymbol{U} = \boldsymbol{U}_0 + \epsilon \boldsymbol{U}_1 + \cdots, \quad P = P_0 + \epsilon P_1 + \cdots; \\ &\quad U_1 = -r \sin 2\theta, \quad V_1 = -r \cos 2\theta, \quad P_1 = 0. \end{aligned} \tag{35}$$

The subscript designates order in the ellipticity parameter ϵ. The augmented term of $O(\epsilon)$ represents a pure shear or a steady quadrupole field.

We superimpose three-dimensional disturbance field $\tilde{\boldsymbol{u}}$ to this two-dimensional steady flow. We express the disturbance velocity field in the form of a power series in two small parameters ϵ and α, and retain, to $O(\alpha^3)$ in amplitude.

$$\boldsymbol{u} = \boldsymbol{U} + \tilde{\boldsymbol{u}} = \boldsymbol{U}_0 + \epsilon \boldsymbol{U}_1 + \alpha \boldsymbol{u}_{01} + \epsilon\alpha \boldsymbol{u}_{11} + \alpha^2 \boldsymbol{u}_{02} + \alpha^3 \boldsymbol{u}_{03} + \cdots, \tag{36}$$

Here, $\boldsymbol{u}_{kl}$ refers to the disturbance velocity field of $O(\epsilon^k \alpha^l)$. The side wall (34) of the cylinder is $r = 1 + \epsilon \cos 2\theta/2 + O(\epsilon^2)$ when the elliptic strain ϵ is small, and the boundary condition to be imposed at the rigid side wall is

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{at} \quad r = 1 + \epsilon \cos 2\theta/2, \tag{37}$$

where $\boldsymbol{n}$ is the unit outward normal vector to the cylinder boundary. The axial wavenumber k of the disturbance field in the form of a normal mode $\tilde{\boldsymbol{u}} \propto \exp[i(m\theta + kz)]$ is expanded in ϵ as

$$k = k_0 + \epsilon k_1 + \cdots. \tag{38}$$

The $O(\alpha)$ field $\alpha \boldsymbol{u}_{01}$ represents a wave on a rotating flow without breaking the SO(2) symmetry. First, we send a pair of Kelvin waves with azimuthal wavenumbers m

and $m+2$. Owing to the symmetry $z \to -z$ of the basic flow, the following combination of waves interact with each other mediated by the basic flow:

$$\boldsymbol{u}_{01} = A_+ \boldsymbol{u}_{A_+} \mathrm{e}^{i\psi_{A+}} + B_+ \boldsymbol{u}_{B_+} \mathrm{e}^{i\psi_{B+}} + A_- \boldsymbol{u}_{A_-} \mathrm{e}^{i\psi_{A-}} + B_- \boldsymbol{u}_{B_-} \mathrm{e}^{i\psi_{B-}} + c.c., \quad (39)$$

where *c.c.* designates complex conjugate and

$$\psi_{A\pm} = m\theta \pm k_0 z, \quad \psi_{B\pm} = (m+2)\theta \pm k_0 z. \quad (40)$$

The number of interacting waves is twice as that of the resonance between a pair of helical waves $(m, m+2) = (-1, 1)$. In the latter case, the given $A_\pm$ terms and their *c.c.* terms are sufficient to cover all the possible interacting waves.

The linear stability is found from an analysis of $O(\epsilon\alpha)$. The wave excited at $O(\epsilon\alpha)$ by the action of elliptical strain of $O(\epsilon)$ consists of

$$\boldsymbol{u}_{11} = \boldsymbol{u}_{11}^{(m-2)}(r,t)\mathrm{e}^{i(\psi_{A+}-2\theta)} + \boldsymbol{u}_{11}^{(m)}(r,t)\mathrm{e}^{i\psi_{A+}} + \boldsymbol{u}_{11}^{(m+2)}(r,t)\mathrm{e}^{i\psi_{B+}}$$
$$+ \boldsymbol{u}_{11}^{(m+4)}(r,t)\mathrm{e}^{i(\psi_{B+}+2\theta)} + [\psi_{A-}, \psi_{B-} \text{ terms}] + c.c., \quad (41)$$

and p_{11} has the similar form. Our choice of the rigid-body rotation for the basic flow makes the linear problem solvable in closed form. For the strained Rankine vortex in an unbounded domain, the solution of the linearized equations at $O(\epsilon\alpha)$ is written out in full in terms of the Bessel and modified Bessel functions without having to include integrals [7]. The situation is simpler for the bounded flow, and the solution for $\boldsymbol{u}_{11}$ is expressed in terms solely of the Bessel functions of the first kind. Enforcement of the boundary condition (37) separately for the m and the $m+2$ waves yields a set of linear algebraic equations with inhomogeneous terms. The requirement for these algebraic equations to have a solution gives rise to the $O(\epsilon)$ correction to the frequency. For the purpose of entering into the weakly nonlinear regime, we write down the solvability condition in the form of a set of differential equations for the amplitude with respect to the slow time $t_{10} = \epsilon t$. The resulting amplitude equations are

$$\frac{\partial A_\pm}{\partial t_{10}} = i\left[p_{11}B_\pm - p_{12}k_1A_\pm\right], \quad \frac{\partial B_\pm}{\partial t_{10}} = i\left[-p_{21}A_\pm + p_{22}k_1B_\pm\right], \quad (42)$$

where

$$p_{11} = -\frac{(\omega_0-m+2)(\omega_0-m)^3(\omega_0-m-2)}{32k_0^2(\omega_0-m-1)[2k_0^2+m(\omega_0+m)]}\frac{J_{m+2}(\eta_{m+2})}{J_m(\eta_m)}f, \quad (43)$$

$$p_{21} = \frac{(\omega_0-m)(\omega_0-m-2)^3(\omega_0-m-4)}{32k_0^2(\omega_0-m-1)[2k_0^2+(m+2)(\omega_0+m+2)]}\frac{J_m(\eta_m)}{J_{m+2}(\eta_{m+2})}f, \quad (44)$$

$$f = \omega_0^2(m+1)\left[k_0^2+m(m+2)\right]$$
$$- 2\omega_0 m(m+2)\left[k_0^2+(m+1)^2\right]$$
$$+ (m+1)\left[k_0^2(m^2+2m-4)+m^2(m+2)^2\right], \quad (45)$$

$$p_{12} = -\frac{(k_0^2+m^2)(\omega_0-m-2)(\omega_0-m)(\omega_0-m+2)}{2k_0(2k_0^2+\omega_0 m+m^2)}, \quad (46)$$

$$p_{22} = \frac{(k_0^2+(m+2)^2)(\omega_0-m-4)(\omega_0-m-2)(\omega_0-m)}{2k_0[2k_0^2+\omega_0(m+2)+(m+2)^2]}. \quad (47)$$

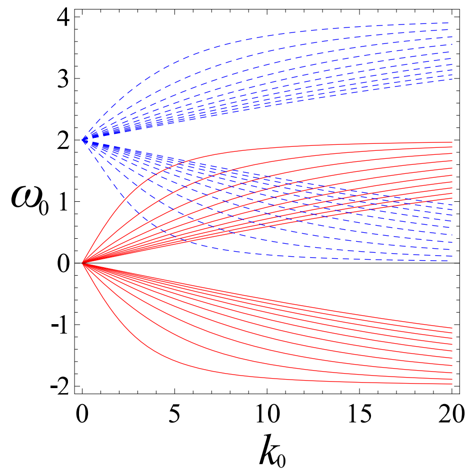

Figure 1. Dispersion relation of Kelvin waves with $m = 0$ (solid lines) and $m = 2$ (dashed lines).

It turns out that the stability criterion depends on the sign of the coefficients p_{11} and p_{21}. Degeneracy of a $(m, m+2)$ Kelvin-wave pair occurs at the intersection points, in the frequency range $m < \omega_0 < m+2$, between curves of the m wave and those of $m+2$ wave. As is read off from the explicit form (43), (44), (46) and (47), the coefficients p_{12} and p_{22} are positive, and p_{11} and p_{21} are of the same signs in this frequency range. The solution of (42) behaves as $A_\pm, B_\pm \propto \exp(-i\omega_1 t_{10}) = \exp(-i\epsilon\omega_1 t)$, and, in case of instability, the growth rate is provided by $\sigma_1 = |\mathrm{Im}[\omega_1]|$. The local maximum of the growth rate is attained at the exact intersection point for which $k_1 = 0$. Hence, the maximum growth rate is

$$\sigma_{1max} = \sqrt{p_{11}p_{21}}, \tag{48}$$

and the corresponding eigenfunction satisfies

$$A_\pm/B_\pm = i\sqrt{p_{11}/p_{21}}. \tag{49}$$

The half-width Δk_1 of the unstable wavenumber band around the intersection point is

$$\Delta k_1 = \sqrt{p_{11}p_{21}/(p_{12}p_{22})}. \tag{50}$$

When specialized to $(m, m+2) = (0, 2)$ pair, (48) and (50) become, respectively,

$$\sigma_{1max}\Big|_{m=0} = \left[\frac{-(\omega_0-4)(\omega_0-2)^6\omega_0^4(\omega_0+2)^3}{4096k_0^2(\omega_0-1)^2(k_0^2+\omega_0+2)}\right]^{1/2}, \tag{51}$$

$$\Delta k_1\Big|_{m=0} = \left|\frac{(\omega_0-2)^2\omega_0(\omega_0+2)}{16(k_0^2+4)^{1/2}(\omega_0-1)}\right|. \tag{52}$$

Notice that the growth rate (51) has a factor $|\omega_0 - 1|^{-1}$. Predominant modes arise at the intersection points located midway between $\omega_0 = 0$ and 2. The corresponding unstable band is wider. The growth rate at the first three such intersection points (k_0, ω_0) of small wavenumbers is $\sigma_{1max} \approx 0.5325$ at $(k_0, \omega_0) = (2.326, 1.038)$, $\sigma_{1max} \approx 0.5509$ at (4.125, 1.014) and $\sigma_{1max} \approx 0.5566$ at (5.928, 1.007). These values of growth rate are almost the same as those of the $(-1, +1)$ stationary modes. The growth rate at intersection points off the line $\omega_0 = 1$ is considerably smaller, for instance, $\sigma_{1max} \approx 0.03942$ at (3.034, 1.241) and $\sigma_{1max} \approx 0.06832$ at (3.075, 0.8029), though the instability occurs at every intersection point.

As prescribed by the Hamiltonian bifurcation theory [22], the instability is characterized by a collision of a pair of purely imaginary eigenvalues, along with a collision at its complex conjugate position. A necessary condition for instability at a non-zero degenerate frequency ($\omega_0 \neq 0$) is that signs of the corresponding wave energy should be opposite. The energy of Kelvin waves per unit length in z is directly calculated through (21), with the volume integral replaced by area integral over the cylindrical cross-section. Upon substitution from (39) supplemented by (5), we find the wave energy for the $(m, m+2)$ pair to be $\alpha^2 H_2/2$,

$$H_2 = C\frac{\omega_0}{2}\left\{\frac{1}{p_{11}}\left(|A_+|^2 + |A_-|^2\right) - \frac{1}{p_{21}}\left(|B_+|^2 + |B_-|^2\right)\right\}, \tag{53}$$

where

$$C = \frac{\pi f J_m(\eta_m) J_{m+2}(\eta_{m+2})}{k_0^2(\omega_0 - m - 1)}. \tag{54}$$

The sign of the wave energy is supplied by that of the coefficients $C\omega_0 p_{11}$ and -$C\omega_0 p_{21}$. We can directly confirm from (43) and (44) that the sign of the m wave with amplitude $A_\pm$ is positive and that of the $m+2$ wave with amplitude $B_\pm$ is negative in the range $m < \omega_0 < m+2$ where these waves can coexist, being in consistent the necessary condition for instability. This is a trend in common with the case of an open geometry [7]. If the $O(\alpha)$ terms are restored, (42) is extended to

$$\frac{\mathrm{d}A_\pm}{\mathrm{d}t} = i\left[-\omega_0 A_\pm + \epsilon\left(p_{11}B_\pm - k_1 p_{12} A_\pm\right)\right], \tag{55}$$

$$\frac{\mathrm{d}B_\pm}{\mathrm{d}t} = i\left[-\omega_0 B_\pm - \epsilon\left(p_{21}A_\pm - k_1 p_{22} B_\pm\right)\right]. \tag{56}$$

These equations do not constitute a Hamiltonian system

$$\frac{\mathrm{d}A_\pm}{\mathrm{d}t} = -2i\frac{\partial H_2}{\partial A_\pm^*}, \quad \frac{\mathrm{d}B_\pm}{\mathrm{d}t} = -2i\frac{\partial H_2}{\partial B_\pm^*}, \tag{57}$$

where asterisk $*$ stands for complex conjugate. With natural choice (53) for the Hamiltonian, (57) is unable to produce, regardless of any choice of normalization of H_2, equations with a common coefficient of the $\omega_0 A_\pm$ and $\omega_0 B_\pm$ terms, as requested by (55) and (56).

This is cured by a normalization of independent variables

$$a_\pm = A_\pm/\sqrt{|p_{11}|}, \quad b_\pm = B_\pm^*/\sqrt{|p_{21}|}, \tag{58}$$

with which (55) and (56) are converted into canonical Hamiltonian equations (57) with the Hamiltonian $\hat{H}_2$ provided by

$$\begin{aligned}\hat{H}_2 = &\frac{\omega_0}{2}\left[|a_+|^2 + |a_-|^2 - \left(|b_+|^2 + |b_-|^2\right)\right]\\ &+ \frac{\epsilon k_1}{2}\left[p_{12}\left(|a_+|^2 + |a_-|^2\right) + p_{22}\left(|b_+|^2 + |b_-|^2\right)\right]\\ &- \epsilon\sigma' \mathrm{Re}\left[a_+ b_+ + a_- b_-\right],\end{aligned} \tag{59}$$

where $\sigma' = \mathrm{sgn}[p_{11}]\sqrt{p_{11}p_{21}}$, with sgn[$\circ$] designating the sign of $\circ$, Re[$\circ$] designates taking the real part of $\circ$, and the fact that p_{11} and p_{21} are of the same sign is to be kept in mind.

In the next section, we demonstrate that the Hamiltonian structure is preserved to higher orders. This desirable fact will lend some support to the correctness of our treatment of the mean flow induced at $O(\alpha^2)$.

6. Amplitude equation

This section exemplifies the effect of nonlinear interaction on the resonance between the $m = 0$ and 2 waves. We start with the linear disturbance velocity (39) and (40) with choosing $m = 0$

$$\boldsymbol{u}_{01} = A_+\boldsymbol{u}_{A_+}e^{ik_0z} + B_+\boldsymbol{u}_{B_+}e^{i(2\theta+k_0z)} + A_-\boldsymbol{u}_{A_-}e^{-ik_0z} + B_-\boldsymbol{u}_{B_-}e^{i(2\theta-k_0z)} + c.c. \tag{60}$$

The linear disturbance pressure p_{01} takes the same form. The mean flow of $O(\alpha^2)$ cannot be determined from the Euler equations, because, for the basic flow with rotation and translation symmetries about the z-axis, the corresponding linear operator degenerates to admit arbitrary azimuthal and axial velocity profiles. Restriction to the isovortical disturbances enables us to unambiguously determine the mean flow from the average of (10) to be

$$\overline{\boldsymbol{u}_{02}} = \overline{\boldsymbol{\xi}_1 \times \frac{\partial \boldsymbol{\xi}_1}{\partial z}} = (0, \overline{v_{02}}, \overline{w_{02}})\,. \tag{61}$$

There is no radial flow, and the azimuthal and the axial components are, respectively,

$$\begin{aligned}\overline{v_{02}} =& -4\left(|A_+|^2 + |A_-|^2\right)\frac{k_0^2\eta_0 J_0(\eta_0 r)J_1(\eta_0 r)}{\omega_0^2(\omega_0-2)(\omega_0+2)} \\ &+4\left(|B_+|^2 + |B_-|^2\right)\frac{k_0^2 J_2(\eta_2 r)}{\omega_0(\omega_0-2)^2}\left[\frac{2J_2(\eta_2 r)}{(\omega_0-2)r} - \frac{\eta_2 J_3(\eta_2 r)}{\omega_0-4}\right],\end{aligned} \tag{62}$$

$$\begin{aligned}\overline{w_{02}} =& -8\left(|A_+|^2 - |A_-|^2\right)\frac{k_0^3 J_1^2(\eta_0 r)}{\omega_0^3(\omega_0-2)(\omega_0+2)} \\ &-8\left(|B_+|^2 - |B_-|^2\right)\frac{k_0}{\omega_0(\omega_0-2)^2}\left[\frac{2J_2^2(\eta_2 r)}{\omega_0 r^2} - \frac{\eta_2 J_2(\eta_2 r)J_3(\eta_2 r)}{\omega_0^2 r}\right. \\ &\left.+\frac{k_0^2 J_3^2(\eta_2 r)}{(\omega_0-4)(\omega_0-2)}\right].\end{aligned} \tag{63}$$

The mean flow (62) and (63) of $O(\alpha^2)$ are gained directly from the ansatz (60) of $O(\alpha)$, without recourse to elliptical strain of $O(\epsilon)$, unlike the Eulerian treatment [28, 21]. The latter treatment suffers from inclusion of undetermined constants, because the solvability condition at $O(\alpha^2\epsilon)$ determines only the time derivative of the amplitude of $\overline{\boldsymbol{u}_{02}}$.

The remaining procedure for deriving the amplitude equations is carried out within the Eulerian framework. We content ourselves with writing down the resulting equations

k_0	2.326	3.034	4.125
ω_0	1.038	1.241	1.014
p_{11}	0.5397	−0.05022	0.5535
p_{21}	0.5254	−0.03094	0.5484
p_{12}	0.3260	0.2515	0.1826
p_{22}	0.3541	0.2271	0.1899
s_{11}	−13.36	−28.33	−84.03
s_{12}	23.21	88.88	141.9
s_{13}	20.58	40.54	133.5
s_{14}	10.91	−47.65	44.62
s_{15}	−5.544	−20.16	−27.70
s_{21}	−22.58	−54.76	−141.3
s_{22}	21.57	85.62	118.5
s_{23}	−10.63	29.35	−44.20
s_{24}	−3.328	14.42	−39.98
s_{25}	5.398	12.42	27.44

Table 1. Numerical values of the coefficients of amplitude equations (64) and (65).

for the complex amplitudes $A_\pm$ and $B_\pm$ of the given Kelvin waves (60), to $O(\alpha^3)$:

$$\begin{aligned}\frac{\mathrm{d}A_\pm}{\mathrm{d}t} &= i\{-\omega_0 A_\pm + \epsilon\,(p_{11}B_\pm - k_1 p_{12}A_\pm) \\ &\quad + \alpha^2\left[A_\pm\left(s_{11}|A_\pm|^2 + s_{12}|B_\pm|^2 + s_{13}|A_\mp|^2 + s_{14}|B_\mp|^2\right)\right. \\ &\quad \left.+ s_{15}A_\mp B_\pm B_\mp^*\right]\},\end{aligned} \tag{64}$$

$$\begin{aligned}\frac{\mathrm{d}B_\pm}{\mathrm{d}t} &= i\{-\omega_0 B_\pm - \epsilon\,(p_{21}A_\pm - k_1 p_{22}B_\pm) \\ &\quad + \alpha^2\left[B_\pm\left(s_{21}|A_\pm|^2 + s_{22}|B_\pm|^2 + s_{23}|A_\mp|^2 + s_{24}|B_\mp|^2\right)\right. \\ &\quad \left.+ s_{25}B_\mp A_\pm A_\mp^*\right]\}.\end{aligned} \tag{65}$$

For three resonance points (k_0, ω_0) of small wavenumber, numerical values of the coefficients p_{ij} and s_{ij} entering in these equations are listed in Table 1. The coefficients p_{11} and p_{21} in the linear terms have the same sign, implying occurrence of the MSTW instability.

We have a numerical evidence that the Hamiltonian structure recovered for the linear equations for the normalized amplitude (58) is highly likely to carry over to $O(\alpha^3)$. For simplification, we further eliminate the first terms of (64) and (65) by introducing new normalized variables

$$z_{1\pm} = A_\pm e^{i\omega_0 t}/\sqrt{|p_{11}|}, \quad z_{2\pm} = B^* e^{-i\omega_0 t}/\sqrt{|p_{21}|}. \tag{66}$$

Then the above amplitude equations (64) and (65) are transformed into

$$\frac{\mathrm{d}z_{1\pm}}{\mathrm{d}t} = i\left\{\epsilon\left(\sigma' z_{2\pm}^* - k_1 p_{12} z_{1\pm}\right) + \alpha^2 c_{15} z_{1\mp} z_{2\pm}^* z_{2\mp}\right.$$

$$+ \alpha^2 z_{1\pm}\left(c_{11}|z_{1\pm}|^2 + c_{12}|z_{2\pm}|^2 + c_{13}|z_{1\mp}|^2 + c_{14}|z_{2\mp}|^2\right)\Big\}, \tag{67}$$

$$\frac{\mathrm{d}z_{2\pm}}{\mathrm{d}t} = i\left\{\epsilon\left(\sigma' z_{1\pm}^* - k_1 p_{22} z_{2\pm}\right) + \alpha^2 c_{25} z_{2\mp} z_{1\pm}^* z_{1\mp}\right.$$
$$\left. + \alpha^2 z_{2\pm}\left(c_{21}|z_{1\pm}|^2 + c_{22}|z_{2\pm}|^2 + c_{23}|z_{1\mp}|^2 + c_{24}|z_{2\mp}|^2\right)\right\}, \tag{68}$$

where

$$c_{11} = s_{11}|p_{11}|,\ c_{13} = s_{13}|p_{11}|,\ c_{12} = s_{12}|p_{21}|,\ c_{14} = s_{14}|p_{21}|,$$
$$c_{15} = s_{15}|p_{21}|,\ c_{21} = -s_{21}|p_{11}|,\ c_{23} = -s_{23}|p_{11}|,\ c_{25} = -s_{25}|p_{11}|,$$
$$c_{22} = -s_{22}|p_{21}|,\ c_{24} = -s_{24}|p_{21}|. \tag{69}$$

The coupled system (67) and (68) constitute canonical Hamiltonian equations for conjugate variables (q_j, p_j) defined by $z_j = q_j + ip_j$ $(j = 1, 2)$, with the Hamiltonian

$$H = \frac{\epsilon k_1}{2}\left[p_{12}\left(|z_{1+}|^2 + |z_{1-}|^2\right) + p_{22}\left(|z_{2+}|^2 + |z_{2-}|^2\right)\right]$$
$$- \epsilon\sigma' \mathrm{Re}\left[z_{1+}z_{2+} + z_{1-}z_{2-}\right]$$
$$- \alpha^2\left\{\frac{c_{11}}{4}\left(|z_{1+}|^4 + |z_{1-}|^4\right) + \frac{c_{22}}{4}\left(|z_{2+}|^4 + |z_{2-}|^4\right) + \frac{c_{13}}{2}|z_{1+}|^2|z_{1-}|^2\right.$$
$$+ \frac{c_{24}}{2}|z_{2+}|^2|z_{2-}|^2 + c_{15}\mathrm{Re}\left[z_{1+}z_{2+}\overline{z_{1-}z_{2-}}\right]$$
$$+ \frac{c_{12}}{2}\left(|z_{1+}|^2|z_{2+}|^2 + |z_{1-}|^2|z_{2-}|^2\right)$$
$$\left. + \frac{c_{14}}{2}\left(|z_{1+}|^2|z_{2-}|^2 + |z_{1-}|^2|z_{2+}|^2\right)\right\}, \tag{70}$$

if and only if the following equalities hold true:

$$c_{12} = c_{21}, \quad c_{14} = c_{23}, \quad c_{15} = c_{25}. \tag{71}$$

Although mathematical proof is unattained, numerical computation shows that all the three hold to significant digits as is read off from Table 1, with the aid of (69). Otherwise stated, we have reached the Hamiltonian normal form [12, 19]. It cannot be overemphasized that the mean-flow velocity (62) and (63) of $O(\alpha^2)$ is requisite for preservation of the Hamiltonian structure.

The system of coupled equations (67) and (68) are shown to have the following first integrals, other than the full Hamiltonian (70),

$$\omega_0\left(|z_{1+}|^2 - |z_{2+}|^2 + |z_{1-}|^2 - |z_{2-}|^2\right), \tag{72}$$

$$k_0\left(|z_{1+}|^2 - |z_{2+}|^2 - |z_{1-}|^2 + |z_{2-}|^2\right). \tag{73}$$

The former amounts to the kinetic energy (59) of the Kelvin waves to $O(\alpha^2)$. The latter amounts to the flux of the axial flow (63) to the same order, being reflective of the translation symmetry along the z-axis. Conceivably, there is no other invariants, because the rotation symmetry about the z-axis is broken. The equations (67) and (68) are canonical Hamiltonian equations of four degrees of freedom, whereas they could possess no more than three first integrals including the Hamiltonian. If this is true, the solution of (67) and (68) would exhibit chaos.

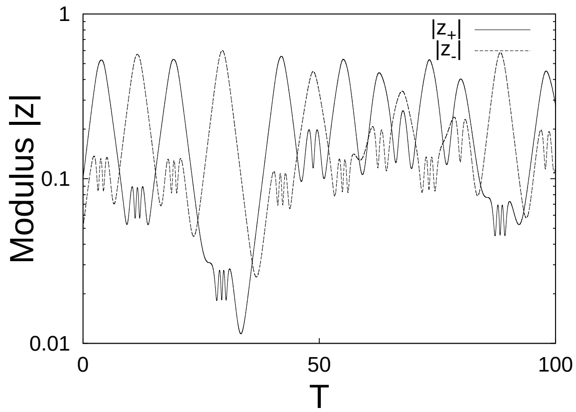

Figure 2. Evolution in $T = \epsilon t$ of the magnitude $|z_+(t)| = |z_{1+}(t)| = |z_{2+}(t)|$ and $|z_-(t)| = |z_{1-}(t)| = |z_{2-}(t)|$ of the normalized Kelvin-wave amplitude $z_{1\pm}$ of $m = 0$ and $z_{2\pm}$ of $m = 2$. The initial conditions are $z_{1+} = 0.1i$, $z_{2+} = 0.1$, $z_{1-} = 0.05i$, $z_{2-} = 0.05$.

When the local maximum of the growth rate around an intersection point of the dispersion curves of the $m = 0$ and 2 waves is attained at $k_1 = 0$, the eigenfunction of the most unstable mode satisfies (49), or equivalently

$$|z_{1\pm}|^2 - |z_{2\pm}|^2 = 0. \tag{74}$$

It is noteworthy that, for the most unstable mode, the disturbance energy (72) and the axial flow flux (73) of $O(\alpha^2)$ both exactly vanish. The form of the first integrals (72) and (73) guarantees $|z_{1\pm}|^2 - |z_{2\pm}|^2 \equiv$ const. at all $t \geq 0$. It follows that, if we set $|z_{1+}| = |z_{2+}|$ and $|z_{1-}| = |z_{2-}|$ at an initial instant, the equalities permanently hold, $|z_{1\pm}(t)| = |z_{2\pm}(t)| = |z_\pm(t)|$ $(t \geq 0)$ say. With this choice of the initial amplitude, we draw, in figure 2, the amplitude $|z_\pm(t)|$ as functions of the stretched time $T = \epsilon t$ for the resonance mode of the lowest wavenumber $(k_0, \omega_0) \approx (2.326, 1.038)$. In the initial stage, a minute disturbance is amplified due to the linear instability, and thereafter nonlinear interaction comes into play as the magnitude of $|z_\pm(t)|$ is increased to some value of order unity. Figure 2 shows that, after the initial exponential growth, the amplitude $|z_\pm(t)|$ behave chaotically, as anticipated. As an overall trend, the amplitude $|z_\pm(t)|$ is bounded.

Rodrigues and Luca [27] found chaotic behavior of the nonlinear evolution for the amplitude of waves in the WKB form that is valid for short-wavelength disturbances, assuming the absence of the mean flow of $O(\alpha^2)$ induced by nonlinear interaction of the waves. Our analysis provides us an illuminating reasoning for the shortage of symmetries behind the occurrence of the chaos. In contrast, the nonlinear evolution of the stationary resonance ($\omega_0 = 0$) between the left- and the right-handed helical waves ($m = \pm 1$) exhibits a smooth saturation as this is a realization of the Hamiltonian pitchfork bifurcation for the Hamiltonian system of two degrees of freedom equipped with two first integrals [25]. In either case, eventual saturation of the wave amplitude growing through the Hamiltonian-Hopf or the Hamiltonian-pitchfork bifurcation is not consistent with the experimental observation [23, 6]; rather, the Hamiltonian Hopf bifurcation triggers excitation of numerous modes in a short time interval, inviting transition to turbulence. To describe this intricate behavior, secondary and tertially instabilities caused by interactions among various combinations of several Kelvin waves should be considered on the way of nonlinear growth [24, 18, 27].

7. Conclusion

We have established a framework of incorporating the wave-induced mean flow, expressed in the Lagrangian variables, into the Eulerian approach for a weakly nonlinear stability analysis of a rotating flow subject to a symmetry breaking perturbation. We cannot overemphasize the advantage of the Lagrangian approach. For the basic flow with translation and rotation symmetries with respect to an axis, the linearized Euler operator is incapable of determining the drift current or the mean flow of quadratic in amplitude induced by nonlinear interaction of a prescribed wave. A sensible treatment is to restrict disturbances to isovortical or irrotational ones by means of the Lagrangian description. In §2, we recalled general form of the disturbance field expressed in terms of the Lagrangian variables. In §3, the obtained nonlinear field was compared with the generalized Lagrangian-mean (GLM) theory. The GLM theory manifests the relation of the wave-induced mean flow with the pseudomomentum and the Stokes drift. As is observed from (10), the mean-flow velocity takes significant values in the region where the vorticity of the basic flow dominates. We speculate that transport and mixture of materials are promoted in such vortical regions.

Exploiting thus obtained mean flow of $O(\alpha^2)$, we derived the weakly nonlinear amplitude equations to $O(\alpha^3)$ in §6. We have a numerical evidence that, with an appropriate normalization (66) of the amplitude, the amplitude equations are reducible to canonical Hamiltonian equations (67) and (68), though mathematical proof is yet to be sought. This serves as a convincing demonstration for consistency of our general framework for weakly nonlinear stability analysis of a steady basic flow endowed with symmetries. At the linear stage, the disturbance vorticity of a resonant pair of Kelvin waves is continuously stretched by the straining field that breaks the rotation symmetry of the basic flow, whereby the disturbance amplitude undergoes exponential amplification [20, 7]. The nonlinear effect, though chaotically, suppresses this exponential growth by deflecting the disturbance vorticity vector from the stretching direction as soon as the amplitude becomes sufficiently large [28, 25].

However this behavior fails to account for the vigorous excitation of a number of waves and the ultimate catastrophic disruption [23, 6]. The nonlinear interaction of a single MSTW mode described here is far from sufficient in describing practical flows. The secondary and the tertiary instabilities of the MSTW mode call for an individual investigation [24, 8].

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