A computer-assisted proof for the pattern formation on reaction-diffusion systems

蔡, 婷
九州大学大学院数理学府

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A computer-assisted proof for the pattern formation on reaction-diffusion systems

Shuting Cai

Graduate School of Mathematics,
Kyushu University

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Abstract

In this paper we give a method by computer-assistance to prove a pattern formation. As a typical model we consider two dimensional time-dependent reaction-diffusion equations with Neumann boundary conditions. For suitable system parameters we solve (approximately) the parabolic problem, hoping for some convergence to some pattern formation stationary (approximate) solution, and improve the approximation to a stationary solution by Newton's method, then enclose the stationary solution by our numerical verification method. Next we prove that the operator linearized at the exact stationary solution is a sectorial operator and compute a bound for the resolvent of the linearized operator which is needed for semigroup estimates. By using the semigroup estimate we analytically compute a domain of attraction for the stationary solution, i.e. some (norm-)neighborhood of the stationary solution such that, for initial data within this neighborhood, the parabolic solution converges to the stationary solution. For suitable initial conditions, if we enclose the solution of the parabolic problem until, for some time T, the enclosing set is a subset of the domain of attraction, then we can conclude that from time T on, convergence to the stationary solution takes place. This gives a complete convergence result, proving a pattern formation, for the initial conditions used for the parabolic problem.

Keyword:

Pattern formation; Computer-assisted proof; Reaction-diffusion system; Domain of attraction.

Chapter 1

Introduction

We consider a time-dependent reaction-diffusion system with Neumann boundary conditions

$$\begin{cases} u_t = \gamma f(u,v) + \Delta u \text{ in } \Omega, \\ v_t = \gamma g(u,v) + d\Delta v \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.0.1}$$

where Ω is a bounded domain in $\Re^2, \gamma, d$ are some positive constants, f, g are nonlinear/linear functions depending on each model and $\nu = (\nu_1, \nu_2)$ is an outer unit normal vector on $\partial\Omega$.

This kind of problem (1.0.1) can be applied in mathematical biology[10, 11]. One simple system is FitzHugh-Nagumo reaction[6, 15]. In [7], u as an activator and v as an inhibitor are interpreted as relative concentrations of two substances known as morphogens. In [26], this kind of problem, called "Excitable media", can be applied in mammalian heart muscle and its cells and Xenopus eggs. For more background, see references in [23].

There are many papers considering the stationary solution of the system. Here we mention some results on the systems of FitzHugh-Nagumo type. There are several results about Dirichlet problem[4, 3, 20, 24]. In [4] the minimization problem associated with the system is considered, and in [3] the peak solutions for the system are investigated. There is also one paper considering the relationship between the parameter and the solution of the system[20]. In [24], by using a numerical verification method, the author encloses the exact solution of the equations. There are also some results for Neumann problem. In [5] some results on the solutions with interior and boundary peaks are shown, and in [21] the relationship between one parameter and the energy minimizers is discussed. In [2], the authors proposed a numerical verification method to enclose a solution of the two dimensional system.

In this paper, we will propose a computer-assisted method to prove a pattern formation. Knowing the pattern formation is very important in biology[11]. When we solve a time-dependent problem, we often see some numerical convergence of them, but analytically there is no proof of it. On the other hand when we compute a steady state problem, we do not know from which initial state the stationary solution appeared. This paper is about how to overcome these two difficulties. As a concrete example, we will apply our method to Schnakenberg equation, which is another reaction-diffusion system.

There are eight chapters in this paper. In chapter 2, we prepare some function spaces and notations and then the fixed-point formulation and the construction of a priori error estimate for the projection are derived. And in Chapter 3, for suitable system parameters we solve (approximately) the parabolic problem and improve the approximation to a stationary (approximate) solution by Newton's method. After that, in Chapter 4, we use Nakao's method to enclose the stationary solution near this approximate solution. This method, which is similar to the method in [2], is based on the infinite dimensional fixed point theorem. First, the time-independent system is rewritten in a fixed point form and then the fixed point equation is decomposed into the finite dimensional part and the infinite dimensional error part. Based on these two parts, we construct a set which satisfies the hypothesis of Schauder's fixed point theorem for a compact map in a suitable Sobolev space. This verification method was originated by Nakao[16] and then has been developed by him and his coworkers[18, 19, 12, 13, 14].

Then we prove that the operator linearized at the exact stationary solution is a sectorial operator and compute a bound for the resolvent of the linearized operator which is needed for semigroup estimates. By using the semigroup estimate we analytically compute a domain of attraction for the stationary solution. And thus, for suitable initial conditions, if we enclose the solution of the parabolic problem until, for some time T, the enclosing set is a subset of the domain of attraction, then we can conclude that from time T on, convergence to the stationary solution takes place. This is showed in Chapter 6. And in order to get the results in Chapter 6, we propose a computer-assisted method to exclude the eigenvalues of the linearized operator in Chapter 5. In Chapter 7, there are some numerical results. We apply our method to an example with some suitable system parameters and then get its domain of attraction. At last there are some conclusions in Chapter 8.

Chapter 2

Some notations and projection error estimation

We consider the domain $\Omega = (0,l) \times (0,l) \subset \Re^2 (l \geq 1)$. Then we choose basis function as

$$\varphi_{i_1 i_2}(x,y) = \cos(i_1 \pi x/l)\cos(i_2 \pi y/l),\ \ i_1, i_2 = 0, 1, 2, \ldots$$

For a fixed non-negative integer N we reorder the basis function

$$\varphi_i = \cos(i_1 \pi x/l)\cos(i_2 \pi y/l), (i = 1, 2, 3, \cdots (N+1)^2)\ (i_1, i_2 = 0, 1, 2, \cdots N)$$

in the following way:

$$i = i_1(N+1) + i_2 + 1.$$

The Sobolev space $W^{k,p}(\Omega)(1 \leq p \leq +\infty)$ is defined as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k\},$$

where α is a multi-index. The natural number k is called the order of the Sobolev space $W^{k,p}(\Omega)$.

And we suppose that for the L^2-Sobolev space of order k on Ω, $H^k(\Omega)$, $\psi \in H^k(\Omega)$ $(k \geq 0)$ is expanded in the Fourier series as

$$\psi = \sum_{i=1}^{\infty} a_i \varphi_i \quad (a_i \in \Re).$$

For a non-negative integer N, let X_N denote a function space as

$$X_N := \left\{ v_N = \sum_{n,m=0}^{N} c_{nm} \varphi_{nm} \mid c_{nm} \in \Re \right\} \subset H^2(\Omega) \subset H^1(\Omega).$$

For $z_1,\, z_2 \in H^1(\Omega)$, define the usual inner product as

$$\langle z_1, z_2\rangle_{H^1(\Omega)} := (z_1, z_2)_{L^2(\Omega)} + (\nabla z_1, \nabla z_2)_{L^2(\Omega)},$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ means the inner product on $L^2(\Omega)$.

For $z \in H^k(\Omega)(1 \le k < \infty, k \neq 2)$, define the usual norm as

$$\|z\|^2_{H^k(\Omega)} := \sum_{|\alpha|\le k} \|D^\alpha z\|^2_{L^2(\Omega)}.$$

When $k = 2$, for $z = \sum\limits_{n,m=0}^{\infty} c_{nm}\varphi_{nm} \in H^2(\Omega)$, we have

$$\begin{aligned}
&\|z\|^2_{L^2(\Omega)} + \|\nabla z\|^2_{L^2(\Omega)} + \|\Delta z\|^2_{L^2(\Omega)} \\
=&\|z\|^2_{L^2(\Omega)} + \|\nabla z\|^2_{L^2(\Omega)} + \sum_{n,m=0}^{\infty} c^2_{nm}(\|(\varphi_{nm})_{xx}\|^2_{L^2(\Omega)} + \|(\varphi_{nm})_{yy}\|^2_{L^2(\Omega)} \\
&+ 2((\varphi_{nm})_{xx}, (\varphi_{nm})_{yy})_{L^2(\Omega)}) \\
=&\|z\|^2_{L^2(\Omega)} + \|\nabla z\|^2_{L^2(\Omega)} + \sum_{n,m=0}^{\infty} c^2_{nm}((n^4 + m^4)\pi^4/l^4 + 2n^2m^2\pi^4/l^4)\|\varphi_{nm}\|^2_{L^2(\Omega)} \\
=&\|z\|^2_{L^2(\Omega)} + \|\nabla z\|^2_{L^2(\Omega)} + \sum_{n,m=0}^{\infty} c^2_{nm}(\|(\varphi_{nm})_{xx}\|^2_{L^2(\Omega)} + \|(\varphi_{nm})_{yy}\|^2_{L^2(\Omega)} \\
&+ 2((\varphi_{nm})_{xy}, (\varphi_{nm})_{xy})_{L^2(\Omega)}) \\
=&\sum_{|\alpha|\le 2} \|D^\alpha z\|^2_{L^2(\Omega)},
\end{aligned}$$

where $(\varphi_{nm})_x$ is the derivative of φ_{nm} with respect to x, $(\varphi_{nm})_{xy}$ is the derivative of $(\varphi_{nm})_x$ with respect to y and $(\varphi_{nm})_{xx}$, $(\varphi_{nm})_{yy}$ are the second derivative of φ_{nm} with respect to x and y respectively, therefore, we define the norm in $H^2(\Omega)$ as

$$\|z\|^2_{H^2(\Omega)} := \|z\|^2_{L^2(\Omega)} + \|\nabla z\|^2_{L^2(\Omega)} + \|\Delta z\|^2_{L^2(\Omega)}.$$

And for $z_1,\, z_2 \in H^2(\Omega)$, define the inner product as

$$\langle z_1, z_2\rangle_{H^2(\Omega)} := (z_1, z_2)_{L^2(\Omega)} + (\nabla z_1, \nabla z_2)_{L^2(\Omega)} + (\Delta z_1, \Delta z_2)_{L^2(\Omega)}.$$

For $z \in H^1(\Omega)$, let $P_N : H^1(\Omega) \to X_N$ denote the H^1-projection defined by the truncation operator:

$$P_N\left(\sum_{n,m=0}^{\infty} c_{nm}\varphi_{nm}\right) = \sum_{n,m=0}^{N} c_{nm}\varphi_{nm},$$

which satisfies

$$\langle z - P_N z, z_N \rangle_{H^1(\Omega)} = 0, \quad \text{for all } z_N \in X_N.$$

When $z \in H^2(\Omega)$, we also have

$$\langle z - P_N z, z_N \rangle_{H^2(\Omega)} = 0, \quad \text{for all } z_N \in X_N.$$

Therefore, we also use P_N as a H^2-projection.

Set $P : H^2(\Omega) \times H^2(\Omega) \to X_N \times X_N$ as

$$P(z_1, z_2) := (P_N z_1, P_N z_2), \; z_1, z_2 \in H^2(\Omega).$$

For Hilbert spaces X and Y, we define the inner product and the norm in $X \times Y$ as

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right)_{X \times Y} := (x_1, x_2)_X + (y_1, y_2)_Y$$

and

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| := \sqrt{\|x\|_X^2 + \|y\|_Y^2}.$$

Defining the operator $L : H^2(\Omega) \to L^2(\Omega)$ by $L\psi := -\triangle\psi + \psi$, we get the following lemma.

Lemma 2.0.1 *For all $\phi \in L^2(\Omega)$, the linear equation*

$$\begin{cases} L\psi = \phi & \text{in } \Omega, \\ \dfrac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.0.1}$$

has the unique solution $\psi \in H^2(\Omega)$.

Proof. Let $\psi = \sum_{n,m=0}^{\infty} \psi_{nm}\varphi_{nm}$ $(\psi_{nm} \in \Re, n, m = 0, 1, 2, \ldots)$.

For $\phi = \sum_{n,m=0}^{\infty} \phi_{nm}\varphi_{nm} \in L^2(\Omega)$ $(\phi_{nm} \in \Re, n, m = 0, 1, 2, \ldots)$, let

$$\psi_{nm} = \frac{\phi_{nm}}{1 + n^2\pi^2/l^2 + m^2\pi^2/l^2},$$

then $L\psi = \phi$ holds. By observing

$$\sum_{n,m=0}^{\infty} (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4)\psi_{nm}^2$$

$$= \sum_{n,m=0}^{\infty} (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4)$$

$$\cdot \left(\frac{\phi_{nm}}{1 + n^2\pi^2/l^2 + m^2\pi^2/l^2} \right)^2$$

$$< \sum_{n,m=0}^{\infty} 3(l^2 + n^4\pi^4/l^2 + m^4\pi^4/l^2) \frac{\phi_{nm}^2}{1 + n^4\pi^4/l^4 + m^4\pi^4/l^4}$$

$$= \sum_{n,m=0}^{\infty} 3\phi_{nm}^2 l^2 < \infty,$$

and

$$\|\psi\|_{H^2(\Omega)}^2$$

$$= \sum_{n,m=0}^{\infty} (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4)\psi_{nm}^2 \|\varphi_{nm}\|_{L^2(\Omega)}^2$$

$$< \sum_{n,m=0}^{\infty} l^2(1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4)\psi_{nm}^2,$$

$\psi \in H^2(\Omega)$ follows.

Noting that the solution of $L\psi = \phi$ is characterized as the solution of

$$\int_\Omega \nabla\psi \cdot \nabla v \; dx + \int_\Omega \psi v \; dx = \int_\Omega \phi v \; dx, \; \forall v \in H^1(\Omega),$$

we have

$$\int_\Omega (-\Delta\psi + \psi) v \; dx + \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu} v \; dx = \int_\Omega \phi v \; dx,$$

and thus

$$\int_{\partial\Omega} \frac{\partial\psi}{\partial\nu} v \; dx = 0.$$

Since v is arbitrary in $H^1(\Omega)$, in case $\psi \in H^2(\Omega)$ we have $\frac{\partial\psi}{\partial\nu} = 0$ in the trace sense.

If there exists another solution $\tilde{\psi} \neq \psi$, we get

$$L\tilde{\psi} = \phi,$$

which means the existence of a non-trivial solution of the equation $L\varphi = 0$. But noting that

$$(L\varphi, \varphi)_{L^2(\Omega)} = \|\varphi\|_{H^1(\Omega)}^2$$

holds, $L\varphi = 0$ derives $\|\varphi\|_{H^1(\Omega)}^2 = 0$, i.e. $\varphi = 0$, which contradicts the assumption and the uniqueness of the solution of $L\psi = \phi$ follows. □

Remark 2.0.2 *For $\phi \in L^2(\Omega)$, we denote the solution of (2.0.1) as $L^{-1}\phi$. Then it is clear that $L^{-1}\phi$ satisfies Neumann boundary conditions.*

Lemma 2.0.3 *For all $\phi \in L^2(\Omega)$, the linear equation*

$$\begin{cases} (\Delta^2 - \Delta + I)\psi = \phi \quad \text{in} \ \ \Omega, \\ \displaystyle\int_{\partial\Omega} \left(\frac{\partial \psi}{\partial \nu} v - \Delta\psi \frac{\partial v}{\partial \nu} + \frac{\partial(\Delta\psi)}{\partial \nu} v \right) dx = 0, \ \text{for} \ \forall v \in H^2(\Omega) \end{cases}$$

has the unique solution $\psi \in H^4(\Omega)$.

Proof. Let $\psi = \sum\limits_{n,m=0}^{\infty} \psi_{nm}\varphi_{nm}$ $(\psi_{nm} \in \Re, n, m = 0, 1, 2, \ldots)$.

For $\phi = \sum\limits_{n,m=0}^{\infty} \phi_{nm}\varphi_{nm} \in L^2(\Omega)$ $(\phi_{nm} \in \Re, n, m = 0, 1, 2, \ldots)$, let

$$\psi_{nm} = \frac{\phi_{nm}}{1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4},$$

then $(\Delta^2 - \Delta + I)\psi = \phi$ holds. By observing

$$\begin{aligned} &\sum_{n,m=0}^{\infty} (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + n^6\pi^6/l^6 + m^6\pi^6/l^6 \\ &+ n^8\pi^8/l^8 + m^8\pi^8/l^8 + 2m^2n^2\pi^4/l^4 + 3m^4n^2\pi^6/l^6 + 3m^2n^4\pi^6/l^6 \\ &+ 4m^2n^6\pi^8/l^8 + 4n^2m^6\pi^8/l^8 + 6n^4m^4\pi^8/l^8)\psi_{nm}^2 \\ =&\sum_{n,m=0}^{\infty} (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + n^6\pi^6/l^6 + m^6\pi^6/l^6 \\ &+ n^8\pi^8/l^8 + m^8\pi^8/l^8 + 2m^2n^2\pi^4/l^4 + 3m^4n^2\pi^6/l^6 + 3m^2n^4\pi^6/l^6 \\ &+ 4m^2n^6\pi^8/l^8 + 4n^2m^6\pi^8/l^8 + 6n^4m^4\pi^8/l^8) \\ &\cdot \left(\frac{\phi_{nm}}{1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4} \right)^2 \\ <&\sum_{n,m=0}^{\infty} 16(l^4 + n^8\pi^8/l^2 + m^8\pi^8/l^2) \frac{\phi_{nm}^2}{1 + n^8\pi^8/l^8 + m^8\pi^8/l^8} \\ =&\sum_{n,m=0}^{\infty} 16\phi_{nm}^2 l^4 < \infty, \end{aligned}$$

and

$$
\begin{aligned}
&\|\psi\|_{H^4}^2 \\
&= \sum_{n,m=0}^{\infty} (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + n^6\pi^6/l^6 + m^6\pi^6/l^6 \\
&\quad + n^8\pi^8/l^8 + m^8\pi^8/l^8 + 2m^2n^2\pi^4/l^4 + 3m^4n^2\pi^6/l^6 + 3m^2n^4\pi^6/l^6 \\
&\quad + 4m^2n^6\pi^8/l^8 + 4n^2m^6\pi^8/l^8 + 6n^4m^4\pi^8/l^8)\psi_{nm}^2\|\varphi_{nm}\|_{L^2}^2 \\
&< \sum_{n,m=0}^{\infty} l^2(1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + n^6\pi^6/l^6 + m^6\pi^6/l^6 \\
&\quad + n^8\pi^8/l^8 + m^8\pi^8/l^8 + 2m^2n^2\pi^4/l^4 + 3m^4n^2\pi^6/l^6 + 3m^2n^4\pi^6/l^6 \\
&\quad + 4m^2n^6\pi^8/l^8 + 4n^2m^6\pi^8/l^8 + 6n^4m^4\pi^8/l^8)\psi_{nm}^2,
\end{aligned}
$$

$\psi \in H^4(\Omega)$ follows.

Noting that the solution of $(\Delta^2 - \Delta + I)\psi = \phi$ is characterized as the solution of

$$\int_\Omega \Delta\psi\Delta v \ dx + \int_\Omega \nabla\psi \cdot \nabla v \ dx + \int_\Omega \psi v \ dx = \int_\Omega \phi v \ dx, \ \forall v \in H^2(\Omega),$$

we have

$$\int_\Omega (\Delta^2\psi - \Delta\psi + \psi)v \ dx + \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu} v \ dx + \int_{\partial\Omega} \frac{\partial(\Delta\psi)}{\partial\nu} v \ dx - \int_{\partial\Omega} \frac{\partial v}{\partial\nu}\Delta\psi \ dx = \int_\Omega \phi v \ dx,$$

and thus

$$\int_{\partial\Omega} \frac{\partial\psi}{\partial\nu} v \ dx + \int_{\partial\Omega} \frac{\partial(\Delta\psi)}{\partial\nu} v \ dx - \int_{\partial\Omega} \frac{\partial v}{\partial\nu}\Delta\psi \ dx = 0.$$

If there exists another solution $\tilde{\psi} \neq \psi$, we get

$$(\Delta^2 - \Delta + I)\tilde{\psi} = \phi,$$

which means the existence of a non-trivial solution of the equation $(\Delta^2 - \Delta + I)\varphi = 0$. But noting that

$$((\Delta^2 - \Delta + I)\varphi, \varphi)_{L^2(\Omega)} = \|\varphi\|_{H^2(\Omega)}^2$$

holds, $(\Delta^2 - \Delta + I)\varphi = 0$ derives $\|\varphi\|_{H^2(\Omega)}^2 = 0$, i.e. $\varphi = 0$, which contradicts the assumption and the uniqueness of the solution of $(\Delta^2 - \Delta + I)\psi = \phi$ follows. □

Now we derive an estimation for the projection P_N, an imbedding constant from $H^1(\Omega)$ to $L^p(\Omega)$ for $2 \leq p < \infty$ and an imbedding constant from $H^2(\Omega)$ to $L^p(\Omega)$ for $2 \leq p \leq \infty$.

Lemma 2.0.4 *For all $z \in H^4(\Omega)$, we have*

$$\|z - P_N z\|_{H^2(\Omega)} \leq C_2(N)\|\triangle^2 z - \triangle z + z\|_{L^2(\Omega)},$$

where $C_2(N) = \sqrt{\frac{1}{1+(N+1)^2\pi^2/l^2+(N+1)^4\pi^4/l^4}}$. *And for $z \in H^2(\Omega)$,*

$$\|z - P_N z\|_{L^2(\Omega)} \leq C_2(N)\|z - P_N z\|_{H^2(\Omega)}$$

holds.

Proof. For $z = \sum_{n,m=0}^{\infty} c_{nm}\varphi_{nm} \in H^2(\Omega)$, we obtain

$$\begin{aligned}
&\|z - P_N z\|^2_{H^2(\Omega)} \\
\leq &\|z - P_N z\|^2_{L^2(\Omega)} + \|(z - P_N z)_x\|^2_{L^2(\Omega)} + \|(z - P_N z)_y\|^2_{L^2(\Omega)} + \|(z - P_N z)_{xx}\|^2_{L^2(\Omega)} \\
&+ \|(z - P_N z)_{yy}\|^2_{L^2(\Omega)} + 2((z - P_N z)_{xx}, (z - P_N z)_{yy})_{L^2(\Omega)} \\
= &\sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} c^2_{nm}\|\varphi_{nm}\|^2_{L^2(\Omega)} + \sum_{n=0}^{N}\sum_{m=N+1}^{\infty} c^2_{nm}\|\varphi_{nm}\|^2_{L^2(\Omega)} \\
&+ \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} c^2_{nm}\|(\varphi_{nm})_x\|^2_{L^2(\Omega)} + \sum_{n=0}^{N}\sum_{m=N+1}^{\infty} c^2_{nm}\|(\varphi_{nm})_x\|^2_{L^2(\Omega)} \\
&+ \sum_{n=0}^{N}\sum_{m=N+1}^{\infty} c^2_{nm}\|(\varphi_{nm})_y\|^2_{L^2(\Omega)} + \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} c^2_{nm}\|(\varphi_{nm})_y\|^2_{L^2(\Omega)} \\
&+ \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} c^2_{nm}\|(\varphi_{nm})_{xx}\|^2_{L^2(\Omega)} + \sum_{n=0}^{N}\sum_{m=N+1}^{\infty} c^2_{nm}\|(\varphi_{nm})_{xx}\|^2_{L^2(\Omega)} \\
&+ \sum_{n=0}^{N}\sum_{m=N+1}^{\infty} c^2_{nm}\|(\varphi_{nm})_{yy}\|^2_{L^2(\Omega)} + \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} c^2_{nm}\|(\varphi_{nm})_{yy}\|^2_{L^2(\Omega)} \\
&+ \sum_{n=0}^{N}\sum_{m=N+1}^{\infty} c^2_{nm}2((\varphi_{nm})_{xx}, (\varphi_{nm})_{yy})_{L^2(\Omega)} + \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} c^2_{nm}2((\varphi_{nm})_{xx}, (\varphi_{nm})_{yy})_{L^2(\Omega)} \\
= &\sum_{n=0}^{N}\sum_{m=N+1}^{\infty} c^2_{nm}(\|\varphi_{nm}\|^2_{L^2(\Omega)} + \|(\varphi_{nm})_x\|^2_{L^2(\Omega)} + \|(\varphi_{nm})_y\|^2_{L^2(\Omega)} + \|(\varphi_{nm})_{xx}\|^2_{L^2(\Omega)} \\
&+ \|(\varphi_{nm})_{yy}\|^2_{L^2(\Omega)} + 2((\varphi_{nm})_{xx}, (\varphi_{nm})_{yy})_{L^2(\Omega)}) + \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} c^2_{nm}(\|\varphi_{nm}\|^2_{L^2(\Omega)} \\
&+ \|(\varphi_{nm})_x\|^2_{L^2(\Omega)} + \|(\varphi_{nm})_y\|^2_{L^2(\Omega)} + \|(\varphi_{nm})_{xx}\|^2_{L^2(\Omega)} + \|(\varphi_{nm})_{yy}\|^2_{L^2(\Omega)} \\
&+ 2((\varphi_{nm})_{xx}, (\varphi_{nm})_{yy})_{L^2(\Omega)})
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} c_{nm}^2(1+n^2\pi^2/l^2+m^2\pi^2/l^2+n^4\pi^4/l^4+m^4\pi^4/l^4+2n^2m^2\pi^4/l^4)\|\varphi_{nm}\|_{L^2(\Omega)}^2 \\
&+\sum_{n=0}^{N}\sum_{m=N+1}^{\infty} c_{nm}^2(1+n^2\pi^2/l^2+m^2\pi^2/l^2+n^4\pi^4/l^4+m^4\pi^4/l^4+2n^2m^2\pi^4/l^4)\|\varphi_{nm}\|_{L^2(\Omega)}^2 \\
&\leq \frac{1}{1+(N+1)^2\pi^2/l^2+(N+1)^4\pi^4/l^4} \\
&\quad\cdot \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(1+n^2\pi^2/l^2+m^2\pi^2/l^2+n^4\pi^4/l^4+m^4\pi^4/l^4+2n^2m^2\pi^4/l^4)^2 c_{nm}^2\|\varphi_{nm}\|_{L^2(\Omega)}^2 \\
&= \frac{1}{1+(N+1)^2\pi^2/l^2+(N+1)^4\pi^4/l^4}\|\Delta^2 z-\Delta z+z\|_{L^2(\Omega)}^2.
\end{aligned}
$$

So we get $C_2(N) = \sqrt{\frac{1}{1+(N+1)^2\pi^2/l^2+(N+1)^4\pi^4/l^4}}$.

For $\|z - P_N z\|_{L^2}$, we use the so-called Aubin-Nitsche technique. We consider the linear equation:

$$
\begin{cases}
(\Delta^2-\Delta+I)\Phi = z - P_N z \quad \text{in } \Omega, \\
\displaystyle\int_{\partial\Omega}\left(\frac{\partial\Phi}{\partial\nu}v - \Delta\Phi\frac{\partial v}{\partial\nu} + \frac{\partial(\Delta\Phi)}{\partial\nu}v\right) dx = 0, \text{for } \forall v\in H^2(\Omega).
\end{cases}
$$

By Lemma 2.0.3, we know that there exists the unique solution $\Phi \in H^4(\Omega)$. Therefore, we have

$$
\begin{aligned}
&\|z-P_N z\|_{L^2(\Omega)}^2 \\
=&(z-P_N z, z-P_N z)_{L^2(\Omega)} = (z-P_N z, (\Delta^2-\Delta+I)\Phi)_{L^2(\Omega)} \\
=&(z-P_N z,\Phi)_{L^2(\Omega)} + (\nabla(z-P_N z),\nabla\Phi)_{L^2(\Omega)} + (\Delta(z-P_N z),\Delta\Phi)_{L^2(\Omega)} \\
&-\int_{\partial\Omega}\left(\frac{\partial\Phi}{\partial\nu}(z-P_N z) - \Delta\Phi\frac{\partial(z-P_N z)}{\partial\nu} + \frac{\partial(\Delta\Phi)}{\partial\nu}(z-P_N z)\right) dx \\
=&\langle z-P_N z,\Phi\rangle_{H^2(\Omega)} \\
=&\langle z-P_N z,\Phi-P_N\phi\rangle_{H^2(\Omega)} \\
\leq&\|z-P_N z\|_{H^2(\Omega)}\|\Phi-P_N\Phi\|_{H^2(\Omega)} \\
\leq&\|z-P_N z\|_{H^2(\Omega)}C_2(N)\|z-P_N z\|_{L^2(\Omega)}.
\end{aligned}
$$

So $\|z-P_N z\|_{L^2(\Omega)} \leq C_2(N)\|z-P_N z\|_{H^2(\Omega)}$ holds. □

Lemma 2.0.5 $\Omega = (a,b)\times(a,b)$, $a<b$. *Then, for all* $u\in H^1(\Omega)$, *we have*

$$
\|u\|_{L^p(\Omega)} \leq K_p\|u\|_{H^1(\Omega)}, (2\leq p<\infty)
$$

where $K_p = \left(\frac{1}{b-a}+\frac{p}{2\sqrt{2}}\right)^{(p-2)/p}$.

Proof. For $(x,y)\in\Omega$, we have

$$\begin{aligned}(x-a)|u(x,y)|^{p/2} &= \int_a^x \frac{\partial}{\partial t}[(t-a)|u(t,y)|^{p/2}]dt\\ &= \int_a^x |u(t,y)|^{p/2}dt + \frac{p}{2}\int_a^x (t-a)|u(t,y)|^{p/2-2}u(t,y)\frac{\partial u}{\partial x}(t,y)dt\\ &\leq \int_a^x |u(t,y)|^{p/2}dt + \frac{p}{2}(b-a)\int_a^x |u(t,y)|^{p/2-1}\left|\frac{\partial u}{\partial x}(t,y)\right|dt,\end{aligned} \tag{2.0.2}$$

and

$$\begin{aligned}(b-x)|u(x,y)|^{p/2} &= -\int_x^b \frac{\partial}{\partial t}[(b-t)|u(t,y)|^{p/2}]dt\\ &= \int_x^b |u(t,y)|^{p/2}dt - \frac{p}{2}\int_x^b (b-t)|u(t,y)|^{p/2-2}u(t,y)\frac{\partial u}{\partial x}(t,y)dt\\ &\leq \int_x^b |u(t,y)|^{p/2}dt + \frac{p}{2}(b-a)\int_x^b |u(t,y)|^{p/2-1}\left|\frac{\partial u}{\partial x}(t,y)\right|dt.\end{aligned} \tag{2.0.3}$$

Adding (2.0.2) and (2.0.3), we get

$$\begin{aligned}(b-a)|u(x,y)|^{p/2} &\leq \int_a^b |u(t,y)|^{p/2}dt + \frac{p}{2}(b-a)\int_a^b |u(t,y)|^{p/2-1}\left|\frac{\partial u}{\partial x}(t,y)\right|dt\\ &=: f(y).\end{aligned} \tag{2.0.4}$$

Analogously, integrating instead in y-direction,

$$\begin{aligned}(b-a)|u(x,y)|^{p/2} &\leq \int_a^b |u(x,t)|^{p/2}dt + \frac{p}{2}(b-a)\int_a^b |u(x,t)|^{p/2-1}\left|\frac{\partial u}{\partial y}(t,y)\right|dt\\ &=: g(x)\end{aligned} \tag{2.0.5}$$

holds.

Multiplying (2.0.4) and (2.0.5), we have

$$(b-a)^2|u(x,y)|^p \leq f(y)g(x)$$

and integrating over Ω we obtain

$$\begin{aligned}&(b-a)^2\int_\Omega |u(x,y)|^p dxdy\\ &\leq \int_\Omega f(y)g(x)dxdy = \left(\int_a^b f(y)dy\right)\left(\int_a^b g(x)dx\right)\\ &= \left(\int_\Omega |u(t,y)|^{p/2}dtdy + \frac{p}{2}(b-a)\int_\Omega |u(t,y)|^{p/2-1}\left|\frac{\partial u}{\partial x}(t,y)\right|dtdy\right)\\ &\quad\cdot\left(\int_\Omega |u(x,t)|^{p/2}dxdt + \frac{p}{2}(b-a)\int_\Omega |u(x,t)|^{p/2-1}\left|\frac{\partial u}{\partial y}(x,t)\right|dxdt\right)\end{aligned}$$

$$\leq \left(\int_\Omega |u|^{p/2} dxdy + \frac{p}{2}(b-a) \left(\int_\Omega |u|^{p-2} dxdy \right)^{1/2} \left(\int_\Omega \left| \frac{\partial u}{\partial x} \right|^2 dxdy \right)^{1/2} \right)$$
$$\cdot \left(\int_\Omega |u|^{p/2} dxdy + \frac{p}{2}(b-a) \left(\int_\Omega |u|^{p-2} dxdy \right)^{1/2} \left(\int_\Omega \left| \frac{\partial u}{\partial y} \right|^2 dxdy \right)^{1/2} \right)$$

and thus, using $(A + B\lambda)(A + B\mu) = A^2 + AB(\lambda + \mu) + B^2\lambda\mu \leq A^2 + \sqrt{2}AB\sqrt{\lambda^2 + \mu^2} + \frac{1}{2}B^2(\lambda^2 + \mu^2) = (A + \frac{1}{\sqrt{2}}B\sqrt{\lambda^2 + \mu^2})^2$, we have

$$(b-a)^2 \int_\Omega |u(x,y)|^p dxdy$$
$$\leq \left(\int_\Omega |u|^{p/2} dxdy + \frac{p}{2\sqrt{2}}(b-a) \left(\int_\Omega |u|^{p-2} dxdy \right)^{1/2} \left(\int_\Omega |\nabla u|^2 dxdy \right)^{1/2} \right)^2$$

i.e. we obtain

$$\int_\Omega |u|^p \leq \left(\frac{1}{b-a} \int_\Omega |u|^{p/2} + \frac{p}{2\sqrt{2}} \left(\int_\Omega |u|^{p-2} \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \right)^{1/2} \right)^2, \quad (2.0.6)$$

here and in the following we omit to write $dxdy$.

When $p = 4$, from (2.0.6), we have

$$\begin{aligned} \int_\Omega |u|^4 &\leq \left(\tfrac{1}{b-a} \int_\Omega |u|^2 + \sqrt{2} \left(\int_\Omega |u|^2 \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \right)^{1/2} \right)^2 \\ &\leq \left(\tfrac{1}{b-a} \int_\Omega |u|^2 + \tfrac{\sqrt{2}}{2} \left(\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \right) \right)^2 \\ &\leq \left(\tfrac{1}{b-a} + \tfrac{\sqrt{2}}{2} \right)^2 \left(\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \right)^2 \\ &= \left(\tfrac{1}{b-a} + \tfrac{\sqrt{2}}{2} \right)^2 \|u\|^4_{H^1(\Omega)}, \end{aligned}$$

therefore, for $2 \leq p < 4$ by using Hölder's inequality, we get

$$\begin{aligned} \int_\Omega |u|^p &= \int_\Omega |u|^{4-p} |u|^{2p-4} \leq \left(\int_\Omega |u|^2 \right)^{(4-p)/2} \left(\int_\Omega |u|^4 \right)^{(p-2)/2} \\ &\leq \|u\|^{4-p}_{H^1(\Omega)} \left(\left(\tfrac{1}{b-a} + \tfrac{\sqrt{2}}{2} \right)^2 \|u\|^4_{H^1(\Omega)} \right)^{(p-2)/2} \\ &\leq \left(\tfrac{1}{b-a} + \tfrac{\sqrt{2}}{2} \right)^{p-2} \|u\|^p_{H^1(\Omega)}, \end{aligned}$$

that is,

$$\|u\|_{L^p(\Omega)} \leq \left(\frac{1}{b-a} + \frac{\sqrt{2}}{2} \right)^{(p-2)/p} \|u\|_{H^1(\Omega)} \quad (2 \leq p < 4) \qquad (2.0.7)$$

holds.

And when $p > 4$, inequality (2.0.6) becomes

$$\begin{aligned}
\|u\|^p_{L^p(\Omega)} &\leq \left(\tfrac{1}{b-a}\left(\int_\Omega |u|^{p-2}\right)^{1/2}\left(\int_\Omega |u|^2\right)^{1/2} + \tfrac{p}{2\sqrt{2}}\left(\int_\Omega |u|^{p-2}\right)^{1/2}\left(\int_\Omega |\nabla u|^2\right)^{1/2}\right)^2 \\
&\leq \left(\tfrac{1}{b-a}\|u\|_{H^1(\Omega)}\|u\|^{(p-2)/2}_{L^{p-2}(\Omega)} + \tfrac{p}{2\sqrt{2}}\|u\|_{H^1(\Omega)}\|u\|^{(p-2)/2}_{L^{p-2}(\Omega)}\right)^2 \\
&\leq \left(\tfrac{1}{b-a} + \tfrac{p}{2\sqrt{2}}\right)^2 \|u\|^2_{H^1(\Omega)}\|u\|^{p-2}_{L^{p-2}(\Omega)},
\end{aligned}$$

then

$$\|u\|_{L^p(\Omega)} \leq \left(\frac{1}{b-a} + \frac{p}{2\sqrt{2}}\right)^{2/p} \|u\|^{2/p}_{H^1(\Omega)}\|u\|^{(p-2)/p}_{L^{p-2}(\Omega)} \tag{2.0.8}$$

holds.

Now iterate (2.0.8) finitely often until only terms

$$\int_\Omega |u|^q \qquad \text{with } \ 2 \leq q < 4$$

are remaining, that is,

$$\|u\|_{L^p(\Omega)} \leq \left(\frac{1}{b-a} + \frac{p}{2\sqrt{2}}\right)^{2(N[p/2]-1)/p} \|u\|^{2(N[p/2]-1)/p}_{H^1(\Omega)}\|u\|^{q/p}_{L^q(\Omega)}, \quad 2 \leq q < 4,$$

where $N[p/2]$ means the integer part of $p/2$ and $q = p - 2(N[p/2] - 1)$. Combining with (2.0.7),

$$\begin{aligned}
\|u\|_{L^p(\Omega)} &\leq \left(\frac{1}{b-a} + \frac{p}{2\sqrt{2}}\right)^{2(N[p/2]-1)/p} \|u\|^{2(N[p/2]-1)/p}_{H^1(\Omega)}\|u\|^{q/p}_{L^q(\Omega)} \\
&\leq \left(\frac{1}{b-a} + \frac{p}{2\sqrt{2}}\right)^{(p-2)/p} \|u\|_{H^1(\Omega)} \quad (p > 4)
\end{aligned}$$

holds.

Consequently, for all $2 \leq p < \infty$ we have

$$\|u\|_{L^p(\Omega)} \leq \left(\frac{1}{b-a} + \frac{p}{2\sqrt{2}}\right)^{(p-2)/p} \|u\|_{H^1(\Omega)}. \quad \square$$

In the following lemma, we use some similar techniques in [1].

Lemma 2.0.6 $\Omega = (a, b) \times (a, b)$, $a < b$. *Then, for all* $u \in H^2(\Omega)$, *we have*

$$\|u\|_{L^p(\Omega)} \leq K_{2,p}\|u\|_{H^2(\Omega)}, (2 \leq p \leq \infty)$$

where $K_{2,p} = \begin{cases} K_p, & p < \infty, \\ \sqrt{12}C_1C_2, & p = \infty, \end{cases}$

$C_1 = \max\{(b-a)^{-2/3}, 3\sqrt{2}(b-a)^{1/3}\}$, $C_2 = (3/2)2^{1/6}(b-a)^{2/3}$.

Proof. Fixing $x \in (a,b)\times(a,b)$, for every $y \in (a,b)\times(a,b)$, we have

$$\begin{aligned} u(x) &= u(y) - \int_0^1 \frac{du(x+t(y-x))}{dt} \\ &= u(y) - \int_0^1 (y-x)\nabla u(x+t(y-x))dt. \end{aligned} \tag{2.0.9}$$

Integrating y of (2.0.9) over $(a,b)\times(a,b)$, we get

$$(b-a)^2 u(x) = \int_\Omega u(y)dy - \int_0^1 \int_\Omega (y-x)\nabla u(x+t(y-x))dtdy. \tag{2.0.10}$$

Set $z := x + t(y-x)$, then (2.0.10) becomes

$$\begin{aligned} (b-a)^2|u(x)| \leq &(b-a)^{4/3}\|u\|_{L^3(\Omega)} + \int_0^1 t^{-2}\int_{\Omega_t} \sqrt{2}(b-a)\nabla u(z)dzdt \\ \leq &(b-a)^{4/3}\|u\|_{L^3(\Omega)} + \sqrt{2}(b-a)\|\nabla u\|_{L^3(\Omega_t)}\int_0^1 t^{-2}(t^2(b-a)^2)^{2/3}dt \\ \leq &(b-a)^{4/3}\|u\|_{L^3(\Omega)} + \sqrt{2}(b-a)^{7/3}\|\nabla u\|_{L^3(\Omega)}\int_0^1 t^{-2/3}dt \\ \leq &(b-a)^{4/3}\|u\|_{L^3(\Omega)} + 3\sqrt{2}(b-a)^{7/3}\|\nabla u\|_{L^3(\Omega)}, \end{aligned}$$

where $\Omega_t := \{x + t(y-x) | y \in \Omega\}$.

Let $C_1 := \max\{(b-a)^{-2/3}, 3\sqrt{2}(b-a)^{1/3}\}$, then we get

$$|u(x)| \leq C_1(\|u\|_{L^3(\Omega)} + \|\nabla u\|_{L^3(\Omega)}). \tag{2.0.11}$$

Fix $(y_1, y_2) = (\frac{a+b}{2}, \frac{a+b}{2}) \in (a,b)\times(a,b)$ (if $\frac{a+b}{2} = 0$, then we choose $(y_1,y_2) = (\frac{a+b}{4}, \frac{a+b}{4})$). For every $(x_1,x_2) \in (a,b)\times(a,b)$,

$$\begin{aligned} &\int_0^1 |u(x_1, x_2 + (1-t)y_2)|^{3/2}dt \\ =&|u(x_1,x_2)|^{3/2} - \frac{3}{2}\int_0^1 t|u(x_1,x_2+(1-t)y_2)|^{1/2}\frac{d}{dt}|u(x_1,x_2+(1-t)y_2)| \end{aligned}$$

holds. Setting $z_2 := x_2 + (1-t)y_2$, we have

$$|u(x_1,x_2)|^{3/2} \leq \int_a^b |u(x_1,z_2)|^{3/2}dz_2 + \frac{3}{2}\int_a^b |u(x_1,z_2)|^{1/2}\left|\frac{\partial u(x_1,z_2)}{\partial z_2}\right| dz_2.$$

Taking supremum of u over $x_2 \in (a,b)$, we have

$$\sup_{x_2\in(a,b)} |u(x_1,x_2)|^{3/2} \leq \int_a^b |u(x_1,z_2)|^{3/2}dz_2 + \frac{3}{2}\int_a^b |u(x_1,z_2)|^{1/2}\left|\frac{\partial u(x_1,z_2)}{\partial z_2}\right| dz_2.$$

Then integrating x_1 over (a,b), we obtain

$$\begin{aligned}
&\int_a^b \sup_{x_2\in(a,b)} |u(x_1,x_2)|^{3/2} dx_1 \\
\leq &\int_a^b \int_a^b |u(x_1,z_2)|^{3/2} dz_2 dx_1 + \frac{3}{2}\int_a^b \int_a^b |u(x_1,z_2)|^{1/2} \left|\frac{\partial u(x_1,z_2)}{\partial z_2}\right| dz_2 dx_1 \\
\leq &\frac{3}{2}\int_\Omega |u(x_1,z_2)|^{1/2} \left(|u(x_1,z_2)| + \left|\frac{\partial u(x_1,z_2)}{\partial z_2}\right|\right) dz_2 dx_1 \\
\leq &\frac{3}{2}\left(\int_\Omega \left(|u(x_1,z_2)| + \left|\frac{\partial u(x_1,z_2)}{\partial z_2}\right|\right)^{6/5} dx_1 dz_2\right)^{5/6} \left(\int_\Omega |u(x_1,z_2)|^3 dx_1 dz_2\right)^{1/6} \\
\leq &(3/2)2^{1/6}(\|u\|_{L^{6/5}(\Omega)} + \|\nabla u\|_{L^{6/5}(\Omega)})\|u\|_{L^3(\Omega)}^{1/2}.
\end{aligned}$$

Similarly,

$$\int_a^b \sup_{x_1\in(a,b)} |u(x_1,x_2)|^{3/2} dx_2 \leq (3/2)2^{1/6}(\|u\|_{L^{6/5}(\Omega)} + \|\nabla u\|_{L^{6/5}(\Omega)})\|u\|_{L^3(\Omega)}^{1/2}$$

holds.

Therefore, we get

$$\begin{aligned}
&\|u\|_{L^3(\Omega)}^3 = \int_\Omega |u(x_1,x_2)|^3 dx_1 dx_2 \\
\leq &\int_\Omega \sup_{x_2\in(a,b)} |u(x_1,x_2)|^{3/2} \sup_{x_1\in(a,b)} |u(x_1,x_2)|^{3/2} dx_1 dx_2 \\
\leq &\int_a^b \sup_{x_2\in(a,b)} |u(x_1,x_2)|^{3/2} dx_1 \int_a^b \sup_{x_1\in(a,b)} |u(x_1,x_2)|^{3/2} dx_2 \\
\leq &((3/2)2^{1/6}(\|u\|_{L^{6/5}(\Omega)} + \|\nabla u\|_{L^{6/5}(\Omega)})\|u\|_{L^3(\Omega)}^{1/2})^2,
\end{aligned}$$

that is,

$$\begin{aligned}
&\|u\|_{L^3(\Omega)} \leq (3/2)2^{1/6}(\|u\|_{L^{6/5}(\Omega)} + \|\nabla u\|_{L^{6/5}(\Omega)}) \\
\leq\ &(3/2)2^{1/6}(b-a)^{2/3}(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) =: C_2(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}).
\end{aligned} \tag{2.0.12}$$

Then from (2.0.11) and (2.0.12), for every $x \in (a,b)\times(a,b)$, we have

$$\begin{aligned}
&|u(x)|^2 \leq C_1^2(\|u\|_{L^3(\Omega)} + \|\nabla u\|_{L^3(\Omega)})^2 \\
\leq &C_1^2C_2^2(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)})^2 \\
\leq &C_1^2C_2^2 3(\|u\|_{L^2(\Omega)}^2 + 4\|\nabla u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2) \\
\leq &12C_1^2C_2^2\|u\|_{H^2(\Omega)}^2.
\end{aligned}$$

Thus, we obtain

$$\|u\|_{L^\infty(\Omega)} \leq \sqrt{12} C_1 C_2 \|u\|_{H^2(\Omega)}.$$

Therefore, we set $K_{2,\infty} = \sqrt{12} C_1 C_2$.

And when $p < \infty$, from Lemma 2.0.5 we have

$$\|u\|_{L^p(\Omega)} \leq K_p \|u\|_{H^1(\Omega)} \leq K_p \|u\|_{H^2(\Omega)}.$$

So $K_{2,p} = K_p$. $\square$

Chapter 3

Approximate solution

In this chapter, we introduce the method how to get the approximate solution of (1.0.1) at each time $t \in [0, T]$ $(T > 0)$ and also get the approximate stationary solution.

As a concrete example for the pattern formation problem, we describe how to obtain an approximate solution for the Schnakenberg equation. We consider the time-dependent Schnakenberg system

$$\begin{cases} u_t = \gamma f(u, v) + \Delta u \text{ in } \Omega, \\ v_t = \gamma g(u, v) + d\Delta v \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ u(x, y, 0) = h_1(x, y), \\ v(x, y, 0) = h_2(x, y), \end{cases} \tag{3.0.1}$$

where $\Omega = (0, l) \times (0, l)$, $f(u, v) = a - u + u^2 v$, $g(u, v) = b - u^2 v$, $h_1, h_2 \in \mathcal{L}(\Re^2, \Re)$ and a, b, d, γ are some positive constants. We write the solution of (3.0.1) at time t as $w(t) = (u(t), v(t))$.

According to Murrey[11], if we want to get a non-trivial stationary solution, a, b, d, l, γ should satisfy the following conditions:

$$\begin{aligned} b - a &< (a + b)^3, \\ b - a &> \frac{1}{d}(a + b)^3 + \frac{2}{\sqrt{d}}(a + b)^2, \end{aligned}$$

and there exists $n \in \mathcal{N}$ such that

$$\gamma M_1(a, b, d) < \left(\frac{n\pi}{l}\right)^2 < \gamma M_2(a, b, d),$$

$$M_1 = \frac{[d(b - a) - (a + b)^3] - \{[d(b - a) - (a + b)^3]^2 - 4d(a + b)^4\}^{1/2}}{2d(a + b)},$$

$$M_2 = \frac{[d(b-a)-(a+b)^3] + \{[d(b-a)-(a+b)^3]^2 - 4d(a+b)^4\}^{1/2}}{2d(a+b)}.$$

After choosing some parameters a, b, d, l, γ satisfying the above conditions (for simplicity, in this paper, we choose $d > 1$), $\forall t \in (0, T)$, set

$$\begin{aligned}
w_h(x,y,t) &:= (u_h(x,y,t), v_h(x,y,t)),\\
u_h(x,y,t) &:= \sum_{i=1}^{(N+1)^2} \xi_{1,i}(t)\varphi_i(x,y), (\xi_{1,i} \in \Re, i = 1, 2, \cdots (N+1)^2)\\
v_h(x,y,t) &:= \sum_{i=1}^{(N+1)^2} \xi_{2,i}(t)\varphi_i(x,y), (\xi_{2,i} \in \Re, i = 1, 2, \cdots (N+1)^2)
\end{aligned}$$

then we get $\frac{\partial u_h}{\partial \nu} = \frac{\partial v_h}{\partial \nu} = 0$. We omit x, y and write $w_h(t), u_h(t), v_h(t)$.

We use Galerkin method to get an approximate solution of (3.0.1) satisfying

$$\begin{cases} (u_t - \gamma f(u,v) - \triangle u, \chi_1)_{L^2(\Omega)} = 0, \ \forall \chi_1 \in X_N, \\ (v_t - \gamma g(u,v) - d\triangle v, \chi_2)_{L^2(\Omega)} = 0, \ \forall \chi_2 \in X_N. \end{cases}$$

So we need to solve the ordinary differential equations

$$\begin{cases} \sum\limits_{i=1}^{(N+1)^2} \frac{d\xi_{1,i}(t)}{dt}((\varphi_i,\varphi_j)_{L^2(\Omega)} + (\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)}) = \sum\limits_{i=1}^{(N+1)^2} \gamma(f(u,v),\varphi_j)_{L^2(\Omega)}, \\ \sum\limits_{i=1}^{(N+1)^2} \frac{d\xi_{2,i}(t)}{dt}((\varphi_i,\varphi_j)_{L^2(\Omega)} + d(\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)}) = \sum\limits_{i=1}^{(N+1)^2} \gamma(g(u,v),\varphi_j)_{L^2(\Omega)}. \end{cases} \tag{3.0.2}$$

Setting

$$\begin{aligned}
A &:= \begin{pmatrix} (\varphi_i,\varphi_j)_{L^2(\Omega)} & 0 \\ 0 & (\varphi_i,\varphi_j)_{L^2(\Omega)} \end{pmatrix} (1 \le i,j \le (N+1)^2),\\
B &:= \begin{pmatrix} (\nabla\varphi_i,\nabla\varphi_j)_{L^2(\Omega)} & 0 \\ 0 & d(\nabla\varphi_i,\nabla\varphi_j)_{L^2(\Omega)} \end{pmatrix} (1 \le i,j \le (N+1)^2),\\
\mathcal{F} &:= \begin{pmatrix} (f(u,v), \ \varphi_j)_{L^2(\Omega)} \\ (g(u,v), \ \varphi_j)_{L^2(\Omega)} \end{pmatrix} (1 \le j \le (N+1)^2),\\
\tau &:= (\xi_{1,1}(t), \xi_{1,2}(t), \cdots, \xi_{1,(N+1)^2}(t), \xi_{2,1}(t), \xi_{1,2}(t), \cdots, \xi_{2,(N+1)^2}(t))^T,
\end{aligned}$$

(3.0.2) is equal to

$$A\frac{d\tau}{dt} + B\tau - \gamma\mathcal{F}^T = 0.$$

We solve this ordinary differential equation by using the solver "ode15s" in Matlab, which is based on numerical differentiation formulas (NDFs).

Now we consider the time-independent system:

$$\begin{cases} -\Delta u = \gamma f(u,v) \text{ in } \Omega, \\ -d\Delta v = \gamma g(u,v) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.0.3}$$

We use Newton's method to obtain an approximate stationary solution $\hat{w}_N = (\hat{u}_N, \hat{v}_N) \in X_N \times X_N$ of (3.0.3) satisfying

$$\begin{cases} (\nabla \hat{u}_N, \nabla \chi_3)_{L^2(\Omega)} - \gamma(f(\hat{u}_N, \hat{v}_N), \chi_3)_{L^2(\Omega)} = 0, \ \forall \chi_3 \in X_N, \\ (\nabla \hat{v}_N, \nabla \chi_4)_{L^2(\Omega)} - \frac{\gamma}{d}(g(\hat{u}_N, \hat{v}_N), \chi_4)_{L^2(\Omega)} = 0, \ \forall \chi_4 \in X_N. \end{cases}$$

Set

$$\begin{aligned} u^{(n)} &:= \sum_{k=1}^{(N+1)^2} a_k^{(n)} \varphi_k, (a_k^{(n)} \in \Re, k = 1, 2, \cdots (N+1)^2) \\ v^{(n)} &:= \sum_{k=1}^{(N+1)^2} b_k^{(n)} \varphi_k, (b_k^{(n)} \in \Re, k = 1, 2, \cdots (N+1)^2) \\ \vec{x}^{(n)} &:= (a_1^{(n)}, \ldots, a_{(N+1)^2}^{(n)}, b_1^{(n)}, \ldots, b_{(N+1)^2)}^{(n)})^t \in \Re^{2(N+1)^2}. \end{aligned}$$

We use the initial value for the Newton's method as $u_h(x,y,T)$ and $v_h(x,y,T)$, which we got from Galerkin method, that is, $a_k^{(0)} = \xi_{1,k}(T)$ and $b_k^{(0)} = \xi_{2,k}(T)$ $(k = 1, 2, \cdots (N+1)^2)$.

We iteratively solve the following linear equation, starting from the initial vector $\vec{x}^{(0)}$,

$$A\vec{x}^{(n+1)} = \vec{r}^{(n)},$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \vec{r}^{(n)} = \begin{pmatrix} \vec{r}_1^{(n)} \\ \vec{r}_2^{(n)} \end{pmatrix},$$

$$\begin{aligned} A_{11}(k,l) =& (\nabla \varphi_k, \nabla \varphi_l)_{L^2(\Omega)} - \gamma(f_u(u^{(n)}, v^{(n)})\varphi_k, \varphi_l)_{L^2(\Omega)} \\ =& (\nabla \varphi_k, \nabla \varphi_l)_{L^2(\Omega)} - \gamma((2u^{(n)}v^{(n)} - 1)\varphi_k, \varphi_l)_{L^2(\Omega)} \ (1 \le k, l \le (N+1)^2), \\ A_{12}(k,l) =& -\gamma(f_v(u^{(n)}), v^{(n)})\varphi_k, \varphi_l)_{L^2(\Omega)} = -\gamma(u^{(n)^2}\varphi_k, \varphi_l)_{L^2(\Omega)} \ (1 \le k, l \le (N+1)^2), \\ A_{21}(k,l) =& -\frac{\gamma}{d}(g_u(u^{(n)}, v^{(n)})\varphi_k, \varphi_l)_{L^2(\Omega)} = \frac{\gamma}{d}(2u^{(n)}v^{(n)}\varphi_k, \varphi_l)_{L^2(\Omega)} \ (1 \le k, l \le (N+1)^2), \\ A_{22}(k,l) =& (\nabla \varphi_k, \nabla \varphi_l)_{L^2(\Omega)} - \frac{\gamma}{d}(g_v(u^{(n)}, v^{(n)})\varphi_k, \varphi_l)_{L^2(\Omega)} \\ =& (\nabla \varphi_k, \nabla \varphi_l)_{L^2(\Omega)} + \frac{\gamma}{d}(u^{(n)^2}\varphi_k, \varphi_l)_{L^2(\Omega)} \ (1 \le k, l \le (N+1)^2), \end{aligned}$$

$$
\begin{aligned}
\vec{r}_1^{(n)}(i) =& \gamma \iint\limits_{\Omega} (f(u^{(n)}, v^{(n)}) - f_u(u^{(n)}, v^{(n)})u^{(n)} - f_v(u^{(n)}, v^{(n)})v^{(n)})\varphi_i dxdy \\
=& \gamma \iint\limits_{\Omega} (-2(u^{(n)})^2 v^{(n)} + a)\varphi_i dxdy \ (1 \leq i \leq (N+1)^2), \\
\vec{r}_2^{(n)}(i) =& \frac{\gamma}{d} \iint\limits_{\Omega} (g(u^{(n)}, v^{(n)}) - g_u(u^{(n)}, v^{(n)})u^{(n)} - g_v(u^{(n)}, v^{(n)})v^{(n)})\varphi_i dxdy \\
=& \frac{\gamma}{d} \iint\limits_{\Omega} (2(u^{(n)})^2 v^{(n)} + b)\varphi_i dxdy \ (1 \leq i \leq (N+1)^2).
\end{aligned}
$$

Chapter 4

Verification

We give a computer-assisted proof for the existence of the stationary solution $w^* = (u^*, v^*)$ of (3.0.3) near the approximate stationary solution which we got in Chapter 3. This method is similar to the method in [2]. First in Section 4.1, a fixed point formulation is derived, then computable verification conditions are given in Section 4.2.

4.1 Fixed point equation

We rewrite the equation (3.0.3) as

$$\begin{cases} Lu = \gamma f(u, v) + u, \\ dLv = \gamma g(u, v) + dv. \end{cases} \tag{4.1.1}$$

Setting $\tilde{u} := u^* - \hat{u}_N$ and $\tilde{v} := v^* - \hat{v}_N$, the equation (4.1.1) becomes

$$\begin{cases} L(\hat{u}_N + \tilde{u}) = \gamma f(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) + (\hat{u}_N + \tilde{u}), \\ dL(\hat{v}_N + \tilde{v}) = \gamma g(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) + d(\hat{v}_N + \tilde{v}). \end{cases}$$

So we have

$$\begin{cases} L\tilde{u} = \gamma f(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) + \tilde{u} + \Delta \hat{u}_N, \\ dL\tilde{v} = \gamma g(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) + d\tilde{v} + d\Delta \hat{v}_N. \end{cases}$$

Thus setting

$$f_1(\tilde{u}, \tilde{v}) := f(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}),$$
$$g_1(\tilde{u}, \tilde{v}) := g(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}),$$

and using the following compact map from $H^2(\Omega)\times H^2(\Omega)$ to $H^2(\Omega)\times H^2(\Omega)$,

$$F(\tilde{u},\tilde{v}) := \begin{pmatrix} L^{-1}\{\gamma f_1(\tilde{u},\tilde{v}) + \Delta\hat{u}_N + \tilde{u}\} \\ \frac{1}{d}L^{-1}\{\gamma g_1(\tilde{u},\tilde{v}) + d\Delta\hat{v}_N + d\tilde{v}\} \end{pmatrix},$$

we have the fixed point equation for $\tilde{w} = (\tilde{u},\tilde{v})$:

$$\tilde{w} = F(\tilde{w}). \tag{4.1.2}$$

By enclosing a fixed point of F, a solution of (3.0.3) can be enclosed as $w^* = (u^*, v^*)$, $u^* = \hat{u}_N + \tilde{u}$ and $v^* = \hat{v}_N + \tilde{v}$.

Now we decompose (4.1.2) into two parts, the finite dimensional part and the infinite dimensional part:

$$\begin{cases} P\tilde{w} = PF(\tilde{w}), \\ (I-P)\tilde{w} = (I-P)F(\tilde{w}). \end{cases} \tag{4.1.3}$$

We use a Newton-like method only for the former part of (4.1.3), that is, we define the Newton-like operator

$$N(\tilde{w}) := P\tilde{w} - [I - F'(0)]_N^{-1}(P\tilde{w} - PF(\tilde{w})).$$

Here, $F'(0)$ is the Fréchet derivative of F at 0 and suppose that restriction to $X_N \times X_N$ of the operator $P[I - F'(0)] : H^2(\Omega) \times H^2(\Omega) \to X_N \times X_N$ has an inverse

$$[I - F'(0)]_N^{-1} : X_N \times X_N \to X_N \times X_N.$$

This assumption can be checked in the actual computation.

Now we define the operator $T : H^2(\Omega) \times H^2(\Omega) \to H^2(\Omega) \times H^2(\Omega)$ as

$$T(\tilde{w}) := N(\tilde{w}) + (I-P)F(\tilde{w}).$$

Then T becomes a compact map on $H^2(\Omega) \times H^2(\Omega)$ and we have the equivalence relation

$$\tilde{w} = F(\tilde{w}) \Leftrightarrow \tilde{w} = T(\tilde{w}).$$

Therefore, if there exists a non-empty, closed, convex and bounded set $W \subset H^2(\Omega)\times H^2(\Omega)$ such that $T(W) \subset W$, then by Schauder's fixed point theorem there exists a solution $\tilde{w} \in W$ of $\tilde{w} = T(\tilde{w})$, i.e. $\tilde{w} = F(\tilde{w})$.

4.2 Verification condition

First we construct several sets:

$$\begin{aligned} W &= U \times V, \\ U &= U_N + U_\perp, \\ V &= V_N + V_\perp, \end{aligned} \tag{4.2.1}$$

with $U_N, U_\perp, V_N, V_\perp$ defined by

$$\begin{aligned} U_N &:= \{\phi_N \in X_N \mid \|\phi_N\|_{H^2(\Omega)} \le \alpha_1\}, \\ U_\perp &:= \{\phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^2(\Omega)} \le \alpha_2\}, \\ V_N &:= \{\phi_N \in X_N \mid \|\phi_N\|_{H^2(\Omega)} \le \beta_1\}, \\ V_\perp &:= \{\phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^2(\Omega)} \le \beta_2\}, \end{aligned}$$

for positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$, where $X_N^\perp$ represents the orthogonal complement of X_N in $H^2(\Omega)$.

A sufficient condition for $T(W) \subset W$ is derived as:

$$\left\{ \begin{array}{l} N(W) \subset PW, \\ (I-P)F(W) \subset (I-P)W, \end{array} \right. \tag{4.2.2}$$

and we construct a set W, in the form of (4.2.1), satisfying (4.2.2).

4.2.1 Finite dimensional part

We consider the former part of (4.2.2).

For all $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in N(W)$, in order to verify $N(W) \subset PW$, we need to find α_1 and β_1 such that

$$\left\{ \begin{array}{l} \|\phi_1\|_{H^2(\Omega)} \le \alpha_1, \\ \|\phi_2\|_{H^2(\Omega)} \le \beta_1, \end{array} \right.$$

where note that the left-hand sides depend on α_1 and β_1.

The operator

$$[I - F'(0)]_N^{-1} : X_N \times X_N \to X_N \times X_N$$

is written as

$$[I - F'(0)]^{-1} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} [I - F'(0)]_{u_N}^{-1} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \\ [I - F'(0)]_{v_N}^{-1} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \end{pmatrix}. \tag{4.2.3}$$

For $\psi_N \in X_N \times X_N$, set $\phi_N := [I - F'(0)]_N^{-1} \psi_N$, then

$$P(I - F'(0))\phi_N = \psi_N$$

that is,

$$\phi_N - PF'(0)\phi_N = \psi_N \tag{4.2.4}$$

holds. Set

$$\phi_N = \begin{pmatrix} \phi_{1N} \\ \phi_{2N} \end{pmatrix}, F'(0) = \begin{pmatrix} F_{1,\tilde{u}} & F_{2,\tilde{u}} \\ F_{1,\tilde{v}} & F_{2,\tilde{v}} \end{pmatrix}, \psi_N = \begin{pmatrix} \psi_{1N} \\ \psi_{2N} \end{pmatrix},$$

where $F_{1,\tilde{u}}$ and $F_{2,\tilde{u}}$ denote the Fréchet derivative with respect to $\tilde{u}$ of the first element and the second element of F respectively, $F_{1,\tilde{v}}$ and $F_{2,\tilde{v}}$ denote the Fréchet derivative with respect to $\tilde{v}$ of the first element and the second element of F respectively.

Then (4.2.4) is written as

$$\begin{cases} \phi_{1N} - P_N F_{1,\tilde{u}}\phi_{1N} - P_N F_{2,\tilde{u}}\phi_{2N} = \psi_{1N}, \\ \phi_{1N} - P_N F_{1,\tilde{v}}\phi_{1N} - P_N F_{2,\tilde{v}}\phi_{2N} = \psi_{2N}. \end{cases} \tag{4.2.5}$$

By the definition of F, we obtain

$$F'(0) = \begin{pmatrix} L^{-1}\{\gamma f_{1\tilde{u}}(0) + 1\} & L^{-1}\{\gamma f_{1\tilde{v}}(0)\} \\ \frac{1}{d}L^{-1}\{\gamma g_{1\tilde{u}}(0)\} & \frac{1}{d}L^{-1}\{\gamma g_{1\tilde{v}}(0) + d\} \end{pmatrix}, \tag{4.2.6}$$

where $f_{1\tilde{u}}(0)$, $g_{1\tilde{u}}(0)$ denote the Fréchet derivative with respect to $\tilde{u}$ at point 0 of f_1 and g_1 respectively, $f_{2\tilde{v}}(0)$, $g_{2\tilde{v}}(0)$ the Fréchet derivative with respect to $\tilde{v}$ at point 0 of f_2 and g_2 respectively. And (4.2.5) is equivalent to

$$\begin{cases} \phi_{1N} - P_N L^{-1}\{\gamma f_{1\tilde{u}}(0) + 1\}\phi_{1N} - P_N L^{-1}\{\gamma f_{1\tilde{v}}(0)\}\phi_{2N} = \psi_{1N}, \\ \phi_{2N} - P_N \frac{1}{d}L^{-1}\{\gamma g_{1\tilde{u}}(0)\}\phi_{1N} - P_N \frac{1}{d}L^{-1}\{\gamma g_{1\tilde{v}}(0) + d\}\phi_{2N} = \psi_{2N}. \end{cases} \tag{4.2.7}$$

For both sides of the first equation in (4.2.7), take a $H^1(\Omega)$ inner product with $\chi_{1,N} \in X_N$,

$$\begin{aligned} \langle \phi_{1N}, \chi_{1,N}\rangle_{H^1(\Omega)} &- \langle P_N L^{-1}\{\gamma f_{1\tilde{u}}(0) + 1\}\phi_{1N}, \chi_{1,N}\rangle_{H^1(\Omega)} \\ &- \langle P_N L^{-1}\{\gamma f_{1\tilde{v}}(0)\}\phi_{2N}, \chi_{1,N}\rangle_{H^1(\Omega)} = \langle \psi_{1N}, \chi_{1,N}\rangle_{H^1(\Omega)}. \end{aligned}$$

Then we have

$$\begin{aligned} \langle \phi_{1N}, \chi_{1,N}\rangle_{H^1(\Omega)} &- \langle L^{-1}\{\gamma f_{1\tilde{u}}(0) + 1\}\phi_{1N}, \phi_N\rangle_{H^1(\Omega)} \\ &- \langle L^{-1}\{\gamma f_{1\tilde{v}}(0)\}\phi_{2N}, \chi_{1,N}\rangle_{H^1(\Omega)} = \langle \psi_{1N}, \chi_{1,N}\rangle_{H^1(\Omega)}, \end{aligned} \tag{4.2.8}$$

and for every $u, v \in H^1(\Omega)$,

$$\begin{aligned} \langle L^{-1}u, v\rangle_{H^1(\Omega)} =& (\nabla L^{-1}u, \nabla v)_{L^2(\Omega)} + (L^{-1}u, v)_{L^2(\Omega)} \\ =& (-\triangle L^{-1}u, v)_{L^2(\Omega)} + \int_{\partial\Omega} \frac{\partial L^{-1}u}{\partial \nu} v ds + (L^{-1}u, v)_{L^2(\Omega)} \\ =& (LL^{-1}u, v)_{L^2(\Omega)} \\ =& (u, v)_{L^2(\Omega)}. \end{aligned}$$

Therefore (4.2.8) can be written as

$$\langle \phi_{1N}, \chi_{1,N}\rangle_{H^1(\Omega)} - (\gamma f_{1\tilde{u}}(0)\phi_{1N} + \phi_{1N} + \gamma f_{1\tilde{v}}(0)\phi_{2N}, \chi_{1,N})_{L^2(\Omega)} = \langle \psi_{1N}, \chi_{1,N}\rangle_{H^1(\Omega)}. \tag{4.2.9}$$

In the same way, for the second equation of (4.2.7), taking a $H^1(\Omega)$ inner product with $\chi_{2,N} \in X_N$, we have

$$\langle \phi_{2N}, \chi_{2,N}\rangle_{H^1(\Omega)} - \frac{1}{d}(\gamma g_{1\tilde{u}}(0)\phi_{1N} + \gamma g_{1\tilde{v}}(0)\phi_{2N} + d\phi_{2N}, \chi_{2,N})_{L^2(\Omega)} = \langle \psi_{2N}, \chi_{2,N}\rangle_{H^1(\Omega)}. \tag{4.2.10}$$

Setting

$$\psi_{1N} = \sum_{i=1}^{(N+1)^2} a_i'\varphi_i,\ \psi_{2N} = \sum_{i=1}^{(N+1)^2} b_i'\varphi_i,\ \phi_{1N} = \sum_{i=1}^{(N+1)^2} a_i\varphi_i,\ \psi_{2N} = \sum_{i=1}^{(N+1)^2} b_i\varphi_i,$$
$$a_i', b_i', a_i, b_i \in \Re(1 \le i \le (N+1)^2),$$

(4.2.9) and (4.2.10) are written as follows

$$\begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix}\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \vec{a'} \\ \vec{b'} \end{pmatrix}$$

where $\vec{a} = (a_1, a_2, \cdots a_{(N+1)^2)})^T$, $\vec{b} = (b_1, b_2, \cdots b_{(N+1)^2})^T$,
$\vec{a'} = (a_1', a_2', \cdots a'_{(N+1)^2})^T$, $\vec{b'} = (b_1', b_2', \cdots b'_{(N+1)^2)})^T$ and

$$\begin{aligned}
G_{ij}^{11} =&(\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)} - (\gamma f_{1\tilde{u}}(0)\varphi_i + \varphi_i, \varphi_j)_{L^2(\Omega)},\\
=&(\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + \gamma(\varphi_i, \varphi_j)_{L^2(\Omega)} - 2\gamma(\hat{u}_N\hat{v}_N\varphi_i, \varphi_j)_{L^2(\Omega)},\\
G_{ij}^{12} =& -\gamma(f_{1\tilde{v}}(0)\varphi_i, \varphi_j)_{L^2(\Omega)} = -\gamma(\hat{u}_N^2\varphi_i, \varphi_j)_{L^2(\Omega)},\\
G_{ij}^{21} =& -\frac{\gamma}{d}(g_{1\tilde{u}}(0)\varphi_i, \varphi_j)_{L^2(\Omega)} = -\frac{\gamma}{d}(-2\hat{u}_N\hat{v}_N\varphi_i, \varphi_j)_{L^2(\Omega)},\\
G_{ij}^{22} =&(\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)} - \frac{1}{d}(\gamma g_{1\tilde{v}}(0)\varphi_i + d\varphi_i, \varphi_j)_{L^2(\Omega)},\\
=&(\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + \frac{\gamma}{d}(\hat{u}_N^2\varphi_i, \varphi_j)_{L^2(\Omega)},\\
D_{ij} =&(\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)},\\
&(1 \le i, j \le (N+1)^2).
\end{aligned}$$

Consequently, $\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix}$ is obtained as

$$\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} \tilde{G}^{11} & \tilde{G}^{12} \\ \tilde{G}^{21} & \tilde{G}^{22} \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \vec{a'} \\ \vec{b'} \end{pmatrix} = \begin{pmatrix} \tilde{G}^{11}D & \tilde{G}^{12}D \\ \tilde{G}^{21}D & \tilde{G}^{22}D \end{pmatrix}\begin{pmatrix} \vec{a'} \\ \vec{b'} \end{pmatrix}$$

with

$$\begin{pmatrix} \tilde{G}^{11} & \tilde{G}^{12} \\ \tilde{G}^{21} & \tilde{G}^{22} \end{pmatrix} := \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix}^{-1}.$$

Set

$$D_{H^2} := ((\Delta\varphi_i, \Delta\varphi_j)_{L^2(\Omega)} + (\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)})_{ij} (1 \le i, j \le (N+1)^2).$$

Now we describe how to estimate (4.2.3). Noting that

$$\begin{aligned}
\|[I - F'(0)]^{-1}_{u_N} \psi_N\|^2_{H^2(\Omega)} =& \|\phi_{1N}\|^2_{H^2(\Omega)} \\
=& \vec{a}^T D_{H^2} \vec{a} \\
=& \vec{a}^T D_{H^2} (\tilde{G}^{11} D\vec{a'} + \tilde{G}^{12} D\vec{b'}) \\
=& \vec{a}^T D_{H^2} \tilde{G}^{11} D\vec{a'} + \vec{a}^T D_{H^2} \tilde{G}^{12} D\vec{b'} \\
=& (D^{T/2}_{H^2} \vec{a})^T D^{T/2}_{H^2} \tilde{G}^{11} D D^{-T/2}_{H^2} (D^{T/2}_{H^2} \vec{a'}) \\
& + (D^{T/2}_{H^2} \vec{a})^T D^{T/2}_{H^2} \tilde{G}^{12} D D^{-T/2}_{H^2} (D^{T/2}_{H^2} \vec{b'}) \\
\le& \|\phi_{1N}\|_{H^2(\Omega)} \|D^{T/2}_{H^2} \tilde{G}^{11} D D^{-T/2}_{H^2}\|_E \|\psi_{1N}\|_{H^2(\Omega)} \\
& + \|\phi_{1N}\|_{H^2(\Omega)} \|D^{T/2}_{H^2} \tilde{G}^{12} D D^{-T/2}_{H^2}\|_E \|\psi_{2N}\|_{H^2(\Omega)}
\end{aligned}$$

holds, where $\|\cdot\|_E$ is the Euclidian norm for a matrix, we obtain

$$\begin{aligned}
& \|[I - F'(0)]^{-1}_{u_N} \psi_N\|_{H^2(\Omega)} \\
\le& \|D^{T/2}_{H^2} \tilde{G}^{11} D D^{-T/2}_{H^2}\|_E \|\psi_{1N}\|_{H^2(\Omega)} + \|D^{T/2}_{H^2} \tilde{G}^{12} D D^{-T/2}_{H^2}\|_E \|\psi_{2N}\|_{H^2(\Omega)} \\
\le& \sqrt{\|D^{T/2}_{H^2} \tilde{G}^{11} D D^{-T/2}_{H^2}\|^2_E + \|D^{T/2}_{H^2} \tilde{G}^{12} D D^{-T/2}_{H^2}\|^2_E} \sqrt{\|\psi_{1N}\|^2_{H^2(\Omega)} + \|\psi_{2N}\|^2_{H^2(\Omega)}} \\
=& \sqrt{\|D^{T/2}_{H^2} \tilde{G}^{11} D D^{-T/2}_{H^2}\|^2_E + \|D^{T/2}_{H^2} \tilde{G}^{12} D D^{-T/2}_{H^2}\|^2_E} \|\psi_N\|_{H^2(\Omega)\times H^2(\Omega)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|[I - F'(0)]^{-1}_{u_N}\|_{\mathcal{L}(H^2(\Omega)\cap X_N, X_N)} \\
=& \sup_{\|\psi_N\|_{H^2(\Omega)\times H^2(\Omega)}=1} \|[I - F'(0)]^{-1}_{u_N} \psi_N\|_{H^2(\Omega)} \\
\le& \sqrt{\|D^{T/2}_{H^2} \tilde{G}^{11} D D^{-T/2}_{H^2}\|^2_E + \|D^{T/2}_{H^2} \tilde{G}^{12} D D^{-T/2}_{H^2}\|^2_E}
\end{aligned}$$

holds.

Similarly, we have

$$\|[I - F'(0)]^{-1}_{v_N}\|_{\mathcal{L}(H^2(\Omega)\cap X_N, X_N)} \le \sqrt{\|D^{T/2}_{H^2} \tilde{G}^{21} D D^{-T/2}_{H^2}\|^2_E + \|D^{T/2}_{H^2} \tilde{G}^{22} D D^{-T/2}_{H^2}\|^2_E}.$$

For all $\tilde{w} \in W$, we know

$$\begin{aligned} N(\tilde{w}) =& P\tilde{w} - [I - F'(0)]_N^{-1}(P\tilde{w} - PF(\tilde{w})) \\ =& w_N - [I - F'(0)]_N^{-1}(w_N - PF(\tilde{w})), \\ =& [I - F'(0)]_N^{-1}P(F(\tilde{w}) - F'(0)w_N), \end{aligned}$$

where $w_N = (u_N, v_N) \in U_N \times V_N$.

Writing $N(\tilde{w}) = \begin{pmatrix} N_1(\tilde{w}) \\ N_2(\tilde{w}) \end{pmatrix}$, we have:

$$\begin{aligned} \|N_1(\tilde{w})\|_{H^2(\Omega)} =& \|[I - F'(0)]_{u_N}^{-1} P(F(\tilde{w}) - F'(0)w_N)\|_{H^2(\Omega)} \\ \leq& \|[I - F'(0)]_{u_N}^{-1}\|_{\mathcal{L}(H^2(\Omega)\cap X_N, X_N)} \\ & \times \|P(F(\tilde{w}) - F'(0)w_N)\|_{H^2(\Omega)\times H^2(\Omega)}. \end{aligned}$$

For every $\tau \in L^2(\Omega)$, from the definition of P_N and the formulation of $L^{-1}\tau$ which we derived from Lemma 2.0.1, we can see

$$L^{-1}P_N\tau = P_N L^{-1}\tau.$$

Then for all $\tilde{w} \in W$,

$$\tilde{w} = \begin{pmatrix} u_N + u_\perp \\ v_N + v_\perp \end{pmatrix},$$

note that

$$\|P(F(\tilde{w}) - F'(0)w_N)\|_{H^2(\Omega)\times H^2(\Omega)} = \|\psi_1\|_{H^2(\Omega)}^2 + \frac{1}{d^2}\|\psi_2\|_{H^2(\Omega)}^2,$$

where,

$$\begin{aligned} \psi_1 &:= L^{-1}\{P_N(\gamma f_1(\tilde{w}) + u_N + u_\perp - \gamma f_{1\tilde{u}}(0)u_N - u_N - \gamma f_{1\tilde{v}}(0)v_N + \triangle \hat{u}_N)\}, \\ \psi_2 &:= L^{-1}\{P_N(\gamma g_1(\tilde{w}) + d(v_N + v_\perp) - \gamma g_{1\tilde{v}}(0)v_N - dv_N - \gamma g_{1\tilde{u}}(0)u_N + d\triangle \hat{v}_N)\}, \end{aligned}$$

$f_{1\tilde{u}}(0), f_{1\tilde{v}}(0), g_{1\tilde{u}}(0), g_{1\tilde{v}}(0)$ are the same as in (4.2.6).

For all $\varsigma = \sum_{n,m=0}^{\infty} \varsigma_{nm}\varphi_{nm} \in L^2(\Omega)$, from Lemma 2.0.1, we have

$$\begin{aligned} \|L^{-1}\varsigma\|_{H^2(\Omega)}^2 =& \left\| \sum_{n,m=0}^{\infty} \frac{\varsigma_{nm}}{1+n^2\pi^2/l^2+m^2\pi^2/l^2} \varphi_{nm} \right\|_{H^2(\Omega)}^2 \\ = & \sum_{n,m=0}^{\infty} (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + m^4\pi^4/l^4 + n^4\pi^4/l^4 + 2m^2n^2\pi^4/l^4) \\ & \cdot \left\| \frac{\varsigma_{nm}}{1+n^2\pi^2/l^2+m^2\pi^2/l^2} \varphi_{nm} \right\|_{L^2(\Omega)}^2 \end{aligned} \tag{4.2.11}$$

$$\leq \sum_{n,m=0}^{\infty} (1+n^2\pi^2/l^2+m^2\pi^2/l^2)^2 \left(\frac{\varsigma_{nm}}{1+n^2\pi^2/l^2+m^2\pi^2/l^2}\right)^2 \|\varphi_{nm}\|^2_{L^2(\Omega)}$$
$$=\|\varsigma\|^2_{L^2(\Omega)}.$$

Therefore, we get

$$\begin{aligned}
\|\psi_1\|_{H^2(\Omega)} &\leq \|P_N(\gamma f_1(\tilde{w}) - \gamma f_{1\tilde{u}}(0)u_N - \gamma f_{1\tilde{v}}(0)v_N + \triangle\hat{u}_N)\|_{L^2(\Omega)},\\
\|\psi_2\|_{H^2(\Omega)} &\leq \|P_N(\gamma g_1(\tilde{w}) - \gamma g_{1\tilde{u}}(0)u_N - \gamma g_{1\tilde{v}}(0)v_N + d\triangle\hat{v}_N)\|_{L^2(\Omega)}.
\end{aligned} \tag{4.2.12}$$

Estimating the right hand-side of (4.2.12) as:

$$\begin{aligned}
\|P_N(\gamma f_1(\tilde{w}) - \gamma f_{1\tilde{u}}(0)u_N - \gamma f_{1\tilde{v}}(0)v_N + \triangle\hat{u}_N)\|_{L^2(\Omega)} &\leq s_1,\\
\|P_N(\gamma g_1(\tilde{w}) - \gamma g_{1\tilde{u}}(0)u_N - \gamma g_{1\tilde{v}}(0)v_N + d\triangle\hat{v}_N)\|_{L^2(\Omega)} &\leq s_2,
\end{aligned}$$

we get the following sufficient conditions for the finite dimensional part of (4.2.2):

$$\begin{aligned}
\|[I-F'(0)]^{-1}_{u_N}\|_{\mathcal{L}(H^2(\Omega)\cap X_N, X_N)} \sqrt{s_1^2+\frac{s_2^2}{d^2}} \leq& \alpha_1,\\
\|[I-F'(0)]^{-1}_{v_N}\|_{\mathcal{L}(H^2(\Omega)\cap X_N, X_N)} \sqrt{s_1^2+\frac{s_2^2}{d^2}} \leq& \beta_1.
\end{aligned}$$

We have the following estimates:

$$\begin{aligned}
&\|(\hat{u}_N+u_N+u_\perp)^2(\hat{v}_N+v_N+v_\perp) - \hat{u}_N^2\hat{v}_N - 2\hat{u}_N\hat{v}_N u_N - \hat{u}_N^2 v_N\|_{L^2(\Omega)}\\
=&\|\hat{u}_N^2 v_\perp + (u_N^2+u_\perp^2+2\hat{u}_N u_\perp + 2u_N u_\perp)(\hat{v}_N+v_\perp+v_N) + 2\hat{u}_N u_N(v_N+v_\perp)\|_{L^2(\Omega)}\\
\leq&\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_\perp\|_{L^2(\Omega)} + \|\hat{v}_N\|_{L^\infty(\Omega)}\|u_N\|^2_{L^4(\Omega)} + \|u_N\|^2_{L^8(\Omega)}\|v_\perp\|_{L^4(\Omega)}\\
&+\|u_N\|^2_{L^8(\Omega)}\|v_N\|_{L^4(\Omega)} + \|u_\perp\|^2_{L^4(\Omega)}\|\hat{v}_N\|_{L^\infty(\Omega)} + \|u_\perp\|^2_{L^8(\Omega)}\|v_N\|_{L^4(\Omega)}\\
&+\|u_\perp\|^2_{L^8(\Omega)}\|v_\perp\|_{L^4(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}\|u_N\|_{L^4(\Omega)}\|v_N\|_{L^4(\Omega)}\\
&+2\|\hat{u}_N\|_{L^\infty(\Omega)}\|u_N\|_{L^4(\Omega)}\|v_\perp\|_{L^4(\Omega)} + 2\|\hat{v}_N\|_{L^\infty(\Omega)}\|u_N\|_{L^4(\Omega)}\|u_\perp\|_{L^4(\Omega)}\\
&+2\|u_N\|_{L^4(\Omega)}\|u_\perp\|_{L^8(\Omega)}\|v_N\|_{L^8(\Omega)} + 2\|u_N\|_{L^4(\Omega)}\|u_\perp\|_{L^8(\Omega)}\|v_\perp\|_{L^8(\Omega)}\\
&+2\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{L^2(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{L^4(\Omega)}\|v_\perp\|_{L^4(\Omega)}\\
&+2\|\hat{u}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{L^4(\Omega)}\|v_N\|_{L^4(\Omega)}\\
\leq&\|\hat{u}_N^2\|_{L^\infty(\Omega)}C_2(N)\beta_2 + \|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,4}^2\alpha_1^2 + K_{2,8}^2\alpha_1^2K_{2,4}\beta_2 + K_{2,8}^2\alpha_1^2K_{2,4}\beta_1\\
&+K_{2,4}^2\alpha_2^2\|\hat{v}_N\|_{L^\infty(\Omega)} + K_{2,8}^2\alpha_2^2K_{2,4}\beta_1 + K_{2,8}^2\alpha_2^2K_{2,4}\beta_2 + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,4}^2\alpha_1\beta_1\\
&+2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,4}^2\alpha_1\beta_2 + 2\|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,4}^2\alpha_1\alpha_2 + 2K_{2,4}K_{2,8}^2\alpha_1\alpha_2\beta_1 + 2K_{2,4}K_{2,8}^2\alpha_1\alpha_2\beta_2\\
&+2\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}C_2(N)\alpha_2 + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,4}^2\alpha_2\beta_2 + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,4}^2\alpha_2\beta_1 =: c_1,
\end{aligned}$$

which derives

$$
\begin{aligned}
&\|P_N(\gamma f_1(\tilde{w}) - \gamma f_{1\tilde{u}}(0)u_N - \gamma f_{1\tilde{v}}(0)v_N + \triangle \hat{u}_N)\|_{L^2(\Omega)}\\
=&\|P_N(\gamma(a - (\hat{u}_N + u_N + u_\perp) + (\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp))\\
&- \gamma(-1 + 2\hat{u}_N\hat{v}_N)u_N - \gamma\hat{u}_N^2 v_N + \triangle \hat{u}_N)\|_{L^2(\Omega)}\\
\leq&\|P_N(\gamma(a - \hat{u}_N + \hat{u}_N^2\hat{v}_N) + \triangle \hat{u}_N)\|_{L^2(\Omega)}\\
&+ \gamma\|((\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp) - \hat{u}_N^2\hat{v}_N - 2\hat{u}_N\hat{v}_N u_N - \hat{u}_N^2 v_N)\|_{L^2(\Omega)}\\
\leq&\|P_N(\gamma(a - \hat{u}_N + \hat{u}_N^2\hat{v}_N) + \triangle \hat{u}_N)\|_{L^2(\Omega)} + \gamma c_1 =: s_1.
\end{aligned}
$$

Similarly we obtain

$$
\begin{aligned}
&\|P_N(\gamma g_1(\tilde{w}) - \gamma g_{1\tilde{v}}(0)v_N - \gamma g_{1\tilde{u}}(0)u_N + d\triangle \hat{v}_N)\|_{L^2(\Omega)}\\
=&\|P_N(\gamma(b - (\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp)) - \gamma(-2\hat{u}_N\hat{v}_N u_N)\\
&- \gamma(-\hat{u}_N^2 v_N) + d\triangle \hat{v}_N)\|_{L^2(\Omega)}\\
\leq&\|P_N(\gamma(b - \hat{u}_N^2\hat{v}_N) + d\triangle \hat{v}_N)\|_{L^2(\Omega)}\\
&+ \gamma\|((\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp) - \hat{u}_N^2\hat{v}_N - 2\hat{u}_N\hat{v}_N u_N - \hat{u}_N^2 v_N)\|_{L^2(\Omega)}\\
\leq&\|P_N(\gamma(b - \hat{u}_N^2\hat{v}_N) + d\triangle \hat{v}_N)\|_{L^2(\Omega)} + \gamma c_1 =: s_2.
\end{aligned}
$$

4.2.2 Infinite dimensional part

As same as the finite dimensional part, we derive a sufficient condition for the latter part of (4.2.2).

For every $\phi \in L^2(\Omega)$, from the definition of P_N and the formulation of $L^{-1}\phi$ which we derived in Lemma 2.0.1, we know

$$L^{-1}(I - P_N)\phi = (I - P_N)L^{-1}\phi.$$

Therefore, for every $\tilde{w} = (\tilde{u}, \tilde{v}) \in W$,

$$(I - P)F(\tilde{w}) = \begin{pmatrix} (I - P_N)L^{-1}\{\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u} + \Delta \hat{u}_N\} \\ (I - P_N)\frac{1}{d}L^{-1}\{\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v} + d\Delta \hat{v}_N\} \end{pmatrix}$$

and

$$
\begin{aligned}
&(I - P_N)L^{-1}\{\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u} + \Delta \hat{u}_N\}\\
=\ & L^{-1}\{(I - P_N)(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u} + \Delta \hat{u}_N)\}\\
=\ & L^{-1}\{(I - P_N)(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u})\},\\
&(I - P_N)\tfrac{1}{d}L^{-1}\{\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v} + d\Delta \hat{v}_N\}\\
=\ & \tfrac{1}{d}L^{-1}\{(I - P_N)(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v} + d\Delta \hat{v}_N)\}\\
=\ & \tfrac{1}{d}L^{-1}\{(I - P_N)(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v})\},
\end{aligned}
$$

hold.

If $\tilde{u}, \tilde{v} \in H^2(\Omega)$ holds, since f_1, g_1 are both polynomial functions, we have $f_1(\tilde{u}, \tilde{v}), g_1(\tilde{u}, \tilde{v}) \in H^2(\Omega)$. And for every $\phi = \sum_{n,m=0}^{\infty} \phi_{nm}\varphi_{nm} \in H^2(\Omega)$, we have

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} \frac{l^2}{4}(1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4)\phi_{nm}^2 \\
<&\|\phi\|_{H^2(\Omega)} < \infty.
\end{aligned}$$

From the formulation of $L^{-1}\phi$ which we derived in Lemma 2.0.1, we know

$$L^{-1}\phi = \sum_{n,m=0}^{\infty} \frac{\phi_{nm}}{1 + n^2\pi^2/l^2 + m^2\pi^2/l^2}\varphi_{nm},$$

therefore, we get

$$\begin{aligned}
&\|L^{-1}\phi\|_{H^4(\Omega)} \\
<& \sum_{n,m=0}^{\infty} l^2(1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4 \\
&+ n^6\pi^6/l^6 + m^6\pi^6/l^6 + 3m^4n^2\pi^6/l^6 + 3m^2n^4\pi^6/l^6 + n^8\pi^8/l^8 + m^8\pi^8/l^8 \\
&+ 4n^2m^6\pi^8/l^8 + 4n^6m^2\pi^8/l^8 + 6n^4m^4\pi^8/l^8)\frac{\phi_{nm}^2}{(1 + n^2\pi^2/l^2 + m^2\pi^2/l^2)^2} \\
\leq& \sum_{n,m=0}^{\infty} l^2((1 + n^2\pi^2/l^2 + m^2\pi^2/l^2) + (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2)^2 \\
&+ (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2)^3 + (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2)^4) \\
&\cdot \frac{\phi_{nm}^2}{(1 + n^2\pi^2/l^2 + m^2\pi^2/l^2)^2} \\
\leq& \sum_{n,m=0}^{\infty} l^2(2 + (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2) + (1 + n^2\pi^2/l^2 + m^2\pi^2/l^2)^2)\phi_{nm}^2 \\
\leq& \sum_{n,m=0}^{\infty} 4l^2(1 + n^2\pi^2/l^2 + m^2\pi^2/l^2 + n^4\pi^4/l^4 + m^4\pi^4/l^4 + 2n^2m^2\pi^4/l^4)\phi_{nm}^2 < \infty,
\end{aligned}$$

so we have $L^{-1}\phi \in H^4(\Omega)$.

Then we obtain

$$
\begin{aligned}
&\|-\Delta^2(L^{-1}\phi)\|_{L^2(\Omega)} = \|-\Delta((\Delta - I)(-\Delta + I)^{-1}\phi + (-\Delta + I)^{-1}\phi)\|_{L^2(\Omega)} \\
=&\|-\Delta\phi - \Delta(-\Delta + I)^{-1}\phi\|_{L^2(\Omega)} \\
\leq&\|\Delta\phi\|_{L^2(\Omega)} + \|(-\Delta + I)(-\Delta + I)^{-1}\phi - (\Delta + I)^{-1}\phi\|_{L^2(\Omega)} \\
\leq&\|\Delta\phi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)} + \|L^{-1}\phi\|_{L^2(\Omega)} \\
\leq&\|\Delta\phi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)} + \|L^{-1}\phi\|_{H^1(\Omega)} \\
\leq&\|\Delta\phi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)} \\
=&\|\Delta\phi\|_{L^2(\Omega)} + 2\|\phi\|_{L^2(\Omega)},
\end{aligned}
$$

therefore,

$$
\begin{aligned}
&\|\Delta^2(L^{-1}\phi) - \Delta(L^{-1}\phi) + L^{-1}\phi\|_{L^2(\Omega)} \\
\leq&\|-\Delta^2(L^{-1}\phi)\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)} \\
\leq&3\|\phi\|_{L^2(\Omega)} + \|\Delta\phi\|_{L^2(\Omega)}
\end{aligned}
$$

holds.

Then replacing z in Lemma 2.0.4 by $(I - P_N)L^{-1}\{(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u})\}$ and $(I - P_N)L^{-1}\{(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v})\}$, we get

$$
\begin{aligned}
&\|(I - P_N)L^{-1}\{\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u}\}\|_{H^2(\Omega)} \\
\leq&C_2(N)(3\|(I - P_N)(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u})\|_{L^2(\Omega)} + \|\Delta(I - P_N)(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u})\|_{L^2(\Omega)}), \\
&\|(I - P_N)\frac{1}{d}L^{-1}\{\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v}\}\|_{H^2(\Omega)} \\
\leq&\frac{1}{d}C_2(N)(3\|(I - P_N)(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v})\|_{L^2(\Omega)} + \|\Delta(I - P_N)(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v})\|_{L^2(\Omega)}).
\end{aligned}
$$

Therefore we obtain the following sufficient conditions for the infinite part of (4.2.2):

$$
\left\{
\begin{array}{l}
C_2(N)(3\|(I - P_N)(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u})\|_{L^2(\Omega)} + \|\Delta(I - P_N)(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u})\|_{L^2(\Omega)}) \leq \alpha_2, \\
\frac{C_2(N)}{d}(3\|(I - P_N)(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v})\|_{L^2(\Omega)} + \|\Delta(I - P_N)(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v})\|_{L^2(\Omega)}) \leq \beta_2.
\end{array}
\right.
\tag{4.2.13}
$$

First we estimate $\|\Delta(\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp)\|_{L^2(\Omega)}$ as following:

$$
\begin{aligned}
&\|(I - P_N)(\Delta(\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp))\|_{L^2(\Omega)} \\
=&\|(I - P_N)(2(\nabla(\hat{u}_N + u_N + u_\perp))^2(\hat{v}_N + v_N + v_\perp) \\
&+ 4(\hat{u}_N + u_N + u_\perp)(\nabla(\hat{v}_N + v_N + v_\perp))(\nabla(\hat{u}_N + u_N + u_\perp)) \\
&+ 2(\hat{u}_N + u_N + u_\perp)(\hat{v}_N + v_N + v_\perp)(\Delta(\hat{u}_N + u_N + u_\perp)) \\
&+ (\hat{u}_N + u_N + u_\perp)^2(\Delta(\hat{v}_N + v_N + v_\perp)))\|_{L^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
&\|(I-P_N)(2(\nabla(\hat{u}_N+u_N+u_\perp))^2(\hat{v}_N+v_N+v_\perp))\|_{L^2(\Omega)}\\
\leq &2(\|(I-P_N)\hat{v}_N(\nabla\hat{u}_N)^2\|_{L^2(\Omega)}+(\|v_N\|_{L^2(\Omega)}+\|v_\perp\|_{L^2(\Omega)})\|(\nabla\hat{u}_N)^2\|_{L^\infty(\Omega)}\\
&+2\|\hat{v}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)},\|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\}\|_{L^\infty(\Omega)}(\|\nabla u_N\|_{L^2(\Omega)}+\|\nabla u_\perp\|_{L^2(\Omega)})\\
&+2(\|v_\perp\|_{L^\infty(\Omega)}+\|v_N\|_{L^\infty(\Omega)})\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)},\|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\}\\
&\cdot(\|\nabla u_N\|_{L^2(\Omega)}+\|\nabla u_\perp\|_{L^2(\Omega)})\\
&+(\|\hat{v}_N\|_{L^\infty(\Omega)}+\|v_N\|_{L^\infty(\Omega)}+\|v_\perp\|_{L^\infty(\Omega)})(\|\nabla u_N\|_{L^4(\Omega)}+\|\nabla u_\perp\|_{L^4(\Omega)})^2)\\
\leq &2(\|(I-P_N)\hat{v}_N(\nabla\hat{u}_N)^2\|_{L^2(\Omega)}+(\beta_1+C_2(N)\beta_2)\|(\nabla\hat{u}_N)^2\|_{L^\infty(\Omega)}\\
&+2\|\hat{v}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)},\|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\}(\alpha_1+\alpha_2)\\
&+2K_{2,\infty}(\beta_2+\beta_1)\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)},\|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\}(\alpha_1+\alpha_2)\\
&+(\|\hat{v}_N\|_{L^\infty(\Omega)}+K_{2,\infty}\beta_1+K_{2,\infty}\beta_2)K_4^2(\alpha_1+\alpha_2)^2=:c_2,
\end{aligned}
$$

$$
\begin{aligned}
&\|(I-P_N)4(\hat{u}_N+u_N+u_\perp)(\nabla(\hat{v}_N+v_N+v_\perp))(\nabla(\hat{u}_N+u_N+u_\perp))\|_{L^2(\Omega)}\\
\leq &4(\|(I-P_N)\hat{u}_N\nabla\hat{v}_N\nabla\hat{u}_N\|_{L^2(\Omega)}+(\|u_N\|_{L^2(\Omega)}+\|u_\perp\|_{L^2(\Omega)})\|\nabla\hat{v}_N\nabla\hat{u}_N\|_{L^\infty(\Omega)}\\
&+\|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{v}_N)_x\|_{L^\infty(\Omega)},\|(\hat{v}_N)_y\|_{L^\infty(\Omega)}\}\\
&\cdot(\|\nabla u_N\|_{L^2(\Omega)}+\|\nabla u_\perp\|_{L^2(\Omega)})+(\|u_N\|_{L^\infty(\Omega)}+\|u_\perp\|_{L^\infty(\Omega)})\\
&\cdot\max\{\|(\hat{v}_N)_x\|_{L^\infty(\Omega)},\|(\hat{v}_N)_y\|_{L^\infty(\Omega)}\}(\|\nabla u_N\|_{L^2(\Omega)}+\|\nabla u_\perp\|_{L^2(\Omega)})\\
&+\|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)},\|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\}(\|\nabla v_N\|_{L^2(\Omega)}+\|\nabla v_\perp\|_{L^2(\Omega)})\\
&+(\|u_N\|_{L^\infty(\Omega)}+\|u_\perp\|_{L^\infty(\Omega)})(\|\nabla v_N\|_{L^2(\Omega)}+\|\nabla v_\perp\|_{L^2(\Omega)})\\
&\cdot\max\{\|(\hat{v}_N)_x\|_{L^\infty(\Omega)},\|(\hat{v}_N)_y\|_{L^\infty(\Omega)}\}+(\|\hat{u}_N\|_{L^\infty(\Omega)}+\|u_N\|_{L^\infty(\Omega)}+\|u_\perp\|_{L^\infty(\Omega)})\\
&\cdot(\|\nabla v_N\|_{L^4(\Omega)}+\|\nabla v_\perp\|_{L^4(\Omega)})(\|\nabla u_\perp\|_{L^4(\Omega)}+\|\nabla u_N\|_{L^4(\Omega)}))\\
\leq &4(\|(I-P_N)\hat{u}_N\nabla\hat{v}_N\nabla\hat{u}_N\|_{L^2(\Omega)}+(\alpha_1+C_2(N)\alpha_2)\|\nabla\hat{v}_N\nabla\hat{u}_N\|_{L^\infty(\Omega)}\\
&+\|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{v}_N)_x\|_{L^\infty(\Omega)},\|(\hat{v}_N)_y\|_{L^\infty(\Omega)}\}(\alpha_1+\alpha_2)\\
&+K_{2,\infty}(\alpha_1+\alpha_2)\max\{\|(\hat{v}_N)_x\|_{L^\infty(\Omega)},\|(\hat{v}_N)_y\|_{L^\infty(\Omega)}\}(\alpha_1+\alpha_2)\\
&+\|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)},\|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\}(\beta_1+\beta_2)\\
&+K_{2,\infty}(\alpha_1+\alpha_2)(\beta_1+\beta_2)\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)},\|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\}\\
&+(\|\hat{u}_N\|_{L^\infty(\Omega)}+K_{2,\infty}\alpha_1+K_{2,\infty}\alpha_2)K_4^2(\beta_1+\beta_2)(\alpha_1+\alpha_2))=:c_3,
\end{aligned}
$$

$$
\begin{aligned}
&\|(I-P_N)2(\hat{u}_N+u_N+u_\perp)(\hat{v}_N+v_N+v_\perp)(\Delta(\hat{u}_N+u_N+u_\perp))\|_{L^2(\Omega)}\\
\leq &2((\|(I-P_N)\hat{u}_N\hat{v}_N\Delta\hat{u}_N\|_{L^2(\Omega)}+\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}(\|\Delta u_N\|_{L^2(\Omega)}+\|\Delta u_\perp\|_{L^2(\Omega)})\\
&+\|\hat{u}_N\Delta\hat{u}_N\|_{L^\infty(\Omega)}(\|v_N\|_{L^2(\Omega)}+\|v_\perp\|_{L^2(\Omega)})+\|\hat{v}_N\Delta\hat{u}_N\|_{L^\infty(\Omega)}(\|u_N\|_{L^2(\Omega)}+\|u_\perp\|_{L^2(\Omega)})\\
&+\|\hat{u}_N\|_{L^\infty(\Omega)}(\|v_N\|_{L^\infty(\Omega)}+\|v_\perp\|_{L^\infty(\Omega)})(\|\Delta u_N\|_{L^2(\Omega)}+\|\Delta u_\perp\|_{L^2(\Omega)})\\
&+\|\hat{v}_N\|_{L^\infty(\Omega)}(\|u_N\|_{L^\infty(\Omega)}+\|u_\perp\|_{L^\infty(\Omega)})(\|\Delta u_N\|_{L^2(\Omega)}+\|\Delta u_\perp\|_{L^2(\Omega)})\\
&+\|\Delta\hat{u}_N\|_{L^\infty(\Omega)}(\|u_N\|_{L^2(\Omega)}+\|u_\perp\|_{L^2(\Omega)})(\|v_N\|_{L^\infty(\Omega)}+\|v_\perp\|_{L^\infty(\Omega)})\\
&+(\|u_N\|_{L^\infty(\Omega)}+\|u_\perp\|_{L^\infty(\Omega)})(\|v_N\|_{L^\infty(\Omega)}+\|v_\perp\|_{L^\infty(\Omega)})(\|\Delta u_N\|_{L^2(\Omega)}+\|\Delta u_\perp\|_{L^2(\Omega)})
\end{aligned}
$$

$$
\begin{aligned}
\le & 2((\|(I-P_N)\hat{u}_N\hat{v}_N\Delta\hat{u}_N\|_{L^2(\Omega)} + \|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}(\alpha_1+\alpha_2) \\
& + \|\hat{u}_N\Delta\hat{u}_N\|_{L^\infty(\Omega)}(\beta_1 + C_2(N)\beta_2) + \|\hat{v}_N\Delta\hat{u}_N\|_{L^\infty(\Omega)}(\alpha_1 + C_2(N)\alpha_2) \\
& + \|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\beta_1+\beta_2)(\alpha_1+\alpha_2) + \|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1+\alpha_2)^2 \\
& + \|\Delta\hat{u}_N\|_{L^\infty(\Omega)}(\alpha_1 + C_2(N)\alpha_2)K_{2,\infty}(\beta_1+\beta_2) + K_{2,\infty}^2(\alpha_1+\alpha_2)^2(\beta_1+\beta_2) =: c_4,
\end{aligned}
$$

$$
\begin{aligned}
& \|(\hat{u}_N + u_N + u_\perp)^2(\Delta(\hat{v}_N + v_N + v_\perp))\|_{L^2(\Omega)} \\
\le & \|(I-P_N)\hat{u}_N^2\Delta\hat{v}_N\|_{L^2(\Omega)} + \|\hat{u}_N^2\|_{L^\infty(\Omega)}(\|\Delta v_N\|_{L^2(\Omega)} + \|\Delta v_\perp\|_{L^2(\Omega)}) \\
& + 2\|\hat{u}_N\Delta\hat{v}_N\|_{L^\infty(\Omega)}(\|u_N\|_{L^2(\Omega)} + \|u_\perp\|_{L^2(\Omega)}) \\
& + 2\|\hat{u}_N\|_{L^\infty(\Omega)}(\|u_N\|_{L^\infty(\Omega)} + \|u_\perp\|_{L^\infty(\Omega)})(\Delta v_N\|_{L^2(\Omega)} + \|\Delta v_\perp\|_{L^2(\Omega)}) \\
& + (\|u_N\|_{L^\infty(\Omega)} + \|u_\perp\|_{L^\infty(\Omega)})^2(\|\Delta\hat{v}_N\|_{L^2(\Omega)} + \|\Delta v_N\|_{L^2(\Omega)} + \|\Delta v_\perp\|_{L^2(\Omega)}) \\
\le & \|(I-P_N)\hat{u}_N^2\Delta\hat{v}_N\|_{L^2(\Omega)} + \|\hat{u}_N^2\|_{L^\infty(\Omega)}(\beta_1+\beta_2) \\
& + 2\|\hat{u}_N\Delta\hat{v}_N\|_{L^\infty(\Omega)}(\alpha_1 + C_2(N)\alpha_2) + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1+\alpha_2)(\beta_1+\beta_2) \\
& + K_{2,\infty}^2(\alpha_1+\alpha_2)^2(\|\Delta\hat{v}_N\|_{L^2(\Omega)} + \beta_1 + \beta_2) =: c_5,
\end{aligned}
$$

here, $(\hat{u}_N)_x, (\hat{v}_N)_x$ denote the Fréchet derivative with respect to x of $\hat{u}_N$ and $\hat{v}_N$ respectively, $(\hat{u}_N)_y, (\hat{v}_N)_y$ denote the Fréchet derivative with respect to y of $\hat{u}_N$ and $\hat{v}_N$ respectively.

Then we can estimate the left hand side of (4.2.13) as follows:

$$
\begin{aligned}
& 3\|(I-P_N)(\gamma f_1(\tilde{u},\tilde{v}) + \tilde{u})\|_{L^2(\Omega)} + \|\Delta(I-P_N)(\gamma f_1(\tilde{u},\tilde{v}) + \tilde{u})\|_{L^2(\Omega)}) \\
= & 3(\|(I-P_N)(\gamma(a - (\hat{u}_N + u_N + u_\perp) + (\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp)) \\
& +(u_N + u_\perp))\|_{L^2(\Omega)}) + \|\Delta(I-P_N)(\gamma(a - (\hat{u}_N + u_N + u_\perp) \\
& +(\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp)) + (u_N + u_\perp))\|_{L^2(\Omega)} \\
\le & 3\|(I-P_N)(\gamma((\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp))\|_{L^2(\Omega)} \\
& +|1-\gamma|(\|\Delta u_\perp\|_{L^2(\Omega)} + 3\|u_\perp\|_{L^2(\Omega)}) \\
& +\|(I-P_N)\Delta(\gamma((\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp))\|_{L^2(\Omega)} \\
\le & 3(\gamma c_1 + \gamma\|(I-P_N)\hat{u}_N^2\hat{v}_N\|_{L^2(\Omega)} + 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_N\|_{L^2(\Omega)} \\
& +\gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_N\|_{L^2(\Omega)}) + |1-\gamma|(1+3C_2(N))\|u_\perp\|_{H^2(\Omega)} + (c_2+c_3+c_4+c_5)\gamma \\
\le & 3(\gamma c_1 + \gamma\|(I-P_N)\hat{u}_N^2\hat{v}_N\|_{L^2(\Omega)} + 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\alpha_1 + \gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\beta_1) \\
& +|1-\gamma|(1+3C_2(N))\alpha_2 + (c_2+c_3+c_4+c_5)\gamma,
\end{aligned}
$$

and

$$
\begin{aligned}
& 1/d(3\|(I-P_N)(\gamma g_1(\tilde{u},\tilde{v}) + d\tilde{v}))\|_{L^2(\Omega)} + \|\Delta(I-P_N)(\gamma g_1(\tilde{u},\tilde{v}) + d\tilde{v}))\|_{L^2(\Omega)}) \\
= & 1/d(3\|(I-P_N)(\gamma(b - (\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp) + d(v_N + v_\perp))\|_{L^2(\Omega)} \\
& + \|\Delta(I-P_N)(\gamma(b - (\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp) + d(v_N + v_\perp))\|_{L^2(\Omega)} \\
\le & 1/d((3d\|v_\perp\|_{L^2(\Omega)} + d\|\Delta v_\perp\|_{L^2(\Omega)}) \\
& + 3\gamma\|(I-P_N)(\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp)\|_{L^2(\Omega)} \\
& + \gamma\|(I-P_N)\Delta(\hat{u}_N + u_N + u_\perp)^2(\hat{v}_N + v_N + v_\perp)\|_{L^2(\Omega)})
\end{aligned}
$$

$$\begin{aligned}
\le & 1/d(d(3C_2(N)+1)\|v_\perp\|_{H^2(\Omega)} + (c_2+c_3+c_4+c_5)\gamma \\
& + 3\gamma(c_1 + \|(I-P_N)\hat{u}_N^2\hat{v}_N\|_{L^2(\Omega)} + 2\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_N\|_{L^2(\Omega)} \\
& + \|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_N\|_{L^2(\Omega)})) \\
\le & 1/d(d(3C_2(N)+1)\beta_2 + \gamma(c_2+c_3+c_4+c_5) + 3\gamma(c_1 + \|(I-P_N)\hat{u}_N^2\hat{v}_N\|_{L^2(\Omega)} \\
& + 2\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\alpha_1 + \|\hat{u}_N^2\|_{L^\infty(\Omega)}\beta_1)).
\end{aligned}$$

4.2.3 Verification Algorithm

In order to enclose the exact solution (u^*, v^*), as described above we construct the sets $W, U_N, U_\perp, V_N, V_\perp$ by

$$\begin{aligned}
W &= U \times V, \\
U &= U_N + U_\perp, \\
V &= V_N + V_\perp,
\end{aligned}$$

with $U_N, U_\perp, V_N, V_\perp$ defined by

$$\begin{aligned}
U_N &= \{\phi_N \in X_N \mid \|\phi_N\|_{H^2(\Omega)} \le \alpha_1\}, \\
U_\perp &= \{\phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^2(\Omega)} \le \alpha_2\}, \\
V_N &= \{\phi_N \in X_N \mid \|\phi_N\|_{H^2(\Omega)} \le \beta_1\}, \\
V_\perp &= \{\phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^2(\Omega)} \le \beta_2\},
\end{aligned}$$

for positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$. Then the exact solutions are obtained as $u^* = \hat{u}_N + u_N + u_\perp$, $v^* = \hat{v}_N + v_N + v_\perp$, where $(\hat{u}_N, \hat{v}_N)$ is the approximate solution we obtained from Newton's method, and $u_N \in U_N$, $v_N \in V_N$, $u_\perp \in U_\perp$, $v_\perp \in V_\perp$. From the discussion in Section 4.1, the residual part $\tilde{w} = (u^* - \hat{u}_N, v^* - \hat{v}_N)$ can be treated as a fixed point of the equation $\tilde{w} = T(\tilde{w})$, with $T(\tilde{w}) = N(\tilde{w}) + (I-P)F(\tilde{w})$. In the finite dimensional part, for $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in N(W)$, we need to find $\alpha_1, \beta_1 > 0$ satisfying

$$\begin{cases} \|\phi_1\|_{H^2(\Omega)} \le \alpha_1, \\ \|\phi_2\|_{H^2(\Omega)} \le \beta_1. \end{cases} \tag{4.2.14}$$

In the infinite dimensional part, by the discussion in Section 4.2.2, we get the sufficient conditions for them:

$$\begin{cases} C_2(N)(3\|(I-P_N)(\gamma f_1(\tilde{u},\tilde{v}) + \tilde{u})\|_{L^2(\Omega)} + \|\Delta(I-P_N)(\gamma f_1(\tilde{u},\tilde{v}) + \tilde{u})\|_{L^2(\Omega)}) \le \alpha_2, \\ \frac{1}{d}C_2(N)(3\|(I-P_N)(\gamma g_1(\tilde{u},\tilde{v}) + d\tilde{v}))\|_{L^2(\Omega)} + \|\Delta(I-P_N)(\gamma g_1(\tilde{u},\tilde{v}) + d\tilde{v}))\|_{L^2(\Omega)}) \le \beta_2. \end{cases} \tag{4.2.15}$$

If we can find positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfying (4.2.14) and (4.2.15), we can enclose the exact solution.

We find $\alpha_1, \alpha_2, \beta_1, \beta_2$ by an iteration, i.e. we consider the following sets depending on $k = 0, 1, \cdots$:

$$\begin{aligned} W^{(k)} &= U^{(k)} \times V^{(k)}, \\ U^{(k)} &= U_N^{(k)} + U_\perp^{(k)}, \\ V^{(k)} &= V_N^{(k)} + V_\perp^{(k)}, \end{aligned}$$

with $U_N^{(k)}, U_\perp^{(k)}, V_N^{(k)}, V_\perp^{(k)}$ defined by

$$\begin{aligned} U_N^{(k)} &= \{\phi_N \in X_N \mid \|\phi_N\|_{H^2(\Omega)} \le \alpha_1^{(k)}\}, \\ U_\perp^{(k)} &= \{\phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^2(\Omega)} \le \alpha_2^{(k)}\}, \\ V_N^{(k)} &= \{\phi_N \in X_N \mid \|\phi_N\|_{H^2(\Omega)} \le \beta_1^{(k)}\}, \\ V_\perp^{(k)} &= \{\phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^2(\Omega)} \le \beta_2^{(k)}\}. \end{aligned}$$

1) Set initial values $\alpha_1^{(0)}, \alpha_2^{(0)}, \beta_1^{(0)}, \beta_2^{(0)} > 0$ and inflation constants $\delta_1, \delta_2, \delta_3, \delta_4 > 0$.

2) Set

$$\begin{aligned} \alpha_1' :=& \sup_{\phi_1 \in N_1(W^{(k)})} \|\phi_1\|_{H^2(\Omega)}, \\ \beta_1' :=& \sup_{\phi_2 \in N_2(W^{(k)})} \|\phi_2\|_{H^2(\Omega)}, \\ \alpha_2' :=& C_2(N) \sup_{(\tilde{u},\tilde{v}) \in W^{(k)}} (3\|(I - P_N)(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u})\|_{L^2(\Omega)} \\ &+ \|\Delta(I - P_N)(\gamma f_1(\tilde{u}, \tilde{v}) + \tilde{u})\|_{L^2(\Omega)}), \\ \beta_2' :=& \frac{1}{d} C_2(N) \sup_{(\tilde{u},\tilde{v}) \in W^{(k)}} (3\|(I - P_N)(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v}))\|_{L^2(\Omega)} \\ &+ \|\Delta(I - P_N)(\gamma g_1(\tilde{u}, \tilde{v}) + d\tilde{v}))\|_{L^2(\Omega)}). \end{aligned}$$

3) If

$$\begin{aligned} \alpha_1' &\le \alpha_1^{(k)}, \\ \alpha_2' &\le \alpha_2^{(k)}, \\ \beta_1' &\le \beta_1^{(k)}, \\ \beta_2' &\le \beta_2^{(k)} \end{aligned}$$

hold, then stop the iteration. Otherwise set

$$\begin{aligned} \alpha_1^{(k+1)} &= (1 + \delta_1)\alpha_1^{(k)}, \\ \alpha_2^{(k+1)} &= (1 + \delta_2)\alpha_2^{(k)}, \\ \beta_1^{(k+1)} &= (1 + \delta_3)\beta_1^{(k)}, \\ \beta_2^{(k+1)} &= (1 + \delta_4)\beta_2^{(k)} \end{aligned}$$

and return to step 2.

4) If k reaches a maximum iteration number then stop and the verification fails. (Then we need to choose some more accurate approximate solutions.)

Chapter 5

Eigenvalue excluding

In this chapter we will establish a computer-assisted method to exclude eigenvalues of the operator linearized at the exact solution. The method is similar to [17, 25]. We use this eigenvalue excluding results to obtain a resolvent estimation of the operator in the next chapter.

5.1 Eigenvalue excluding theorem

We recall that $(\hat{u}_N, \hat{v}_N)$ is an approximate solution of the following equations:

$$\begin{cases} -\Delta u = \gamma(a - u + u^2 v) \text{ in } \Omega, \\ -d\Delta v = \gamma(b - u^2 v) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

First we consider the eigenvalue problem for the operator linearized at the approximate solution:

$$\begin{cases} -\Delta u - \gamma(-1 + 2\hat{u}_N \hat{v}_N)u - \gamma \hat{u}_N^2 v = \lambda u, \\ -d\Delta v - \gamma(-2\hat{u}_N \hat{v}_N)u + \gamma \hat{u}_N^2 v = \lambda v. \end{cases} \tag{5.1.1}$$

Let $\mu \in \mathcal{C}$ be a given *candidate* excluding point which is suspected that no eigenvalue of Eq. (5.1.1) is close to μ, then by defining a linear operator $\hat{L} : H^2(\Omega) \times H^2(\Omega) \to H^2(\Omega) \times H^2(\Omega)$ as

$$\hat{L}(u, v) := \begin{pmatrix} u + L^{-1}\{-(1+\mu)u - \gamma(-1 + 2\hat{u}_N \hat{v}_N)u - \gamma \hat{u}_N^2 v\} \\ dv + L^{-1}\{-(d+\mu)v + 2\gamma \hat{u}_N \hat{v}_N u + \gamma \hat{u}_N^2 v\} \end{pmatrix},$$

the equations (5.1.1) can be rewritten as

$$\hat{L}(u, v) = (\lambda - \mu) \begin{pmatrix} L^{-1}u \\ L^{-1}v \end{pmatrix}.$$

Then we have the following eigenvalue excluding theorem.

Theorem 5.1.1 *Suppose that* $\hat{L}$ *has an inverse* $\hat{L}^{-1} : H^2(\Omega) \times H^2(\Omega) \to H^2(\Omega) \times H^2(\Omega)$ *and there exists* $\hat{M}_\mu > 0$ *such that*

$$\|\hat{L}^{-1}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} \le \hat{M}_\mu \|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}, \tag{5.1.2}$$

then there is no eigenvalue $\tilde{\lambda}$ *of Eq. (5.1.1) in the disk given by* $|\tilde{\lambda} - \mu| < \frac{1}{\hat{M}_\mu}$.

Proof. For any eigenpair $(u_1, v_1, \tilde{\lambda})^T \in H^2(\Omega) \times H^2(\Omega) \times \mathcal{C}$ of eq. (5.1.1) which satisfies

$$\hat{L}(u,v) = (\lambda - \mu)\begin{pmatrix} L^{-1}u \\ L^{-1}v \end{pmatrix},$$

where $u_1, v_1 \neq 0$, taking $(u,v) \in H^2(\Omega) \times H^2(\Omega)$ as $\hat{L}(u_1, v_1)$ in (5.1.2), we have

$$\begin{aligned} \|(u_1,v_1)\|_{H^2(\Omega)\times H^2(\Omega)} &\le \hat{M}_\mu \|\hat{L}(u_1,v_1)\|_{H^2(\Omega)\times H^2(\Omega)} \\ &= \hat{M}_\mu |\tilde{\lambda} - \mu| \cdot \|(L^{-1}u_1, L^{-1}v_1)\|_{H^2(\Omega)\times H^2(\Omega)}. \end{aligned} \tag{5.1.3}$$

Same as in (4.2.11), for all $\phi \in L^2(\Omega)$,

$$\|L^{-1}\phi\|_{H^2(\Omega)} \le \|\phi\|_{L^2(\Omega)}$$

holds, therefore, (5.1.3) becomes

$$\begin{aligned} &\|(\tilde{u},\tilde{v})\|_{H^2(\Omega)\times H^2(\Omega)} \\ \le&\hat{M}_\mu |\tilde{\lambda} - \mu| \cdot \|(\tilde{u},\tilde{v})\|_{L^2(\Omega)\times L^2(\Omega)} \\ \le&\hat{M}_\mu |\tilde{\lambda} - \mu| \cdot \|(\tilde{u},\tilde{v})\|_{H^2(\Omega)\times H^2(\Omega)}. \qquad \square \end{aligned}$$

5.2 Invertibility condition of $\hat{L}$

In order to show the invertibility of $\hat{L} : H^2(\Omega) \times H^2(\Omega) \to H^2(\Omega) \times H^2(\Omega)$, we now give a condition that the problem $\hat{L}(u,v) = 0$ has only unique solution $(u,v) = 0$.

By defining a compact map $F : H^2(\Omega) \times H^2(\Omega) \to H^2(\Omega) \times H^2(\Omega)$ as

$$F(u,v) := \begin{pmatrix} L^{-1}\{(1+\mu)u + \gamma(-1 + 2\hat{u}_N\hat{v}_N)u + \gamma\hat{u}_N^2 v\} \\ \frac{1}{d}L^{-1}\{-2\gamma\hat{u}_N\hat{v}_N u + (-\gamma\hat{u}_N^2 + d + \mu)v\} \end{pmatrix},$$

the problem $\hat{L}(u,v) = 0$ can be rewritten equivalently in the fixed-point form $(u,v) = F(u,v)$.

Set $w = (u, v)$. The fixed point equation $w = Fw$ can be decomposed as

$$\begin{cases} Pw = PFw, \\ (I-P)Fw = (I-P)Fw. \end{cases}$$

Now we define the Newton-like operator $N : H^2(\Omega) \times H^2(\Omega) \to X_N \times X_N$ by

$$N(w) = Pw - [I - F]_N^{-1}(Pw - PFw) = \begin{pmatrix} N_1 w \\ N_2 w \end{pmatrix}.$$

Setting the compact map $T : H^2(\Omega) \times H^2(\Omega) \to H^2(\Omega) \times H^2(\Omega)$ defined by

$$Tw = Nw + (I-P)Fw,$$

we find that the two fixed-point problems $w = Fw$ and $w = Tw$ are equivalent.

Next we construct several sets:

$$\begin{aligned} &W = U \times V, \\ &U = U_N + U_\perp, \\ &V = V_N + V_\perp, \\ &W_N = U_N \times V_N, \\ &W_\perp = U_\perp \times V_\perp \end{aligned}$$

with $U_N, U_\perp, V_N, V_\perp$ defined by

$$\begin{aligned} U_N &:= \{\phi_N \in X_N \mid \|\phi_N\|_{H^2(\Omega)} \le \alpha_3\}, \\ U_\perp &:= \{\phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^2(\Omega)} \le \alpha_4\}, \\ V_N &:= \{\phi_N \in X_N \mid \|\phi_N\|_{H^2(\Omega)} \le \beta_3\}, \\ V_\perp &:= \{\phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^2(\Omega)} \le \beta_4\}, \end{aligned}$$

for positive constants $\alpha_3, \alpha_4, \beta_3, \beta_4$, where $X_N^\perp$ represents the orthogonal complement of X_N in $H^2(\Omega)$.

Then a sufficient condition for the invertibility of $\hat{L}$ is as follows.

Lemma 5.2.1 *When an inclusion*

$$\overline{TW} \subset \mathring{W}$$

holds, $\hat{L}$ is invertible.

Proof. If there exists $w = (u, v) \in H^2(\Omega) \times H^2(\Omega)$ such that $\hat{L}(u, v) = 0$ and $(u, v) \neq 0$, then (u, v) also satisfies $(u, v) = T(u, v)$. Since T is a linear operator, for each $t \in \Re$, we have

$$T(t(u, v)) = tT(u, v) = t(u, v).$$

Then, we can choose $\hat{t} \in \Re$ satisfying $\hat{t}w \in \partial W$. However, this contradicts with $\overline{TW} \subset \mathring{W}$ and $T(tw) = tw$. Therefore, $w = 0$. That is, $w = 0$ is a unique solution of $\hat{L}w = 0$. $\square$

The finite dimensional part of the inclusion, $\overline{NW} \subset \mathring{W}_N$ can be written as

$$\begin{aligned} \sup_{w \in W} \|N_1 w\|_{H^2(\Omega)} < \alpha_3, \\ \sup_{w \in W} \|N_2 w\|_{H^2(\Omega)} < \beta_3. \end{aligned} \tag{5.2.1}$$

On the other hand, we set

$$\begin{aligned} f_1(u,v) &:= (1+\mu)u + \gamma(-1 + 2\hat{u}_N \hat{v}_N)u + \gamma \hat{u}_N^2 v, \\ f_2(u,v) &:= -2\gamma \hat{u}_N \hat{v}_N u + (-\gamma \hat{u}_N^2 + d + \mu)v. \end{aligned}$$

Replacing z in Lemma 2.0.4 by $(I-P_N)(L^{-1}f_1(u,v))$ and $(I-P_N)(L^{-1}f_2(u,v))$, we know that

$$\begin{aligned} &\|(I-P_N)(L^{-1}f_1(u,v))\|_{H^2(\Omega)} \\ \le &C_2(N)\|(\Delta^2 - \Delta + I)((I-P_N)(L^{-1}f_1(u,v)))\|_{L^2(\Omega)} \\ \le &C_2(N)(3\|(I-P_N)f_1(u,v)\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_1(u,v)\|_{L^2(\Omega)}), \\ &\|(I-P_N)(L^{-1}f_2(u,v))\|_{H^2(\Omega)} \\ \le &\frac{C_2(N)}{d}(3\|(I-P_N)f_2(u,v)\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_2(u,v)\|_{L^2(\Omega)}). \end{aligned}$$

Therefore, the infinite dimensional part of the inclusion $\overline{(I-P)FW} \subset \mathring{W}_\perp$ means

$$\begin{aligned} C_2(N) \sup_{w \in W} (3\|(I-P_N)f_1(u,v)\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_1(u,v)\|_{L^2(\Omega)}) < \alpha_4, \\ \tfrac{C_2(N)}{d} \sup_{w \in W} (3\|(I-P_N)f_2(u,v)\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_2(u,v)\|_{L^2(\Omega)}) < \beta_4. \end{aligned} \tag{5.2.2}$$

Lemma 5.2.2 *If (5.2.1) and (5.2.2) hold, then $\hat{L} : H^2(\Omega) \times H^2(\Omega) \to H^2(\Omega) \times H^2(\Omega)$ is invertible.*

5.3 Computable criterion for the invertibility of $\hat{L}$

Set

$$\begin{aligned}
\hat{\alpha}_3 &:= \sup_{w\in W}\|N_1 w\|_{H^2(\Omega)}, \quad \hat{\beta}_3 := \sup_{w\in W}\|N_2 w\|_{H^2(\Omega)},\\
\hat{\alpha}_4 &:= \sup_{w\in W}(\|3(I-P_N)f_1(w)\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_1(w)\|_{L^2(\Omega)}),\\
\hat{\beta}_4 &:= \sup_{w\in W}\frac{1}{d}(3\|(I-P_N)f_2(w)\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_2(w)\|_{L^2(\Omega)}).
\end{aligned}$$

First we need some constants.

Set $\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6$ satisfying

$$\begin{aligned}
&\|P_N(L^{-1}f_1(u_\perp,v_\perp))\|_{H^2(\Omega)} \le \vartheta_1(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}),\\
&\tfrac{1}{d}\|P_N(L^{-1}f_2(u_\perp,v_\perp))\|_{H^2(\Omega)} \le \vartheta_2(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}),\\
&\|(I-P_N)(3f_1(u,v))\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_1(u,v)\|_{L^2(\Omega)})\\
\le\ &\vartheta_3(\|u_N\|_{H^2(\Omega)} + \|v_N\|_{H^2(\Omega)}) + \vartheta_4(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}),\\
&\tfrac{1}{d}(\|(I-P_N)(3f_2(u,v))\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_2(u,v)\|_{L^2(\Omega)}\\
\le\ &\vartheta_5(\|u_N\|_{H^2(\Omega)} + \|v_N\|_{H^2(\Omega)}) + \vartheta_6(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}).
\end{aligned} \tag{5.3.1}$$

Now we explain how to get $\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6$.

Because

$$\begin{aligned}
&\|P_N(L^{-1}f_1(u_\perp,v_\perp))\|_{H^2(\Omega)}\\
\le&\|P_N((1+\mu)u_\perp + \gamma(-1+2\hat{u}_N\hat{v}_N)u_\perp + \gamma\hat{u}_N^2 v_\perp)\|_{L^2(\Omega)}\\
\le&C_2(N)(2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{H^2(\Omega)} + \gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_\perp\|_{H^2(\Omega)}),
\end{aligned}$$

holds, we let

$$\vartheta_1 := C_2(N)\max\{2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}, \gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\}.$$

And we also have

$$\begin{aligned}
&\|P_N(L^{-1}f_2(u_\perp,v_\perp))\|_{H^2(\Omega)}\\
\le&\|P_N(-2\gamma\hat{u}_N\hat{v}_N u_\perp + (-\gamma\hat{u}_N^2 + d + \mu)v_\perp)\|_{L^2(\Omega)}\\
\le&C_2(N)(\gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_\perp\|_{H^2(\Omega)} + 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{H^2(\Omega)}),
\end{aligned}$$

therefore, we set

$$\vartheta_2 := \frac{C_2(N)}{d}\max\{\gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}, 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\}.$$

Since

$$\begin{aligned}
&\|(I-P_N)(3f_1(u,v))\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_1(u,v)\| \\
=&3\|(I-P_N)((1+\mu)u + \gamma(-1+2\hat{u}_N\hat{v}_N)u + \gamma\hat{u}_N^2 v)\|_{L^2(\Omega)} \\
&+ \|(I-P_N)\Delta((1+\mu)u + \gamma(-1+2\hat{u}_N\hat{v}_N)u + \gamma\hat{u}_N^2 v)\|_{L^2(\Omega)} \\
\le&|1+\mu-\gamma|(3C_2(N)+1)\|u_\perp\|_{H^2(\Omega)} + 2\gamma\|3\hat{u}_N\hat{v}_N u + \Delta(\hat{u}_N\hat{v}_N u)\|_{L^2(\Omega)} \\
&+ \gamma\|3\hat{u}_N^2 v + \Delta(\hat{u}_N^2 v)\|_{L^2(\Omega)},
\end{aligned}$$

$$\begin{aligned}
&\|3\hat{u}_N\hat{v}_N u + \Delta(\hat{u}_N\hat{v}_N u)\|_{L^2(\Omega)} \\
=&\|3\hat{u}_N\hat{v}_N u + 2\nabla\hat{u}_N\nabla\hat{v}_N u + 2\hat{v}_N\nabla u\nabla\hat{u}_N + 2\hat{u}_N\nabla u\nabla\hat{v}_N + \hat{v}_N u\Delta\hat{u}_N \\
&+ \hat{u}_N u\Delta\hat{v}_N + \hat{u}_N\hat{v}_N\Delta u\|_{L^2(\Omega)} \\
\le&\|3\hat{u}_N\hat{v}_N + 2\nabla\hat{u}_N\nabla\hat{v}_N + \hat{v}_N\Delta\hat{u}_N + \hat{u}_N\Delta\hat{v}_N\|_{L^\infty(\Omega)}(\|u_N\|_{L^2(\Omega)} + \|u_\perp\|_{L^2(\Omega)}) \\
&+ 2(\|\hat{v}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\} \\
&+ \|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{v}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{v}_N)_y\|_{L^\infty(\Omega)}\}) \cdot (\|\nabla u_N\|_{L^2(\Omega)} + \|\nabla u_\perp\|_{L^2(\Omega)}) \\
&+ \|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}(\|\Delta u_N\|_{L^2(\Omega)} + \|\Delta u_\perp\|_{L^2(\Omega)}) \\
\le&(\|3\hat{u}_N\hat{v}_N + 2\nabla\hat{u}_N\nabla\hat{v}_N + \hat{v}_N\Delta\hat{u}_N + \hat{u}_N\Delta\hat{v}_N\|_{L^\infty(\Omega)} \\
&+ 2(\|\hat{v}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\} \\
&+ \|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{v}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{v}_N)_y)\|_{L^\infty(\Omega)}\}) + \|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)})\|u_N\|_{H^2(\Omega)} \\
&+ (\|3\hat{u}_N\hat{v}_N + 2\nabla\hat{u}_N\nabla\hat{v}_N + \hat{v}_N\Delta\hat{u}_N + \hat{u}_N\Delta\hat{v}_N\|_{L^\infty(\Omega)}C_2(N) \\
&+ 2(\|\hat{v}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\} \\
&+ \|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{v}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{v}_N)_y)\|_{L^\infty(\Omega)}\}) + \|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)})\|u_\perp\|_{H^2(\Omega)} \\
=:&c_{un}\|u_N\|_{H^2(\Omega)} + c_{up}\|u_\perp\|_{H^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
&\|3\hat{u}_N^2 v + \Delta(\hat{u}_N^2 v)\|_{L^2(\Omega)} \\
=&\|3\hat{u}_N^2 v + 2v(\nabla\hat{u}_N)^2 + 4\hat{u}_N\nabla v\nabla\hat{u}_N + 2\hat{u}_N v\Delta\hat{u}_N + \hat{u}_N^2\Delta v\|_{L^2(\Omega)} \\
\le&\|3\hat{u}_N^2 + 2(\nabla\hat{u}_N)^2 + 2\hat{u}_N\Delta\hat{u}_N\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)} \\
&+ 4\|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\}\|\nabla v\|_{L^2(\Omega)} + \|\hat{u}_N^2\|_{L^\infty(\Omega)}\|\Delta v\|_{L^2(\Omega)} \\
\le&(\|3\hat{u}_N^2 + 2(\nabla\hat{u}_N)^2 + 2\hat{u}_N\Delta\hat{u}_N\|_{L^\infty(\Omega)} \\
&+ 4\|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\} + \|\hat{u}_N^2\|_{L^\infty(\Omega)})\|v_N\|_{H^2(\Omega)} \\
&+ (\|3\hat{u}_N^2 + 2(\nabla\hat{u}_N)^2 + 2\hat{u}_N\Delta\hat{u}_N\|_{L^\infty(\Omega)}C_2(N) \\
&+ 4\|\hat{u}_N\|_{L^\infty(\Omega)}\max\{\|(\hat{u}_N)_x\|_{L^\infty(\Omega)}, \|(\hat{u}_N)_y\|_{L^\infty(\Omega)}\} + \|\hat{u}_N^2\|_{L^\infty(\Omega)})\|v_\perp\|_{H^2(\Omega)} \\
=:&c_{vn}\|v_N\|_{H^2(\Omega)} + c_{vp}\|v_\perp\|_{H^2(\Omega)}
\end{aligned}$$

hold, so we can set

$$\vartheta_3 := \max\{2\gamma c_{un}, \gamma c_{vn}\},$$
$$\vartheta_4 := \max\{|1+\mu-\gamma|(3C_2(N)+1) + 2\gamma c_{up}, \gamma c_{vp}\}.$$

And we have

$$\begin{aligned}
&\|(I-P_N)(3f_2(u,v))\|_{L^2(\Omega)} + \|(I-P_N)(\Delta f_2(u,v))\|_{L^2(\Omega)} \\
=&3\|(I-P_N)(-2\gamma\hat{u}_N\hat{v}_N u + (-\gamma\hat{u}_N^2 + d + \mu)v)\|_{L^2(\Omega)} \\
&+ \|\Delta(I-P_N)(-2\gamma\hat{u}_N\hat{v}_N u + (-\gamma\hat{u}_N^2 + d + \mu)v)\|_{L^2(\Omega)} \\
\le&3C_2(N)|d+\mu|\cdot\|v_\perp\|_{H^2(\Omega)} + |d+\mu|\cdot\|v_\perp\|_{H^2(\Omega)} + 2\gamma\|3\hat{u}_N\hat{v}_N u + \Delta(\hat{u}_N\hat{v}_N u)\|_{L^2(\Omega)} \\
&+ \gamma\|3\hat{u}_N^2 v + \Delta(\hat{u}_N^2 v)\|_{L^2(\Omega)},
\end{aligned}$$

therefore, we set

$$\vartheta_5 := \frac{\vartheta_3}{d},$$
$$\vartheta_6 := \frac{1}{d}\max\{2\gamma c_{up}, |d+\mu|(3C_2(N)+1) + \gamma c_{vp}\}.$$

Now we start to get an invertibility condition for $\hat{L}$. We know

$$Nw = Pw - [I-F]_N^{-1}(Pw - PFw) = [I-F]_N^{-1}(PFw - PFw_N) = [I-F]_N^{-1}PFw_\perp.$$

Therefore, for every $\chi_1 \in X_N$, $\chi_2 \in X_N$, we have

$$\begin{cases} (L(I-F)_{u_N}N_1 w, \chi_1)_{L^2(\Omega)} = (L(PFw_\perp)_{u_N}, \chi_1)_{L^2(\Omega)}, \\ (L(I-F)_{v_N}N_2 w, \chi_2)_{L^2(\Omega)} = (L(PFw_\perp)_{v_N}, \chi_2)_{L^2(\Omega)}. \end{cases} \tag{5.3.2}$$

Set

$$PFw_\perp = S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{(N+1)^2} s_i^1\varphi_i \\ \sum_{i=1}^{(N+1)^2} s_i^2\varphi_i \end{pmatrix} = \begin{pmatrix} \vec{S^1} \\ \vec{S^2} \end{pmatrix},$$

$$Nw = \begin{pmatrix} N_1 w \\ N_2 w \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{(N+1)^2} a_i\varphi_i \\ \sum_{i=1}^{(N+1)^2} b_i\varphi_i \end{pmatrix} = \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix},$$

then, (5.3.2) is equivalent to

$$\begin{cases} \sum_{i=1}^{(N+1)^2} a_i((\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (-\mu+\gamma)(\varphi_i, \varphi_j)_{L^2(\Omega)} - 2\gamma(\hat{u}_N\hat{v}_N\varphi_i, \varphi_j)_{L^2(\Omega)}) \\ -\gamma \sum_{i=1}^{(N+1)^2} b_i(\hat{u}_N^2\varphi_i, \varphi_j)_{L^2(\Omega)} = \sum_{i=1}^{(N+1)^2} s_i^1((\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)}), \\ \frac{2\gamma}{d} \sum_{i=1}^{(N+1)^2} a_i(\hat{u}_N\hat{v}_N\varphi_i, \varphi_j)_{L^2(\Omega)} + \frac{1}{d} \sum_{i=1}^{(N+1)^2} b_i(d(\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} - \mu(\varphi_i, \varphi_j)_{L^2(\Omega)} \\ +\gamma(\hat{u}_N^2\varphi_i, \varphi_j)_{L^2(\Omega)}) = \sum_{i=1}^{(N+1)^2} s_i^2((\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)}). \end{cases}$$

Therefore, defining

$$G := \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix},$$

where

$$\begin{array}{ll} G_{ij}^{11} & := (\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (-\mu+\gamma)(\varphi_i, \varphi_j)_{L^2(\Omega)} - 2\gamma(\hat{u}_N\hat{v}_N\varphi_i, \varphi_j)_{L^2(\Omega)} \\ G_{ij}^{12} & := -\gamma(\hat{u}_N^2\varphi_i, \varphi_j)_{L^2(\Omega)}, \\ G_{ij}^{21} & := \frac{2\gamma}{d}(\hat{u}_N\hat{v}_N\varphi_i, \varphi_j)_{L^2(\Omega)}, \\ G_{ij}^{22} & := (\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} - \frac{\mu}{d}(\varphi_i, \varphi_j)_{L^2(\Omega)} + \frac{\gamma}{d}(\hat{u}_N^2\varphi_i, \varphi_j)_{L^2(\Omega)}, \\ D_{ij} & := (\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)}, \\ & (1 \le i, j \le (N+1)^2) \end{array} \tag{5.3.3}$$

we have

$$\begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \vec{S^1} \\ \vec{S^2} \end{pmatrix}.$$

So we get

$$\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} \tilde{G}^{11} & \tilde{G}^{12} \\ \tilde{G}^{21} & \tilde{G}^{22} \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \vec{S^1} \\ \vec{S^2} \end{pmatrix} = \begin{pmatrix} \tilde{G}^{11}D & \tilde{G}^{12}D \\ \tilde{G}^{21}D & \tilde{G}^{22}D \end{pmatrix} \begin{pmatrix} \vec{S^1} \\ \vec{S^2} \end{pmatrix}$$

with

$$\begin{pmatrix} \tilde{G}^{11} & \tilde{G}^{12} \\ \tilde{G}^{21} & \tilde{G}^{22} \end{pmatrix} = \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix}^{-1}. \tag{5.3.4}$$

Set

$$\rho_1 := \|D_{H^2}^{T/2}\tilde{G}^{11}DD_{H^2}^{-T/2}\|_E, \quad \rho_2 := \|D_{H^2}^{T/2}\tilde{G}^{12}DD_{H^2}^{-T/2}\|_E,$$
$$\rho_3 := \|D_{H^2}^{T/2}\tilde{G}^{21}DD_{H^2}^{-T/2}\|_E, \quad \rho_4 := \|D_{H^2}^{T/2}\tilde{G}^{22}DD_{H^2}^{-T/2}\|_E,$$

where $D_{H^2} = ((\Delta\varphi_i, \Delta\varphi_j)_{L^2(\Omega)} + (\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)})_{ij}(1 \le i, j \le (N+1)^2)$. Since

$$\begin{aligned}
\|N_1 w\|^2_{H^2(\Omega)} &= \vec{a}^T D_{H^2}\vec{a} \\
&= \vec{a}^T D_{H^2}(\tilde{G}^{11} D\vec{S^1} + \tilde{G}^{12} D\vec{S^2}) \\
&= \vec{a}^T D_{H^2}\tilde{G}^{11} D D_{H^2}^{-T/2} D_{H^2}^{T/2}\vec{S^1} + \vec{a}^T D_{H^2}\tilde{G}^{12} D D_{H^2}^{-T/2} D_{H^2}^{T/2}\vec{S^2} \\
&= (D_{H^2}^{T/2}\vec{a})^T D_{H^2}^{T/2}\tilde{G}^{11} D D_{H^2}^{-T/2}(D_{H^2}^{T/2}\vec{S^1}) \\
&+ (D_{H^2}^{T/2}\vec{a})^T D_{H^2}^{T/2}\tilde{G}^{12} D D_{H^2}^{-T/2}(D_{H^2}^{T/2}\vec{S^2}) \\
&\le \|N_1 w\|_{H^2(\Omega)} \|D_{H^2}^{T/2}\tilde{G}^{11} D D_{H^2}^{-T/2}\|_E \|S_1\|_{H^2(\Omega)} \\
&\quad + \|N_2 w\|_{H^2(\Omega)} \|D_{H^2}^{T/2}\tilde{G}^{12} D D_{H^2}^{-T/2}\|_E \|S_2\|_{H^2(\Omega)}
\end{aligned}$$

holds, we obtain

$$\begin{aligned}
\|N_1 w\|_{H^2(\Omega)} &\le \|D_{H^2}^{T/2}\tilde{G}^{11} D D_{H^2}^{-T/2}\|_E \|S_1\|_{H^2(\Omega)} + \|D_{H^2}^{T/2}\tilde{G}^{12} D D_{H^2}^{-T/2}\|_E \|S_2\|_{H^2(\Omega)} \\
&\le \sqrt{\|D_{H^2}^{T/2}\tilde{G}^{11} D D_{H^2}^{-T/2}\|_E^2 + \|D^{T/2}\tilde{G}^{12} D D_{H^2}^{-T/2}\|_E^2}\sqrt{\|S_1\|^2_{H^2(\Omega)} + \|S_2\|^2_{H^2(\Omega)}} \\
&= \sqrt{\|D_{H^2}^{T/2}\tilde{G}^{11} D D_{H^2}^{-T/2}\|_E^2 + \|D_{H^2}^{T/2}\tilde{G}^{12} D D_{H^2}^{-T/2}\|_E^2}\|S\|_{H^2(\Omega)\times H^2(\Omega)} \\
&\le \sqrt{\rho_1^2 + \rho_2^2}\|S\|_{H^2(\Omega)\times H^2(\Omega)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|N_2 w\|_{H^2(\Omega)} &\le \|D_{H^2}^{T/2}\tilde{G}^{21} D D_{H^2}^{-T/2}\|_E \|S_1\|_{H^2(\Omega)} + \|D_{H^2}^{T/2}\tilde{G}^{22} D D_{H^2}^{-T/2}\|_E \|S_2\|_{H^2(\Omega)} \\
&\le \sqrt{\|D_{H^2}^{T/2}\tilde{G}^{21} D D_{H^2}^{-T/2}\|_E^2 + \|D_{H^2}^{T/2}\tilde{G}^{22} D D_{H^2}^{-T/2}\|_E^2}\|S\|_{H^2(\Omega)\times H^2(\Omega)} \\
&\le \sqrt{\rho_3^2 + \rho_4^2}\|S\|_{H^2(\Omega)\times H^2(\Omega)}.
\end{aligned}$$

Using parameters $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6$ in (5.3.1), we have

$$\begin{aligned}
\|S_1\|_{H^2(\Omega)} \le \vartheta_1(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}) \le \vartheta_1(\alpha_4 + \beta_4), \\
\|S_2\|_{H^2(\Omega)} \le \vartheta_2(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}) \le \vartheta_2(\alpha_4 + \beta_4).
\end{aligned}$$

Hence,

$$\|S\|^2_{H^2(\Omega)\times H^2(\Omega)} = \|S_1\|^2_{H^2(\Omega)} + \|S_2\|^2_{H^2(\Omega)} \le (\vartheta_1^2 + \vartheta_2^2)(\alpha_4 + \beta_4)^2$$

holds. Set $\vartheta := \sqrt{\vartheta_1^2 + \vartheta_2^2}$, then we get

$$\|N_1 w\|_{H^2(\Omega)} \le \vartheta(\alpha_4 + \beta_4)\sqrt{\rho_1^2 + \rho_2^2},$$

$$\|N_2 w\|_{H^2(\Omega)} \le \vartheta(\alpha_4 + \beta_4)\sqrt{\rho_3^2 + \rho_4^2}.$$

On the other hand, we know

$$
\begin{aligned}
&\|(I-P_N)(3f_1(u,v))\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_1(u,v)\|_{L^2(\Omega)}\\
\le&\vartheta_3(\|u_N\|_{H^2(\Omega)} + \|v_N\|_{H^2(\Omega)}) + \vartheta_4(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)})\\
\le&\vartheta_3(\alpha_3+\beta_3) + \vartheta_4(\alpha_4+\beta_4)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{d}(\|(I-P_N)f_2(u,v)\|_{L^2(\Omega)} + \|\Delta(I-P_N)f_2(u,v)\|_{L^2(\Omega)})\\
\le&\vartheta_5(\|u_N\|_{H^2(\Omega)} + \|v_N\|_{H^2(\Omega)}) + \vartheta_6(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)})\\
\le&\vartheta_5(\alpha_3+\beta_3) + \vartheta_6(\alpha_4+\beta_4).
\end{aligned}
$$

Therefore, the following criterion for verification holds.

Theorem 5.3.1 *If*

$$
\kappa_1 := 2C_2(N)\left(2\vartheta(\vartheta_3+\vartheta_5)\left(\sqrt{\rho_1^2+\rho_2^2}+\sqrt{\rho_3^2+\rho_4^2}\right)+\vartheta_4+\vartheta_6\right) < 1 \tag{5.3.5}
$$

holds, then the operator $\hat{L} : H^2(\Omega)\times H^2(\Omega) \to H^2(\Omega)\times H^2(\Omega)$ *has an inverse.*

Proof. Since we have

$$
\begin{aligned}
\hat{\alpha}_3 &\le \vartheta(\alpha_4+\beta_4)\sqrt{\rho_1^2+\rho_2^2},\\
\hat{\beta}_3 &\le \vartheta(\alpha_4+\beta_4)\sqrt{\rho_3^2+\rho_4^2},\\
\hat{\alpha}_4 &\le C_2(N)(\vartheta_3(\alpha_3+\beta_3)+\vartheta_4(\alpha_4+\beta_4)),\\
\hat{\beta}_4 &\le C_2(N)(\vartheta_5(\alpha_3+\beta_3)+\vartheta_6(\alpha_4+\beta_4)),
\end{aligned}
$$

in order to obtain conditions $\hat{\alpha}_3 < \alpha_3$, $\hat{\beta}_3 < \beta_3$, $\hat{\alpha}_4 < \alpha_4$, $\hat{\beta}_4 < \beta_4$, we have to check inequalities

$$
\begin{array}{l}
\vartheta(\alpha_4+\beta_4)\sqrt{\rho_1^2+\rho_2^2} < \alpha_3,\\
\vartheta(\alpha_4+\beta_4)\sqrt{\rho_3^2+\rho_4^2} < \beta_3,\\
C_2(N)(\vartheta_3(\alpha_3+\beta_3)+\vartheta_4(\alpha_4+\beta_4)) < \alpha_4,\\
C_2(N)(\vartheta_5(\alpha_3+\beta_3)+\vartheta_6(\alpha_4+\beta_4)) < \beta_4,
\end{array} \tag{5.3.6}
$$

for some $\alpha_3, \beta_3, \alpha_4, \beta_4 > 0$.

We choose some $\alpha_3, \beta_3, \alpha_4, \beta_4$ satisfying

$$
\alpha_3 = \beta_3, \quad \alpha_4 = \beta_4.
$$

Then (5.3.6) becomes

$$\begin{aligned}&2\vartheta\alpha_4\sqrt{\rho_1^2+\rho_2^2}<\alpha_3,\\&2\vartheta\alpha_4\sqrt{\rho_3^2+\rho_4^2}<\alpha_3,\\&C_2(N)(2\vartheta_3\alpha_3+2\vartheta_4\alpha_4)<\alpha_4,\\&C_2(N)(2\vartheta_5\alpha_3+2\vartheta_6\alpha_4)<\alpha_4.\end{aligned} \tag{5.3.7}$$

If α_3, α_4 satisfy

$$\begin{aligned}&2\vartheta\alpha_4\left(\sqrt{\rho_1^2+\rho_2^2}+\sqrt{\rho_3^2+\rho_4^2}\right)<\alpha_3,\\&2C_2(N)((\vartheta_3+\vartheta_5)\alpha_3+(\vartheta_4+\vartheta_6)\alpha_4)<\alpha_4,\end{aligned} \tag{5.3.8}$$

then (5.3.7) holds.

Using the assumption (5.3.5), we have

$$1-2C_2(N)\left(2\vartheta(\vartheta_3+\vartheta_5)\left(\sqrt{\rho_1^2+\rho_2^2}+\sqrt{\rho_3^2+\rho_4^2}\right)+\vartheta_4+\vartheta_6\right)>0,$$

then for any fixed $\delta > 0$, positive number α_4 can be taken satisfying

$$\begin{aligned}&\alpha_4\left(1-2C_2(N)\left(2\vartheta(\vartheta_3+\vartheta_5)\left(\sqrt{\rho_1^2+\rho_2^2}+\sqrt{\rho_3^2+\rho_4^2}\right)+\vartheta_4+\vartheta_6\right)\right)\\&>2C_2(N)(\vartheta_3+\vartheta_5)\delta.\end{aligned} \tag{5.3.9}$$

Now setting

$$\alpha_3:=2\vartheta\alpha_4\left(\sqrt{\rho_1^2+\rho_2^2}+\sqrt{\rho_3^2+\rho_4^2}\right)+\delta,$$

we have

$$\delta=\alpha_3-2\vartheta\alpha_4\left(\sqrt{\rho_1^2+\rho_2^2}+\sqrt{\rho_3^2+\rho_4^2}\right)>0 \tag{5.3.10}$$

that is, the first inequality of (5.3.8) holds.

Substituting δ in (5.3.10) into (5.3.9), we get

$$\begin{aligned}&\alpha_4\left(1-2C_2(N)\left(2\vartheta(\vartheta_3+\vartheta_5)\left(\sqrt{\rho_1^2+\rho_2^2}+\sqrt{\rho_3^2+\rho_4^2}\right)+\vartheta_4+\vartheta_6\right)\right)\\&>2C_2(N)(\vartheta_3+\vartheta_5)\left(\alpha_3-2\vartheta\alpha_4\left(\sqrt{\rho_1^2+\rho_2^2}+\sqrt{\rho_3^2+\rho_4^2}\right)\right),\end{aligned}$$

then we get the last inequality of (5.3.8). □

5.4 Direct computation of upper bound for $\hat{L}^{-1}$

Let

$$\begin{aligned} A_{ij} &:= (\varphi_i, \varphi_j)_{L^2(\Omega)} (1 \le i, j \le (N+1)^2), \\ \hat{\rho}_1 &:= \| D_{H^2}^{T/2} \tilde{G}^{11} A^{1/2} \|_E, \quad \hat{\rho}_2 := \| D_{H^2}^{T/2} \tilde{G}^{12} A^{1/2} \|_E, \\ \hat{\rho}_3 &:= \| D_{H^2}^{T/2} \tilde{G}^{21} A^{1/2} \|_E, \quad \hat{\rho}_4 := \| D_{H^2}^{T/2} \tilde{G}^{22} A^{1/2} \|_E, \end{aligned}$$

where $\tilde{G}^{11}, \tilde{G}^{12}, \tilde{G}^{21}, \tilde{G}^{22}$ are the same as in (5.3.4).

Then we obtain the following theorem.

Theorem 5.4.1 *Under the assumption* $\kappa_1 < 1$, *if* $\kappa_2 := C_2(N)((\vartheta_3+\vartheta_5)(\hat{\rho}_1\vartheta_1 + \hat{\rho}_2\vartheta_2 + \hat{\rho}_3\vartheta_1 + \hat{\rho}_4\vartheta_2) + \vartheta_4 + \vartheta_6) < 1$ *holds, then* $\hat{M}_\mu > 0$ *can be taken as*

$$\hat{M}_\mu = \frac{\sqrt{2(2\hat{\rho}^2(1 - C_2(N)(\vartheta_4+\vartheta_6) + \vartheta_1 + \vartheta_2)^2 + (\sqrt{2}\hat{\rho}(\vartheta_3+\vartheta_5)C_2(N)+1)^2)}}{1-\kappa_2},$$

here $\hat{\rho} = \max\{\hat{\rho}_1 + \hat{\rho}_3, \hat{\rho}_2 + \hat{\rho}_4\}$.

Proof. Since $\hat{L}$ is invertible, for each $(u_1, v_1) \in H^2(\Omega) \times H^2(\Omega)$, there exists $(u, v) \in H^2(\Omega) \times H^2(\Omega)$, such that

$$\hat{L}(u,v) = \begin{pmatrix} u + L^{-1}\{-(1+\mu)u - \gamma(-1+2\hat{u}_N\hat{v}_N)u - \gamma\hat{u}_N^2 v\} \\ dv + L^{-1}\{-(d+\mu)v + 2\gamma\hat{u}_N\hat{v}_N u + \gamma\hat{u}_N^2 v\} \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}. \tag{5.4.1}$$

(5.4.1) is equivalent to

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} L^{-1}\{(\mu+1)u + \gamma(-1+2\hat{u}_N\hat{v}_N)u + \gamma\hat{u}_N^2 v + Lu_1\} \\ \frac{1}{d}L^{-1}\{-2\gamma\hat{u}_N\hat{v}_N u + (-\gamma\hat{u}_N^2 + d + \mu)v + Lv_1\} \end{pmatrix} \\ &= \begin{pmatrix} L^{-1}\{f_1(u,v) + Lu_1\} \\ \frac{1}{d}L^{-1}\{f_2(u,v) + Lv_1\} \end{pmatrix} =: J(u,v) = Jw. \end{aligned}$$

Same as before, we rewrite $w = Jw$ as

$$\begin{cases} Pw = PJw, \\ (I-P)w = (I-P)Jw. \end{cases}$$

Then for the finite dimensional part, for all $\phi_{1,N}, \phi_{2,N} \in X_N$, we get

$$\begin{aligned} (u_N, \phi_{1,N})_{H^1(\Omega)} &= (f_1(u_N, v_N) + f_1(u_\perp, v_\perp) + Lu_1, \phi_{1,N})_{L^2(\Omega)}, \\ (v_N, \phi_{2,N})_{H^1(\Omega)} &= \frac{1}{d}(f_2(u_N, v_N) + f_2(u_\perp, v_\perp) + Lv_1, \phi_{2,N})_{L^2(\Omega)}. \end{aligned}$$

So we obtain

$$\begin{cases} & (\nabla u_N, \nabla \phi_{1,N})_{L^2(\Omega)} + (u_N, \phi_{1,N})_{L^2(\Omega)} - (f_1(u_N, v_N), \phi_{1,N})_{L^2(\Omega)} \\ = & (f_1(u_\perp, v_\perp), \phi_{1,N})_{L^2(\Omega)} + (Lu_1, \phi_{1,N})_{L^2(\Omega)}, \\ & (\nabla v_N, \nabla \phi_{2,N})_{L^2(\Omega)} + (v_N, \phi_{2,N})_{L^2(\Omega)} - \frac{1}{d}(f_2(u_N, v_N), \phi_{2,N})_{L^2(\Omega)} \\ = & \frac{1}{d}(f_2(u_\perp, v_\perp), \phi_{2,N})_{L^2(\Omega)} + \frac{1}{d}(Lv_1, \phi_{2,N})_{L^2(\Omega)}. \end{cases} \tag{5.4.2}$$

By setting

$$\begin{aligned} u_N &:= \sum_{i=1}^{(N+1)^2} a_i \varphi_i, \ v_N := \sum_{i=1}^{(N+1)^2} b_i \varphi_i, \\ \vec{a} &:= (a_1, a_2, \cdots a_{(N+1)^2})^T, \vec{b} := (b_1, b_2, \cdots b_{(N+1)^2})^T, \\ g_1(i) &:= (u_1, \varphi_i)_{L^2(\Omega)} + (-\Delta u_1, \varphi_i)_{L^2(\Omega)} + (f_1(u_\perp, v_\perp), \varphi_i)_{L^2(\Omega)}, \\ g_2(i) &:= \frac{1}{d}(v_1, \varphi_i)_{L^2(\Omega)} + \frac{1}{d}(-\Delta v_1, \varphi_i)_{L^2(\Omega)} + \frac{1}{d}(f_2(u_\perp, v_\perp), \varphi_i)_{L^2(\Omega)}, \\ & (1 \le i \le (N+1)^2) \end{aligned}$$

(5.4.2) can be written as

$$\begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} \vec{g_1} \\ \vec{g_2} \end{pmatrix},$$

where $G^{11}, G^{12}, G^{21}, G^{22}$ are the same as in (5.3.3), therefore, we get

$$\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} \tilde{G}^{11}\vec{g_1} + \tilde{G}^{12}\vec{g_2} \\ \tilde{G}^{21}\vec{g_1} + \tilde{G}^{22}\vec{g_2} \end{pmatrix}.$$

Now, defining the L^2-projection $P_0 : L^2(\Omega) \to X_N$ as

$$(s - P_0 s, s_N)_{L^2(\Omega)} = 0, \quad \forall s_N \in X_N.$$

It is easily seen that

$$\begin{aligned} \|P_0(f_1(u_\perp, v_\perp) + Lu_1)\|_{L^2(\Omega)} &= \|A^{-1/2}\vec{g_1}\|_E, \\ \frac{1}{d}\|P_0(f_2(u_\perp, v_\perp) + Lv_1)\|_{L^2(\Omega)} &= \|A^{-1/2}\vec{g_2}\|_E. \end{aligned}$$

And for every $\xi \in L^2(\Omega)$, we also have

$$\begin{aligned} &\|L\xi\|^2_{L^2(\Omega)} = \|(-\Delta + I)\xi\|^2_{L^2(\Omega)} \le (\|\Delta\xi\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)})^2 \\ \le& 2(\|\Delta\xi\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}) \le 2\|\xi\|^2_{H^2(\Omega)}. \end{aligned}$$

Therefore,

$$\begin{array}{rl}
& \|u_N\|_{H^2(\Omega)} = \|D_{H^2}^{T/2}\vec{a}\|_E \\
= & \|D_{H^2}^{T/2}\tilde{G}^{11}A^{1/2}A^{-1/2}\vec{g_1} + D_{H^2}^{T/2}\tilde{G}^{12}A^{1/2}A^{-1/2}\vec{g_2}\|_E \\
\leq & \|D_{H^2}^{T/2}\tilde{G}^{11}A^{1/2}\|_E\|A^{-1/2}\vec{g_1}\|_E + \|D_{H^2}^{T/2}\tilde{G}^{12}A^{1/2}\|_E\|A^{-1/2}\vec{g_2}\|_E \\
\leq & \hat{\rho}_1\|P_0(f_1(u_\perp, v_\perp) + Lu_1)\|_{L^2(\Omega)} + \frac{\hat{\rho}_2}{d}\|P_0(f_2(u_\perp, v_\perp) + Lv_1)\|_{L^2(\Omega)} \\
\leq & \hat{\rho}_1(\|P_0(f_1(u_\perp, v_\perp)\|_{L^2(\Omega)} + \|Lu_1\|_{L^2(\Omega)}) \\
& +\frac{\hat{\rho}_2}{d}(\|P_0(f_2(u_\perp, v_\perp))\|_{L^2(\Omega)} + \|Lv_1\|_{L^2(\Omega)}), \\
\leq & \hat{\rho}_1(\vartheta_1(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}) + \sqrt{2}\|u_1\|_{H^2(\Omega)}) \\
& +\frac{\hat{\rho}_2}{d}(\vartheta_2(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}) + \sqrt{2}\|v_1\|_{H^2(\Omega)})
\end{array} \tag{5.4.3}$$

and

$$\begin{array}{rl}
& \|v_N\|_{H^2(\Omega)} \\
\leq & \hat{\rho}_3(\|P_0(f_1(u_\perp, v_\perp)\|_{L^2(\Omega)} + \|Lu_1\|_{L^2(\Omega)}) \\
& +\frac{\hat{\rho}_4}{d}(\|P_0(f_2(u_\perp, v_\perp))\|_{L^2(\Omega)} + \|Lv_1\|_{L^2(\Omega)}), \\
\leq & \hat{\rho}_3(\vartheta_1(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}) + \sqrt{2}\|u_1\|_{H^2(\Omega)}) \\
& +\frac{\hat{\rho}_4}{d}(\vartheta_2(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}) + \sqrt{2}\|v_1\|_{H^2(\Omega)})
\end{array} \tag{5.4.4}$$

hold. And we know that

$$u_\perp = (I - P_N)L^{-1}\{f_1(u, v) + Lu_1\},$$
$$v_\perp = \frac{1}{d}(I - P_N)L^{-1}\{f_2(u, v) + Lv_1\},$$

so we have

$$\begin{array}{rl}
& \|u_\perp\|_{H^2(\Omega)} \\
\leq & C_2(N)(3\|(I - P_N)f_1(u, v)\|_{L^2(\Omega)} + \|\Delta(I - P_N)f_1(u, v)\|_{L^2(\Omega)}) + \|u_1\|_{H^2(\Omega)} \\
\leq & C_2(N)(3\|(I - P_N)f_1(u, v)\|_{L^2(\Omega)} + \|\Delta(I - P_N)f_1(u, v)\|_{L^2(\Omega)}) + \|u_1\|_{H^2(\Omega)} \\
\leq & C_2(N)(\vartheta_3(\|u_N\|_{H^2(\Omega)} + \|v_N\|_{H^2(\Omega)}) + \vartheta_4(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)})) + \|u_1\|_{H^2(\Omega)}, \\
& \|v_\perp\|_{H^2(\Omega)} \\
\leq & C_2(N)(\vartheta_5(\|u_N\|_{H^2(\Omega)} + \|v_N\|_{H^2(\Omega)}) + \vartheta_6(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)})) + \frac{1}{d}\|v_1\|_{H^2(\Omega)}.
\end{array} \tag{5.4.5}$$

Substituting (5.4.3) and (5.4.4) into (5.4.5) and recalling that in this paper, we always choose $d > 1$, we get

$$\begin{array}{l}
\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)} \\
\leq C_2(N)((\vartheta_3 + \vartheta_5)(\hat{\rho}_1\vartheta_1 + \hat{\rho}_2\vartheta_2 + \hat{\rho}_3\vartheta_1 + \hat{\rho}_4\vartheta_2) + (\vartheta_4 + \vartheta_6)) \\
\quad \cdot (\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)}) + (\sqrt{2}\hat{\rho}(\vartheta_3 + \vartheta_5)C_2(N) + 1)(\|u_1\|_{H^2(\Omega)} + \|v_1\|_{H^2(\Omega)}),
\end{array}$$

that is,

$$\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)} \leq \frac{\sqrt{2}\hat{\rho}(\vartheta_3+\vartheta_5)C_2(N)+1}{1-\kappa_2}(\|u_1\|_{H^2(\Omega)} + \|v_1\|_{H^2(\Omega)}). \tag{5.4.6}$$

And also substituting (5.4.6) into (5.4.3) and (5.4.4), we have

$$\begin{aligned}
&\|u_N\|_{H^2(\Omega)} + \|v_N\|_{H^2(\Omega)}\\
\leq&(\hat{\rho}_1\vartheta_1+\hat{\rho}_2\vartheta_2+\hat{\rho}_3\vartheta_1+\hat{\rho}_4\vartheta_2)(\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)})\\
&+\sqrt{2}(\hat{\rho}_1+\hat{\rho}_3)\|u_1\|_{H^2(\Omega)} + \sqrt{2}(\hat{\rho}_2+\hat{\rho}_4)\|v_1\|_{H^2(\Omega)}\\
\leq&\left((\hat{\rho}_1\vartheta_1+\hat{\rho}_2\vartheta_2+\hat{\rho}_3\vartheta_1+\hat{\rho}_4\vartheta_2)\frac{\sqrt{2}\hat{\rho}(\vartheta_3+\vartheta_5)C_2(N)+1}{1-\kappa_2}+\sqrt{2}\hat{\rho}\right)(\|u_1\|_{H^2(\Omega)} + \|v_1\|_{H^2(\Omega)})\\
\leq&\sqrt{2}\hat{\rho}\left(1+\frac{\kappa_2-C_2(N)(\vartheta_4+\vartheta_6)+(\vartheta_1+\vartheta_2)}{1-\kappa_2}\right)(\|u_1\|_{H^2(\Omega)} + \|v_1\|_{H^2(\Omega)})\\
\leq&\frac{\sqrt{2}\hat{\rho}(1-C_2(N)(\vartheta_4+\vartheta_6)+(\vartheta_1+\vartheta_2))}{1-\kappa_2}(\|u_1\|_{H^2(\Omega)} + \|v_1\|_{H^2(\Omega)}).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\|u\|^2_{H^2(\Omega)} + \|v\|^2_{H^2(\Omega)}\\
\leq&(\|u_N\|_{H^2(\Omega)} + \|v_N\|_{H^2(\Omega)})^2 + (\|u_\perp\|_{H^2(\Omega)} + \|v_\perp\|_{H^2(\Omega)})^2\\
\leq&\left((\tfrac{\sqrt{2}\hat{\rho}(1-C_2(N)(\vartheta_4+\vartheta_6)+(\vartheta_1+\vartheta_2))}{1-\kappa_2})^2 + (\tfrac{\sqrt{2}\hat{\rho}(\vartheta_3+\vartheta_5)C_2(N)+1}{1-\kappa_2})^2 \right)(\|u_1\|_{H^2(\Omega)} + \|v_1\|_{H^2(\Omega)})^2\\
\leq&2\frac{(\sqrt{2}\hat{\rho}(1-C_2(N)(\vartheta_4+\vartheta_6)+(\vartheta_1+\vartheta_2)))^2+(\sqrt{2}\hat{\rho}(\vartheta_3+\vartheta_5)C_2(N)+1)^2}{(1-\kappa_2)^2}\\
&\cdot(\|u_1\|^2_{H^2(\Omega)} + \|v_1\|^2_{H^2(\Omega)}).
\end{aligned}$$

And so

$$\|u\|^2_{H^2(\Omega)} + \|v\|^2_{H^2(\Omega)} \leq \hat{M}^2_\mu \|(u_1,v_1)\|^2_{H^2(\Omega)\times H^2(\Omega)}$$

holds. □

5.5 Eigenvalue problem of the linearized operator at the exact solution

Recall that $w^* = (u^*, v^*)$ is the exact solution of (3.0.3) and $u^* = \hat{u}_N + \tilde{u}$, $v^* = \hat{v}_N + \tilde{v}$, where $\tilde{u}, \tilde{v}$ are the residual part of $\hat{u}_N$ and $\hat{v}_N$ respectively. By using the verification method in Chapter 4, we get $\|\tilde{u}\|_{H^2(\Omega)} \leq \alpha_1 + \alpha_2$ and $\|\tilde{v}\|_{H^2(\Omega)} \leq \beta_1 + \beta_2$.

Now we consider the following eigenvalue problem

$$\begin{cases} -\Delta u - \gamma(-1 + 2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v}))u - \gamma(\hat{u}_N^2 + \tilde{u})v = \tilde{\lambda} u, \\ -d\Delta v - \gamma(-2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v}))u + \gamma(\hat{u}_N + \tilde{u})^2 v = \tilde{\lambda} v. \end{cases} \tag{5.5.1}$$

We know the eigenvalue λ of the equation (5.5.1) can be written as

$$\begin{aligned}\lambda =& ((\nabla u, \nabla u)_{L^2(\Omega)} - \gamma((-1 + 2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v}))u, u)_{L^2(\Omega)} - \gamma((\hat{u}_N + \tilde{u})^2 v, u)_{L^2(\Omega)} \\ &+ d(\nabla v, \nabla v)_{L^2(\Omega)} + 2\gamma((\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})u, v)_{L^2(\Omega)} + \gamma((\hat{u}_N + \tilde{u})^2 v, v)_{L^2(\Omega)}) \\ &/((u, u)_{L^2(\Omega)} + (v, v)_{L^2(\Omega)}).\end{aligned}$$

So we have

$$\begin{aligned} &Re(\lambda) = Re((\nabla u, \nabla u)_{L^2(\Omega)} - \gamma((-1 + 2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v}))u, u)_{L^2(\Omega)} \\ &- \gamma((\hat{u}_N + \tilde{u})^2 v, u)_{L^2(\Omega)} + d(\nabla v, \nabla v)_{L^2(\Omega)} + 2\gamma((\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})u, v)_{L^2(\Omega)} \\ &+ \gamma((\hat{u}_N + \tilde{u})^2 v, v)_{L^2(\Omega)})/((u, u)_{L^2(\Omega)} + (v, v)_{L^2(\Omega)}) \\ \geq& \gamma((u, u)_{L^2(\Omega)} - 2\|(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} \\ &- \|(\hat{u}_N + \tilde{u})^2\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} - 2\|(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &- \|(\hat{u}_N + \tilde{u})^2\|_{L^\infty(\Omega)}\|v\|^2_{L^2(\Omega)})/(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}) \\ \geq& \gamma(\|u\|^2_{L^2(\Omega)} - 2(\|\hat{u}_N\hat{v}\|_{L^\infty(\Omega)} + \|\hat{u}_N\|_{L^\infty(\Omega)}\|\tilde{v}\|_{L^\infty(\Omega)} + \|\hat{v}_N\|_{L^\infty(\Omega)}\|\tilde{u}\|_{L^\infty(\Omega)} \\ &+ \|\tilde{u}\|_{L^\infty(\Omega)}\|\tilde{v}\|_{L^\infty(\Omega)})(\|u\|^2_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}) \\ &- (\|\hat{u}_N^2\|_{L^\infty(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}\|\tilde{u}\|_{L^\infty(\Omega)} + \|\tilde{u}\|^2_{L^\infty(\Omega)})(\|v\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)})) \\ &/(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}) \\ \geq& \gamma(\|u\|^2_{L^2(\Omega)} - 2(\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)} + \|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\beta_1 + \beta_2) + \|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1 + \alpha_2) \\ &+ K^2_{2,\infty}(\alpha_1 + \alpha_2)(\beta_1 + \beta_2))(\|u\|^2_{L^2(\Omega)} + 1/2(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)})) \\ &- (\|\hat{u}_N^2\|_{L^\infty(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1 + \alpha_2) + K^2_{2,\infty}(\alpha_1 + \alpha_2)^2) \\ &\cdot (1/2(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}) + \|v\|^2_{L^2(\Omega)}))/(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}). \end{aligned}$$

Set $b_1 := \min\{1-3(\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}+\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\beta_1+\beta_2)+\|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1+\alpha_2) + K^2_{2,\infty}(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)) - \frac{1}{2}(\|\hat{u}_N^2\|_{L^\infty(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1 + \alpha_2) + K^2_{2,\infty}(\alpha_1+\alpha_2)^2), -(\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}+\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\beta_1+\beta_2)+\|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1+\alpha_2) + K^2_{2,\infty}(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)) - \frac{3}{2}(\|\hat{u}_N^2\|_{L^\infty(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1 + \alpha_2) + K^2_{2,\infty}(\alpha_1 + \alpha_2)^2)\}$, then we obtain

$$Re(\lambda) \geq \gamma b_1.$$

And

$$
\begin{aligned}
&Im(\lambda) = Im(-\gamma(2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})u, u)_{L^2(\Omega)} - \gamma((\hat{u}_N + \tilde{u})^2 v, u)_{L^2(\Omega)} \\
&\quad + 2\gamma((\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})u, v)_{L^2(\Omega)} + \gamma((\hat{u}_N + \tilde{u})^2 v, v)_{L^2(\Omega)})/((u, u)_{L^2(\Omega)} + (v, v)_{L^2(\Omega)}) \\
&\geq \gamma(-2\|(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} - \|(\hat{u}_N + \tilde{u})^2\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\quad - 2\|(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} - \|(\hat{u}_N + \tilde{u})^2\|_{L^\infty(\Omega)}\|v\|^2_{L^2(\Omega)}) \\
&\quad /(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}) \\
&\geq \gamma(-2(\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)} + \|\hat{u}_N\|_{L^\infty(\Omega)}\|\tilde{v}\|_{L^\infty(\Omega)} + \|\hat{v}_N\|_{L^\infty(\Omega)}\|\tilde{u}\|_{L^\infty(\Omega)} + \|\tilde{u}\|_{L^\infty(\Omega)}\|\tilde{v}\|_{L^\infty(\Omega)}) \\
&\quad \cdot (\|u\|^2_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}) - (\|\hat{u}_N^2\|_{L^\infty(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}\|\tilde{u}\|_{L^\infty(\Omega)} + \|\tilde{u}\|^2_{L^\infty(\Omega)}) \\
&\quad \cdot (\|v\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}))/(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}) \\
&\geq \gamma(-2(\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)} + \|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\beta_1 + \beta_2) + \|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1 + \alpha_2) \\
&\quad + K^2_{2,\infty}(\alpha_1 + \alpha_2)(\beta_1 + \beta_2))(\|u\|^2_{L^2(\Omega)} + 1/2(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)})) \\
&\quad - (\|\hat{u}_N^2\|_{L^\infty(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1 + \alpha_2) + K^2_{2,\infty}(\alpha_1 + \alpha_2)^2) \\
&\quad \cdot (1/2(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}) + \|v\|^2_{L^2(\Omega)}))/(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)})
\end{aligned}
$$

holds.

Set $b_2 := \min\{-3(\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)} + \|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\beta_1+\beta_2) + \|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1+\alpha_2) + K^2_{2,\infty}(\alpha_1+\alpha_2)(\beta_1+\beta_2)) - 1/2(\|\hat{u}_N^2\|_{L^\infty(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1+\alpha_2) + K^2_{2,\infty}(\alpha_1+\alpha_2)^2), -(\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)} + \|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\beta_1+\beta_2) + \|\hat{v}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1+\alpha_2) + K^2_{2,\infty}(\alpha_1+\alpha_2)(\beta_1+\beta_2)) - 3/2(\|\hat{u}_N^2\|_{L^\infty(\Omega)} + 2\|\hat{u}_N\|_{L^\infty(\Omega)}K_{2,\infty}(\alpha_1+\alpha_2) + K^2_{2,\infty}(\alpha_1+\alpha_2)^2)\} < 0$, then we obtain

$$Im(\lambda) \geq \gamma b_2. \tag{5.5.2}$$

It is clear that if λ is an eigenvalue of problem (5.5.1), then $\bar{\lambda}$ is also an eigenvalue of problem (5.5.1), therefore, from (5.5.2), we have

$$Im(\lambda) \leq -\gamma b_2.$$

And thus, the eigenvalues of problem (5.5.1) are in the domain

$$\{x + iy | x \geq \gamma b_1, |y| \leq -\gamma b_2\}. \tag{5.5.3}$$

Set

$$L_\mu(u, v) := \begin{pmatrix} u + L^{-1}\{-(1+\mu)u - \gamma(-1 + 2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})u - \gamma(\hat{u}_N + \tilde{u})^2 v\} \\ dv + L^{-1}\{(-d+\mu)v + 2\gamma(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})u + \gamma(\hat{u}_N + \tilde{u})^2 v\} \end{pmatrix}.$$

Same as the proof of Theorem 5.1.1, we have the following theorem.

Theorem 5.5.1 *Suppose that L_μ has an inverse $L_\mu^{-1}: H^2(\Omega)\times H^2(\Omega) \to H^2(\Omega)\times H^2(\Omega)$ and there exists $M_\mu > 0$ such that*

$$\|L_\mu^{-1}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} \leq M_\mu \|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}, \quad (5.5.4)$$

then there is no eigenvalue $\tilde{\lambda}$ of Eq. (5.5.1) in the disk given by $|\tilde{\lambda}-\mu| < \frac{1}{M_\mu}$.

Now we discuss how to get M_μ from $\hat{M}_\mu$ in Theorem 5.1.1. Note that

$$\begin{aligned}
&\|L_\mu(u,v) - \hat{L}(u,v)\|^2_{H^2(\Omega)\times H^2(\Omega)}\\
=&\|L^{-1}\{-2\gamma(\hat{u}_N\tilde{v}+\hat{v}_N\tilde{u}+\tilde{u}\tilde{v})u - \gamma(2\hat{u}_N\tilde{u}+\tilde{u}^2)v\}\|^2_{H^2(\Omega)}\\
&+\|L^{-1}\{2\gamma(\hat{u}_N\tilde{v}+\hat{v}_N\tilde{u}+\tilde{u}\tilde{v})u + \gamma(2\hat{u}_N\tilde{u}+\tilde{u}^2)v\}\|^2_{H^2(\Omega)}\\
\leq&\|-2\gamma(\hat{u}_N\tilde{v}+\hat{v}_N\tilde{u}+\tilde{u}\tilde{v})u - \gamma(2\hat{u}_N\tilde{u}+\tilde{u}^2)v\|^2_{L^2(\Omega)}\\
&+\|2\gamma(\hat{u}_N\tilde{v}+\hat{v}_N\tilde{u}+\tilde{u}\tilde{v})u + \gamma(2\hat{u}_N\tilde{u}+\tilde{u}^2)v\|^2_{L^2(\Omega)}\\
\leq&2\gamma^2(4\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|\tilde{v}\|^2_{L^4(\Omega)}\|u\|^2_{L^4(\Omega)} + 4\|\hat{v}_N^2\|_{L^\infty(\Omega)}\|\tilde{u}\|^2_{L^4(\Omega)}\|u\|^2_{L^4(\Omega)}\\
&+4\|\tilde{u}\|^2_{L^4(\Omega)}\|\tilde{v}\|^2_{L^8(\Omega)}\|u\|^2_{L^8(\Omega)} + 4\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|\tilde{u}\|^2_{L^4(\Omega)}\|v\|^2_{L^4(\Omega)} + \|\tilde{u}\|^4_{L^8(\Omega)}\|v\|^2_{L^4(\Omega)})\\
\leq&2\gamma^2(4\|\hat{u}_N^2\|_{L^\infty(\Omega)}K^4_{2,4}(\beta_1+\beta_2)^2\|u\|^2_{H^2(\Omega)} + 4\|\hat{v}_N^2\|_{L^\infty(\Omega)}K^4_{2,4}(\alpha_1+\alpha_2)\|u\|^2_{H^2(\Omega)}\\
&+4K^2_{2,4}K^4_{2,8}(\alpha_1+\alpha_2)^2(\beta_1+\beta_2)^2\|u\|^2_{H^2(\Omega)} + 4\|\hat{u}_N^2\|_{L^\infty(\Omega)}K^4_{2,4}(\alpha_1+\alpha_2)^2\|v\|^2_{H^2(\Omega)}\\
&+K^4_{2,8}K^2_{2,4}(\alpha_1+\alpha_2)^3\|v\|^2_{H^2(\Omega)}).
\end{aligned}$$

Hence, if we set $\varsigma := \max\{2(4\|\hat{u}_N^2\|_{L^\infty(\Omega)}K^4_{2,4}(\beta_1+\beta_2)^2+4\|\hat{v}_N^2\|_{L^\infty(\Omega)}K^4_{2,4}(\alpha_1+\alpha_2)+4K^2_{2,4}K^4_{2,8}(\alpha_1+\alpha_2)^2(\beta_1+\beta_2)^2), 2(4\|\hat{u}_N^2\|_{L^\infty(\Omega)}K^4_{2,4}(\alpha_1+\alpha_2)^2+K^4_{2,8}K^2_{2,4}(\alpha_1+\alpha_2)^3)\}$, then we have

$$\begin{aligned}
&\|L_\mu(u,v) - \hat{L}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}\\
\leq&\gamma\sqrt{\varsigma}(\|u\|_{H^2(\Omega)} + \|v\|_{H^2(\Omega)})\\
\leq&\gamma\sqrt{\varsigma}\sqrt{2(\|u\|^2_{H^2(\Omega)} + \|v\|^2_{H^2(\Omega)})}\\
=&\gamma\sqrt{2\varsigma}\|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}.
\end{aligned}$$

And therefore, we obtain

$$\begin{aligned}
&\|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} \leq \hat{M}_\mu\|\hat{L}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}\\
\leq&\hat{M}_\mu(\|L_\mu(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} + \|L_\mu(u,v)-\hat{L}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)})\\
\leq&\hat{M}_\mu(\|L_\mu(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} + \gamma\sqrt{2\varsigma}\|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}).
\end{aligned}$$

If $1-\gamma\sqrt{2\varsigma}\hat{M}_\mu > 0$ holds, we have

$$\|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} \leq M_\mu\|L_\mu(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}, \quad (5.5.5)$$

where $M_\mu := \frac{\hat{M}_\mu}{1-\gamma\sqrt{2\varsigma}\hat{M}_\mu}$.

Remark 5.5.2 *When μ' satisfies $|\mu' - \mu| \leq \frac{1}{2M_\mu}$, we get*

$$\begin{aligned}
&\|(L_\mu - L_{\mu'})(u,v)\|^2_{H^2(\Omega)\times H^2(\Omega)}\\
=&\|L^{-1}\{(\mu' - \mu)u\}\|^2_{H^2(\Omega)} + \|L^{-1}\{(\mu' - \mu)v\}\|^2_{H^2(\Omega)}\\
=&|\mu' - \mu|^2\|(L^{-1}u, L^{-1}v)\|^2_{H^2(\Omega)\times H^2(\Omega)}\\
\leq&\frac{1}{4M_\mu^2}\|(u,v)\|^2_{L^2(\Omega)\times L^2(\Omega)}\\
\leq&\frac{1}{4M_\mu^2}\|(u,v)\|^2_{H^2(\Omega)\times H^2(\Omega)},
\end{aligned}$$

therefore, from (5.5.4)*, we have*

$$\begin{aligned}
&\|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} \leq M_\mu\|L_\mu(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}\\
\leq&M_\mu(\|(L_\mu - L_{\mu'})(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} + \|L_{\mu'}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)})\\
\leq&M_\mu\left(\frac{1}{2M_\mu}\|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} + \|L_{\mu'}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}\right),
\end{aligned}$$

thus,

$$\|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} \leq 2M_\mu\|L_{\mu'}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}$$

holds.

Therefore, in actual computation, if we use $B(\mu, \frac{1}{2M_\mu})$ to cover the candidate excluding domain, then every point $\mu' \in B(\mu, \frac{1}{2M_\mu})$ satisfies $\|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} \leq 2M_\mu\|L_{\mu'}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}$.

Chapter 6

The domain of attraction

$w = (u, v)$ is the solution of the equation

$$\begin{cases} w_t = K\Delta w + R(w) \text{ in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.0.1}$$

where

$$K := \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad R(w) := \begin{pmatrix} \gamma f(u, v) \\ \gamma g(u, v) \end{pmatrix}.$$

And $w^* = (u^*, v^*)$ is the solution of the equation

$$\begin{cases} 0 = K\Delta w^* + R(w^*), \\ \dfrac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.0.2}$$

Set $\hat{w}(t) := w(t) - w^*$. From (6.0.1) and (6.0.2), we have

$$\begin{aligned} w_t &= K\Delta(w - w^*) + R(w) - R(w^*) + R'(w^*)\hat{w} - R'(w^*)\hat{w} \\ &= K\Delta\hat{w} + R'(w^*)\hat{w} + R(\hat{w} + w^*) - R(w^*) - R'(w^*)\hat{w}. \end{aligned}$$

Let $-L_{w^*} : D(-L_{w^*}) = \left\{ w \middle| w \in H^2(\Omega) \times H^2(\Omega), \dfrac{\partial w}{\partial \nu} = 0 \right\} \to L^2(\Omega) \times L^2(\Omega)$ as $-L_{w^*}(\hat{w}) := K\Delta\hat{w} + R'(w^*)\hat{w}$. Set $H(\hat{w}) := R(\hat{w} + w^*) - R(w^*) - R'(w^*)\hat{w}$, then we have

$$w_t = -L_{w^*}(\hat{w}) + H(\hat{w}).$$

And so

$$\hat{w}_t = -L_{w^*}(\hat{w}) + H(\hat{w}) \tag{6.0.3}$$

holds.

It is easily seen that if λ is an eigenvalue of (5.5.1), then $-\lambda$ is an eigenvalue of $-L_{w^*}$. We assume that after we use the method in Chapter 5, we get the resolvent set of $-L_{w^*}$ containing a sector $S_{\theta_0,-\sigma} = \{\lambda \in \mathcal{C} : \lambda \neq -\sigma, |\arg(\lambda+\sigma)| < \theta_0\}$ ($\sigma > 0$, $\frac{\pi}{2} < \theta_0 < \pi$).

In order to solve (6.0.3), we need a proposition of ordinary differential equation.

Proposition 6.0.3 *([8], Prop. 1.2.3) For Cauchy problem*

$$\begin{cases} \hat{w}_t = -L_{w^*}(\hat{w}) + H(\hat{w}), \\ \hat{w}(0) = w_0, \end{cases}$$

there is a unique solution in $[0, T]$*, given by*

$$\hat{w}(t) = e^{-tL_{w^*}}\hat{w}(0) + \int_0^t e^{-(t-s)L_{w^*}} H(\hat{w}(s))ds.$$

Before we continue, we need some information of sectorial operators.

Definition 6.0.4 *([9], Def. 2.0.1) X is a Banach space. $A : D(A) \to X$ is said to be sectorial if there are constants $\omega \in \Re$, $\theta \in (\frac{\pi}{2}, \pi)$, $M > 0$ such that*

$$\begin{cases} (i)\ \rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathcal{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ (ii)\ \|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda-\omega|},\ \forall \lambda \in S_{\theta,\omega}. \end{cases} \tag{6.0.4}$$

For every $t > 0$, (6.0.4) allows us to define a linear bounded operator e^{tA} in X, by means of the Dunford integral

$$e^{tA} = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A)d\lambda, \quad t > 0,$$

where $r > 0$, $\eta \in (\pi/2, \theta)$, and $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathcal{C} : |\arg\lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathcal{C} : |\arg\lambda| \leq \eta, |\lambda| = r\}$, oriented counterclockwise. We also set

$$e^{0A}x = x,\ \forall x \in X.$$

In the following two propositions, we suppose $A : D(A) \subset X \to X$ is a sectorial operator satisfying (6.0.4).

Proposition 6.0.5 *([9], Prop. 2.1.1, Prop. 2.1.4)*

(i) $e^{tA}x \in D(A)$ for each $t > 0$, $x \in X$.

(ii) For every $x \in X$ and $t \geq 0$, the integral $\int_0^t e^{sA}xds$ belongs to $D(A)$.

Proposition 6.0.6 *([9], Prop. 2.1.1) There is a constant M_0, such that*

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}, \ t > 0, \tag{6.0.5}$$

where $M_0 = \frac{M}{2\pi}(2\int_r^\infty \rho^{-1} e^{\rho\cos\theta} d\rho + \int_{-\theta}^{\theta} e^{r\cos\eta} d\eta)$ with M is the same as the one in (6.0.4) and $r > 0$.

Remark 6.0.7 *If we set $M_0 = \frac{M}{2\pi}(2\int_r^\infty \rho^{-1} e^{\rho\cos\theta} d\rho + \int_{-\theta}^{\theta} e^{r\cos\eta} d\eta)$, then for every $r > 0$, (6.0.5) holds.*

Now we consider the resolvent bound of $-L_{w^*}$.

Theorem 6.0.8 *For all $\lambda \in S_{\theta_0,-\sigma}$, there exists a constant M, such that*

$$\|(\lambda I + L_{w^*})^{-1}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))} \leq \frac{M}{|\lambda + \sigma|}.$$

Proof. For every $w = (u, v) = (\sum_{n,m=0}^{\infty} u_{nm}\phi_{nm}, \sum_{n,m=0}^{\infty} v_{nm}\phi_{nm}) \in D(-L_{w^*}) \subset L^\infty(\Omega) \times L^\infty(\Omega)$, set $\lambda w + L_{w^*} w =: F$, then we have

$$w = (-K\triangle - R'(w^*) + \lambda)^{-1} F,$$

and

$$(-K\triangle + \lambda)w = F + R'(w^*)w,$$

therefore,

$$w = (-K\triangle + \lambda)^{-1} F + (-K\triangle + \lambda)^{-1} R'(w^*)w$$

holds. Then, for all $\lambda \in \mathcal{C}$ ($\lambda \neq -\sigma$), if $\|(-K\triangle+\lambda)^{-1}R'(w^*)\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))} < 1$ holds, we have

$$\|w\|_{L^\infty(\Omega)} \leq \frac{\|(-K\triangle + \lambda)^{-1}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))}}{1 - \|(-K\triangle + \lambda)^{-1} R'(w^*)\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))}} \|F\|_{L^\infty(\Omega)\times L^\infty(\Omega)}.$$

Note that

$$R'(w^*) = \gamma \begin{pmatrix} -1 + 2u^*v^* & u^{*2} \\ -2u^*v^* & -u^{*2} \end{pmatrix},$$

therefore,

$$\|R'(w^*)\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))} \leq \gamma \max\{\|-1+2u^*v^*\|_{L^\infty(\Omega)}, 2\|u^*v^*\|_{L^\infty(\Omega)}, \|u^{*2}\|_{L^\infty(\Omega)}\} =: R$$

holds.

And we know that when $\lambda \neq 0$,

$$(-K\triangle+\lambda)^{-1}w = \left(\sum_{n,m=0}^{\infty} \frac{u_{nm}\phi_{nm}}{\lambda + n^2\pi^2/l^2 + m^2\pi^2/l^2}, \sum_{n,m=0}^{\infty} \frac{v_{nm}\phi_{nm}}{\lambda + d(n^2\pi^2/l^2 + m^2\pi^2/l^2)} \right),$$

holds, therefore, we get

$$\|(-K\triangle + \lambda)^{-1}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))} \leq \frac{1}{|\lambda|}. \ (\lambda \neq 0)$$

Thus, when $|\lambda| > 2R$ holds, we have $\|(-K\triangle + \lambda)^{-1}R'(w^*)\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))} \leq \frac{R}{|\lambda|} < \frac{R}{2R} = \frac{1}{2} < 1$ and then we get

$$\|(\lambda I + L_{w^*})^{-1}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))} \leq \frac{2}{|\lambda|}.$$

And since

$$\frac{|\lambda+\sigma|}{|\lambda|} \leq \frac{|\lambda|+\sigma}{|\lambda|} \leq 1 + \frac{\sigma}{|\lambda|} \leq 1 + \frac{\sigma}{2R}$$

holds, we have

$$\|(\lambda I + L_{w^*})^{-1}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))} \leq \frac{2R+\sigma}{R|\lambda+\sigma|}.$$

Now we consider the case when $|\lambda| \leq 2R$ and $\lambda \in S_{\theta_0,-\sigma}$. Reminding that

$$L_\mu(u,v) = \begin{pmatrix} u + L^{-1}\{-(1+\mu)u - \gamma(-1 + 2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})u - \gamma(\hat{u}_N + \tilde{u})^2 v\} \\ dv + L^{-1}\{(-d+\mu)v + 2\gamma(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})u + \gamma(\hat{u}_N + \tilde{u})^2 v\} \end{pmatrix}$$

and $Lu = -\triangle u + u$, we know

$$L_{w^*} = (L, L)L_\mu + \mu I,$$

therefore, we get

$$L_\mu^{-1} = (L_{w^*} - \mu I)^{-1}(L, L).$$

Set $\lambda = -\mu$. From the eigenvalue excluding result, there are finite circles $B(\lambda_i, \frac{1}{2M_{\lambda_i}})(i = 1, 2, \cdots m)$ which are inside the domain $\{\lambda \in \mathcal{C} : |\lambda| \leq 2R\}$ or have intersection with the domain. Then for every $\lambda \in \{\lambda \in \mathcal{C} : |\lambda| \leq 2R\}$, we find a circle $B(\lambda_{i_0}, \frac{1}{2M_{\lambda_{i_0}}})(1 \leq i_0 \leq m)$ such that $\lambda \in B(\lambda_{i_0}, \frac{1}{2M_{\lambda_{i_0}}})$. And so, by Remark 5.5.2, for $(u,v) \in H^2(\Omega) \times H^2(\Omega)$, we have

$$\|L_\lambda^{-1}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)} \leq 2M_{\lambda_{i_0}} \|(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}.$$

Thus, we get

$$
\begin{aligned}
&\|(L_{w^*}+\lambda I)^{-1}(u,v)\|_{L^\infty(\Omega)\times L^\infty(\Omega)}\\
\le& K_{2,\infty}\|(L_{w^*}+\lambda I)^{-1}(u,v)\|_{H^2(\Omega)\times H^2(\Omega)}\\
=& K_{2,\infty}\|L_\lambda^{-1}(L^{-1}u,L^{-1}v)\|_{H^2(\Omega)\times H^2(\Omega)}\\
\le& 2K_{2,\infty}M_{\lambda_{i_0}}\|(L^{-1}u,L^{-1}v)\|_{H^2(\Omega)\times H^2(\Omega)}\\
\le& 2K_{2,\infty}M_{\lambda_{i_0}}\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)}\\
\le& 2K_{2,\infty}M_{\lambda_{i_0}}|\Omega|\cdot\|(u,v)\|_{L^\infty(\Omega)\times L^\infty(\Omega)}.
\end{aligned}
$$

Therefore, we obtain $\|(L_{w^*}+\lambda I)^{-1}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))}\le 2K_{2,\infty}M_{\lambda_{i_0}}|\Omega|$.

Set $M_1 := \max\limits_{i=1,2,\cdots m}\{2K_{2,\infty}M_{\lambda_i}|\Omega|\}$, then when $|\lambda|\le 2R$ holds, we have

$$
\|(\lambda I+L_{w^*})^{-1}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))}\le M_1\le M_1\frac{|\lambda+\sigma|}{|\lambda+\sigma|}\le\frac{M_1(2R+\sigma)}{|\lambda+\sigma|}.
$$

Let $M:=\max\left\{\frac{2R+\sigma}{R}, M_1(2R+\sigma)\right\}$, then for all $\lambda\in S_{\theta_0,-\sigma}$, we have $\|(\lambda I+L_{w^*})^{-1}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))}\le\frac{M}{|\lambda+\sigma|}$. □

Then we have the following theorem.

Theorem 6.0.9 *Set*

$$
D(A_0)=\{w\in\bigcap_{p\ge2}W^{2,p}(\Omega)\times W^{2,p}(\Omega):\ \frac{\partial w}{\partial\nu}\Big|_{\partial\Omega}=0\},
$$
$$
A_0: D(A_0)\to L^\infty(\Omega)\times L^\infty(\Omega),\ A_0w=-L_{w^*}w.
$$

Then A_0 is a sectorial operator.

Proof. From the eigenvalue excluding result, we know all eigenvalues λ of $-L_{w^*}: H^2(\Omega)\times H^2(\Omega)\to L^2(\Omega)\times L^2(\Omega)$ satisfying $\lambda\in\mathcal{C}\setminus S_{\theta_0,-\sigma}$.

If λ is an eigenvalue of A_0, then there exists an eigenfunction $w\in D(A_0)$ such that $A_0w=\lambda w$. Since it is easily seen that $D(A_0)\subset D(-L_{w^*})$, $w\in H^2(\Omega)\times H^2(\Omega)$ holds, and thus λ is also an eigenvalue of $-L_{w^*}$, which means all eigenvalues of A_0 are the eigenvalues of $-L_{w^*}$, therefore, we know all eigenvalues λ of A_0 satisfy $\lambda\in\mathcal{C}\setminus S_{\theta_0,-\sigma}$.

From the definition of A_0, when we consider $w\in D(A_0)\subset D(-L_{w^*})\subset L^\infty(\Omega)\times L^\infty(\Omega)$, it is clear that we can replace $-L_{w^*}$ in Theorem 6.0.8 by A_0. Therefore by using Definition 6.0.4, A_0 is a sectorial operator. □

Remark 6.0.10 *From Proposition 6.0.6, there is a constant M_0 satisfying $\|e^{tA_0}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))}\le M_0e^{-\sigma t}$.*

Now we can get the domain of attraction.

Theorem 6.0.11 *There exist $\delta_0, C > 0$, $\omega < 0$ such that when $\|\hat{w}(0)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \leq \delta_0$, we have*

$$\|\hat{w}(t)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \leq Ce^{\omega t}\|\hat{w}(0)\|_{L^\infty(\Omega)\times L^\infty(\Omega)}.$$

And therefore, we can say when $t \to \infty$, $\hat{w}(t) \to 0$, which means $w(t) \to w^*$.

Proof. From Proposition 6.0.3, the solution in $[0, T]$ of the equation (6.0.3) can be written as

$$\hat{w}(t) = e^{-tL_{w^*}}\hat{w}(0) + \int_0^t e^{-(t-s)L_{w^*}}H(\hat{w}(s))ds.$$

Then, if $\hat{w}(t) \in D(A_0)(t \in [0, T])$ holds (from Proposition 6.0.5, this condition can be easily satisfied if $\hat{w}(0) \in D(A_0)$), by Remark 6.0.10, we have

$$\begin{aligned} & \|\hat{w}(t)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \\ \leq\ & \|e^{tA_0}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))}\|\hat{w}(0)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \\ & + \textstyle\int_0^t \|e^{-(t-s)L_{w^*}}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))}\|H(\hat{w}(s))\|_{L^\infty(\Omega)\times L^\infty(\Omega)}ds \\ \leq\ & M_0e^{-\sigma t}\|\hat{w}(0)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} + \textstyle\int_0^t M_0e^{-\sigma(t-s)}\|H(\hat{w}(s))\|_{L^\infty(\Omega)\times L^\infty(\Omega)}ds. \end{aligned} \tag{6.0.6}$$

Note that

$$H(\hat{w}) = \gamma \begin{pmatrix} f(\hat{w} + w^*) - f(w^*) - f_u(w^*)\hat{u} - f_v(w^*)\hat{v} \\ g(\hat{w} + w^*) - g(w^*) - g_u(w^*)\hat{u} - g_v(w^*)\hat{v} \end{pmatrix},$$

so

$$\begin{aligned} \|H(\hat{w})\|^2_{L^\infty(\Omega)\times L^\infty(\Omega)} = \ & \gamma^2\|f(\hat{w} + w^*) - f(w^*) - f_u(w^*)\hat{u} - f_v(w^*)\hat{v}\|^2_{L^\infty(\Omega)} \\ & +\gamma^2\|g(\hat{w} + w^*) - g(w^*) - g_u(w^*)\hat{u} - g_v(w^*)\hat{v}\|^2_{L^\infty(\Omega)} \\ & = \gamma^2(\|a - (\hat{u} + u^*) + (\hat{u} + u^*)^2(\hat{v} + v^*) - (a - u^* + u^{*2}v^*) \\ & -(-1 + 2u^*v^*)\hat{u} - u^{*2}\hat{v})\|^2_{L^\infty(\Omega)} + \|b - (\hat{u} + u^*)^2(\hat{v} + v^*) \\ & -(b - u^{*2}v^*) + 2u^*v^*\hat{u} + u^{*2}\hat{v}\|^2_{L^\infty(\Omega)}) \\ & = \gamma^2(\|\hat{u}^2\hat{v} + \hat{u}^2v^* + 2\hat{u}\hat{v}u^*\|^2_{L^\infty(\Omega)} + \|\hat{u}^2\hat{v} + \hat{u}^2v^* + 2\hat{u}\hat{v}u^*\|^2_{L^\infty(\Omega)}) \end{aligned}$$

holds. Therefore, we have

$$\begin{aligned} & \|H(\hat{w})\|_{L^\infty(\Omega)\times L^\infty(\Omega)} = \sqrt{2}\gamma\|\hat{u}^2\hat{v} + \hat{u}^2v^* + 2\hat{u}\hat{v}u^*\|_{L^\infty(\Omega)} \\ \leq & \sqrt{2}\gamma(\|\hat{u}\|^2_{L^\infty(\Omega)}\|\hat{v}\|_{L^\infty(\Omega)} + \|\hat{u}\|^2_{L^\infty(\Omega)}\|v^*\|_{L^\infty(\Omega)} + 2\|\hat{u}\|_{L^\infty(\Omega)}\|\hat{v}\|_{L^\infty(\Omega)}\|u^*\|_{L^\infty(\Omega)}) \\ \leq & \sqrt{2}\gamma\left(\frac{\|\hat{w}\|^3_{L^\infty(\Omega)\times L^\infty(\Omega)}}{2} + \|\hat{w}\|^2_{L^\infty(\Omega)\times L^\infty(\Omega)}\|v^*\|_{L^\infty(\Omega)} + \|\hat{w}\|^2_{L^\infty(\Omega)\times L^\infty(\Omega)}\|u^*\|_{L^\infty(\Omega)}\right). \end{aligned}$$

For all $\zeta > 0$, after solving inequality

$$\sqrt{2}\gamma\left(\frac{\delta^2}{2}+\delta(\|u^*\|_{L^\infty(\Omega)}+\|v^*\|_{L^\infty(\Omega)})\right)-\zeta<0,$$

we have $\delta < \delta_1(\zeta) = \frac{-\sqrt{2}\gamma(\|u^*\|_{L^\infty(\Omega)}+\|v^*\|_{L^\infty(\Omega)})+\sqrt{2\gamma^2(\|u^*\|_{L^\infty(\Omega)}+\|v^*\|_{L^\infty(\Omega)})^2+2\sqrt{2}\gamma\zeta}}{\sqrt{2}\gamma}$.

Then when we have $\|\hat{w}(s)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \le \delta_1(\zeta)$, inequality (6.0.6) can be replaced by

$$\|\hat{w}(t)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \le M_0 e^{-\sigma t}\|\hat{w}(0)\|_{L^\infty(\Omega)\times L^\infty(\Omega)}+\int_0^t M_0 e^{-\sigma(t-s)}\zeta\|\hat{w}(s)\|_{L^\infty(\Omega)\times L^\infty(\Omega)}ds.$$

Set $p(t) := e^{\sigma t}\|\hat{w}(t)\|_{L^\infty(\Omega)\times L^\infty(\Omega)}$, then we have

$$p(t) \le \quad M_0 p(0) + M_0\zeta \int_0^t p(s)ds.$$

By using Gronwall inequality, we get

$$p(t) \le M_0 p(0) e^{M_0\zeta t},$$

that is,

$$\|\hat{w}(t)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \le M_0 e^{(-\sigma+M_0\zeta)t}\|\hat{w}(0)\|_{L^\infty(\Omega)\times L^\infty(\Omega)}$$

holds.

Therefore, we choose $\zeta = \frac{\sigma}{(1+\xi)M_0}$ (where ξ is a very small positive constant), and then we find some corresponding $\delta(\zeta)$. If $\|\hat{w}(0)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \le \frac{\delta}{M_0} =: \delta_0$ holds, then we have $\|\hat{w}(t)\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \le \delta$. Set $C = M_0$ and $\omega = -\sigma + M_0\zeta = -\frac{\xi\sigma}{1+\xi}$, then the statement holds. □

Remark 6.0.12 *From the proof of Theorem 6.0.11, we need to choose some initial value $\hat{w}(0) \in D(A_0)$. Since we have $\hat{w}(0) = w(0) - \hat{w}_N - \tilde{w}$, we choose $w(0) = w_h(0) + \tilde{w}$, then we get $\hat{w}(0) = w_h(0) - \hat{w}_N \in D(A_0)$. And from Chapter 3, we see that adding $\tilde{w}$ to $w_h(0)$ will not affect the result of the approximate solutions and the approximate stationary solutions, thus it will not change the stationary solution.*

Chapter 7

Numerical Results

We made use of an interval arithmetic based on the interval library([22]) to avoid the effects of rounding errors in the floating-point computations. The computations were carried out on a SONY VPCZ11AFJ(Intel(R) Core(TM) i5 M520 2.40GHz) using Matlab(Ver.7.5.0).

We apply our method to an example with some suitable system parameters and get the following numerical results.

Example 1. We choose $a = 0.01$, $b = 1$, $d = 6.5$, $\gamma = 0.5$, $\Omega = (0, 10) \times (0, 10)$ and $N = 25$. We use initial values as $u_h(0) = 1 - 0.4 \cos \pi x \cos \pi y$, $v_h(0) = 1 + 0.3 \cos \pi x \cos \pi y$ (Fig. 7.1), then by using Galerkin method, we get the approximate solutions $u_h(T)$ and $v_h(T)$ $(T = 20)$ in Fig. 7.2.

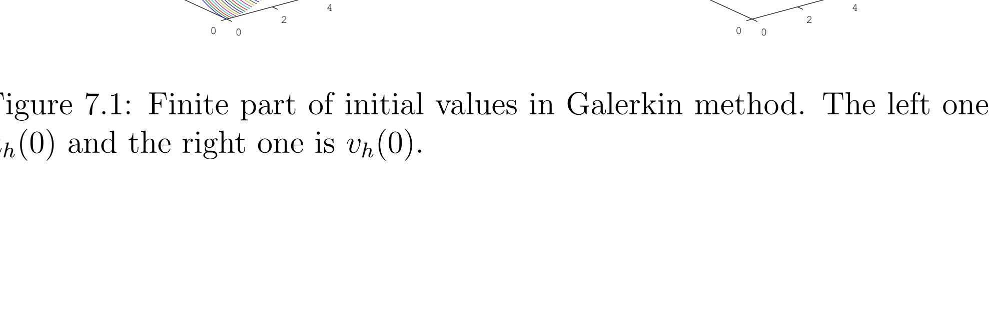

Figure 7.1: Finite part of initial values in Galerkin method. The left one is $u_h(0)$ and the right one is $v_h(0)$.

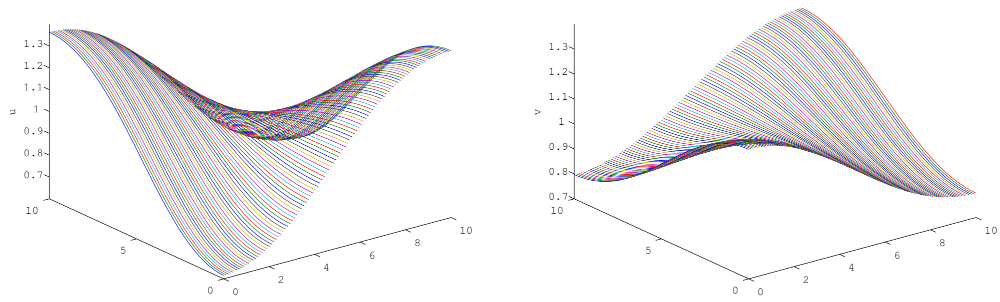

Figure 7.2: The approximate solutions of Example 1 which we got from Galerkin method. The left one is $u_h(T)$ and the right one is $v_h(T)$.

Then we use Newton's method to improve the solutions in Fig.7.2. The approximate stationary solutions are shown in Fig. 7.3.

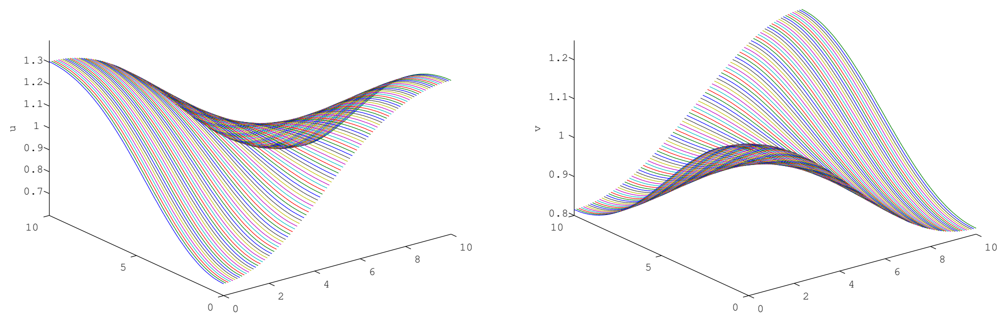

Figure 7.3: The approximate stationary solutions of Example 1 which we got from Newton's method. The left one is $\hat{u}_N$ and the right one is $\hat{v}_N$.

By using the verification method in Chapter 4, we get the following verification results for the stationary solution in Table 7.1.

α_1	β_1	α_2	β_2	iteration number
1.2516E-11	6.2896E-12	3.0810E-12	4.7800E-13	1

Table 7.1: Verification results

For the eigenvalue problem (5.1.1), we get the following approximate eigenvalues (Fig. 7.4). The points are the approximate eigenvalues and the rectangle domain is the domain of the eigenvalues of problem (5.5.1), where $\gamma b_1 = -1.9160$, $\gamma b_2 = -1.9160$.

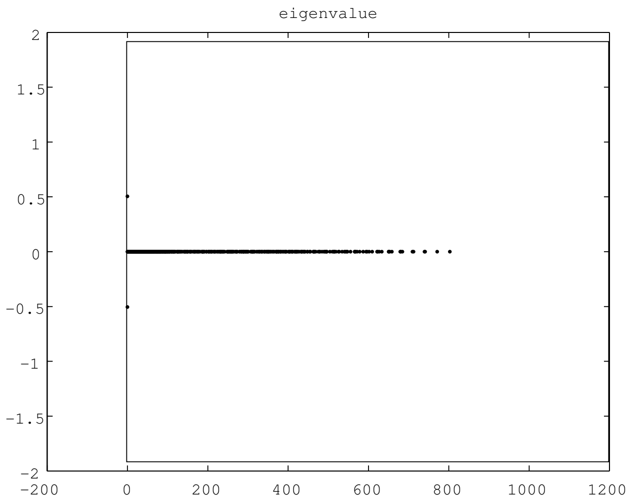

Figure 7.4: The approximate eigenvalues.

It is clear that if λ is the eigenvalue of (5.5.1), then $\overline{\lambda}$ is also an eigenvalue of (5.5.1), therefore, we only need to do eigenvalue excluding in the area $\{\lambda|Im(\lambda) \geq 0\}$. By Theorem 5.1.1, we get $\hat{M}_\mu$, then from (5.5.5) and Remark 5.5.2, we have the eigenvalue excluding results in Fig. 7.5. The right-hand side of Fig. 7.5 indicates a zoom in of the left-hand side. The line passes through the point $(0.001, 0)$ and $(0.016, 0.50525)$. And the line is inside the eigenvalue excluding area, which means the eigenvalues of L_{w^*} are in the right-hand side of the line. Therefore, the spectrum of L_{w^*} is contained in the sector $S_{\pi-\theta_0,\sigma}$, where $\theta_0 = \pi - \arctan(0.50525/0.015)$ and $\sigma = 0.001$.

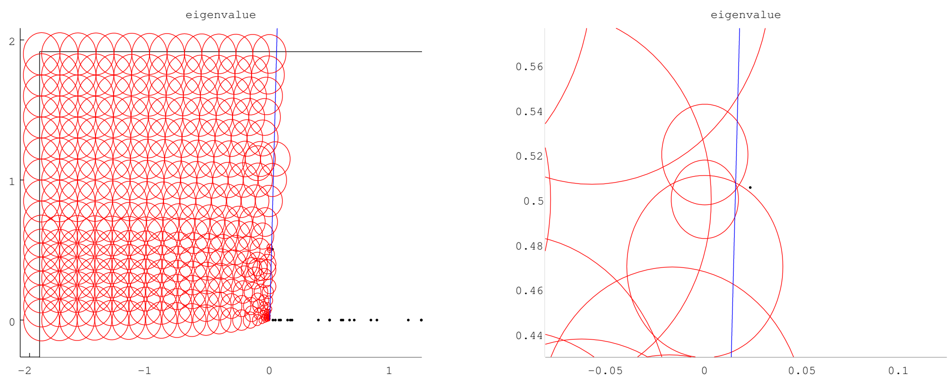

Figure 7.5: The eigenvalue excluding results. The right one is a zoom in of the left one.

From Fig. 7.5, we get the resolvent set of $-L_{w^*}$ containing the sector $S_{\theta_0,-\sigma} = \{\lambda \in \mathcal{C} : \lambda \neq -\sigma, |\arg(\lambda+\sigma)| < \theta_0\}$, where $\sigma = 0.001$, $\theta_0 = 1.6005$.

Then from Theorem 6.0.8, we get the value of M, so by using Proposition 6.0.6 and Remark 6.0.7, when we set $M_0 = \frac{M}{2\pi}(2\int_r^\infty \rho^{-1}e^{\rho\cos\theta_0}d\rho + \int_{-\theta_0}^{\theta_0} e^{r\cos\eta}d\eta)$, for every $r > 0$, we have $\|e^{-tL_{w^*}}\|_{\mathcal{L}(L^\infty(\Omega)\times L^\infty(\Omega))} \leq M_0 e^{-\sigma t}, t > 0$.

Here, we choose some $r > 0$ and get the numerical value of M_0. Then set

$$\zeta = \frac{\sigma}{(1+\xi)M_0}(\xi = 1E-10)$$

and

$$\delta = \frac{-\sqrt{2}\gamma(\|u^*\|_{L^\infty(\Omega)} + \|v^*\|_{L^\infty(\Omega)}) + \sqrt{2\gamma^2(\|u^*\|_{L^\infty(\Omega)} + \|v^*\|_{L^\infty(\Omega)})^2 + 2\sqrt{2}\gamma\zeta}}{\sqrt{2}\gamma}.$$

By using Theorem 6.0.11, if we enclose the solution of (1.0.1) until T and the enclosing set is a subset of the domain $\{w| \; \|w - w^*\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \leq \frac{\delta}{M_0}\}$, then we can conclude that from time T on, the solution at last will converge to w^*.

Table 7.2 shows the numerical results related to the domain of attraction.

M	r	M_0	δ	δ/M_0
1.3155E+7	0.61	2.4422E+7	2.2642E-11	9.2635E-19

Table 7.2: Numerical results related to the domain of attraction

Chapter 8

Conclusions

We proposed a method by computer-assistance to prove a pattern formation on reaction-diffusion systems. We obtained the eigenvalue excluding results for the linearized operator at the exact stationary solution and then by using the semigroup estimate, we computed a domain of attraction for the stationary solution, i.e. some (norm-)neighborhood of the stationary solution, which assures the convergence of the parabolic solution to the stationary solution. This is a constructive way to prove a pattern formation and well supports a numerical behavior with a convergence proof in mathematically rigorous sense.

As seen in the numerical results in our example, the norm-neighborhood of the stationary solution is still very small, and for the moment this prevents us from enclosing the parabolic solution until the enclosing set is a subset of the domain of attraction. But since the final result for the domain of attraction heavily depends on the choice of all parameters in the system, if we could find other parameters which provide a better distribution of eigenvalues for the linearized operator (for example, $\sigma > 0$ is far away from 0), then we will have a good chance to enlarge the domain of attraction which will be more interesting to see the dynamics of the pattern formation.

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