The kappa function

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The kappa function

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Abstract: The kappa function is introduced as the function $\kappa$ satisfying $J(\kappa(\tau)) = \lambda(\tau)$, where $J$ and $\lambda$ are the elliptic modular functions. A Fourier expansion of $\kappa$ is studied.

Keywords: covariant function, hypergeometric function, Schwarz’s $s$-function, elliptic modular function.

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1 Introduction

Let $G$ and $G'$ be discrete subgroups of the group $PGL_2(\mathbb{C})$ of linear fractional transformations, and $r : G \to G'$ a surjective homomorphism. A holomorphic function $f(z)$ is said to be covariant of type $(G, r, G')$ if

$$f\left(\frac{az + b}{cz + d}\right) = \frac{a'f(z) + b'}{c'f(z) + d'}, \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G, \quad r(g) = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in G'.
$$

When $G'$ is trivial, a covariant function is a $G$-automorphic function. We are interested in the following cases:

1. $G = G'$ is a finite group ($r$ is the identity map). See [OY].

2. $G = G'$ is a triangle Fuchsian group ($r$ is the identity map). An example of covariant functions for $G = PSL_2(\mathbb{Z})$ is given in [KK].

3. $G$ and $G'$ are triangle Fuchsian groups, and $\text{Ker}(r)$ and $G/\text{Ker}(r)$ are both infinite groups.

In this paper, as a typical example of the third case, we introduce the kappa function $\kappa$ defined by $J(\kappa(\tau)) = \lambda(\tau)$, where $J$ and $\lambda$ are the elliptic modular functions, and study its Fourier expansion at $i\infty$. 

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2 The Schwarz map of the hypergeometric equation

We briefly recall in this section a classical theory of Schwarz maps (cf. [Yos]). Let $E(a, b, c)$ be the hypergeometric differential equation

$$x(1-x)u'' + (c-(a+b+1)x)u' - abu = 0,$$

where $a, b$ and $c$ are parameters. Its Schwarz map is defined by

$$s : X = \mathbb{C} - \{0, 1\} \ni x \mapsto z = u_1(x) : u_2(x) \in Z = \mathbb{P}^1 := \mathbb{C} \cup \{\infty\},$$

where $u_1$ and $u_2$ are two linearly independent solutions of $E(a, b, c)$. The local exponents of the equation $E(a, b, c)$ at 0, 1 and $\infty$ are given as $\{0, 1-c\}, \{0, c-a-b\}$ and $\{a, b\}$, respectively. Denote the differences of the local exponents by

$$\mu_0 = 1 - c, \quad \mu_1 = c - a - b, \quad \mu_\infty = a - b,$$

and the monodromy group by Monod($\mu_0, \mu_1, \mu_\infty$). Then the Schwarzian derivative $s; x$ of $s$ with respect to $x$ is given as

$$-4\{s; x\} = \frac{2s's'' - 3(s'')^2}{(s')^2} = \frac{1 - \mu_0^2}{x^2} + \frac{1 - \mu_1^2}{(1-x)^2} + \frac{1 + \mu_\infty^2 - \mu_0^2 - \mu_1^2}{x(1-x)}.$$ 

We assume that the parameters $a, b$ and $c$ are rational numbers such that

$$k_0 := \frac{1}{|\mu_0|}, \quad k_1 := \frac{1}{|\mu_1|}, \quad k_\infty := \frac{1}{|\mu_\infty|} \in \{2, 3, \ldots\} \cup \{\infty\},$$

and $1/k_0 + 1/k_1 + 1/k_\infty < 1$. Then the Schwarz map

$$s = s(k_0, k_1, k_\infty) : X \rightarrow H = \{z \in \mathbb{C} \mid \Im(z) > 0\}$$

gives the developing map of the universal branched covering with ramification indices $(k_0, k_1, k_\infty)$; its inverse map

$$s^{-1} : H - \text{Fix}(\Delta) \rightarrow X$$

is single-valued, and induces the isomorphism $(H - \text{Fix}(\Delta))/\Delta \cong X$, where $\Delta = \Delta(k_0, k_1, k_\infty)$ is the monodromy group Monod($k_0, k_1, k_\infty$) regarded as a transformation group (Schwarz’s triangle group) of $H$, and $\text{Fix}(\Delta)$ is the set of fixed points of $\Delta$.

3 Covariant functions of type $(\Gamma(2), r, \Delta)$

In particular, when $(k_0, k_1, k_\infty) = (\infty, \infty, \infty)$, the monodromy group $\Delta(\infty, \infty, \infty)$ is isomorphic to the principal congruence subgroup

$$\Gamma(2) = \{g \in SL_2(\mathbb{Z}) \mid g \equiv \text{id} \mod 2\}/\{\pm 1\},$$
which has no fixed points, and the inverse of the Schwarz map is known by the
name of the lambda function \( \lambda(z) \) defined on \( \mathbb{H} \). Since \( \lambda : \mathbb{H} \to X \) is the universal
covering of \( X \), for any \((k_0, k_1, k_\infty)\) satisfying \( 1/k_0 + 1/k_1 + 1/k_\infty < 1 \), the branched
covering
\[
s^{-1} = s^{-1}_{(k_0, k_1, k_\infty)} : \mathbb{H} - \text{Fix}(\Delta) \to X
\]
factors \( \lambda \), that is, there is a unique map \( f = f_{(k_0, k_1, k_\infty)} : \mathbb{H} \to \mathbb{H} - \text{Fix}(\Delta) \) such that
\[
s^{-1}(f(z)) = \lambda(z), \quad z \in \mathbb{H},
\]
where \( \Delta = \Delta_{(k_0, k_1, k_\infty)} \). The Galois correspondence can be illustrated as

\[
\begin{array}{c|cc}
\{1\} & \mathbb{H} & \text{The universal covering of } X \\
\downarrow & \downarrow f & \\
N & \mathbb{H} - \text{Fix}(\Delta) & \text{The universal branched covering of } X \\
\downarrow & \downarrow s^{-1} & \\
\pi_1(X) & X & \\
\end{array}
\]

Here \( N \) is a normal subgroup of the fundamental group \( \pi_1(X) \) of \( X \) corresponding
to the middle cover \( \mathbb{H} - \text{Fix}(\Delta) \). Actually, \( N \) is given as follows: Let \( \gamma_0 \) (resp. \( \gamma_1 \)
and \( \gamma_\infty \)) be a simple loop around \( x = 0 \) (resp. 1 and \( \infty \)), and regard these loops as
elements of \( \pi_1(X) \). Then \( N \) is the smallest subgroup of \( \pi_1(X) \) containing
\( \gamma_0^{k_0}, \gamma_1^{k_1} \) and \( \gamma_\infty^{k_\infty} \).

At any rate, we have\n
\[
\pi_1(X)/N \cong \Delta;
\]
let \( r : \pi_1(X) \to \Delta \) denote the projection.

For a point \( z \in \mathbb{H} \), put \( w = f(z) \) and \( x = \lambda(z) \). Let \( \gamma \) be a loop in \( X \) with base \( x \). The lift of \( \gamma \) under \( \lambda \) is a path in \( \mathbb{H} \) connecting \( z \) and \( g(z) \) for some \( g \in \Gamma(2) \);
this gives the isomorphism
\[
\pi_1(X, x) \cong \Gamma(2).
\]
The lift of \( \gamma \) under the Schwarz map \( s \) is a path in \( \mathbb{H} - \text{Fix}(\Delta) \) connecting \( w \) and \( g'(w) \) for some \( g' \in \Delta \); the correspondence
\[
\Gamma(2) \ni g \mapsto g' = r(g) \in \Delta
\]
is the homomorphism \( r \) via the identification \( \pi_1(X, x) \cong \Gamma(2) \).

**Proposition 1** Our function \( f \) is covariant of type \((\Gamma(2), r, \Delta)\).

**Proof.** We have
\[
f(g(z)) = g'(w) = g'(f(z)), \quad g \in \Gamma(2).
\]
\[\square\]
The following illustration may help the reader.

\[
\begin{array}{ccc}
H & \ni & z \xrightarrow{\lambda^*(\gamma)} g(z) \\
\downarrow & \downarrow & \downarrow \\
H - \text{Fix}(\Delta) & \ni & w \xrightarrow{s^*(\gamma)} r(g)(w) \\
\downarrow & \downarrow & \\
X & \ni & x \xrightarrow{\gamma} x
\end{array}
\]

### 3.1 The kappa function

We are especially interested in the case 

\[(k_0, k_1, k_\infty) = (3, 2, \infty)\].

The monodromy group \(\Delta_{(3, 2, \infty)}\) is isomorphic to \(\Gamma(1) = \text{PSL}_2(\mathbb{Z})\), and the map \(s^{-1}\) is usually denoted by \(J\). We name the function \(f\) as the \textit{kappa function} \(\kappa\); this is because the letter \(k\) is situated between \(j\) and \(l\) in the alphabetic sequence. So we have

\[J(\kappa(z)) = \lambda(z)\].

We normalize the maps in question as

\[
\begin{array}{ccc}
\{1\} & H & \ni \quad z = 0 \quad \infty \quad 1 \\
| & \downarrow \kappa & \downarrow \downarrow \downarrow \\
N & H - \Gamma(1)\{i, \rho\} & w = \rho \quad i \quad \infty \\
| & \downarrow J & \downarrow \downarrow \downarrow \\
\Gamma(2) & X & x = 0 \quad 1 \quad \infty
\end{array}
\]

where \(\rho = \exp(2\pi i/6)\). Let \(\gamma_0\) and \(\gamma_1\) be the simple loops (with base point in the lower half \(x\)-plane) around 0 and 1 as are shown in Figure 1. According to the normalization above, \(\gamma_0\) and \(\gamma_1\), as elements of \(\pi_1(X)\), are identified respectively with the two generators

\[g_0 : z \mapsto \frac{z}{-2z + 1} \quad \text{and} \quad g_\infty : z \mapsto z + 2\]

of \(\Gamma(2)\); they fix 0 and \(\infty\), respectively. Then the subgroup \(N\) is the smallest normal subgroup of \(\Gamma(2)\) containing

\[g_0^3 : z \mapsto \frac{z}{-6z + 1} \quad \text{and} \quad g_\infty^2 : z \mapsto z + 4,\]

and the isomorphism \(\Gamma(2)/N \cong \Gamma(1)\) is given by the surjective homomorphism \(r : \Gamma(2) \rightarrow \Gamma(1)\) defined by

\[g_0 \mapsto \left(w \mapsto \frac{1}{1 - w}\right) \quad \text{and} \quad g_\infty \mapsto \left(w \mapsto \frac{-1}{w}\right).
\]
Thus our function $\kappa$ satisfies
\[
\kappa\left(\frac{z}{-2z+1}\right) = \frac{1}{1 - \kappa(z)} \quad \text{and} \quad \kappa(z+2) = -\frac{1}{\kappa(z)}.
\]

Figure 1: A geometric explanation of the correspondence: $\gamma_0 \leftrightarrow g_0, \gamma_1 \leftrightarrow g_\infty$

### 3.2 A fundamental domain for $N$

Recall that the map $\kappa : \mathbb{H} \rightarrow \mathbb{H} - \Gamma(1)\{i, \rho\}$ is the universal cover (of the infinitely punctured upper half $w$-plane $\mathbb{H} - \Gamma(1)\{i, \rho\}$) with the transformation group $N \subseteq \Gamma(2)$. To obtain a fundamental domain of $N$ in the upper half $z$-plane, we cut the punctured upper half $w$-plane so that it becomes simply connected.

Our cut shown in Figure 2 is invariant under the action of $\Gamma(2)$, where $\Gamma(2)$ is here regarded as the subgroup of $\Gamma(1)$ acting on the $w$-space. In the figure, a fundamental domain of $\Gamma(2)$ is shown as the union of twelve triangles $1, \ldots, 6, 1', \ldots, 6'$, each of which is a fundamental domain of the extended triangle group of $\Gamma(1)$. Our cuts are now given by

\[
1 \cap 6, \quad 1' \cap 2', \quad 3' \cap 4', \quad 5' \cap 6', \quad 6' \cap 1'.
\]

It is easy to check that the complement of the $\Gamma(2)$-orbits of these cuts is connected and simply connected. If we draw this connected net of triangles on the $z$-plane
through $\kappa$, shown in Figure 1, making use of the Schwarz reflection principle, we eventually obtain a fundamental domain of $N$ bounded by infinitely many arcs as is shown in Figure 3.

3.3 A Fourier expansion of the kappa function

In this section we compute the Fourier development of $\kappa(z)$ at $z = i\infty$. Since $\kappa(z + 4) = \kappa(z)$ and $\kappa(\infty) = i$ by definition, the Fourier series of $\kappa(z)$ has the form

$$\kappa(z) = i(1 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots),$$

where

$$q := \exp \frac{\pi i z}{2}.$$
Proposition 2 1) The $n$th Fourier coefficient $a_n$ of $\kappa(z)$ can be expressed as a polynomial of degree $n$ in $a := a_1$ with rational coefficients, starting with $a^n/2^{n-1} + \cdots$ and having no constant term. The polynomial is even or odd according as $n$ is even or odd.

2) The value of $a$ is explicitly given by

$$a = -i \frac{32}{\sqrt{3}} \frac{\pi^2}{\Gamma(1/4)^4} = -1.0552729262852 \ldots \times i.$$

Example 1

\begin{align*}
a_1 &= a, \\
a_2 &= \frac{1}{2} a^2, \\
a_3 &= \frac{1}{4} a^3 - \frac{16}{27} a, \\
a_4 &= \frac{1}{8} a^4 - \frac{16}{27} a^2, \\
a_5 &= \frac{1}{16} a^5 - \frac{4}{9} a^3 + \frac{98}{1215} a, \\
a_6 &= \frac{1}{32} a^6 - \frac{8}{27} a^4 + \frac{934}{3645} a^2, \\
a_7 &= \frac{1}{64} a^7 - \frac{5}{27} a^5 + \frac{787}{2430} a^3 - \frac{1504}{6561} a, \\
a_8 &= \frac{1}{128} a^8 - \frac{1}{9} a^6 + \frac{41}{135} a^4 - \frac{9088}{32805} a^2.
\end{align*}

Proof. For 1), we shall establish recursion relations among $a_n$’s. First, by the identity

$$\kappa(z + 2) = -\frac{1}{\kappa(z)},$$

we immediately obtain the recursion with which the even index coefficients are determined by the previous ones.

Lemma 1 For each even integer $n \geq 2$, we have

$$a_n = \sum_{i=1}^{n/2-1} (-1)^{i-1} a_i a_{n-i} + (-1)^{n/2-1} a_{n/2}^2. \quad (1)$$

In particular, $a_2 = a_1^2/2, a_4 = a_1 a_3 - a_2^2/2, a_6 = a_1 a_5 - a_2 a_4 + a_3^2/2, \ldots$.

Proof. Since we have $\kappa(z + 2) = i(1 - a_1 q + a_2 q^2 - a_3 q^3 + \cdots)$, we get the recursion by expanding $\kappa(z + 2) \kappa(z)$ and equating the coefficient of $q^n$ with 0. 

Note $\kappa(z + 2) \kappa(z)$ is the even function of $q$ and so for odd $n$ the coefficient is automatically 0. To determine $a_n$ for odd $n$, we make use of the explicit formula
for the Schwarzian derivative \{\kappa; z\}. To describe this, we introduce Jacobi’s theta constants;

\[
\theta_0(z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{2n^2}, \quad \theta_2(z) = \sum_{n \in \mathbb{Z}} q^{(2n+1)^2/4}, \quad \theta_3(z) = \sum_{n \in \mathbb{Z}} q^{2n^2}.
\]

They satisfy the famous identity \(\theta_0(z)^4 + \theta_2(z)^4 = \theta_3(z)^4\), which will be used later. By these theta’s, our \(\lambda\) function can be expressed as

\[
\lambda(z) = \frac{\theta_0(z)^4}{\theta_3(z)^4} = 1 - 16q^2 + 128q^4 - 704q^6 + \cdots.
\]

In fact, the \(\Gamma(2)\)-invariance is classical and the only thing we have to check is the values \(\lambda(\infty) = 1, \lambda(0) = 0\) and \(\lambda(1) = \infty\) that we have chosen to normalize \(\lambda\). But this is readily seen by the above and the following expansions

\[
\lambda\left(-\frac{1}{z}\right) = \frac{\theta_2(z)^4}{\theta_3(z)^4} = 16q^2 - 128q^4 + 704q^6 + \cdots,
\]

\[
\lambda\left(1 - \frac{1}{z} + 1\right) = \frac{\theta_3(-1/(z + 1))}{\theta_0(-1/(z + 1))}^4 = \frac{\theta_3(z)^4}{\theta_0(z)^4} = \frac{1}{16q^2} + \frac{2}{3} + \frac{5}{4}q^2 + \cdots,
\]

which can be derived from the well-known transformation formulae (cf. [Mum])

\[
\begin{align*}
\theta_0(z + 1) &= \theta_3(z), & \theta_0(-1/z) &= \sqrt{z/i} \theta_2(z), \\
\theta_2(z + 1) &= e^{\pi i/4} \theta_2(z), & \theta_2(-1/z) &= \sqrt{z/i} \theta_0(z), \\
\theta_3(z + 1) &= \theta_0(z), & \theta_3(-1/z) &= \sqrt{z/i} \theta_3(z).
\end{align*}
\]

**Lemma 2** We have

\[
\frac{2\kappa'\kappa''' - 3\kappa''^2}{\kappa^2} = -\frac{1}{9} \left(5\theta_0(z)^4 \theta_3(z)^4 + 4\theta_3(z)^8\right),
\]

where \(\kappa' = q \frac{d}{dq} = \frac{2}{\pi i} \frac{d}{dz}\).

**Proof.** Since we have

\[
\begin{align*}
-4\{\lambda^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\
-4\{J^{-1}; x\} &= \frac{1 - (1/3)^2}{x^2} + \frac{1 - (1/2)^2}{(1-x)^2} + \frac{1 - (1/3)^2 - (1/2)^2}{x(1-x)},
\end{align*}
\]

the connection formula of the Schwarzian derivative

\[
\{\kappa; z\} = \{J^{-1} \circ \lambda; z\} = \{\lambda; z\} + \{J^{-1}; x\} \left(\frac{dx}{dz}\right)^2
\]

\[
= (-\{\lambda^{-1}; x\} + \{J^{-1}; x\}) \left(\frac{dx}{dz}\right)^2
\]

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allows us to express the Schwarzian \( \{ \kappa; z \} \) as a rational function of \( x = \lambda(z) \) and its derivative (we multiply \((2/\pi i)^2 \) on both sides to have a formula with \( \ell = q \, d/dq \):

\[
\frac{2\kappa' \kappa''' - 3\kappa''^2}{\kappa'^2} = -\frac{\lambda'(z)^2}{36} \left( \frac{5\lambda(z) + 4}{\lambda(z)^2(1 - \lambda(z))^2} \right).
\]

The lemma then follows from the identities

\[
\lambda'(z) = -2\theta_2(z)^4 \lambda(z) \quad \text{and} \quad 1 - \lambda(z) = \frac{\theta_2(z)^4}{\theta_3(z)^4}.
\]

Now we use (2) to obtain another recursion for \( a_n \).

\[
\frac{1}{9} \left( 5\theta_0(z)^4 \theta_3(z)^4 + 4\theta_3(z)^8 \right) = \sum_{n=0}^{\infty} b_n q^n.
\]

By the formulas

\[
\theta_0(z)^4 \theta_3(z)^4 = 1 + 16 \sum_{n=1}^{\infty} \left( \sum_{d|n} (-1)^d d^3 \right) q^{4n},
\]

\[
\theta_3(z)^8 = 1 + 16 \sum_{n=1}^{\infty} (-1)^n \left( \sum_{d|n} (-1)^d d^3 \right) q^{2n},
\]

the \( b_n \) is explicitly given by \( b_0 = 1 \) and

\[
b_n = \begin{cases} 
0, & \text{for } n \text{ odd,} \\
(-1)^{n/2} \frac{64}{9} \sum_{d|n/2} (-1)^d d^3, & \text{for } n \equiv 2 \text{ mod } 4, \\
(-1)^{n/2} \frac{64}{9} \sum_{d|n/2} (-1)^d d^3 + \frac{80}{9} \sum_{d|n/4} (-1)^d d^3, & \text{for } n \equiv 0 \text{ mod } 4.
\end{cases}
\]

Equating the coefficients of \( q^{n+1} \) on both sides of

\[
2\kappa' \kappa''' - 3\kappa''^2 = -\kappa'^2 \sum_{n=0}^{\infty} b_n q^n,
\]

we obtain, after some manipulation, the recursive relation

\[
2n(n-1)(n-2)a \cdot a_n = - \sum_{i=2}^{n-1} i(n+1-i) \left( 2(n+1)^2 - 7i(n+1) + 5i^2 + 1 \right) a_i a_{n+1-i} - \sum_{j=1}^{n-1} b_j \sum_{i=1}^{n-j} i(n+1-j-i) a_i a_{n+1-j-i}.
\]

With this recursion and \( a_1 = a, a_2 = a^2/2, \) we can deduce all the assertions in 1) of Proposition 2 by induction. For parity result we should note that \( b_j = 0 \) for \( j \) odd, and for the top term we use the identity

\[
\sum_{i=2}^{n-1} i(n+1-i) \left( 2(n+1)^2 - 7i(n+1) + 5i^2 + 1 \right) = -2n(n-1)(n-2).
\]
and note the second sum on the right has lower degree.

Next we evaluate \( a \). Differentiating the identity \( J(\kappa(z)) = \lambda(z) \) twice and multiplying both sides by \( \left( \frac{2}{\pi i} \right)^2 \), we have

\[
\frac{d^2 J}{dw^2}(\kappa(z)) \left( q \frac{d\kappa}{dq}(z) \right)^2 + \frac{dJ}{dw}(\kappa(z)) \left( q \frac{d}{dq} \right)^2 \kappa(z) = \left( q \frac{d}{dq} \right)^2 \lambda(z) = -64q^2 + \cdots (3)
\]

After dividing this by \( q^2 \), we look at the limit when \( z \to i \infty \) (so \( w \to i \) and \( q \to 0 \)). Since

\[
\left( q \frac{d\kappa}{dq}(z) \right)^2 = -a^2 q^2 + \cdots, \quad \left( q \frac{d}{dq} \right)^2 \kappa(z) = iaq + \cdots,
\]

we need the limiting values of \( d^2 J(w)/dw^2 \) and \( (dJ(w)/dw)/q \) as \( w \to i \) (\( w = \kappa(z) \)). To compute these, we use the classical Eisenstein series

\[
E_2(w) = 1 - 24 \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) e^{2\pi i n w},
\]

\[
E_4(w) = 1 + 240 \sum_{n=1}^{\infty} \left( \sum_{d|n} d^3 \right) e^{2\pi i n w},
\]

\[
E_6(w) = 1 - 504 \sum_{n=1}^{\infty} \left( \sum_{d|n} d^5 \right) e^{2\pi i n w},
\]

and the cusp form

\[
\Delta(w) = e^{2\pi i w} \prod_{n=1}^{\infty} (1 - e^{2\pi i n w})^{24}.
\]

**Lemma 3** We have

\[
\frac{dJ}{dw}(w) \frac{1}{q} \to -2\pi^2 i a E_4(i) \quad (\text{as } w \to i)
\]

and

\[
\frac{d^2 J}{dw^2}(i) = -2\pi^2 E_4(i).
\]

**Proof.** We use the formula

\[
\frac{dE_6}{dw}(w) = \pi i (E_2(w)E_6(w) - E_4(w)^2)
\]

as well as the value \( E_6(i) = 0 \) and \( dw/dz = d\kappa(z)/dz = -\pi aq/2 + \cdots \) to obtain (use de L'Hôpital's rule)

\[
\lim_{z \to i\infty} \frac{E_6(w)}{q} = \lim_{z \to i\infty} \frac{\pi i (E_2(w)E_6(w) - E_4(w)^2) \frac{dw}{dz}}{\pi i q} = \pi a E_4(i)^2.
\]
Hence by
\[ \frac{dJ}{dw}(w) = -2\pi i \frac{E_6(w)}{E_4(w)} J(w) \text{ and } J(i) = 1, \]
we obtain
\[ \lim_{z \to \infty} \frac{dJ}{dw}(w) \frac{q^{-i}}{q} \longrightarrow -2\pi^2 ia E_4(i). \]

For the second value, we compute
\[ \frac{d^2 J}{dw^2}(w) = -2\pi i \left( \frac{d}{dw} \left( \frac{J(w)}{E_4(w)} \right) E_6(w) + \frac{J(w)}{E_4(w)} \cdot \pi i (E_2(w) E_6(w) - E_4(w)^2) \right) \]
and use \( E_6(i) = 0, J(i) = 1. \)

Applying this lemma to the identity (3) together with the evaluation
\[ E_4(i) = \frac{3}{64} \frac{\Gamma(1/4)^8}{\pi^4}, \]
we obtain
\[ a^2 = \frac{1024}{3} \frac{\pi^4}{\Gamma(1/4)^8}. \]

Since \( \kappa(z) \) tends to \( i \) from the right on the unit circle as \( z \) goes up to infinity along the pure-imaginary axis, \( ia \) must be positive. This proves 2) of Proposition 2.

References


