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Abstract

This paper is concerned with a multivariate growth curve model for observations obtained by simultaneously measuring m response variables at each of p time points, on samples from multiple groups. The objective is to develop a test for determining whether the $m_2 = m - m_1$ response variables carry no additional information (are redundant) for a comparison between the groups, given the presence of the first m_1 response variables. We obtain some equivalent hypotheses for redundancy by extending the technique of Rao (1970). The likelihood ratio (LR) test is discussed. However, because its null distribution is complicated, we propose a more practical approximate test using a conditional LR criterion.

Key Words and Phrases: Additional information, Comparison between groups, Multivariate growth curve model, Likelihood ratio test.

1. Introduction

Many phenomena are described by multiple characteristic values. In clinical trials, target diseases are often characterized by numerous primary variables (or primary endpoints) and symptomatic states. For example, if we conduct a clinical trial involving migraines, we will be assessing four primary variables (Walter et al. (2007)). In addition, multiple time points of longitudinal data must be analyzed in clinical trials that evaluate changes in the effects of treatment of a chronic disease over time.

In recent years, many trials have conducted using multiple primary variables (or multiple endpoints), without attempting to reduce the number of variables. The usual analysis procedures in this situation are O'Brien's procedure (O'Brien (1984)) and the closed testing procedure (Marcus et al. (1967)).

On the other hand, the guideline presented in *Statistical Principles for Clinical Trials* (1998), developed by the International Conference on Harmonisation (ICH), states that it is generally preferable to have only one primary variable. If we follow this guideline, we must strive to reduce the number of variables in clinical trials planned for multiple primary variables, and it is important to devise a statistical method for accomplishing this.

This paper is concerned with a multivariate growth curve model for observations obtained by simultaneously measuring m response variables at each of p time points, on

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samples from multiple groups. The objective is to develop a test for determining whether the $m - m_1$ response variables carry no additional information (are redundant) for a comparison between the groups, given the presence of the first m_1 response variables.

In Section 2, we explain the multivariate growth curve model, together with some notations used in this paper, and discuss the maximum likelihood estimators for the multivariate growth curve model, as well as the likelihood ratio (LR) criterion for the general linear hypothesis. In Section 3, we discuss some equivalent hypotheses for redundancy in the case of two response variables and then derive a LR criterion. However, since the null distribution is complicated, we present a more practical approximate test for the hypothesis of redundancy. Then, we extend these results to the case of multiple response variables. A numerical example is given to illustrate the procedure

2. Multivariate Growth Curve Model

2.1. Model

In this section, we discuss the maximum likelihood estimators for the multivariate growth curve model, as well as the likelihood ratio (LR) criterion for the general linear hypothesis. The results are well known that a theory of the LR test and maximum likelihood estimator is essentially the same as that in a univariate growth curve model. However, we state these derivations together with some notations used in this paper as preliminaries for subsequent section.

Let $y_{\ell jk}^{(i)}$ denote the ℓth response measurement at time t_k on the jth subject from the ith group, for $i = 1, \ldots, g+1$, $j = 1, \cdots, N_i$, $k = 1, \ldots, p$ and $\ell = 1, \ldots, m$. Let $\boldsymbol{y}_j^{(i)} = (y_{1j1}^{(i)}, \ldots, y_{1jp}^{(i)}, \ldots, y_{mj1}^{(i)}, \ldots, y_{mjp}^{(i)})'$ denote the vector of observations on the jth subject from the ith group. We assume that the $\boldsymbol{y}_j^{(i)}$ are independent and have a multivariate normal distribution with means and covariances given by

$$E[\boldsymbol{y}_j^{(i)}] = X_m \boldsymbol{\theta}^{(i)} : mp \times 1, \quad Var[\boldsymbol{y}_j^{(i)}] = \Sigma : mp \times mp, \tag{2.1}$$

where $X_m = (I_m \otimes X)$, X is a $p \times q$ within-subject design matrix with $\mathrm{rank}(X) = q \leq p$, $\boldsymbol{\theta}^{(i)}$ is an $mq \times 1$ parameter vector on the ith group, and $\otimes$ is the Kronecker product. The covariance matrix Σ is assumed to be unknown but positive definite. Similarly, let $Y = [\boldsymbol{y}_1^{(1)}, \cdots, \boldsymbol{y}_{N_1}^{(1)}, \ldots, \boldsymbol{y}_1^{(g+1)}, \cdots, \boldsymbol{y}_{N_{(g+1)}}^{(g+1)}]'$ denote the $N \times pm$ matrices of all observations, where N is the total sample size (i.e., $N = \sum_{i=1}^{g+1} N_i$). Then, the distribution of Y is normal with

$$E[Y] = A\boldsymbol{\Theta}' X_m', \quad Var[Y] = \Sigma \otimes I_N, \tag{2.2}$$

where A is an $N \times (g+1)$ between-subject design matrix, $\boldsymbol{\Theta} = [\boldsymbol{\theta}^{(1)}, \cdots, \boldsymbol{\theta}^{(g+1)}]$ is an $mq \times (g+1)$ parameter matrix, and $Var[Y]$ is the covariance matrix for the vector obtained from Y by using the $\mathrm{vec}(\cdot)$ operator. Hence, $Var[Y] = Var[\mathrm{vec}(Y)]$.

The model (2.2) is called a multivariate growth curve model (See, for example, Kshirsagar and Smith (1995), Nummi and Möttönen (2000)). In this model, the mean has the same form for each group. (For instance, it might be a polynomial of degree $q-1$ for each group.)

2.2. Maximum Likelihood Estimators

In this section, we discuss some maximum likelihood estimators for a multivariate growth curve model. In order to simplify the model, we consider a transformation defined by

$$\begin{aligned} Z &= [\boldsymbol{z}_1^{(1)}, \cdots, \boldsymbol{z}_{N_1}^{(1)}, \ldots, \boldsymbol{z}_1^{(g+1)}, \cdots, \boldsymbol{z}_{N_{(g+1)}}^{(g+1)}]' \\ &= [Y(I_m \otimes Q_1), Y(I_m \otimes Q_2)] = (U, V), \end{aligned}$$

where $Q = (Q_1, Q_2)$ is any $p \times p$ nonsingular matrix such that $Q_1 = X(X'X)^{-1} : p \times q$, $Q_2'X = O : p \times (p-q)$, and $Q_2'Q_2 = I_{p-q}$ (see, Siotani et al. (1985)). Then,

$$E[Z] = (A\boldsymbol{\Theta}', O), \quad Var[Z] = \Psi \otimes I_N. \tag{2.3}$$

By (2.1), the entries in each row of Z are independent and have a multivariate normal distribution with

$$E[\boldsymbol{z}_j^{(i)}] = \boldsymbol{\mu}^{(i)} = \begin{pmatrix} \boldsymbol{\theta}^{(i)} \\ \mathbf{0} \end{pmatrix} : mp \times 1, \tag{2.4}$$

$$Var[\boldsymbol{z}_j^{(i)}] = \Psi = \begin{pmatrix} \Psi_{uu} & \Psi_{uv} \\ \Psi_{vu} & \Psi_{vv} \end{pmatrix}, \quad \Psi_{uv} : mq \times m(p-q). \tag{2.5}$$

This transformation may be regarded as a representation of $\boldsymbol{y}$ as an mp-dimensional random variable $\boldsymbol{y} = (y_{11}, \ldots, y_{1p}, \ldots, y_{m1}, \ldots, y_{mp})'$, which is decomposed into an mq-dimensional main random variable $\boldsymbol{u} = (u_{11}, \ldots, u_{1q}, \ldots, u_{m1}, \ldots, u_{mq})'$ and an $m(p-q)$-dimensional covariate random variable $\boldsymbol{v} = (v_{11}, \ldots, v_{1(p-q)}, \ldots, v_{m1}, \ldots, v_{m(p-q)})'$. Moreover, the covariate $\boldsymbol{v}$ is independent of the mean parameters. From (2.4) and (2.5) , the conditional distribution of U given V and the marginal distribution of V are multivariate normal with means and covariances given by

$$E[U|V] = A^*\boldsymbol{\Xi}', \quad Var[U|V] = \Psi_{uu\cdot v} \otimes I_N, \tag{2.6}$$

$$E[V] = O, \quad Var[V] = \Psi_{vv} \otimes I_N, \tag{2.7}$$

where

$$\begin{aligned} &A^* = (A, V), \;\; \boldsymbol{\Xi} = (\boldsymbol{\Theta}, \Gamma), \;\; \Gamma = \Psi_{uv}\Psi_{vv}^{-1}, \\ &\Psi_{uu\cdot v} = \Psi_{uu} - \Psi_{uv}\Psi_{vv}^{-1}\Psi_{vu}. \end{aligned}$$

The conditional model of U given V is a multivariate linear model. The maximum likelihood estimator for the multivariate growth curve model is then obtained by applying the maximum likelihood estimator for the multivariate linear model to the model specified by (2.6) and (2.7).

$$\widehat{\boldsymbol{\Xi}}' = (A^{*\prime}A^*)^{-1}A^{*\prime}U, \quad N\widehat{\Psi}_{uu\cdot v} = U'(I_N - A^*(A^{*\prime}A^*)^{-1}A^{*\prime})U, \tag{2.8}$$

$$N\widehat{\Psi}_{vv} = V'V. \tag{2.9}$$

Let

$$nS = Y'(I_N - A(A'A)^{-1}A')Y, \tag{2.10}$$

where $n = N-(g+1)$. Then, nS has the Wishart distribution with n degrees of freedom. By the same derivation used to obtain the maximum likelihood estimator for a univariate growth curve model, the maximum likelihood estimator of $\boldsymbol{\Theta}'$ can be expressed as

$$\widehat{\boldsymbol{\Theta}}' \quad = \quad (A'A)^{-1}A'YS^{-1}X_m(X_m'S^{-1}X_m)^{-1}. \tag{2.11}$$

Similarly, the maximum likelihood estimator of $\Psi_{uu\cdot v}$ is given by

$$N\widehat{\Psi}_{uu\cdot v} \quad = \quad (X_m'S^{-1}X_m)^{-1}. \tag{2.12}$$

2.3. Likelihood Ratio Tests

We now turn our attention to the problem of testing the linear hypothesis

$$H_0 : C\boldsymbol{\Theta}' = O,$$

where C is a known $c \times (g+1)$ matrix with $\mathrm{rank}(C) = c(\leq g+1)$. Because all the information about $\boldsymbol{\Theta}$ is contained in U, and the covariate $V = (V_1, V_2)$ depends on the hypothesis through the covariance matrix, we can start with the conditional model (2.6), and the hypothesis is then expressed as

$$H_0 : C^*\boldsymbol{\Xi}' = O, \tag{2.13}$$

where $C^* = (C, O) : c \times \{g + m(p-q) + 1\}$. Let B, W and $T = W + B$ be the matrices containing the sums of squares and products of transformed between-group observations, transformed within-group observations, and the total of the transformed observation vectors $\boldsymbol{z}_j^{(i)}$, respectively. Hence

$$B = \sum_{i=1}^{g+1} N_i(\bar{\boldsymbol{z}}^{(i)} - \bar{\boldsymbol{z}})(\bar{\boldsymbol{z}}^{(i)} - \bar{\boldsymbol{z}})', \quad W = \sum_{i=1}^{g+1}\sum_{j=1}^{N_i}(\boldsymbol{z}_j^{(i)} - \bar{\boldsymbol{z}}^{(i)})(\boldsymbol{z}_j^{(i)} - \bar{\boldsymbol{z}}^{(i)})', \tag{2.14}$$

where $\bar{\boldsymbol{z}}^{(i)} = (1/N_i)\sum_{j=1}^{N_i}\boldsymbol{z}_j^{(i)}$ and $\bar{\boldsymbol{z}} = (1/N)\sum_{i=1}^{g+1}N_i\bar{\boldsymbol{z}}^{(i)}$. We partition B, W, and T in the same manner as Ψ:

$$B = \begin{pmatrix} B_{uu} & B_{uv} \\ B_{vu} & B_{vv} \end{pmatrix}, \quad W = \begin{pmatrix} W_{uu} & W_{uv} \\ W_{vu} & W_{vv} \end{pmatrix}, \quad T = \begin{pmatrix} T_{uu} & T_{uv} \\ T_{vu} & T_{vv} \end{pmatrix}.$$

Then, in the conditional model (2.6), the LR criterion for the hypothesis of equality of the vectors of growth curve coefficients for the $g+1$ groups (i.e., $\boldsymbol{\theta}^{(1)} = \cdots = \boldsymbol{\theta}^{(g+1)}$) is given by

$$\Lambda_0 = \frac{|W_{uu\cdot v}|}{|T_{uu\cdot v}|},$$

where

$$W_{uu\cdot v} = W_{uu} - W_{uv}W_{vv}^{-1}W_{vu}, \quad T_{uu\cdot v} = T_{uu} - T_{uv}T_{vv}^{-1}T_{vu}.$$

The LR criterion is obtained by reducing the conditional model (2.6) to a conditional multivariate analysis of variance (MANOVA) model. When the hypothesis is true, the statistic Λ_0 has a Wilks lambda distribution $\Lambda_{mq,g,n-m(p-q)}$.

The hypothesis of equality is thus equivalent to the linear hypothesis in which the first g columns of C constitute an $g \times g$ identity matrix, and all the entries of the last column are -1, i.e. $c = g$ and

$$C = (I_g, -\mathbf{1}_g) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

Then, the LR criterion for the linear hypothesis is the same as Λ_0 (see, Kshirsagar and Smith (1995), Tamhane (2009)).

The LR criterion for the general linear hypothesis (2.13) is obtained by the usual procedure for multivariate linear models:

$$\Lambda = \lambda^{2/n} = \frac{|S_e|}{|S_e + S_h|}, \tag{2.15}$$

where

$$S_e = U'(I_N - A^*(A^{*\prime}A^*)^{-1}A^{*\prime})U, \quad S_h = (C^*\widehat{\mathbf{\Xi}}')'\{C^*(A^{*\prime}A^*)^{-1}C^{*\prime}\}^{-1}(C^*\widehat{\mathbf{\Xi}}'),$$

are the matrices containing the respective sums of squares and products due to error and hypothesis (2.13) (see, Siotani et al. (1985)). Using maximum likelihood estimators, we can express these statistics as

$$S_e = (X_m'S^{-1}X_m)^{-1}, \tag{2.16}$$
$$S_h = (C\widehat{\mathbf{\Theta}}')'(CRC')^{-1}(C\widehat{\mathbf{\Theta}}'), \tag{2.17}$$

where

$$\begin{aligned} R &= (A'A)^{-1} + (A'A)^{-1}A'YS^{-1} \\ &\quad \times\{S - X_m(X_m'S^{-1}X_m)^{-1}X_m'\}S^{-1}Y'A(A'A)^{-1}. \end{aligned}$$

When the hypothesis (2.13) is true, the statistic (2.15) has a Wilks lambda distribution $\Lambda_{mq,c,n-m(p-q)}$, and the limiting null distribution of the following statistic corrected by a Bartlett factor:

$$-\left\{n - m(p-q) - \frac{1}{2}(mq - c + 1)\right\}\log\Lambda \tag{2.18}$$

is a chi-squared distribution with cmq degrees of freedom (See, for example, Timm (2002)). The above results are summarized in the following well-known theorem, which will be used in the next section.

THEOREM 2.1. *The LR criterion for the linear hypothesis* (2.13) *in the multivariate growth curve model is given by* (2.15)*, where* S_e, S_h *are defined by* (2.16) *and* (2.17)*. If the hypothesis is true, the statistic* (2.15) *has a Wilks lambda distribution* $\Lambda_{qm,c,n-m(p-q)}$*, and the limiting null distribution of* (2.18) *is a chi-squared distribution with cqm degrees of freedom.*

3. Tests for Redundancy

3.1. Redundancy in the case of two response variables

In this subsection, we consider the case $m = 2$, and formulate the redundancy of y_2 in the presence of y_1.

We partition the observations $\boldsymbol{y}_j^{(i)}$ as

$$\boldsymbol{y}_j^{(i)} = \begin{pmatrix} \boldsymbol{y}_{1j}^{(i)} \\ \boldsymbol{y}_{2j}^{(i)} \end{pmatrix}, \quad i = 1, \ldots, g+1; j = 1, \ldots, N_i,$$

and $\boldsymbol{\theta}^{(i)}$ and Σ in the same manner as $\boldsymbol{y}_j^{(i)}$:

$$\boldsymbol{\theta}^{(i)} = \begin{pmatrix} \boldsymbol{\theta}_1^{(i)} \\ \boldsymbol{\theta}_2^{(i)} \end{pmatrix}, \quad i = 1, \ldots, g+1, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \Sigma_{ij} : p \times p, (i, j = 1, 2),$$

where $\boldsymbol{\theta}_\ell^{(i)}$ is a $q \times 1$ parameter vector for the ℓth response variable of the ith group. Following the technique of Rao (1970), we define the redundancy of y_2 in the presence of y_1 for the multivariate growth curve model by

$$X\boldsymbol{\theta}_2^{(1)} - \Sigma_{21}\Sigma_{11}^{-1}X\boldsymbol{\theta}_1^{(1)} = \cdots = X\boldsymbol{\theta}_2^{(g+1)} - \Sigma_{21}\Sigma_{11}^{-1}X\boldsymbol{\theta}_1^{(g+1)}. \tag{3.1}$$

This formulation is equivalent to the conditional mean of $\boldsymbol{y}_2 = (y_{21}, \cdots, y_{2p})'$ given $\boldsymbol{y}_1 = (y_{11}, \cdots, y_{1p})'$ are the same for all the groups.

We will now rewrite the hypothesis of redundancy in terms of the transformed random variables z_2 in the presence of z_1 and prove that it is equivalent to hypothesis (3.1). We partition $\boldsymbol{z}_j^{(i)}$ as

$$\boldsymbol{z}_j^{(i)} = \begin{pmatrix} Q_1'\boldsymbol{y}_{1j}^{(i)} \\ Q_1'\boldsymbol{y}_{2j}^{(i)} \\ Q_2'\boldsymbol{y}_{1j}^{(i)} \\ Q_2'\boldsymbol{y}_{2j}^{(i)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_{1j}^{(i)} \\ \boldsymbol{u}_{2j}^{(i)} \\ \boldsymbol{v}_{1j}^{(i)} \\ \boldsymbol{v}_{2j}^{(i)} \end{pmatrix}, \quad i = 1, \ldots, g+1; j = 1, \ldots, N_i$$

and $\boldsymbol{\mu}^{(i)}$ and Ψ in the same manner as $\boldsymbol{z}_j^{(i)}$:

$$\boldsymbol{\mu}^{(i)} = \begin{pmatrix} \boldsymbol{\theta}_1^{(i)} \\ \boldsymbol{\theta}_2^{(i)} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad i = 1, \ldots, g+1, \quad \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{pmatrix}.$$

Let $\boldsymbol{z}_1 = (\boldsymbol{u}_1', \boldsymbol{v}_1')'$ and $\boldsymbol{z}_2 = (\boldsymbol{u}_2', \boldsymbol{v}_2')'$. Then, we can write the hypothesis for the redundancy of $\boldsymbol{z}_2$ in the presence of $\boldsymbol{z}_1$ as

$$H : \boldsymbol{\mu}_2^{(1)} - \Psi_{(24)(13)}\Psi_{(13)(13)}^{-1}\boldsymbol{\mu}_1^{(1)} = \cdots = \boldsymbol{\mu}_2^{(g+1)} - \Psi_{(24)(13)}\Psi_{(13)(13)}^{-1}\boldsymbol{\mu}_1^{(g+1)}, \tag{3.2}$$

where

$$\boldsymbol{\mu}_1^{(1)} = \begin{pmatrix} \boldsymbol{\theta}_1^{(i)} \\ \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\mu}_2^{(1)} = \begin{pmatrix} \boldsymbol{\theta}_2^{(i)} \\ \mathbf{0} \end{pmatrix}, \quad i = 1, \ldots, g+1,$$

$$\Psi_{(24)(13)} = \begin{pmatrix} \Psi_{21} & \Psi_{23} \\ \Psi_{41} & \Psi_{43} \end{pmatrix}, \quad \Psi_{(13)(13)} = \begin{pmatrix} \Psi_{11} & \Psi_{13} \\ \Psi_{31} & \Psi_{33} \end{pmatrix}.$$

This hypothesis can be expressed as

$$Q'(X\boldsymbol{\theta}_2^{(1)} - \Sigma_{21}\Sigma_{11}^{-1}X\boldsymbol{\theta}_1^{(1)}) = \cdots = Q'(X\boldsymbol{\theta}_2^{(g+1)} - \Sigma_{21}\Sigma_{11}^{-1}X\boldsymbol{\theta}_1^{(g+1)}).$$

The derivation from the hypothesis (3.2) to the above expression can be provided by using the following transformation,

$$\Psi_{(24)(13)}\Psi_{(13)(13)}^{-1}\boldsymbol{\mu}_1^{(i)} = \begin{pmatrix} \Psi_{21\cdot 3} \\ \Psi_{41\cdot 3} \end{pmatrix} \Psi_{11\cdot 3}^{-1}\boldsymbol{\theta}_1^{(i)}, \quad i = 1, \ldots, g+1$$

and

$$\begin{aligned} &\Psi_{11\cdot 3}^{-1} = X'\Sigma_{11}^{-1}X, \quad \begin{pmatrix} \Psi_{21\cdot 3} \\ \Psi_{41\cdot 3} \end{pmatrix} = Q'\Sigma_{21}\Sigma_{11}^{-1}X(X'\Sigma_{11}^{-1}X)^{-1}, \\ &\boldsymbol{\mu}_2^{(i)} = Q'X\boldsymbol{\theta}_2^{(i)}, \quad i = 1, \ldots, g+1. \end{aligned}$$

Because the matrix Q is a nonsingular matrix, the hypothesis (3.1) is equivalent to the hypothesis (3.2) after the transformation (2.3). Hence we can consider the redundancy of the subset of main variables $(\boldsymbol{u}_1', \boldsymbol{u}_2')'$ and covariates $(\boldsymbol{v}_1', \boldsymbol{v}_2')'$ as being the same as the redundancy of y_2.

Fujikoshi and Khatri (1990) discussed the redundancy of the subset of main variables and covariates in covariate discriminant analysis with multiple groups. We extend their results to ones of redundancy in the multivariate growth curve model. Specifically, they showed that the hypothesis of redundancy (3.2) can be decomposed into

$$H_1 : \boldsymbol{\theta}_2^{(1)} - \Psi_{21\cdot 34}\Psi_{11\cdot 34}^{-1}\boldsymbol{\theta}_1^{(1)} = \cdots = \boldsymbol{\theta}_2^{(g+1)} - \Psi_{21\cdot 34}\Psi_{11\cdot 34}^{-1}\boldsymbol{\theta}_1^{(g+1)}, \tag{3.3}$$

$$H_2 : \Psi_{41\cdot 3}\Psi_{11\cdot 3}^{-1}\boldsymbol{\theta}_1^{(1)} = \cdots = \Psi_{41\cdot 3}\Psi_{11\cdot 3}^{-1}\boldsymbol{\theta}_1^{(g+1)}, \tag{3.4}$$

where

$$\begin{aligned} &\Psi_{21\cdot 34} = \Psi_{21} - \Psi_{2(34)}\Psi_{(34)(34)}^{-1}\Psi_{(34)1}, \quad && \Psi_{11\cdot 34} = \Psi_{11} - \Psi_{1(34)}\Psi_{(34)(34)}^{-1}\Psi_{(34)1}, \\ &\Psi_{41\cdot 3} = \Psi_{41} - \Psi_{43}\Psi_{33}^{-1}\Psi_{31}, && \Psi_{11\cdot 3} = \Psi_{11} - \Psi_{13}\Psi_{33}^{-1}\Psi_{31}. \end{aligned}$$

The hypotheses (3.3) and (3.4), which are equivalent to the hypothesis (3.2) is equivalent to

$$\boldsymbol{\theta}_2^{(1)} - \Psi_{21\cdot 3}\Psi_{11\cdot 3}^{-1}\boldsymbol{\theta}_1^{(1)} = \cdots = \boldsymbol{\theta}_2^{(g+1)} - \Psi_{21\cdot 3}\Psi_{11\cdot 3}^{-1}\boldsymbol{\theta}_1^{(g+1)} \tag{3.5}$$

and

$$\Psi_{41\cdot 3}\Psi_{11\cdot 3}^{-1}\boldsymbol{\theta}_1^{(1)} = \cdots = \Psi_{41\cdot 3}\Psi_{11\cdot 3}^{-1}\boldsymbol{\theta}_1^{(g+1)}. \tag{3.6}$$

The equivalence of the set of (3.3) and (3.4) and the set of (3.5) and (3.6) is proved by using

$$\Psi_{21\cdot 34} = \Psi_{21\cdot 3} - \Psi_{24\cdot 3}\Psi_{44\cdot 3}^{-1}\Psi_{41\cdot 3},$$

and the following result (See, for example Schott (2005)):

$$\Psi_{11\cdot 34}^{-1} = \Psi_{11\cdot 3}^{-1} + \Psi_{11\cdot 3}^{-1}\Psi_{14\cdot 3}(\Psi_{44\cdot 3} - \Psi_{41\cdot 3}\Psi_{11\cdot 3}^{-1}\Psi_{14\cdot 3})^{-1}\Psi_{41\cdot 3}\Psi_{11\cdot 3}^{-1}.$$

3.2. Likelihood ratio test for H_1

We partition $Y, \boldsymbol{\Theta}$, and S in the same manner as $\boldsymbol{y}_j^{(i)}$:

$$Y = (Y_1, Y_2), \quad \boldsymbol{\Theta} = \begin{pmatrix} \boldsymbol{\Theta}_1 \\ \boldsymbol{\Theta}_2 \end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

Likewise, Z is partitioned in the same manner as $\boldsymbol{z}_j^{(i)}$:

$$Z = (Y_1Q_1, Y_2Q_1, Y_1Q_2, Y_2Q_2) = (U_1, U_2, V_1, V_2).$$

The LR test for the hypothesis (3.3) is provided by the conditional distribution of U_2 given (U_1, V), and this is obtained from the conditional distribution of U given V

$$U_2|(U_1, V) \sim N(A_1\boldsymbol{\Xi}_1', \Psi_{22\cdot 134} \otimes I_N), \tag{3.7}$$

where

$$\begin{aligned}
&U = (U_1, U_2), \quad V = (V_1, V_2), \quad A_1 = (A, U_1, V), \quad \boldsymbol{\Xi}_1 = (\boldsymbol{\Theta}_{2\cdot 1}, \Gamma_1, \Gamma_2), \\
&\boldsymbol{\Theta}_{2\cdot 1} = \boldsymbol{\Theta}_2 - \Gamma_1\boldsymbol{\Theta}_1, \quad \Gamma_1 = \Psi_{21\cdot 34}\Psi_{11\cdot 34}^{-1}, \quad \Gamma_2 = \Psi_{2(34)}\Psi_{(34)(34)}^{-1} - \Gamma_1\Psi_{1(34)}\Psi_{(34)(34)}^{-1}, \\
&\Psi_{22\cdot(134)} = \Psi_{22} - \Psi_{2(134)}\Psi_{(134)(134)}^{-1}\Psi_{(134)2}.
\end{aligned}$$

In this way, we obtained the LR criterion for the hypothesis of equality of the parameters $\boldsymbol{\theta}_{2\cdot 1}^{(i)} (i = 1, \ldots, g+1)$ for the $g+1$ groups, where $\boldsymbol{\theta}_{2\cdot 1}^{(i)}$ is the ith column vector of $\boldsymbol{\Theta}_{2\cdot 1}$, i.e., $\boldsymbol{\Theta}_{2\cdot 1} = (\boldsymbol{\theta}_{2\cdot 1}^{(1)}, \cdots, \boldsymbol{\theta}_{2\cdot 1}^{(g+1)})$. We partition B and W, defined by (2.14), in the same manner as Ψ:

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix}, \quad W = \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix}.$$

Because the test of (3.3) under a conditional model of U_2 given (U_1, V) is the test for a hypothesis of equality of mean parameters in a multivariate linear model, we obtain the LR criterion λ_1 given by

$$\Lambda_1 = \lambda_1^{2/n} = \frac{|W_{22\cdot 134}|}{|T_{22\cdot 134}|}, \tag{3.8}$$

where

$$\begin{aligned}
W_{22\cdot 134} &= W_{22} - W_{2(134)}W_{(134)(134)}^{-1}W_{(134)2}, \\
W_{(134)(134)} &= \begin{pmatrix} W_{11} & W_{13} & W_{14} \\ W_{31} & W_{33} & W_{34} \\ W_{41} & W_{43} & W_{44} \end{pmatrix}, \text{etc.}
\end{aligned}$$

Because (3.7) is the conditional MANOVA model, hypothesis (3.3) can be expressed as the linear hypothesis

$$H_1 : C_1\boldsymbol{\Xi}_1' = O, \tag{3.9}$$

where $C_1 = (C, O, O) : g \times (g + 2p - q + 1), C = (I_g, -\mathbf{1}_g)$ and $\mathbf{1}_g$ is a $g \times 1$ vector, all of whose components are 1. From Theorem 2.1, the LR criterion for the linear hypothesis (3.9) is expressed as

$$\Lambda_1 = \lambda_1^{2/n} = \frac{|S_{e1}|}{|S_{e1} + S_{h1}|}, \tag{3.10}$$

where

$$S_{e1} = U_2'(I_N - A_1(A_1'A_1)^{-1}A_1')U_2,$$
$$S_{h1} = (C_1\widehat{\mathbf{\Xi}}_1')'\{C_1(A_1'A_1)^{-1}C_1'\}^{-1}(C_1\widehat{\mathbf{\Xi}}_1').$$

To obtain the LR criterion using maximum likelihood estimators, we refer to the following Lemma (see, Siotani et al. (1985)):

LEMMA 3.1. *Let $G_1 : p \times q$ and $G_2 : p \times (p - q)$ have the respective ranks q and $(p - q)$, and let $G_2'G_1 = O$. Then, if S is a symmetric positive definite matrix,*

$$S^{-1} - S^{-1}G_1(G_1'S^{-1}G_1)^{-1}G_1'S^{-1} = G_2(G_2'SG_2)^{-1}G_2'.$$

We first express $A_1(A_1'A_1)^{-1}A_1'$ and $(A_1'A_1)^{-1}$ as the matrix A and the complete covariate (U_1, V), respectively. Then we decompose the partitioned matrix of U_1 and V and apply Lemma 3.1 to it. Finally, S_{e1} and S_{h1} are written in terms of maximum likelihood estimators as

$$S_{e1} = (X'S_{22\cdot 1}^{-1}X)^{-1}, \tag{3.11}$$
$$S_{h1} = \{C(\widehat{\mathbf{\Theta}}_2 - \widehat{\Gamma}_1\widehat{\mathbf{\Theta}}_1)'\}'(CR_1C')^{-1}\{C(\widehat{\mathbf{\Theta}}_2 - \widehat{\Gamma}_1\widehat{\mathbf{\Theta}}_1)'\}, \tag{3.12}$$

where (3.11) and (3.12) are the matrices containing the respective sums of squares and products due to error and hypothesis for (3.9), and

$$\begin{aligned}
\widehat{\Gamma}_1 &= (X'S_{22\cdot 1}^{-1}X)^{-1}X'S_{22\cdot 1}^{-1}S_{21}S_{11}^{-1}X, \\
R_1 &= R + (A'A)^{-1}A'YS^{-1}X_2 \\
&\quad \times \begin{pmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{21}^*S_{11}^{*-1}S_{12}^* \end{pmatrix} X_2'S^{-1}Y'A(A'A)^{-1}, \\
R &= (A'A)^{-1} + (A'A)^{-1}A'YS^{-1} \\
&\quad \times\{S - X_2(X_2'S^{-1}X_2)^{-1}X_2'\}S^{-1}Y'A(A'A)^{-1}.
\end{aligned}$$

Here $X_2 = (I_2 \otimes X)$ and

$$S^* = (X_2'S^{-1}X_2)^{-1} = \begin{pmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{pmatrix}, \quad S_{ij}^* : q \times q, (i, j = 1, 2).$$

The null distribution of the LR criterion (3.10) is a Wilks lambda distribution $\Lambda_{q,g,n-2p+q}$, which is independent of (U_1, V_1, V_2), and the limiting null distribution of the statistic corrected by a Bartlett factor:

$$-\left\{n - 2p + q - \frac{1}{2}(q - g + 1)\right\}\log\Lambda_1$$

is a chi-squared distribution with gq degrees of freedom.

3.3. Likelihood ratio test for H_2

We consider the conditional distribution of (U_1, V_2) given V_1. If the parameters are unrestricted, we can obtain maximum likelihood by using the conditional density expressed as the product of the conditional density of U_1 given V and the conditional density of V_2 given V_1. On the other hand, when the hypothesis (3.4) is true, we use the expression given by

$$\begin{aligned} f(V_2|(U_1,V_1))f(U_1|V_1) &= (2\pi)^{pN/2}|\Psi_{44\cdot 13}|^{-N/2} \\ &\quad \times \mathrm{etr}\left\{-\frac{1}{2}\Psi_{44\cdot 13}^{-1}\left[V_2-(\mathcal{B}_1,\mathcal{B}_2)\begin{pmatrix} U_1 \\ V_1\end{pmatrix}\right]'\right. \\ &\quad \left.\times\left[V_2-(\mathcal{B}_1,\mathcal{B}_2)\begin{pmatrix} U_1 \\ V_1\end{pmatrix}\right]\right\} \\ &\quad \times|\Psi_{11\cdot 3}|^{-N/2}\mathrm{etr}\left\{-\frac{1}{2}\Psi_{11\cdot 3}^{-1}\left[U_1-A\mathbf{\Theta}_1-\Psi_{13}\Psi_{33}^{-1}V_1\right]'\right. \\ &\quad \left.\times\left[U_1-A\mathbf{\Theta}_1-\Psi_{13}\Psi_{33}^{-1}V_1\right]\right\}, \end{aligned} \tag{3.13}$$

where $\mathcal{B}_1 = \Psi_{41\cdot 3}\Psi_{11\cdot 3}^{-1}, \mathcal{B}_2 = \Psi_{43}\Psi_{33}^{-1} - \mathcal{B}_1\Psi_{13}\Psi_{33}^{-1}$. Using Fujikoshi and Khatri (1990), the conditional LR test for (3.13) is based on

$$\Lambda_{(2)} = \frac{|W_{11\cdot 34}||T_{44\cdot 3}|}{|W_{11\cdot 3}||T_{44\cdot 3}|\prod_{i=q-r+1}^{q} f_i} = \frac{\prod_{i=1}^{q}(1-\rho_i^2)}{\prod_{i=q-r+1}^{q} f_i}, \tag{3.14}$$

where $r = \mathrm{rank}(\Psi_{41\cdot 3}\Psi_{11\cdot 3}^{-1})$, $f_1 \geq \cdots \geq f_{q-r} > 1 > f_{q-r+1} \geq \cdots \geq f_q > 0$ are the eigenvalues of $T_{11\cdot 34}W_{11\cdot 3}^{-1}$ and $\rho_1^2 \geq \cdots \geq \rho_q^2$ are the eigenvalues of $W_{14\cdot 3}W_{44\cdot 3}^{-1}W_{41\cdot 3}W_{11\cdot 3}^{-1}$. Because the null distribution of $\Lambda_{(2)}$ is complicated, we turn to an approximate test.

For the sake of practicality, we consider the test for the redundancy of v_2 in the presence of u_1 and v_1 only. We begin with the conditional distribution of V_2 given (U_1, V_1),

$$V_2|(U_1,V_1) \sim N(A_2\mathbf{\Xi}_2', \Psi_{44\cdot 13}\otimes I_N), \tag{3.15}$$

where

$$\begin{aligned} &A_2 = (A, U_1, V_1), \quad \mathbf{\Xi}_2 = (-\mathbf{\Theta}_1^*, \mathcal{B}_1, \mathcal{B}_2), \quad \mathbf{\Theta}_1^* = \mathcal{B}_1\mathbf{\Theta}_1, \\ &\Psi_{44\cdot 13} = \Psi_{44} - \Psi_{4(13)}\Psi_{(13)(13)}^{-1}\Psi_{(13)4}. \end{aligned}$$

Thus, we consider the problem of testing of the hypothesis of equality of the parameters $\boldsymbol{\theta}_1^{*(i)} (i = 1, \ldots, g+1)$ for the $g+1$ groups, where $\mathbf{\Theta}_1^* = (\boldsymbol{\theta}_1^{*(1)}, \cdots, \boldsymbol{\theta}_1^{*(g+1)})$. We assume that the dimensions of the parameter $\mathbf{\Theta}_1$ are not reduced by multiplying by the matrix $\mathcal{B}_1$. Then, the conditional LR criterion can be written as

$$\Lambda_2 = \lambda_2^{2/n} = \frac{|W_{44\cdot 13}|}{|T_{44\cdot 13}|}$$

instead of (3.14), where

$$\begin{aligned} &W_{44\cdot 13} = W_{44} - W_{4(13)}W_{(13)(13)}^{-1}W_{(13)4}, \\ &W_{(13)(13)} = \begin{pmatrix} W_{11} & W_{13} \\ W_{31} & W_{33}\end{pmatrix}, \text{etc.} \end{aligned}$$

Because (3.15) is the conditional MANOVA model, hypothesis (3.4) can be expressed as the linear hypothesis

$$H_2 : C_2\mathbf{\Xi}_2' = O, \tag{3.16}$$

where $C_2 = (C, O, O) : g \times (g + p + 1)$ and C is the same as in the previous section. By Theorem 2.1, the LR criterion for the linear hypothesis (3.16) is expressed as

$$\Lambda_2 = \lambda_2^{2/n} = \frac{|S_{e2}|}{|S_{e2} + S_{h2}|}, \tag{3.17}$$

where

$$S_{e2} = V_2'(I_N - A_2(A_2'A_2)^{-1}A_2')V_2,$$
$$S_{h2} = (C_2\widehat{\mathbf{\Xi}}_2')'\{C_2(A_2'A_2)^{-1}C_2'\}^{-1}(C_2\widehat{\mathbf{\Xi}}_2').$$

By again applying Lemma 3.1, with $A_2(A_2'A_2)^{-1}A_2'$ and $(A_2'A_2)^{-1}$ expressed as the matrix A and the covariate V, respectively, S_{e2} and S_{h2} can be written in terms of maximum likelihood estimators as

$$S_{e2} = Q_2'S_{22\cdot 1}Q_2, \tag{3.18}$$
$$S_{h2} = \{C(\widehat{\mathcal{B}}_1\widehat{\mathbf{\Theta}}_1)'\}'(CR_2C')^{-1}\{C(\widehat{\mathcal{B}}_1\widehat{\mathbf{\Theta}}_1)'\}, \tag{3.19}$$

where (3.18) and (3.19) are the matrices containing the respective sums of squares and products due to error and hypothesis (3.16). Moreover,

$$\begin{aligned} \widehat{\mathcal{B}}_1 &= Q_2'S_{21}S_{11}^{-1}X, \\ R_2 &= (A'A)^{-1} + (A'A)^{-1}A'Y_1S_{11}^{-1}Y_1'A(A'A)^{-1}. \end{aligned}$$

By Theorem 2.1, the null distribution of the LR criterion (3.17) is a Wilks lambda distribution $\Lambda_{p-q,g,n-p}$, and the limiting null distribution of the statistic corrected by a Bartlett factor:

$$-\left\{n - p - \frac{1}{2}(p - q - g + 1)\right\}\log\Lambda_2$$

is a chi-squared distribution with $g(p-q)$ degrees of freedom.

The joint density of (U_1, U_2, V_1, V_2) can be partitioned as follows:

$$f(U_1, U_2, V_1, V_2) = f(U_2|U_1, V_1, V_2) \cdot f(V_2|U_1, V_1) \cdot f(U_1, V_1).$$

Two statistics Λ_1 and Λ_2 are derived from the conditional density of U_2 given (U_1, V_1, V_2) and the conditional density of V_2 given (U_1, V_1), respectively. We propose $\Lambda_1 \cdot \Lambda_2$ for testing (3.3) and (3.4) as an approximate LR test. We note that $\Lambda_1 \cdot \Lambda_2$ is not an exact LR test, since the marginal of U_1 does depend on the hypothesis. The approximate test for (3.2) can thus be written in closed form. From a result concerning the distribution of a product of independent Lambda distributions (found in Siotani et al. (1985)), the distribution of

$$\Lambda^* = -gp\left(\sum_{i=1}^{2}\frac{\alpha_i\beta_i}{\gamma_i + (1/2)(\beta_i - \alpha_i - 1)}\right)^{-1}\log\prod_{i=1}^{2}\Lambda_i \tag{3.20}$$

is asymptotically a chi-squared distribution with gp degrees of freedom, where

$$\begin{array}{lll} \alpha_1 = q, & \beta_1 = g, & \gamma_1 = n-2p+q, \\ \alpha_2 = p-q, & \beta_2 = g, & \gamma_2 = n-p. \end{array}$$

3.4. Redundancy in the case of multiple response variables

In this subsection, we extend the results of the previous subsections to the case of multiple response variables. We partition $\boldsymbol{y}$ into the first m_1 response variables $\boldsymbol{y}_1$ and the $m_2 = m - m_1$ remaining response variables $\boldsymbol{y}_2$. We also partition the other parameters in accordance with the number of response variables of $\boldsymbol{y}_1, \boldsymbol{y}_2$. Then, the hypothesis for the redundancy of the remaining $m_2 = m - m_1$ response variables can be formulated in much the same way as in Subsection 3.1. Let $C_1 = (C, O, O) : g \times (g + mp - m_2 q + 1)$ and $C_2 = (C, O, O) : g \times (g + m_1 p + 1)$.

We recalculate the statistics (3.11) and (3.12) of hypothesis (3.9), and obtain

$$S_{e1} = (X'_{m_2} S^{-1}_{22\cdot 1} X_{m_2})^{-1}, \tag{3.21}$$

$$S_{h1} = \{C(\widehat{\boldsymbol{\Theta}}_2 - \widehat{\Gamma}_1 \widehat{\boldsymbol{\Theta}}_1)'\}'(CR_1C')^{-1}\{C(\widehat{\boldsymbol{\Theta}}_2 - \widehat{\Gamma}_1 \widehat{\boldsymbol{\Theta}}_1)'\}, \tag{3.22}$$

where $X_{m_1} = (I_{m_1} \otimes X), X_{m_2} = (I_{m_2} \otimes X)$,

$$\begin{aligned} \widehat{\Gamma}_1 &= (X'_{m_2} S^{-1}_{22\cdot 1} X_{m_2})^{-1} X'_{m_2} S^{-1}_{22\cdot 1} S_{21} S^{-1}_{11} X_{m_1}, \\ R_1 &= R + (A'A)^{-1} A' Y S^{-1} X_m \\ &\quad \times \begin{pmatrix} S^*_{11} & S^*_{12} \\ S^*_{21} & S^*_{21} S^{*-1}_{11} S^*_{12} \end{pmatrix} X'_m S^{-1} Y' A (A'A)^{-1}, \\ R &= (A'A)^{-1} + (A'A)^{-1} A' Y S^{-1} \\ &\quad \times \{S - X_m (X'_m S^{-1} X_m)^{-1} X'_m\} S^{-1} Y' A (A'A)^{-1}, \end{aligned}$$

where $X_m = (I_m \otimes X)$.

The null distribution of the LR criterion is a Wilks lambda distribution

$$\Lambda_{m_2 q, g, n - mp + m_2 q},$$

and the limiting null distribution of the statistic corrected by a Bartlett factor:

$$-\left\{n - mp + m_2 q - \frac{1}{2}(m_2 q - g + 1)\right\} \log \Lambda_1$$

is a chi-squared distribution with $g m_2 q$ degrees of freedom, where $n = N - (g+1)$.

Equivalently, we recalculate the statistics (3.18) and (3.19) of hypothesis (3.16) for an approximate test, and we this time get

$$S_{e2} = (I_{m_2} \otimes Q'_2) S_{22\cdot 1} (I_{m_2} \otimes Q_2), \tag{3.23}$$

$$S_{h2} = \{C(\widehat{\mathcal{B}}_1 \widehat{\boldsymbol{\Theta}}_1)'\}'(CR_2C')^{-1}\{C(\widehat{\mathcal{B}}_1 \widehat{\boldsymbol{\Theta}}_1)'\}, \tag{3.24}$$

where

$$\begin{aligned} \widehat{\mathcal{B}}_1 &= (I_{m_2} \otimes Q'_2) S_{21} S^{-1}_{11} X_{m_1}, \\ R_2 &= (A'A)^{-1} + (A'A)^{-1} A' Y_1 S^{-1}_{11} Y'_1 A (A'A)^{-1}, \end{aligned}$$

The null distribution of the LR criterion (3.17) is a Wilks lambda distribution $\Lambda_{m_2(p-q),g,n-m_1p}$, and the limiting null distribution of the statistic corrected by a Bartlett factor:

$$-\left[n - m_1p - \frac{1}{2}\{m_2(p-q) - g + 1\}\right] \log \Lambda_2$$

is a chi-squared distribution with $gm_2(p-q)$ degrees of freedom.

Using the aforementioned result for the distribution of a product of independent Lambda distributions, the distribution of

$$\Lambda^* = -gm_2p \left(\sum_{i=1}^{2} \frac{\alpha_i \beta_i}{\gamma_i + (1/2)(\beta_i - \alpha_i - 1)} \right)^{-1} \log \prod_{i=1}^{2} \Lambda_i \tag{3.25}$$

is asymptotically a chi-squared distribution with gm_2p degrees of freedom, where

$$\begin{aligned} &\alpha_1 = qm_2, &&\beta_1 = g, &&\gamma_1 = n - mp + m_2q, \\ &\alpha_2 = m_2(p-q), &&\beta_2 = g, &&\gamma_2 = n - pm_1. \end{aligned}$$

THEOREM 3.2. *The hypothesis for redundancy in a multivariate growth curve model is given by* (3.1)*, and this hypothesis is expressed in closed form by* (3.3) *and* (3.4)*. It is assumed that the dimensions of the parameter* $\boldsymbol{\Theta}_1$ *are not reduced by multiplying by the matrix* $\mathcal{B}_1$*. The conditional LR criterion for these hypotheses is given by* (3.25)*. If the hypothesis is true, the distribution of the statistic* (3.25) *is asymptotically a chi-squared distribution with* gm_2p *degrees of freedom.*

4. A numerical example

The following data set (used by Timm (1980)), was obtained in a study concerning the relative effectiveness of two orthopedic adjustments of the mandible. Nine subjects were assigned to each of two different orthopedic treatments, called activator treatments. Two variables were used to indicate the size of the mandible.

For this data set, we assume a multivariate growth curve model, in which the time trend for each group is described by a first-degree polynomial. We are testing whether the second response variable Y_2:[ANS-Me(mm)] carries no additional information (is redundant) for the comparison between the two groups in the presence of the first response variable Y_1:[Sor-Me(mm)]. The estimates are as follows:

$$\widehat{\mathbf{\Theta}}' = \begin{bmatrix} 121.4296 & 1.5203 & 64.4045 & 1.0392 \\ 126.0568 & 1.9227 & 65.9698 & 1.2001 \end{bmatrix},$$

$$S = \begin{bmatrix} 48.3160 & 47.7830 & 49.1076 & 28.5087 & 27.8594 & 30.6250 \\ 47.7830 & 48.5382 & 49.6597 & 28.5313 & 28.4531 & 31.4497 \\ 49.1076 & 49.6597 & 51.1111 & 29.2135 & 29.0937 & 32.2431 \\ 28.5087 & 28.5313 & 29.2135 & 20.7569 & 20.7656 & 22.3507 \\ 27.8594 & 28.4531 & 29.0937 & 20.7656 & 23.1250 & 23.2656 \\ 30.6250 & 31.4497 & 32.2431 & 22.3507 & 23.2656 & 25.0556 \end{bmatrix}$$

$$S_{e1} = \begin{bmatrix} 4.0179 & 0.1368 \\ 0.1368 & 0.0749 \end{bmatrix}, \quad S_{e2} = 1.0940,$$

$$S_{h1} = \begin{bmatrix} 1.2227 & 0.1850 \\ 0.1850 & 0.0280 \end{bmatrix}, \quad S_{h2} = 0.01043,$$

$$\Lambda_1 = 0.6477, \quad \Lambda_2 = 0.9906,$$

$$\begin{aligned} \alpha_1 = 2, \quad \beta_1 = 1, \quad \gamma_1 = 12, \\ \alpha_2 = 1, \quad \beta_2 = 1, \quad \gamma_2 = 13. \end{aligned}$$

$$\Lambda^* = -3\left(\sum_{i=1}^{2} \frac{\alpha_i\beta_i}{\gamma_i + (1/2)(\beta_i - \alpha_i - 1)}\right)^{-1} \log \prod_{i=1}^{2} \Lambda_i = 5.0854 < \chi_3^2(0.05)$$

Hence, the null hypothesis is not rejected at the 5% significance level, and we cannot say that the second response variable Y_2 is not redundant.

5. Conclusion

In this paper, we developed an LR test for redundancy in a comparison between multiple groups, using a multivariate growth curve model. In order to accomplish this, we transformed the original variables $\boldsymbol{y}$ to the main variables $\boldsymbol{u}$ and covariates $\boldsymbol{v}$, and verified that the hypothesis of the redundancy is independent of this transformation. Moreover, we presented some equivalent hypotheses and derived their LR criteria. However, the test statistics has a complicated null distribution. We therefore provided a conditional LR criterion which is given by a product of the LR criterion when thinking only about each hypothesis as a more practical approximate test and prove that the corrected test statistics has a chi-squared distribution.

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