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Abstract

Sato and Kawasaki (Preprint) introduced a class of n-person games called partially monotone games, and showed that any partially monotone game has a pure-strategy Nash equilibrium. Further, they proved that partial monotonicity is necessary for the existence of a pure-strategy Nash equilibrium in the case of two persons. In this paper, we present an algorithm for determining whether a two-person game belongs to the class. Our algorithm requires $O(m^2n^2)$ time, where m and n are the number of pure strategies of players 1 and 2, respectively.

Key Words and Phrases: pure strategy, Nash equilibrium, monotonicity of the best responses

1. Introduction

A Nash equilibrium is one of the most important solution concepts in non-cooperative games. Nash (1950) and Nash (1951) showed that if each player uses a mixed-strategy, any non-cooperative game has a Nash equilibrium. A pure-strategy Nash equilibrium, on the other hand, does not always exist. Therefore, some authors dealt with sufficient conditions for the existence of a pure-strategy Nash equilibrium. The first result is due to Topkis (1979). He introduced the so-called supermodular games. He first got monotonicity of the greatest and least element of each player's best response by assuming the property of increasing differences for each player's payoff function. Next, relying on Tarski's fixed point theorem, he showed the existence in supermodular games. Sato and Kawasaki (2009) introduced the so-called monotone game. They provided a discrete fixed point theorem based on monotonicity of a set-valued mapping, and as its application, showed that any monotone game has a pure-strategy Nash equilibrium. The common idea of Sato and Kawasaki (2009) and Topkis (1979) is monotonicity of the best responses.

In Sato and Kawasaki (Preprint), they introduced the so-called partially monotone game (see Definition 2.1), and showed this game has a pure-strategy Nash equilibrium (see Theorem 2.2), which is an extension of the result of Sato and Kawasaki (2009). They also showed that partial monotonicity of the best responses is necessary for the

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existence in the case of two persons. Thus, the class of partially monotone games can be regarded as a wide range of two-person games having a pure-strategy Nash equilibrium.

However, it is not efficient to check whether a game belongs to this class directly from its definition. Hence the aim of this paper is presenting an efficient algorithm for determining whether a two-person game is a partially monotone game.

In this paper, we consider the following two-person game $G = \{\{S_1, S_2\}, \{A, B\}\}$:

- $S_1 := \{1, \ldots, m\}$ is the set of pure strategies of player 1, where $m \in \mathbb{N}$.
- $S_2 := \{1, \ldots, n\}$ is the set of pure strategies of player 2, where $n \in \mathbb{N}$.
- $A = (a_{ij})$ is a payoff matrix of player 1.
- $B = (b_{ij})$ is a payoff matrix of player 2.

If we impromptu check whether a two-person game is a partially monotone game, we need $O(2^{m+n}m!n!)$ time in general. In Section 3, we prove any partially monotone game contains a small-size subgame that is also a partially monotone game. Then our algorithm requires only $O(m^2n^2)$ time in total.

This paper consists of five sections. In Section 2, we recall the class of partially monotone games and the existence theorem of a pure-strategy Nash equilibrium given by Sato and Kawasaki (Preprint). Section 3 provides a useful property of the class for computing. Section 4 develops an algorithm for determining whether a two-person game belongs to the class. Finally, Section 5 is concluding remarks.

Notation Throughout this paper, let $F_1(j)$ and $F_2(i)$ be the set of best responses of players 1 and 2, respectively, that is,

$$F_1(j) := \left\{ i' \in S_1 \, : \, a_{i'j} = \max_{i \in S_1} a_{ij} \right\} \text{ for any } j \in S_2,$$
$$F_2(i) := \left\{ j' \in S_2 \, : \, b_{ij'} = \max_{j \in S_2} b_{ij} \right\} \text{ for any } i \in S_1.$$

Further, $F(i,j) := F_1(j) \times F_2(i)$ denotes the set of best responses of $(i,j) \in S_1 \times S_2$. Then a pair (i^*, j^*) is a pure-strategy Nash equilibrium if and only if $(i^*, j^*) \in F(i^*, j^*)$.

2. Preliminaries

We first recall the definition of a partially monotone game from Sato and Kawasaki (Preprint). Although Sato and Kawasaki (Preprint) dealt with n-person games, we consider only two-person games in this paper.

Let T_k be a non-empty subset of S_k. For any permutation σ_k on T_k, we define a total order $s_k \leqq_{\sigma_k} t_k$ on T_k by $\sigma_k(s_k) \leqq \sigma_k(t_k)$. Further, $s_k <_{\sigma_k} t_k$ means $\sigma_k(s_k) < \sigma_k(t_k)$. For any $\sigma = (\sigma_1, \sigma_2)$, $s = (s_1, s_2)$ and $t = (t_1, t_2)$, $s \trianglelefteq_\sigma t$ means $s_k \leqq_{\sigma_k} t_k$ for all $k = 1, 2$. The symbol $s \preceq_\sigma t$ means $s \trianglelefteq_\sigma t$ and $s \neq t$.

DEFINITION 2.1. We call G a partially monotone game if there exist a selection f of F, non-empty subsets $T_k \subset S_k$ and permutations σ_k on T_k ($k = 1, 2$) such that at least one of T_k's has two or more elements, $f(T_1 \times T_2) \subset T_1 \times T_2$,

$$j^0 <_{\sigma_2} j^1 \;\Rightarrow\; f_1(j^0) \leqq_{\sigma_1} f_1(j^1) \text{ for any } j^0, j^1 \in T_2,$$

and

$$i^0 <_{\sigma_1} i^1 \;\Rightarrow\; f_2(i^0) \leqq_{\sigma_2} f_2(i^1) \text{ for any } i^0, i^1 \in T_1.$$

For the existence of a pure-strategy Nash equilibrium, we recall the following:

THEOREM 2.2. *Any partially monotone game has a pure-strategy Nash equilibrium in* $T := T_1 \times T_2$.

3. A key lemma on partially monotone games

We first define the term of "isolated Nash equilibrium."

DEFINITION 3.1. A Nash equilibrium (i^*, j^*) is said to be *isolated* if $(i^*, j^*) \in F(i, j)$ implies $(i, j) = (i^*, j^*)$.

Since F has a separated form $F_1 \times F_2$, isolated Nash equilibria are characterized as the lemma below. Further, the following implies that when (i^*, j^*) is an isolated Nash equilibrium, the best response operation is closed in $(S_1 \setminus \{i^*\}) \times (S_2 \setminus \{j^*\})$.

LEMMA 3.2. *A Nash equilibrium* (i^*, j^*) *is isolated if and only if both* $i^* \notin F_1(j)$ *for any* $j \neq j^*$, *and* $j^* \notin F_2(i)$ *for any* $i \neq i^*$ *hold. Then it holds that*

$$F((S_1 \setminus \{i^*\}) \times (S_2 \setminus \{j^*\})) \subset (S_1 \setminus \{i^*\}) \times (S_2 \setminus \{j^*\}).$$

PROOF. Since (i^*, j^*) is a Nash equilibrium, it holds that $i^* \in F_1(j^*)$ and $j^* \in F_2(i^*)$.

("only if" part) Suppose that $i^* \in F_1(j)$ for some $j \neq j^*$. Then $(i^*, j^*) \in F_1(j) \times F_2(i^*)$ and $(i^*, j) \neq (i^*, j^*)$, which contradicts the assumption. Hence $i^* \notin F_1(j)$ for any $j \neq j^*$. Similarly, $j^* \notin F_2(i)$ for any $i \neq i^*$. ("if" part) If $(i^*, j^*) \in F_1(j) \times F_2(i)$, then, by the assumption, we have $j = j^*$ and $i = i^*$, that is, (i^*, j^*) is isolated. The second claim is evident from the first claim. □

The next lemma will be a key in the proof of Theorem 3.5 below.

LEMMA 3.3. *Let* G *be a partially monotone game, and* T_k $(k = 1, 2)$ *be corresponding subsets of pure strategies mentioned in Definition* 2.1. *If* G *has an isolated pure-strategy Nash equilibrium in* $T = T_1 \times T_2$, *then it has another pure-strategy Nash equilibrium in* T, *which neither first nor second element coincides with that of the isolated equilibrium.*

PROOF. Let (i^*, j^*) be an isolated pure-strategy Nash equilibrium in T. Then it follows from Lemma 3.2 that $F(T') \subset T'$, where $T_1' := T_1 \setminus \{i^*\}$, $T_2' := T_2 \setminus \{j^*\}$ and $T' := T_1' \times T_2'$. Hence $\#T_1 \geq 2$ and $\#T_2 \geq 2$.

We first consider the case where $\#T_1 = \#T_2 = 2$, then T' is a single point set. Since $F(T') \subset T'$, the point is the pure-strategy Nash equilibrium we want.

Next, we consider the case where $\#T_k \geq 3$ for some $k \in \{1, 2\}$. By the definition of a partially monotone game, there exist a selection f of F and permutations σ_k on T_k $(k = 1, 2)$ such that $f(T) \subset T$,

$$j^0 <_{\sigma_2} j^1 \;\Rightarrow\; f_1(j^0) \leqq_{\sigma_1} f_1(j^1) \text{ for any } j^0, j^1 \in T_2, \tag{1}$$

and

$$i^0 <_{\sigma_1} i^1 \;\Rightarrow\; f_2(i^0) \leqq_{\sigma_2} f_2(i^1) \text{ for any } i^0, i^1 \in T_1. \tag{2}$$

Align the elements of T_1 as $i_1 < i_2 < \cdots < i_{m_1}$. Then, since i^* and $\sigma(i^*)$ belong to T_1, they can be expressed as $i^* = i_p$ and $\sigma(i^*) = i_q$ for some $1 \leq p$, $q \leq m_1$, respectively. Here we define a permutation $\hat{\sigma}_1$ on T_1' as follows.

Case 1: When $p = q$, we define $\hat{\sigma}_1(i) := \sigma_1(i)$ for any $i \in T_1'$.

Case 2: When $p > q$, we define

$$\hat{\sigma}_1(\sigma_1^{-1}(i_m)) := \begin{cases} i_{m-1}, & \text{if } q+1 \leq m \leq p \\ i_m, & \text{otherwise.} \end{cases}$$

Note that the meaning of $\hat{\sigma}_1$ is illustrated in Example 3.4 below.

Case 3: When $p < q$, we define

$$\hat{\sigma}_1(\sigma_1^{-1}(i_m)) := \begin{cases} i_{m+1}, & \text{if } p \leq m \leq q-1 \\ i_m, & \text{otherwise.} \end{cases}$$

In the same way, we define a permutation $\hat{\sigma}_2$ on T_2', and by (1) and (2), it follows that

$$j^0 <_{\hat{\sigma}_2} j^1 \;\Rightarrow\; f_1(j^0) \leqq_{\hat{\sigma}_1} f_1(j^1) \text{ for any } j^0, j^1 \in T_2',$$
$$i^0 <_{\hat{\sigma}_1} i^1 \;\Rightarrow\; f_2(i^0) \leqq_{\hat{\sigma}_2} f_2(i^1) \text{ for any } i^0, i^1 \in T_1'.$$

Then the subgame $G' = \{\{T_1', T_2'\}, \{A', B'\}\}$ is a partially monotone game. Therefore, by Theorem 2.2, there exists a pure-strategy Nash equilibrium, say, (i', j') in T'. Since $i^* \notin T_1'$ and $j^* \notin T_2'$, (i', j') is a pure-strategy Nash equilibrium we want. $\square$

EXAMPLE 3.4. Let $T_1 = \{2, 4, 6, 8, 10, 12, 14\}$, $i^* = 12$ and

$$\sigma_1 = \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 14 & 2 & 10 & 12 & 8 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 12 & 14 & 10 & 6 & 8 & 2 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 \end{pmatrix}.$$

Then $T_1' := \{2, 4, 6, 8, 10, 14\}$, $p = 6 > 2 = q$, $i_p = 12$ and $i_q = \sigma_1(i_p) = 4$. This is Case 2 in the proof of Lemma 3.3, and our goal is deleting number 12 from the rage of σ_1 by renumbering. First, since there is no $i \in T_1'$ such that $\sigma_1(i) = 4$, we define $\hat{\sigma}_1(\sigma_1^{-1}(6)) = \hat{\sigma}_1(14) := 4$. By this procedure, since there does not exist i such that $\sigma_1(i) = 6$, we define $\hat{\sigma}_1(\sigma_1^{-1}(8)) = \hat{\sigma}(10) := 6$. Repeating this procedure until we meet i such that $\sigma_1(i) = 12$, we can delete number 12 from the range of σ_1. We define $\hat{\sigma}_1 = \sigma_1$ for numbers not appearing the procedure above. Then $\hat{\sigma}$ is as follows:

$$\hat{\sigma}_1 = \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 14 \\ 14 & 2 & 8 & 10 & 6 & 4 \end{pmatrix}.$$

Any partially monotone game has the property that it contains at least one 1×2-, 2×1- or 2×2-subgame that is also a partially monotone game. This is a key property for developing our algorithm in the next section.

THEOREM 3.5. *A game G is a partially monotone game if and only if there exist a selection f of F, $R_1 \subset S_1$ and $R_2 \subset S_2$ such that one of the following holds:*

(i) *$\#R_1 = 1$, $\#R_2 = 2$ and $f(R_1 \times R_2) \subset R_1 \times R_2$;*

(ii) *$\#R_1 = 2$, $\#R_2 = 1$ and $f(R_1 \times R_2) \subset R_1 \times R_2$;*

(iii) *there exists a permutation σ_2 on R_2 such that $\#R_1 = \#R_2 = 2$, $f(R_1 \times R_2) \subset R_1 \times R_2$,*

$$j <_{\sigma_2} j' \ \Rightarrow\ f_1(j) \leqq_{id} f_1(j') \text{ for any } j, j' \in R_2,$$

and

$$i <_{id} i' \ \Rightarrow\ f_2(i) \leqq_{\sigma_2} f_2(i') \text{ for any } i, i' \in R_1.$$

PROOF. Since "if" part is evident from the definition of a partially monotone game, we only prove "only if" part. Suppose that $G = \{\{A, B\}, \{S_1, S_2\}\}$ is a partially monotone game. Then, by Theorem 2.2, there exists a pure-strategy Nash equilibrium in T, say, (i^*, j^*). Define

$$f_1(j^*) = i^*,\ f_2(i^*) = j^*.$$

Case 1: When (i^*, j^*) is isolated, by Lemma 3.3, there exists another pure-strategy Nash equilibrium $(i^\star, j^\star)$ in T such that $i^* \neq i^\star$ and $j^* \neq j^\star$. Taking $R_1 = \{i^*, i^\star\}$, $R_2 = \{j^*, j^\star\}$, $f_1(j^\star) = i^\star$ and $f_2(i^\star) = j^\star$, we have

$$\begin{aligned}
f(i^*, j^*) &:= (f_1(j^*), f_2(i^*)) = (i^*, j^*) \in F(i^*, j^*),\\
f(i^*, j^\star) &:= (f_1(j^\star), f_2(i^*)) = (i^\star, j^*) \in F(i^*, j^\star),\\
f(i^\star, j^*) &:= (f_1(j^*), f_2(i^\star)) = (i^*, j^\star) \in F(i^\star, j^*),\\
f(i^\star, j^\star) &:= (f_1(j^\star), f_2(i^\star)) = (i^\star, j^\star) \in F(i^\star, j^\star).
\end{aligned}$$

Hence f is a selection of F and $f(R_1 \times R_2) \subset R_1 \times R_2$. Moreover, by taking $\sigma_1 = id$ and

$$\sigma_2 = \begin{cases} id, & \text{if } i^* < i^\star \text{ and } j^* < j^\star, \text{ or } i^* > i^\star \text{ and } j^* > j^\star \\ (j^*, j^\star), & \text{otherwise,} \end{cases}$$

we get (iii).

Case 2: When (i^*, j^*) is not isolated, there exists $(i^\star, j^\star) \neq (i^*, j^*)$ such that $(i^*, j^*) \in F(i^\star, j^\star)$.

Case 2-1: When $i^\star \neq i^*$ and $j^\star \neq j^*$, by defining R_1, R_2, f and σ_2 in the same way as Case 1, we get (iii).

Case 2-2: When $i^\star = i^*$ and $j^\star \neq j^*$, by taking $R_1 = \{i^*\}$, $R_2 = \{j^*, j^\star\}$ and $f_1(j^\star) = i^*$, we get (i).

Case 2-3: When $i^\star \neq i^*$ and $j^\star = j^*$, we easily obtain (ii) as well as Case 2-2. □

4. An algorithm

In this section, we present an algorithm for determining if a game is a partially monotone game.

Algorithm. MAIN

Input: Payoff matrices A and B.

Output: TRUE or FALSE.

Step 0: Construct best response tables R_A and R_B as follows.

(0-1): For each $j = 1, 2, \ldots, n$, $R_A(j) := \{i' \in S_1 \colon a_{i'j} = \max_{i \in S_1} a_{ij}\}$.

(0-2): For each $i = 1, 2, \ldots, m$, $R_B(i) := \{j' \in S_2 \colon b_{ij'} = \max_{j \in S_2} b_{ij}\}$.

Step 1: For each $T_1 \subset S_1$ and $T_2 \subset S_2$ with $\#T_1 = \#T_2 = 2$, repeat the following.

(1-1): Check if $T_1 \times T_2$ is a game or not by Algorithm CHECK-GAME. Note that we describe procedure CHECK-GAME later. If it is true, then perform Step 1-2.

(1-2): Check if there exist a selection f of F and permutation σ_2 on T_2 such that the following hold:

$$j <_{\sigma_2} j' \;\Rightarrow\; f_1(j) \leqq_{id} f_1(j') \text{ for any } j, j' \in T_2,$$
$$i <_{id} i' \;\Rightarrow\; f_2(i) \leqq_{\sigma_2} f_2(i') \text{ for any } i, i' \in T_1.$$

or not by Algorithm CHECK-MONOTONE. If it is true, then output TRUE and halt.

Step 2: For each $T_1 \subset S_1$ and $T_2 \subset S_2$ with $\#T_1 = 2$ and $\#T_2 = 1$, repeat the following.

(2-1): Check if $T_1 \times T_2$ is a game or not by Algorithm CHECK-GAME. If it is true, then output TRUE and halt.

Step 3: For each $T_1 \subset S_1$ and $T_2 \subset S_2$ with $\#T_1 = 1$ and $\#T_2 = 2$, repeat the following.

(3-1): Check if $T_1 \times T_2$ is a game or not by Algorithm CHECK-GAME. If it is true, then output TRUE and halt.

Step 4: Output FALSE.

Algorithm. CHECK-GAME

Input: T_1, T_2, R_A and R_B.

Output: TRUE or FALSE.

Step 1: If for any $j \in T_2$, $R_A(j) \cap T_1 \neq \emptyset$ is satisfied, then go to Step 2. Otherwise output FALSE and halt.

Step 2: If for any $i \in T_1$, $R_B(i) \cap T_2 \neq \emptyset$ is satisfied, then go to Step 3. Otherwise output FALSE and halt.

Step 3: Output TRUE.

Algorithm. CHECK-MONOTONE

Input: $T_1 := \{t_{11}, t_{12}\}$ $(t_{11} < t_{12})$, $T_2 := \{t_{21}, t_{22}\}$ $(t_{21} < t_{22})$, R_A and R_B.

Output: TRUE or FALSE.

Step 1: (1-1) If $R_A(t_{21}) = \{t_{11}\}$ and $R_A(t_{22}) = \{t_{12}\}$, then let $\Sigma_A = \{id\}$ and go to Step 2.

(1-2) If $R_A(t_{21}) = \{t_{12}\}$ and $R_A(t_{22}) = \{t_{11}\}$, then let $\Sigma_A = \{(t_{21}, t_{22})\}$ and go to Step 2.

(1-3) Let $\Sigma_A = \{id, (t_{21}, t_{22})\}$

Step 2: (2-1) If $R_B(t_{11}) = \{t_{21}\}$ and $R_A(t_{12}) = \{t_{22}\}$, then let $\Sigma_B = \{id\}$ and go to step 3.

(2-2) If $R_A(t_{11}) = \{t_{22}\}$ and $R_A(t_{12}) = \{t_{21}\}$, then let $\Sigma_B = \{(t_{21}, t_{22})\}$ and go to step 3.

(2-3) Let $\Sigma_B = \{id, (t_{21}, t_{22})\}$

Step 3: If $\Sigma_A \cap \Sigma_B \neq \emptyset$, output TRUE. Otherwise output FALSE.

THEOREM 4.1. *This algorithm determines whether a two-person game is a partially monotone game.*

PROOF. Thanks to Theorem 3.5, it suffices to check 2×2-, 2×1- and 1×2-subgames to determine whether a two-person game is a partially monotone game. This yields our claim. □

THEOREM 4.2. *This algorithm requires $O(m^2n^2)$ time in total.*

PROOF. Since the numbers of combinations of T_1 and T_2 satisfying $\#T_1 = \#T_2 = 2$, $\#T_1 = 2$ and $\#T_2 = 1$, and $\#T_1 = 2$ and $\#T_2 = 1$ are

$$\frac{m(m-1)}{2} \times \frac{n(n-1)}{2}, \frac{m(m-1)}{2} \times n \text{ and } m \times \frac{n(n-1)}{2},$$

respectively, Algorithm MAIN requires $O(m^2n^2)$ time. Further, Algorithms CHECK-GAME and CHECK-MONOTONE require constant time. Thus, this algorithm requires $O(m^2n^2)$ time in total. □

5. Concluding remarks

At the beginning of this paper, we quoted an existence theorem of a pure-strategy Nash equilibrium based on monotonicity of the best responses. On the other hand, Iimura (2003) gave a class of games having a pure-strategy Nash equilibrium as an application of another type of discrete fixed point theorems from Iimura et al. (2005). Their theorem is based on Brouwer's fixed point theorem and relies on an integrally convex set. In Iimura (2003), he introduced direction preserving property for mappings, and presented the existence theorem by using it. However, this property is also not simple, so it is another interesting theme to create an algorithm for determining whether a game belongs to his class.

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