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MULTIRELATIONAL MODELS OF LAZY, MONODIC TREE, AND PROBABILISTIC KLEENE ALGEBRAS

By

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Abstract

This paper studies basic properties of multirelations, and then shows that classes of multirelations provides models of three weaker variants of Kleene algebras, namely, lazy, monodic tree, and probabilistic Kleene algebras. Also it is shown that these classes of up-closed multirelations need not be models of Kozen's Kleene algebras unlike the case of ordinary binary relations.

Key Words and Phrases: Multirelations, Lazy Kleene algebras, Monodic tree Kleene algebras, Probabilistic Kleene algebras

1. Introduction

A notion of Kleene algebras is introduced by Kozen (1994) as a complete axiomatisation of regular events. Three weaker variants of Kleene algebras have been independently introduced for different purposes.

- Möller (2004) has introduced lazy Kleene algebras to handle both of finite and infinite streams. A lazy Kleene algebra subsumes Dijkstra's computation calculus (cf. Dijkstra (2000)), Cohen's omega algebra (cf. Cohen (2000)) and von Wright's demonic refinement algebra (cf. von Wright (2004)).
- A notion of monodic tree Kleene algebras has been introduced by Takai and Furusawa (2006) to develop Kleene-like algebras for a class of tree languages, which is called monodic. Though, as reported by Takai and Furusawa (2008), the proof of their completeness result contains some mistakes, the set of monodic tree languages over a signature still forms a monodic tree Kleene algebra.
- McIver and Weber (2005) have introduced a notion of probabilistic Kleene algebras. Using probabilistic Kleene algebras, McIver, Cohen, and Morgan (2006) have generalised Cohen's separation theorems (cf. Cohen (2000)) for probabilistic distributed systems and the general separation results are applied to Rabin's solution (cf. Rabin (1982)) to distributed mutual exclusion with bounded waiting.

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This result shows that probabilistic Kleene algebras are useful to simplify models of probabilistic distributed systems without numerical calculations which makes difficult to analyse.

Up-closed multirelations are studied as a semantic domain of programs. They serve predicate transformer semantics with both of angelic and demonic nondeterminism in the same framework (cf. Martin, Curtis, and Rewitzky (2004), Rewitzky (2003), Rewitzky and Brink (2006)). Constructions of reflexive and transitive closure of up-closed multirelations are studied by Tsumagari, Nishizawa, and Furusawa (2008). Also up-closed multirelations provide models of game logic introduced by Parikh (1985). Pauly and Parikh (2003) have given an overview of this research area. Operations of the game logic have been studied from an algebraic point of view by Goranko (2003) and Venema (2003). They have given complete axiomatisation of iteration-free game logic.

In this paper, we give multirelational models of three weaker variants of Kleene algebras. Since these are independently introduced, they have not been investigated from the unified point of view. Our giving models may reveal both of difference and commonality of these. Though it is known that the set of (usual) binary relations on a set forms a Kleene algebra, models here need not be. Essentially, this paper is reorganising and revising results (cf. Furusawa, Tsumagari, and Nishizawa (2008), Tsumagari, Nishizawa, and Furusawa (2008)) presented at the International Conference on Relational Methods in Computer Science, Frauenwörth, Germany, 2008.

We begin in Section 2 and 3 recalling definitions of three weaker variants of Kleene algebras and study basic notions and properties of up-closed multirelations. In Section 4, we show that the set of up-closed multirelations on a set forms a lazy Kleene algebra. Though Möller (2004) has proved the fact via correspondence between up-closed multirelations and monotone predicate transformers, we prove it without using the correspondence. In the proof, right residue plays an important rôle. We also give an example which shows that the set of up-closed multirelations need not form a monodic tree Kleene algebra. We introduce the notion of finitary up-closed multirelations in the beginning of Section 5. Then we show that the set of finitary up-closed multirelations on a set forms a monodic tree Kleene algebra. Tarski's least fixed point theorem for continuous mappings is used to prove it. Assuming a notion called totality, which is introduced by Rewitzky and Brink (2006), we obtain a multirelational model of probabilistic Kleene algebras.

2. Lazy, Monodic Tree, and Probabilistic Kleene Algebra

We recall the definition of lazy Kleene algebras introduced by Möller (2004).

Definition 2.1. A *lazy Kleene algebra* is a tuple $(K, +, \cdot, {}^*, 0, 1)$ of a set K, two binary operations $+$ and $\cdot$ on K, a unary operation * on K, and $0, 1 \in K$ satisfying the following conditions:

$$0 + a = a \tag{1}$$

$$a + b = b + a \tag{2}$$

$$a + a = a \tag{3}$$

$$a + (b + c) = (a + b) + c \tag{4}$$

$$
\begin{array}{rcll}
a(bc) & = & (ab)c & (5) \\
0a & = & 0 & (6) \\
1a & = & a & (7) \\
a1 & = & a & (8) \\
ab + ac & \leq & a(b + c) & (9) \\
ac + bc & = & (a + b)c & (10) \\
1 + aa^* & \leq & a^* & (11) \\
ab \leq b & \Longrightarrow & a^*b \leq b & (12)
\end{array}
$$

for all $a, b, c \in K$, where $\cdot$ is omitted and the order $\leq$ is defined by $a \leq b$ iff $a + b = b$. □

The notion of monodic tree Kleene algebras introduced by Takai and Furusawa (2006) is as follows.

Definition 2.2. A lazy Kleene algebra $(K, +, \cdot, ^*, 0, 1)$ satisfying

$$a(b + 1) \leq a \quad \Longrightarrow \quad ab^* \leq a \tag{13}$$

for all $a, b \in K$ is called a *monodic tree Kleene algebra.* □

The notion of probabilistic Kleene algebras introduced by McIver and Weber (2005) is as follows.

Definition 2.3. A monodic tree Kleene algebra $(K, +, \cdot, ^*, 0, 1)$ satisfying

$$a0 \quad = \quad 0 \tag{14}$$

for all $a \in K$ is called a *probabilistic Kleene algebra.* □

Kozen's Kleene algebras require stronger conditions

$$ab + ac = a(b + c) \tag{9'}$$

and

$$ab \leq a \Longrightarrow ab^* \leq a \tag{13'}$$

instead of (9) and (13), respectively. Note that a probabilistic Kleene algebra satisfying (9') is a Kleene algebra in the sense of Kozen (1994).

3. Up-Closed Multirelation

In this section we recall definitions and basic properties of multirelations and their operations. More precise information on these can be obtained from Rewitzky (2003), Martin, Curtis, and Rewitzky (2004), Rewitzky and Brink (2006).

A *multirelation* over a set A is a subset of the Cartesian product $A \times \wp(A)$ of A and the power set $\wp(A)$ of A. A multirelation R is called *up-closed* if $(x, X) \in R$ and $X \subseteq Y$ imply $(x, Y) \in R$ for each $x \in A$, $X, Y \subseteq A$. The null multirelation $\emptyset$ and the universal multirelation $A \times \wp(A)$ are up-closed, and will be denoted by 0 and ∇, respectively. The set of up-closed multirelations over A will be denoted by $\mathsf{UMRel}(A)$.

For a family $\{R_i \mid i \in I\}$ of up-closed multirelations the union $\bigcup_{i\in I} R_i$ is up-closed since

$$\begin{array}{lll} & (x,X) \in \bigcup_{i\in I} R_i \text{ and } X \subseteq Y & \\ \iff & \exists i \in I.\,(x,X) \in R_i \text{ and } X \subseteq Y & \\ \implies & \exists i \in I.\,(x,Y) \in R_i & (R_i \text{ is up-closed}) \\ \iff & (x,Y) \in \bigcup_{i\in I} R_i \ . & \end{array}$$

So $\mathsf{UMRel}(A)$ is closed under arbitrary union $\bigcup$. Then it is immediate that a tuple $(\mathsf{UMRel}(A), \bigcup)$ is a sup-semilattice equipped with the least element 0 with respect to the inclusion ordering $\subseteq$.

REMARK. $\mathsf{UMRel}(A)$ is also closed under arbitrary intersection $\bigcap$. So, $\mathsf{UMRel}(A)$ forms a complete lattice together with the union and the intersection. $\square$

$R + S$ denotes $R \cup S$ for a pair of up-closed multirelations R and S. Then the following holds.

PROPOSITION 3.1. *A tuple* $(\mathsf{UMRel}(A), +, 0)$ *satisfies conditions* (1), (2), (3), *and* (4) *in* Definition 2.1. $\square$

For a pair of multirelations $R, S \subseteq A \times \wp(A)$ the composition $R; S$ is defined by

$$(x,X) \in R;S \ \text{ iff } \ \exists Y \subseteq A.((x,Y) \in R \text{ and } \forall y \in Y.(y,X) \in S) \ .$$

It is immediate from the definition that one of the zero laws

$$0 = 0; R$$

is satisfied. The other zero law

$$R; 0 = 0$$

need not hold.

EXAMPLE 3.2. Consider the universal multirelation ∇ on a singleton set $\{x\}$. Then, since $(x, \emptyset) \in \nabla$, $\nabla; 0 = \nabla \neq 0$. $\square$

Also the composition ; preserves the inclusion ordering $\subseteq$, that is,

$$P \subseteq P' \text{ and } R \subseteq R' \implies P; R \subseteq P'; R'$$

since

$$\begin{array}{rcl} (x,X) \in P;R & \iff & \exists Y \subseteq A.((x,Y) \in P \text{ and } \forall y \in Y.(y,X) \in R) \\ & \implies & \exists Y \subseteq A.((x,Y) \in P' \text{ and } \forall y \in Y.(y,X) \in R') \\ & \iff & (x,X) \in P'; R' \ . \end{array}$$

If R and S are up-closed, so is the composition $R; S$ since

$$\begin{array}{lll} & (x,X) \in R;S \text{ and } X \subseteq Z & \\ \implies & \exists Y \subseteq A.((x,Y) \in R \text{ and } \forall y \in Y.(y,Z) \in S) & (S \text{ is up-closed}) \\ \iff & (x,Z) \in R;S \ . & \end{array}$$

In other words, the set $\mathsf{UMRel}(A)$ is closed under the composition ;.

LEMMA 3.3. *Up-closed multirelations are associative under the composition* ;.

PROOF. Let P, Q, and R be up-closed multirelations over a set A. We prove $(P;Q);R \subseteq P;(Q;R)$.

$$\begin{array}{ll}
 & (x,X) \in (P;Q);R \\
\iff & \exists Y \subseteq A.((x,Y) \in P;Q \text{ and } \forall y \in Y.(y,X) \in R) \\
\iff & \exists Y \subseteq A.(\exists Z \subseteq A.((x,Z) \in P \text{ and } \forall z \in Z.(z,Y) \in Q) \text{ and } \\
 & \qquad \forall y \in Y.(y,X) \in R) \\
\implies & \exists Z \subseteq A.((x,Z) \in P \text{ and } \\
 & \qquad \forall z \in Z.\exists Y \subseteq A.((z,Y) \in Q \text{ and } \forall y \in Y.(y,X) \in R)) \\
\iff & \exists Z \subseteq A.((x,Z) \in P \text{ and } \forall z \in Z.(z,X) \in Q;R) \\
\iff & (x,X) \in P;(Q;R) \ .
\end{array}$$

For $P;(Q;R) \subseteq (P;Q);R$ it is sufficient to show

$$\begin{array}{ll}
 & \exists Z \subseteq A.((x,Z) \in P \text{ and } \\
 & \qquad \forall z \in Z.\exists Y \subseteq A.((z,Y) \in Q \text{ and } \forall y \in Y.(y,X) \in R)) \\
\implies & \exists Y \subseteq A.(\exists Z \subseteq A.((x,Z) \in P \text{ and } \forall z \in Z.(z,Y) \in Q) \text{ and } \\
 & \qquad \forall y \in Y.(y,X) \in R) \ .
\end{array}$$

Suppose that there exists a set Z such that

$$(x,Z) \in P \text{ and } \forall z \in Z.\exists Y \subseteq A.((z,Y) \in Q \text{ and } \forall y \in Y.(y,X) \in R) \ .$$

If Z is empty, it is obvious since we can take the empty set as Y. Otherwise, take a set Y_z satisfying

$$(z,Y_z) \in Q \text{ and } \forall y \in Y_z.(y,X) \in R$$

for each $z \in Z$. Then set $Y_0 = \bigcup_{z \in Z} Y_z$. Since Q is up-closed, $(z,Y_0) \in Q$ for each z. Also $(y,X) \in R$ for each $y \in Y_0$ by the definition of Y_0. Thus Y_0 satisfies

$$\exists Z \subseteq A.((x,Z) \in P \text{ and } \forall z \in Z.(z,Y_0) \in Q) \text{ and } \forall y \in Y_0.(y,X) \in R \ .$$

□

We used the fact that Q is up-closed to show $P;(Q;R) \subseteq (P;Q);R$. Multirelations need not be associative under the composition.

EXAMPLE 3.4. Consider multirelations

$$\begin{array}{rcl}
R & = & \{(x,\{x,y,z\}),(y,\{x,y,z\}),(z,\{x,y,z\})\} \text{ and} \\
Q & = & \{(x,\{y,z\}),(y,\{x,z\}),(z,\{x,y\})\}
\end{array}$$

on a set $\{x,y,z\}$. Here, R is up-closed but Q is not. Since $R;Q = 0$, $(R;Q);R = 0$. On the other hand, $R;(Q;R) = R$ since $Q;R = R$ and $R;R = R$. Therefore

$$(R;Q);R \subseteq R;(Q;R)$$

but

$$R;(Q;R) \not\subseteq (R;Q);R \ .$$

Replacing Q with an up-closed multirelation Q' defined by $Q' = Q + R$,

$$R;(Q';R) = (R;Q');R$$

holds since $Q';R = R = R;Q'$. □

The *identity* $1 \in \mathsf{UMRel}(A)$ is defined by

$$(x, X) \in 1 \text{ iff } x \in X \ .$$

LEMMA 3.5. *The identity satisfies the unit laws, that is,*

$$1; R = R \text{ and } R; 1 = R$$

for each $R \in \mathsf{UMRel}(A)$.

PROOF. First, we prove $1; R \subseteq R$.

$$\begin{array}{rcl} (x, X) \in 1; R & \iff & \exists Y \subseteq A.((x, Y) \in 1 \text{ and } \forall y \in Y.(y, X) \in R) \\ & \iff & \exists Y \subseteq A.(x \in Y \text{ and } \forall y \in Y.(y, X) \in R) \\ & \implies & (x, X) \in R \ . \end{array}$$

Conversely, if $(x, X) \in R$, then $(x, X) \in 1; R$ since $(x, \{x\}) \in 1$. Next, we prove $R; 1 \subseteq R$.

$$\begin{array}{rcl} (x, X) \in R; 1 & \iff & \exists Y \subseteq A.((x, Y) \in R \text{ and } \forall y \in Y.(y, X) \in 1) \\ & \iff & \exists Y \subseteq A.((x, Y) \in R \text{ and } \forall y \in Y.y \in X) \\ & \iff & \exists Y \subseteq A.((x, Y) \in R \text{ and } Y \subseteq X) \\ & \implies & (x, X) \in R \end{array}$$

since R is up-closed. Conversely, if $(x, X) \in R$, then $(x, X) \in R; 1$ since, by the definition of 1, $(y, X) \in 1$ for each $y \in X$. □

Therefore the following property holds.

PROPOSITION 3.6. *A tuple* $(\mathsf{UMRel}(A), ;, 0, 1)$ *satisfies conditions* (5)*,* (6)*,* (7)*, and* (8) *in* Definition 2.1. □

As Example 3.2 has shown, the condition (14) need not be satisfied. We discuss about this condition in Section 6.

Since the composition ; preserves the inclusion ordering $\subseteq$, we have

$$\bigcup_{i \in I} R; S_i \subseteq R; (\bigcup_{i \in I} S_i)$$

for each up-closed multirelation R and a family $\{S_i \mid i \in I\}$. Also

$$\bigcup_{i \in I} R_i; S = (\bigcup_{i \in I} R_i); S$$

holds for each up-closed multirelation S and a family $\{R_i \mid i \in I\}$ since

$$\begin{array}{rcl} (x, X) \in \bigcup_{i \in I} R_i; S & \iff & \exists k.((x, X) \in R_k; S) \\ & \iff & \exists k.(\exists Y \subseteq A.((x, Y) \in R_k \text{ and } \forall y \in Y.(y, X) \in S)) \\ & \iff & \exists Y \subseteq A.(\exists k.((x, Y) \in R_k \text{ and } \forall y \in Y.(y, X) \in S)) \\ & \iff & \exists Y \subseteq A.((x, Y) \in \bigcup_{i \in I} R_i \text{ and } \forall y \in Y.(y, X) \in S)) \\ & \iff & (x, X) \in (\bigcup_{i \in I} R_i); S \ . \end{array}$$

PROPOSITION 3.7. *A tuple* $(\mathsf{UMRel}(A), +, ;)$ *satisfies conditions* (9) *and* (10) *in* Definition 2.1. $\square$

We give an example showing that the equation (9') need not hold in $\mathsf{UMRel}(A)$.

EXAMPLE 3.8. Consider the up-closed multirelation

$$R = \{(x, W) \mid z \in W\} \cup \{(y, W) \mid \{x, z\} \subseteq W\} \cup \{(z, W) \mid \{x, z\} \subseteq W\}$$

on a set $\{x, y, z\}$. Clearly, this R is up-closed. Then, $R; (1 + R) \not\subseteq R; 1 + R; R$ since $(y, \{z\}) \notin R; 1 + R; R$ though $(y, \{z\}) \in R; (1 + R)$. $\square$

4. Multirelational Model of Lazy Kleene Algebra

For $R \in \mathsf{UMRel}(A)$, a mapping $\varphi_R \colon \mathsf{UMRel}(A) \to \mathsf{UMRel}(A)$ is defined by

$$\varphi_R(\xi) = R; \xi + 1 \ .$$

Since $(\mathsf{UMRel}(A), \cup, \cap)$ is a complete lattice and the mapping φ_R preserves the ordering $\subseteq$, φ_R has the least fixed point, given by $\bigcap\{\xi \mid \varphi_R(\xi) \subseteq \xi\}$.

For an up-closed multirelation R we define R^* as

$$R^* = \bigcap\{\xi \mid \varphi_R(\xi) \subseteq \xi\} \ .$$

Then the following (15) and (16) hold since R^* is the least fixed point of φ_R.

$$1 + R; R^* \subseteq R^* \tag{15}$$

$$1 + R; P \subseteq P \implies R^* \subseteq P \tag{16}$$

Thus, we have already shown the following proposition.

PROPOSITION 4.1. *A tuple* $(\mathsf{UMRel}(A), +, ;, ^*, 0, 1)$ *satisfies the condition* (11) *in* Definition 2.1. $\square$

For $P, Q \in \mathsf{UMRel}(A)$ we define P/Q as

$$P/Q = \bigcup\{\xi \mid \xi; Q \subseteq P\} \ .$$

LEMMA 4.2. *For* $P, Q, R \in \mathsf{UMRel}(A)$ *it holds that*

$$R \subseteq P/Q \iff R; Q \subseteq P \ .$$

PROOF. Suppose that $R \subseteq P/Q$. By the left distributivity we have

$$\begin{aligned} R; Q \subseteq (P/Q); Q &= \bigcup\{\xi \mid \xi; Q \subseteq P\}; Q \\ &= \bigcup\{\xi; Q \mid \xi; Q \subseteq P\} \\ &\subseteq P \ . \end{aligned}$$

Conversely, suppose that $R; Q \subseteq P$. Since $R \in \{\xi \mid \xi; Q \subseteq P\}$, $R \subseteq P/Q$ holds. $\square$

PROPOSITION 4.3. *For* $P, R \in \mathsf{UMRel}(A)$ *it holds that*

$$R; P \subseteq P \implies R^* P \subseteq P \ .$$

Proof. Suppose that $R; P \subseteq P$. Then we have

$$\begin{array}{rcl} (1 + R; (P/P)); P & = & P + R; (P/P); P \\ & \subseteq & P + R; P \\ & \subseteq & P \end{array}$$

since $(P/P); P \subseteq P$. So $1 + R; (P/P) \subseteq (P/P)$ holds. By (16) we have $R^* \subseteq P/P$. Therefore $R^*; P \subseteq P$ holds. □

Theorem 4.4. *A tuple* $(\mathsf{UMRel}(A), +, ;, ^*, 0, 1)$ *is a lazy Kleene algebra.* □

$(\mathsf{UMRel}(A), +, ;, ^*, 0, 1)$ need not satisfy the condition (13).

Example 4.5. Consider up-closed multirelations

$$P = \{(n, X) \mid X \text{ is infinite}\} \text{ and}$$
$$R = \{(0, \emptyset)\} \cup \{(n, X) \mid \exists m \in X. n \leq m + 1\}$$

over the set $\mathbf{N}$ of natural numbers. It can be proved that $\varphi_R(\xi) = R; \xi + 1 \subseteq \xi$ implies $\forall m \in \mathbf{N}.(m, \{0\}) \in \xi$ by induction on m. So, $\forall m \in \mathbf{N}.(m, \{0\}) \in R^*$ holds since R^* is the least fixed point of φ_R. Moreover, $(n, \mathbf{N}) \in P$ holds for a natural number n. Therefore, $(n, \{0\}) \in P; R^*$ holds. Since $(n, \{0\}) \notin P$, we have $P; R^* \not\subseteq P$. However, $P; (R + 1) \subseteq P$ holds. □

Therefore $(\mathsf{UMRel}(A), +, ;, ^*, 0, 1)$ need not be a monodic tree Kleene algebra.

5. Multirelational Model of Monodic Tree Kleene Algebra

For monodic tree Kleene algebras, we consider a subclass of up-closed multirelations.

Definition 5.1. An up-closed multirelation R is called *finitary* if $(x, Y) \in R$ implies that there exists a finite set Z such that $Z \subseteq Y$ and $(x, Z) \in R$. □

Clearly all up-closed multirelations over a finite set are finitary. The set of finitary up-closed multirelations over a set A will be denoted by $\mathsf{UMRel}_f(A)$.

Remark. An up-closed multirelation R is called *disjunctive* (cf. Pauly and Parikh (2003)) or *angelic* (cf. Martin, Curtis, and Rewitzky (2004)) if, for each $x \in A$ and each $V \subseteq \wp(A)$,

$$(x, \bigcup V) \in R \quad \text{iff} \quad \exists Y \in V.(x, Y) \in R \ .$$

Let R be disjunctive and $(x, X) \in R$. And let V be the set of finite subsets of X. Then $\bigcup V = X$. By disjunctivity, there exists $Y \in V$ such that $(x, Y) \in R$. Also Y is finite by the definition of V. Therefore disjunctive up-closed multirelations are finitary. However, finitary up-closed multirelations need not be disjunctive. Consider a finitary up-closed multirelation $R = \{(x, \{x, y\})\}$ on a set $\{x, y\}$. Then $\bigcup\{\{x\}, \{y\}\} = \{x, y\}$ and $(x, \{x, y\}) \in R$ but $(x, \{x\}), (x, \{y\}) \notin R$. □

It is obvious that $0, 1 \in \mathsf{UMRel}_f(A)$. Also the set $\mathsf{UMRel}_f(A)$ is closed under arbitrary union $\bigcup$.

PROPOSITION 5.2. *The set* $\mathsf{UMRel}_f(A)$ *is closed under the composition* ;.

PROOF. Let P and R be finitary up-closed multirelations. Suppose $(x, X) \in P; R$. Then, by the definition of the composition, there exists $Y \subseteq A$ such that

$$(x, Y) \in P \text{ and } \forall y \in Y.(y, X) \in R \ .$$

Since P is finitary, there exists a finite set $Y_0 \subseteq Y$ such that

$$(x, Y_0) \in P \text{ and } \forall y \in Y_0.(y, X) \in R \ .$$

Also, since R is finitary, there exists a finite set $X_y \subseteq X$ such that $(y, X_y) \in R$ for each $y \in Y_0$. Then the set $\bigcup_{y \in Y_0} X_y$ is a finite subset of X such that

$$(x, \bigcup_{y \in Y_0} X_y) \in P; R$$

since $(y, \bigcup_{y \in Y_0} X_y) \in R$ for each $y \in Y_0$. Therefore $P; R$ is finitary. □

Thus, if R and ξ are finitary, then so is $\varphi_R(\xi)$.

The set $\mathsf{UMRel}_f(A)$ need not be closed under arbitrary intersection $\bigcap$.

EXAMPLE 5.3. For each natural number i, consider the finitary up-closed multirelation $R_i = \{(1, X) \mid i \in X\}$ over the set $\mathbf{N}$ of natural numbers. Then, $\bigcap\{R_i \mid i \in \mathbf{N}\}$ is not finitary since $\bigcap\{R_i \mid i \in \mathbf{N}\} = \{(1, \mathbf{N})\}$. □

For a family $\{P_i \mid i \in I\}$ of $P_i \in \mathsf{UMRel}_f(A)$ we define that

$$\bigwedge\{P_i \mid i \in I\} = \bigcup\{R \in \mathsf{UMRel}_f(A) \mid \forall i \in I.R \subseteq P_i\} \ .$$

Then, in a poset $(\mathsf{UMRel}_f(A), \subseteq)$, $\bigwedge\{P_i \mid i \in I\}$ is the greatest lower bound of a family $\{P_i \mid i \in I\}$.

For a finitary up-closed multirelation R we define R^* as

$$R^* = \bigwedge\{\xi \mid \varphi_R(\xi) \subseteq \xi\} \ .$$

Then, as the case of $\mathsf{UMRel}(A)$ in the last section, it may be shown that a tuple $(\mathsf{UMRel}_f(A), +, ;, ^*, 0, 1)$ is a Lazy Kleene algebra.

Moreover, for a finitary up-closed multirelation R, we obtain bottom-up construction of R^*. Proving the fact, we use the following lemma.

LEMMA 5.4. *Let* $\mathcal{D}$ *be a directed subset of* $\mathsf{UMRel}_f(A)$ *and let* $R \in \mathsf{UMRel}_f(A)$. *Then it holds that*

$$R; (\bigcup \mathcal{D}) = \bigcup\{R; P \mid P \in \mathcal{D}\}.$$

PROOF. $\bigcup\{R; P \mid P \in \mathcal{D}\} \subseteq R; (\bigcup \mathcal{D})$ holds by the monotonicity of composition. Suppose $(x, X) \in R; (\bigcup \mathcal{D})$. Then, by the definition of composition, there exists $Y \subseteq A$ such that

$$(x, Y) \in R \text{ and } \forall y \in Y.(y, X) \in \bigcup \mathcal{D} \ .$$

Since R is finitary, there exists a finite set $Y_0 \subseteq Y$ such that

$$(x, Y_0) \in R \text{ and } \forall y \in Y_0.(y, X) \in \bigcup \mathcal{D} \ .$$

Thus there exists $P_y \in \mathcal{D}$ such that $(y, X) \in P_y$ for each $y \in Y_0$. Since $\mathcal{D}$ is directed and Y_0 is finite, there exists $P_0 \in \mathcal{D}$ such that $P_y \subseteq P_0$ for each $y \in Y_0$. Therefore $(x, X) \in R; P_0$, and then $(x, X) \in \bigcup\{R; P \mid P \in \mathcal{D}\}$. □

PROPOSITION 5.5. *Let R be a finitary up-closed multirelation. Then*

$$R^* = \bigcup_{n \geq 0} \varphi_R^n(0) \ ,$$

where φ_R^0 is the identity mapping and $\varphi_R^{n+1} = \varphi_R \circ \varphi_R^n$.

PROOF. Since R^* is the least fixed point of φ_R, it is sufficient to show that φ_R is continuous, that is,

$$\bigcup\{\varphi_R(P) \mid P \in \mathcal{D}\} = \varphi_R(\bigcup \mathcal{D})$$

for each directed subset $\mathcal{D}$ of $\mathsf{UMRel}_f(A)$. $\bigcup\{\varphi_R(P) \mid P \in \mathcal{D}\} \subseteq \varphi_R(\bigcup \mathcal{D})$ holds by the monotonicity of φ_R. On the other hand, it is obvious that

$$\begin{array}{l} 1 \subseteq \bigcup\{\varphi_R(P) \mid P \in \mathcal{D}\} \text{ and} \\ \bigcup\{R; P \mid P \in \mathcal{D}\} \subseteq \bigcup\{\varphi_R(P) \mid P \in \mathcal{D}\} \end{array}$$

by the definition of φ_R. Also, $R; (\bigcup \mathcal{D}) = \bigcup\{R; P \mid P \in \mathcal{D}\}$ holds by Lemma 5.4. Therefore it holds that $\varphi_R(\bigcup \mathcal{D}) \subseteq \bigcup\{\varphi_R(P) \mid P \in \mathcal{D}\}$. □

REMARK. The bottom-up construction does not work in the case of $\mathsf{UMRel}(A)$. Let $\mathbf{N}$ be the set of natural numbers and let ω satisfy

$$\forall n \in \mathbf{N}.n < \omega \ .$$

Now consider an up-closed multirelation

$$R = \{(x, X) \mid y < x \implies y \in X\}$$

over $\mathbf{N} \cup \{\omega\}$, which is not finitary. Then $\bigcup_{n \geq 0} \varphi_R^n(0)$ is not a fixed point of φ_R since $(\omega, \emptyset) \notin \bigcup_{n \geq 0} \varphi_R^n(0)$ and $(\omega, \emptyset) \in \varphi_R(\bigcup_{n \geq 0} \varphi_R^n(0))$. □

A condition related to the operator * is left to check.

PROPOSITION 5.6. *Let $P, R \in \mathsf{UMRel}_f(A)$. Then the following implication holds.*

$$P; (R + 1) \subseteq P \implies P; R^* \subseteq P$$

PROOF. It will be follow from $P; \varphi_R^n(0) \subseteq P$ since

$$P; R^* \subseteq P; (\bigcup_{n \geq 0} \varphi_R^n(0)) = \bigcup_{n \geq 0} P; \varphi_R^n(0) \subseteq P$$

by Lemma 5.4. Supposing that $P;(R+1) \subseteq P$, we show that $P;\varphi_R^n(0) \subseteq P$ by induction on n. For $n=0$ it holds since φ_R^0 is the identity. For $n=1$

$$P;\varphi_R(0) = P;(R;0+1) \subseteq P;(R+1) \subseteq P \ .$$

Assume that $P;\varphi_R^n(0) \subseteq P$ for $n \geq 1$. Then we have

$$\begin{array}{rcl} P;\varphi_R^{n+1}(0) & = & P;(R;\varphi_R^n(0)+1) \\ & \subseteq & P;(R;\varphi_R^n(0)+\varphi_R^n(0)) \\ & = & P;(R+1);\varphi_R^n(0) \\ & \subseteq & P;\varphi_R^n(0) \\ & \subseteq & P \end{array}$$

since $1 \subseteq R;\varphi_R^{n-1}(0)+1 = \varphi_R^n(0)$ for $n \geq 1$. □

Remark. Kozen's Kleene algebras requires the condition (13')

$$ab \leq a \implies ab^* \leq a$$

instead of (13). The following example shows that the condition (13') need not hold for finitary up-closed multirelations. Consider the up-closed multirelation R appeared in Example 3.8. Then $R;R \subseteq R$ since

$$R;R = \{(w,W) \mid w \in \{x,y,z\},\ \{x,z\} \subseteq W\} \subseteq R \ .$$

Also, we have already seen that $(y,\{z\}) \in R;(R+1)$ in Example 3.8. Since

$$R;(R+1) \subseteq R;\varphi_R^2(0) \subseteq R;(\bigcup_{n\geq 0} \varphi_R^n(0)) = R;R^* \ ,$$

$(y,\{z\}) \in R;R^*$. But $(y,\{z\}) \notin R$. So, $R;R^* \not\subseteq R$ in spite of $R;R \subseteq R$. □

We have already shown the following.

Theorem 5.7. *A tuple* $(\mathsf{UMRel}_f(A),+,;,^*,0,1)$ *is a monodic tree Kleene algebras.* □

Example 3.2 shows that $(\mathsf{UMRel}_f(A),+,;,^*,0,1)$ need not be a probabilistic Kleene algebra.

6. Multirelational Model of Probabilistic Kleene Algebra

It has been shown by Rewitzky and Brink (2006) that the following notion ensures the right zero law.

Definition 6.1. A multirelation R on a set A is called *total* if $(x,\emptyset) \notin R$ for each $x \in A$. □

Clearly, the null multirelation 0 and the identity 1 are total.

The set of finitary total up-closed multirelations will be denoted by $\mathsf{UMRel}_f^+(A)$. Then $\mathsf{UMRel}_f^+(A)$ is closed under $\bigcup$, $\bigwedge$, ;, and *.

Theorem 6.2. *A tuple* $(\mathsf{UMRel}_f^+(A),+,;,^*,0,1)$ *is a probabilistic Kleene algebra.* □

$(\mathsf{UMRel}_f^+(A),+,;,^*,0,1)$ need not be a Kozen's Kleene algebra. It is induced from either Example 3.8 or the last remark in which we consider only finitary total up-closed multirelations.

Table 1: Summary

	$\mathsf{UMRel}(A)$	$\mathsf{UMRel}_f(A)$	$\mathsf{UMRel}_f^+(A)$
lazy KA?	○	○	○
monodic tree KA?	×	○	○
probabilistic KA?	×	×	○
KA?	×	×	×

○ : Yes
× : Not always

7. Conclusion

This paper has studied up-closed multirelations carefully. Then we have shown that classes of up-closed multirelations provides models of three weaker variants of Kleene algebras:

- the set $\mathsf{UMRel}(A)$ of up-closed multirelations forms a lazy Kleene algebra,
- the set $\mathsf{UMRel}_f(A)$ of finitary up-closed multirelations forms a monodic tree Kleene algebra,
- and the set $\mathsf{UMRel}_f^+(A)$ of finitary total up-closed multirelations forms a probabilistic Kleene algebra.

Also we have shown that

- (13) need not hold in $\mathsf{UMRel}(A)$ and
- (14) need not hold in $\mathsf{UMRel}(A)$ nor $\mathsf{UMRel}_f(A)$.

Table 1 summarises the results of this paper.

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