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Certain automorphism groups of pro-l fundamental groups of punctured Riemann surfaces[*)]

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Introduction.

In this paper we present some results on certain automorphism groups of pro-l fundamental groups of punctured Riemann surfaces.

Let l be a prime, $g\geq 1$, $r\geq 0$ be integers and $G=G_{g,r}$ be the pro-l completion of the fundamental group of Riemann surface of genus g with r-points deleted. Assume $r\geq 1$. Then G is a pro-l free group of rank $2g+r-1$ having a standard presentation:

$$G=\left\langle \begin{matrix} x_1, x_2, \cdots, x_{2g} \\ z_1, \cdots, z_r \end{matrix} \middle| [x_1, x_{g+1}][x_2, x_{g+2}]\cdots[x_g, x_{2g}]z_1\cdots z_r=1 \right\rangle_{\text{pro-}l}.$$

We give G a central filtration $\{G(m)\}_{m\geq 1}$ such that the elements $x_1, \cdots, x_{2g}$ are of degree 1, the elements $z_1, \cdots, z_r$ are of degree 2 and generally the degree of a commutator $[x, y]$ is the sum of degrees of x and y. For this filtration let $grG=\bigoplus_{m\geq 1} G(m)/G(m+1)$. Then, by a standard method, grG turns out to be a free Lie algebra generated by the classes of $x_1, \cdots, x_{2g}, z_1, \cdots, z_{r-1}$. By using this we first establish a "successive approximation lemma" to construct automorphisms of G. Then we study some basic properties of the subgroup

$$\tilde{\Gamma}=\tilde{\Gamma}_{g,r}=\{\sigma\in \mathrm{Aut}\, G \mid z_j^{\sigma}\sim z_j^{\alpha_j},\ \exists\alpha_j\in Z_l^{\times},\ 1\leq j\leq r\}$$

of the automorphism group of G. Such type of groups arise naturally in the context of "large Galois representations" (cf. [2] [3]). These studies are viewed as a continuation of our previous study [2] in which we treated exclusively the case of $r=0$. A new ingredient is the filtration of G explained above. The author owes the idea of introducing such filtration to study the group $\tilde{\Gamma}$ to Professor Takayuki Oda. It seems

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*) This paper is a part of the author's doctoral dissertation submitted to the University of Tokyo (1988).

that the lower central series used in [2] does not work well to study $\tilde{\Gamma}$ when $r \geq 2$.

As a consequence of these studies, we establish the following

THEOREM. *Suppose $r > s \geq 0$. The naturally induced homomorphism*

$$\tilde{\Gamma}_{g,r} \longrightarrow \tilde{\Gamma}_{g,s}$$

is surjective.

This is a pro-l analogue of the classical theorem of Dehn-Nielsen. We can derive from this a result on conjugacy classes of $\tilde{\Gamma}_{g,r}/\mathrm{Int}\, G$ as in [2].

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1. Filtration of the fundamental group and Lie algebra.

We fix a prime number l throughout the paper. Let g and r be two integers greater than or equal to 1. We denote by $G_{g,r}$ the pro-l completion of the fundamental group of r-punctured Riemann surface of genus g,

$$G_{g,r} = \left\langle \begin{matrix} x_1, x_2, \cdots, x_{2g} \\ z_1, \cdots, z_r \end{matrix} \middle| [x_1, x_{g+1}][x_2, x_{g+2}] \cdots [x_g, x_{2g}] z_1 \cdots z_r = 1 \right\rangle_{\text{pro-}l}.$$

We fix g and r throughout the sections 1, 2 and 3, so we write G for $G_{g,r}$ in these sections. In this section we define a certain central filtration of G and study its associated Lie algebra.

Let $N = N_{g,r}$ be the closed subgroup of G normally generated by the parabolic elements $z_1, \cdots, z_r$. For each $m \geq 1$, we define inductively a subset Σ_m of the set of all closed normal subgroups of G as follows:

$$\Sigma_1 = \{G\},\ \Sigma_2 = \{[G, G], N\},$$
$$\Sigma_m = \{[H_i, H_j] \mid H_i \in \Sigma_i,\ H_j \in \Sigma_j,\ i+j=m\} \quad (m \geq 3).$$

Here, [,] denotes the closure of algebraic commutator. Then the sequence $\{G(m)\}_{m \geq 1}$ of closed normal subgroups of G is defined by putting

$G(m)$ = the minimal closed normal subgroup of G containing all elements in Σ_m.

It is easy to check that the sequence $\{G(m)\}_{m\geq 1}$ has the properties

$$G=G(1)\supset G(2)\supset\cdots\supset G(m)\supset G(m+1)\supset\cdots$$

and

$$[G(m), G(n)]\subset G(m+n) \qquad (m, n\geq 1).$$

In particular, we have $G(m+1)\supset[G, G(m)]\supset[G(m), G(m)]$, i.e., the quotient $gr^mG=G(m)/G(m+1)$ is abelian, hence a $\boldsymbol{Z}_l$-module.

PROPOSITION 1. *Equipped with the bracket operator* $[\ ,\ ]$, *the* $\boldsymbol{Z}_l$*-module* $grG=\bigoplus_{m\geq 1} gr^mG$ *is a free Lie algebra over* $\boldsymbol{Z}_l$ *generated by the elements* $x_i \bmod G(2)$ $(1\leq i\leq 2g)$ *and* $z_j \bmod G(3)$ $(1\leq j\leq r-1)$. *The module* gr^mG *is a finitely generated free* $\boldsymbol{Z}_l$*-module whose rank* $\rho(m)$ *is given by the formula*

$$\prod_{m=1}^{\infty}(1-t^m)^{\rho(m)}=1-2gt-(r-1)t^2.$$

PROOF. As was pointed out by J. Labute in an abstract case ([4, Proposition 1]), this can be shown by a standard argument which, in the case of lower central series and pro-l, was indicated in [3, p. 58]. Likewise, the point is to show that there exists a representation of the Lie algebra grG into the free associative $\boldsymbol{Z}_l$-algebra generated by $2g+r-1$ elements $X_1, \cdots, X_{2g}, Z_1, \cdots, Z_{r-1}$, which maps x_i to X_i $(1\leq i\leq 2g)$ and z_j to Z_j $(1\leq j\leq r-1)$. Here, we regard the associative algebra as given the graduation which assign X_i degree 1 and Z_j degree 2. Such a representation is obtained by the Magnus embedding

$$G \longrightarrow \boldsymbol{Z}_l[[X_1, \cdots, X_{2g}, Z_1, \cdots, Z_{r-1}]]_{n.c}=\Lambda$$

of G into the non-commutative formal power series algebra $(x_i\mapsto 1+X_i$, $z_j\mapsto 1+Z_j)$. Here again the degree of each X_i $(1\leq i\leq 2g)$ is 1 and that of each Z_j $(1\leq j\leq r-1)$ is 2. Let I_m be the ideal of Λ consisting of all power series whose lowest degree is greater than or equal to m. Then $G(m)$ is mapped into $1+I_m$ $(m\geq 1)$ and we can associate to each element of gr^mG a homogeneous polynomial of degree m in $X_1, \cdots, X_{2g}, Z_1, \cdots, Z_{r-1}$ $(\deg(X_i)=1$, $\deg(Z_j)=2)$. This gives the desired representation of grG. Calculation of the rank is also carried out in the similar manner as that in [7]. ∎

2. Filtration of "braid type" automorphism group.

Put

$$\tilde{\Gamma}=\tilde{\Gamma}_{g,r}=\{\sigma\in\operatorname{Aut}G_{g,r}\mid z_j^\sigma\sim z_j^{\alpha_j},\ \exists\alpha_j\in Z_l^\times,\ 1\le j\le r\},$$

where $\sim$ denotes conjugacy in $G=G_{g,r}$. Since each element in $\tilde{\Gamma}$ stabilizes $N=N_{g,r}$, $\tilde{\Gamma}$ acts on $G/G(2)\simeq Z_l^{2g}$. Taking the class of x_i $(1\le i\le 2g)$ in $G/G(2)$ as coordinates, we get a representation

$$\tilde{\lambda}:\tilde{\Gamma}\longrightarrow \mathrm{GL}(2g;\ Z_l).$$

PROPOSITION 2. *The representation $\tilde{\lambda}$ induces an exact sequence*

$$1\longrightarrow\tilde{\Gamma}(1)\longrightarrow\tilde{\Gamma}\xrightarrow{\tilde{\lambda}}\mathrm{GS_p}(2g;\ Z_l)\longrightarrow 1,$$

where $\tilde{\Gamma}(1)=\{\sigma\in\tilde{\Gamma}\mid x_i^\sigma\cdot x_i^{-1}\in G(2),\ 1\le i\le 2g\}$ *and*

$$\mathrm{GS_p}(2g;\ Z_l)=\left\{A\in\mathrm{GL}(2g;\ Z_l)\,\middle|\,{}^tAJ_gA=\mu(A)J_g,\ \mu(A)\in Z_l^\times,\ J_g=\begin{pmatrix}0 & -1_g\\ 1_g & 0\end{pmatrix}\right\}.$$

Moreover, for $\sigma\in\tilde{\Gamma}$, we have $z_j^\sigma\sim z_j^{\mu(\tilde{\lambda}(\sigma))}$ $(1\le j\le r)$.

PROOF. The fact that the image of $\tilde{\lambda}$ is contained in $\mathrm{GS_p}(2g;\ Z_l)$ and the relation $z_j^\sigma\sim z_j^{\mu(\tilde{\lambda}(\sigma))}$ are easily seen by a calculation modulo $G(3)$ of the effect of σ on the relation

$$[x_1,x_{g+1}][x_2,x_{g+2}]\cdots[x_g,x_{2g}]z_1\cdots z_r=1.$$

The crucial part is to show that the image of $\tilde{\lambda}$ coincides with $\mathrm{GS_p}(2g;\ Z_l)$. As in the proof of Proposition 1 in [2], the essential tool for that is the "successive approximation lemma" presented below. Once established the lemma, the proof of Proposition 2 is totally the same as that of Proposition 1 in [2]. ■

For $A\in\mathrm{GS_p}(2g;\ Z_l)$, let $\boldsymbol{a}_i$ denote the i-th column vector of A $(1\le i\le 2g)$ and $\boldsymbol{x}^{\boldsymbol{a}_i}$ denote $x_1^{a_{1i}}x_2^{a_{2i}}\cdots x_{2g}^{a_{2gi}}$, where $\boldsymbol{a}_i={}^t(a_{1i},a_{2i},\cdots,a_{2gi})\in Z_l^{2g}$.

LEMMA 3 (Successive approximation). *Let $m\ge 1$ and $A=(\boldsymbol{a}_i)_{1\le i\le 2g}\in\mathrm{GS_p}(2g;\ Z_l)$. Suppose the elements $s_1^{(m)},\cdots,s_{2g}^{(m)}\in G(2)$ and $t_1^{(m)},\cdots,t_r^{(m)}\in G$ satisfy a congruence*

$$(\sharp_m)\qquad [s_1^{(m)}\boldsymbol{x}^{\boldsymbol{a}_1},s_{g+1}^{(m)}\boldsymbol{x}^{\boldsymbol{a}_{g+1}}]\cdots[s_g^{(m)}\boldsymbol{x}^{\boldsymbol{a}_g},s_{2g}^{(m)}\boldsymbol{x}^{\boldsymbol{a}_{2g}}]t_1^{(m)}z_1t_1^{(m)^{-1}}\cdots t_r^{(m)}z_rt_r^{(m)^{-1}}\equiv 1 \mod G(m+2).$$

Then, there exist $s_1, \cdots, s_{2g} \in G(2)$ *and* $t_1, \cdots, t_r \in G$ *such that*

$$s_i \equiv s_i^{(m)} \quad \bmod G(m+1) \ (1 \leq i \leq 2g),$$
$$t_j \equiv t_j^{(m)} \quad \bmod G(m) \ (1 \leq j \leq r),$$

and

$$[s_1 x^{a_1}, s_{g+1} x^{a_{g+1}}] \cdots [s_g x^{a_g}, s_{2g} x^{a_{2g}}] t_1 z_1 t_1^{-1} \cdots t_r z_r t_r^{-1} = 1.$$

PROOF. The proof is similar to that of Lemma 1 in [2]. So consult [2] as for the detail of the following calculation. Now, it suffices to prove that there exist $s_i^{(m+1)} \equiv s_i^{(m)} \bmod G(m+1)$ $(1 \leq i \leq 2g)$ and $t_j^{(m+1)} \equiv t_j^{(m)} \bmod G(m)$ $(1 \leq j \leq r)$ satisfying the next higher congruence $(\sharp_{m+1})$. Put $s_i^{(m+1)} = S_i s_i^{(m)}$ with $S_i \in G(m+1)$ $(1 \leq i \leq 2g)$ and $t_j^{(m+1)} = T_j t_j^{(m)}$ with $T_j \in G(m)$ $(1 \leq j \leq r)$. We shall show that we can choose S_i and T_j suitably so that $s_i^{(m+1)}$ $(1 \leq i \leq 2g)$ and $t_j^{(m+1)}$ $(1 \leq j \leq r)$ satisfy the congruence $(\sharp_{m+1})$. By the same calculation as in [2], we obtain

$$[s_i^{(m+1)} x^{a_i}, s_{g+i}^{(m+1)} x^{a_{g+i}}] \equiv [x^{a_i}, S_{g+i}][S_i, x^{a_{g+i}}][s_i^{(m)} x^{a_i}, s_{g+i}^{(m)} x^{a_{g+i}}] \quad \bmod G(m+3),$$
$$t_j^{(m+1)} z_j t_j^{(m+1)^{-1}} \equiv [T_j, z_j] t_j^{(m)} z_j t_j^{(m)^{-1}} \quad \bmod G(m+3).$$

Put

$$\rho = [s_1^{(m)} x^{a_1}, s_{g+1}^{(m)} x^{a_{g+1}}] \cdots [s_g^{(m)} x^{a_g}, s_{2g}^{(m)} x^{a_{2g}}] \times t_1^{(m)} z_1 t_1^{(m)^{-1}} \cdots t_r^{(m)} z_r t_r^{(m)^{-1}} \in G(m+2).$$

Then the left hand side of $(\sharp_{m+1})$ is congruent modulo $G(m+3)$ to

$$\rho \cdot \prod_{i=1}^{g} [x^{a_i}, S_{g+i}][S_i, x^{a_{g+i}}] \cdot \prod_{j=1}^{r} [T_j, z_j].$$

Since $x_i \bmod G(2)$ $(1 \leq i \leq 2g)$ and $z_j \bmod G(3)$ $(1 \leq j \leq r-1)$ are the generators of grG and since A is invertible, we have

$$gr^{m+2}G = \sum_{i=1}^{g} [x^{a_i} \bmod G(2), gr^{m+1}G] + \sum_{i=1}^{g} [gr^{m+1}G, x^{a_{g+i}} \bmod G(2)]$$
$$+ \sum_{j=1}^{r-1} [gr^m G, z_j \bmod G(3)].$$

Therefore, we can choose $S_1, \cdots, S_{2g}, T_1, \cdots, T_r$ such that the congruence

$$\rho^{-1} \equiv \prod_{i=1}^{g} [x^{a_i}, S_{g+i}][S_i, x^{a_{g+i}}] \cdot \prod_{j=1}^{r} [T_j, z_j] \qquad \bmod G(m+3)$$

holds. (Actually we can take $T_r = 1$.) Then, $s_i^{(m+1)} = S_i s_i^{(m)}$ $(1 \leq i \leq 2g)$ and $t_j^{(m+1)} = T_j t_j^{(m)}$ $(1 \leq j \leq r)$ satisfy the congruence $(\sharp_{m+1})$. ■

For $m\geq 1$, put

$$\tilde{\Gamma}_{g,r}(m)=\tilde{\Gamma}(m)=\{\sigma\in\tilde{\Gamma}\mid x_i^{\sigma}\cdot x_i^{-1}\in G(m+1)\ (1\leq i\leq 2g),\ z_j^{\sigma}\overset{m}{\sim} z_j\ (1\leq j\leq r)\}$$

where $\overset{m}{\sim}$ denotes conjugacy by an element in $G(m)$. Let $\tilde{f}_m$ denote the following surjective Z_l-linear homomorphism

$$\tilde{f}_m:(gr^{m+1}G)^{2g}\times(gr^mG)^r\ni(s_i)_{1\leq i\leq 2g}\times(t_j)_{1\leq j\leq r}$$
$$\longmapsto\sum_{i=1}^{g}([\bar{x}_i,s_{g+i}]+[s_i,\bar{x}_{g+i}])+\sum_{j=1}^{r}[t_j,\bar{z}_j]\in gr^{m+2}G$$

where $\bar{x}_i=x_i \bmod G(2)$, $\bar{z}_j=z_j \bmod G(3)$. Assume $m\neq 2$. We can define an injective homomorphism from $\tilde{\Gamma}(m)/\tilde{\Gamma}(m+1)$ to $(gr^{m+1}G)^{2g}\times(gr^mG)^r$ by

$$\sigma\longmapsto(x_i^{\sigma}\cdot x_i^{-1})_{1\leq i\leq 2g}\times(t_j)_{1\leq j\leq r},$$

where $z_j^{\sigma}=t_jz_jt_j^{-1}$, $t_j\in G(m)$ $(1\leq j\leq r)$. At this point we use, to confirm that this is well defined, the fact that the centralizer of z_j in G is $\langle z_j\rangle$, the (topologically) cyclic group generated by z_j $(1\leq j\leq r)$. (See [3, p. 55].)

PROPOSITION 4. (1) $[\tilde{\Gamma}(m),\tilde{\Gamma}(n)]\subset\tilde{\Gamma}(m+n)$ $m,n\geq 1$. (2) *Assume* $m\neq 2$. *The* Z_l-*module* $\tilde{\Gamma}(m)/\tilde{\Gamma}(m+1)$ *is identified with the kernel of* $\tilde{f}_m$.

COROLLARY. *For* $m\geq 1$, $m\neq 2$, $\tilde{\Gamma}(m)/\tilde{\Gamma}(m+1)$ *is a finitely generated free* Z_l-*module of rank* $2g\rho(m+1)+r\rho(m)-\rho(m+2)$ $(\rho(m)=\mathrm{rank}(gr^mG))$.

PROOF. The proof is essentially the same as that of Theorem 1 in [2]. Successive approximation (Lemma 3) will play the crucial role. We omit the details here. ■

REMARK. With slight modification, the case of $m=2$ can be described similarly and $\tilde{\Gamma}(2)/\tilde{\Gamma}(3)$ turns out to be a finitely generated free Z_l-module whose rank can be given explicitly.

3. Outer automorphism group.

Put

$$\Gamma=\Gamma_{g,r}=\tilde{\Gamma}_{g,r}/\mathrm{Int}\,G_{g,r},\qquad \Gamma(1)=\Gamma_{g,r}(1)=\tilde{\Gamma}_{g,r}(1)/\mathrm{Int}\,G_{g,r},$$

where Int denotes inner automorphism group.

LEMMA 5. $\mathrm{Int}\,G\cap\tilde{\Gamma}(m)=\mathrm{Int}_G\,G(m)$, *where* $\mathrm{Int}_G\,G(m)=\{\sigma\in\mathrm{Int}\,G\mid \exists g\in G(m),\ x^{\sigma}=gxg^{-1},\ \forall x\in G\}$.

PROOF. The inclusion $\supset$ is obvious. Conversely, let $g \in G$ satisfy $[g, x_i] \in G(m+1)$, $[g, z_j] \in G(m+2)$. When g belongs to $G(k)$ for $k \leq m-1$, put $g_k = g \bmod G(k+1)$. Then in grG, $[\bar{x}_i, g_k] = [\bar{z}_j, g_k] = 0$. Since grG is a free Lie algebra generated by $\bar{x}_i$ $(1 \leq i \leq 2g)$ and $\bar{z}_j$ $(1 \leq j \leq r-1)$, we have $g_k = 0$, i.e., $g \in G(k+1)$ ([5, Theorem 5.10]). Hence $g \in G(m)$. ∎

For $m \geq 1$, put

$$\Gamma(m) = \Gamma_{g,r}(m) = (\tilde{\Gamma}_{g,r}(m) \cdot \mathrm{Int}\, G)/\mathrm{Int}\, G.$$

According to Lemma 5, we have

$$\Gamma(m) = \tilde{\Gamma}_{g,r}(m)/\mathrm{Int}_G\, G(m).$$

There is an exact sequence induced from that in Proposition 2;

$$1 \longrightarrow \Gamma(1) \longrightarrow \Gamma \longrightarrow \mathrm{GS_p}(2g;\ \boldsymbol{Z}_l) \longrightarrow 1,$$

and we have the following proposition analogous to Proposition 4.

PROPOSITION 6. (1) $[\Gamma(m), \Gamma(n)] \subset \Gamma(m+n)$ $(m, n \geq 1)$. (2) *Assume* $m \neq 2$. *The* $\boldsymbol{Z}_l$*-module* $\Gamma(m)/\Gamma(m+1)$ *is identified with the kernel of* f_m, *where* f_m *is a homomorphism from* $\{(gr^{m+1}G)^{2g} \times (gr^m G)^r\}/gr^m G$ *to* $gr^{m+2}G$ *induced by* $\tilde{f}_m$. *Here,* $gr^m G$ *is embedded in* $(gr^{m+1}G)^{2g} \times (gr^m G)^r$ *by the map*

$$g \bmod G(m+1) \longmapsto ([g, x_i])_{1 \leq i \leq 2g} \times (g, g, \cdots, g).$$

PROOF. This is a consequence of Proposition 4 and Lemma 5. Again the proof is essentially the same as that of Theorem 2 in [2]. Since G is free, present case is a little easier. ∎

COROLLARY. *For* $m \geq 1$, $m \neq 2$, $\Gamma(m)/\Gamma(m+1)$ *is a finitely generated free* $\boldsymbol{Z}_l$*-module of rank* $2g\rho(m+1) + (r-1)\rho(m) - \rho(m+2)$ $(\rho(m) = \mathrm{rank}(gr^m G))$.

PROOF. It suffices to show that the module $\{(gr^{m+1}G)^{2g} \times (gr^m G)^r\}/gr^m G$ is l-torsion free. Take an element $(s_i)_{1 \leq i \leq 2g} \times (t_j)_{1 \leq j \leq r}$ in $(gr^{m+1}G)^{2g} \times (gr^m G)^r$ such that

$$(ls_i)_{1 \leq i \leq 2g} \times (lt_j)_{1 \leq j \leq r} = ([g, x_i])_{1 \leq i \leq 2g} \times (g, g, \cdots, g)$$

for some $g \in gr^m G$. As $gr^m G$ is torsion free, we must have $t_1 = t_2 = \cdots = t_r$ and $g = lt_j$. Then $ls_i = [lt_j, x_i] = l[t_j, x_i]$. Hence $s_i = [t_j, x_i]$. ∎

4. Pro-l version of the theorem of Dehn-Nielsen.

In this section, we prove the surjectivity of the natural homomorphism

$$\tilde{\Gamma}_{g,r} \longrightarrow \tilde{\Gamma}_{g,s} \qquad (r>s\geq 0).$$

Here, we understand by $\tilde{\Gamma}_{g,0}$ the full automorphism group of the pro-l fundamental group of Riemann surface of genus g, which was studied in [2]. In the classical case, corresponding statement is known as the theorem of Dehn-Nielsen (cf. e.g. [8, 5.6]).

First we define the above homomorphism. Consider the homomorphism $G_{g,r}\to G_{g,s}$ defined by $x_i\mapsto x_i$ $(1\leq i\leq 2g)$, $z_j\mapsto z_j$ $(1\leq j\leq s)$ and $z_j\mapsto 1$ $(s+1\leq j\leq r)$. As each element of $\tilde{\Gamma}_{g,r}$ stabilizes the closed normal subgroup generated normally by $z_{s+1},\cdots,z_r$, the above homomorphism induces a homomorphism $\phi=\phi_{r,s}$;

$$\phi:\tilde{\Gamma}_{g,r} \longrightarrow \tilde{\Gamma}_{g,s}.$$

THEOREM 7 (Pro-l analogue of the theorem of Dehn-Nielsen). *For each $r>s\geq 0$, the homomorphism*

$$\phi:\tilde{\Gamma}_{g,r} \longrightarrow \tilde{\Gamma}_{g,s}$$

is surjective.

PROOF. First we prove the theorem in the case where $r>s\geq 1$. Next, by using some results in [2] and [4], we prove the surjectivity of $\tilde{\Gamma}_{g,1}\to\tilde{\Gamma}_{g,0}$. We follow the argument in [6] where the corresponding result in the case of $g=0$ is proved.

So first assume $r>s\geq 1$. In view of the commutative diagram below, it suffices to show that the induced homomorphism $\phi_1:\tilde{\Gamma}_{g,r}(1)\to\tilde{\Gamma}_{g,s}(1)$ is surjective.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{\Gamma}_{g,r}(1) & \longrightarrow & \tilde{\Gamma}_{g,r} & \longrightarrow & \mathrm{GS_p}(2g;\ Z_l) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id.}} & & \\
1 & \longrightarrow & \tilde{\Gamma}_{g,s}(1) & \longrightarrow & \tilde{\Gamma}_{g,s} & \longrightarrow & \mathrm{GS_p}(2g;\ Z_l) & \longrightarrow & 1.
\end{array}$$

For that purpose, we only need to check that the induced homomorphisms

$$gr^m\phi_1:\tilde{\Gamma}_{g,r}(m)/\tilde{\Gamma}_{g,r}(m+1) \longrightarrow \tilde{\Gamma}_{g,s}(m)/\tilde{\Gamma}_{g,s}(m+1)$$

are surjective for all $m\geq 1$. Consider the following commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{\Gamma}_{g,r}(m)/\tilde{\Gamma}_{g,r}(m+1) & \longrightarrow & (gr^{m+1}G_{g,r})^{2g}\times(gr^mG_{g,r})^r & \xrightarrow{\tilde{f}_m} & gr^{m+2}G_{g,r} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle gr^m\phi_1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\
1 & \longrightarrow & \tilde{\Gamma}_{g,s}(m)/\tilde{\Gamma}_{g,s}(m+1) & \longrightarrow & (gr^{m+1}G_{g,s})^{2g}\times(gr^mG_{g,s})^s & \xrightarrow{\tilde{f}_m} & gr^{m+2}G_{g,s} & \longrightarrow & 1
\end{array}$$

where α and β are naturally induced homomorphisms. By the snake lemma, the sequence

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \operatorname{coker}(gr^m\phi_1) \longrightarrow 1$$

is exact. Hence our task is to show that $\ker\alpha\to\ker\beta$ is surjective. We let denote by $\mathfrak{A}_m$ the kernel of the surjective homomorphism $gr^mG_{g,r}\to gr^mG_{g,s}$ $(m\geq 1)$. Then $\ker\alpha$ (resp. $\ker\beta$) is equal to $\mathfrak{A}_{m+1}^{2g}\oplus\mathfrak{A}_m^s\oplus(gr^mG_{g,r})^{r-s}$ (resp. $\mathfrak{A}_{m+2}$). Since the ideal $\mathfrak{A}=\bigoplus_{m\geq 1}\mathfrak{A}_m$ is generated by $z_{s+1},\cdots,z_r$ and grG is generated by x_i $(1\leq i\leq 2g)$ and z_j $(1\leq j\leq r-1)$, we have

$$\mathfrak{A}_{m+2}=[\mathfrak{A}_{m+1}, gr^1G]+\sum_{j=1}^{s}[\mathfrak{A}_m, z_j]+\sum_{j=s+1}^{r}[gr^mG, z_j].$$

From this and the definition of the map $\tilde{f}_m$, we conclude that $\ker\alpha\to\ker\beta$ is surjective.

Next we consider the case where $r=1$, $s=0$. As similar as the case treated above, we need to look at the following diagram and to show that $\ker\alpha\to\ker\beta$ is surjective:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{\Gamma}_{g,1}(m)/\tilde{\Gamma}_{g,1}(m+1) & \longrightarrow & (gr^{m+1}G_{g,1})^{2g}\times gr^mG_{g,1} & \xrightarrow{\tilde{f}_m} & gr^{m+2}G_{g,1} & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle gr^m\phi_1} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\
1 & \longrightarrow & \tilde{\Gamma}_{g,0}(m)/\tilde{\Gamma}_{g,0}(m+1) & \longrightarrow & (gr^{m+1}G_{g,0})^{2g} & \xrightarrow{\tilde{f}_m} & gr^{m+2}G_{g,0} & \longrightarrow & 1.
\end{array}$$

Let $\mathfrak{A}_m$ be the kernel of the homomorphism $gr^mG_{g,1}\to gr^mG_{g,0}$. Also in this case, owing to a theorem of J. Labute [4], the ideal $\mathfrak{A}=\bigoplus_{m\geq 1}\mathfrak{A}_m$ is generated by $z_1=[x_1, x_{g+1}]+\cdots+[x_g, x_{2g}]$. Hence the argument as above also works in this case. ■

As a direct consequence of the above theorem and Theorem 3 in [2], we get the following

COROLLARY. *Suppose* $g\geq 3$. *Let* A *be an element of* $\mathrm{GS_p}(2g;\,\mathbf{Z}_l)$ *satisfying the following conditions*:

$$A\equiv 1_{2g}\begin{cases}\bmod l & l\neq 2\\ \bmod l^2 & l=2,\end{cases}$$

and C *be the* $\mathrm{GS_p}(2g;\,\mathbf{Z}_l)$*-conjugacy class of* A. *Then* $\lambda^{-1}(C)$ *contains more than one* $\Gamma_{g,r}$*-conjugacy class. Here,* λ *is the map induced from the action of* $\Gamma_{g,r}$ *on* $G_{g,r}/G_{g,r}(2)\simeq\mathbf{Z}_l^{2g}$.

REMARK. By a result of M. Asada [1], the above corollary holds for $g=2$ in a slightly weaker form.

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