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Certain automorphism groups of pro-$l$ fundamental groups of punctured Riemann surfaces$^*$

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Introduction.

In this paper we present some results on certain automorphism groups of pro-$l$ fundamental groups of punctured Riemann surfaces.

Let $l$ be a prime, $g \geq 1$, $r \geq 0$ be integers and $G = G_{g,r}$ be the pro-$l$ completion of the fundamental group of Riemann surface of genus $g$ with $r$-points deleted. Assume $r \geq 1$. Then $G$ is a pro-$l$ free group of rank $2g + r - 1$ having a standard presentation:

$$G = \left\langle x_1, x_2, \ldots, x_{2g}, z_1, \ldots, z_r \mid [x_1, x_{g+1}] [x_2, x_{g+2}] \cdots [x_r, x_{2g}] z_1 \cdots z_r = 1 \right\rangle_{\text{pro-}l}.$$

We give $G$ a central filtration $\{G(m)\}_{m \geq 1}$ such that the elements $x_1, \ldots, x_{2g}$ are of degree 1, the elements $z_1, \ldots, z_r$ are of degree 2 and generally the degree of a commutator $[x, y]$ is the sum of degrees of $x$ and $y$. For this filtration let $grG = \bigoplus_{m \geq 1} G(m)/G(m+1)$. Then, by a standard method, $grG$ turns out to be a free Lie algebra generated by the classes of $x_1, \ldots, x_{2g}, z_1, \ldots, z_{r-1}$. By using this we first establish a “successive approximation lemma” to construct automorphisms of $G$. Then we study some basic properties of the subgroup

$$\bar{G} = \bar{G}_{g,r} = \{ \sigma \in \text{Aut } G \mid z_j^i \sim z_j^{i^\prime}, \exists \alpha_j \in \mathbb{Z}^r, 1 \leq j \leq r \}$$

of the automorphism group of $G$. Such type of groups arise naturally in the context of “large Galois representations” (cf. [2] [3]). These studies are viewed as a continuation of our previous study [2] in which we treated exclusively the case of $r = 0$. A new ingredient is the filtration of $G$ explained above. The author owes the idea of introducing such filtration to study the group $\bar{G}$ to Professor Takayuki Oda. It seems

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that the lower central series used in [2] does not work well to study $\tilde{F}$ when $r \geq 2$.

As a consequence of these studies, we establish the following

**Theorem.** Suppose $r > s \geq 0$. The naturally induced homomorphism

$$\tilde{F}_{s,r} \longrightarrow \tilde{F}_{s,s},$$

is surjective.

This is a pro-$l$ analogue of the classical theorem of Dehn-Nielsen. We can derive from this a result on conjugacy classes of $\tilde{F}_{s,r}/\text{Int} \ G$ as in [2].

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1. Filtration of the fundamental group and Lie algebra.

We fix a prime number $l$ throughout the paper. Let $g$ and $r$ be two integers greater than or equal to 1. We denote by $G_{s,r}$, the pro-$l$ completion of the fundamental group of $r$-punctured Riemann surface of genus $g$,

$$G_{s,r} = \langle x_1, x_2, \ldots, x_{2g} \mid [x_1, x_{s+1}][x_{s+2}, x_{s+3}] \cdots [x_g, x_{2g}]z_1 \cdots z_r = 1 \rangle_{\text{pro}-l}.$$  

We fix $g$ and $r$ throughout the sections 1, 2 and 3, so we write $G$ for $G_{s,r}$ in these sections. In this section we define a certain central filtration of $G$ and study its associated Lie algebra.

Let $N = N_{s,r}$ be the closed subgroup of $G$ normally generated by the parabolic elements $z_1, \ldots, z_r$. For each $m \geq 1$, we define inductively a subset $\Sigma_m$ of the set of all closed normal subgroups of $G$ as follows:

$$\Sigma_1 = [G], \quad \Sigma_2 = [[G, G], N],$$

$$\Sigma_m = [[H_i, H_j] \mid H_i \in \Sigma_i, H_j \in \Sigma_j, i + j = m \rangle \quad (m \geq 3).$$

Here, $[ , ]$ denotes the closure of algebraic commutator. Then the sequence $\{G(m)\}_{m \geq 1}$ of closed normal subgroups of $G$ is defined by putting

$G(m) = \text{the minimal closed normal subgroup of } G \text{ containing all elements in } \Sigma_m.$
It is easy to check that the sequence \( \{G(m)\}_{m \geq 1} \) has the properties

\[
G = G(1) \supset G(2) \supset \cdots \supset G(m) \supset G(m+1) \supset \cdots
\]

and

\[
[G(m), G(n)] \subset G(m+n) \quad (m, n \geq 1).
\]

In particular, we have \( G(m+1) \supset [G, G(m)] \supset [G(m), G(m)] \), i.e., the quotient \( gr^*G = G(1)/G(m+1) \) is abelian, hence a \( \mathbb{Z}_t \)-module.

**Proposition 1.** Equipped with the bracket operator \([ , ]\), the \( \mathbb{Z}_t \)-module \( grG = \bigoplus_{m \geq 1} gr^mG \) is a free Lie algebra over \( \mathbb{Z}_t \) generated by the elements \( x_i \mod G(2) \) (\( 1 \leq i \leq 2g \)) and \( z_j \mod G(3) \) (\( 1 \leq j \leq r - 1 \)). The module \( gr^*G \) is a finitely generated free \( \mathbb{Z}_t \)-module whose rank \( \rho(m) \) is given by the formula

\[
\prod_{m=1}^{\infty} (1-t^m)^{\rho(m)} = 1 - 2gt - (r-1)t^2.
\]

**Proof.** As was pointed out by J. Labute in an abstract case ([4, Proposition 1]), this can be shown by a standard argument which, in the case of lower central series and pro-\( l \), was indicated in [3, p. 58]. Likewise, the point is to show that there exists a representation of the Lie algebra \( grG \) into the free associative \( \mathbb{Z}_t \)-algebra generated by \( 2g + r - 1 \) elements \( X_i, \ldots, X_{2g}, Z_1, \ldots, Z_{r-1} \), which maps \( x_i \) to \( X_i \) (\( 1 \leq i \leq 2g \)) and \( z_j \) to \( Z_j \) (\( 1 \leq j \leq r - 1 \)). Here, we regard the associative algebra as given the graduation which assign \( X_i \) degree 1 and \( Z_j \) degree 2. Such a representation is obtained by the Magnus embedding

\[
G \longrightarrow \mathbb{Z}_t[[X_1, \ldots, X_{2g}, Z_1, \ldots, Z_{r-1}]]_{*}=A
\]

of \( G \) into the non-commutative formal power series algebra \( \{x_i \mapsto 1 + X_i, \quad z_j \mapsto 1 + Z_j\} \). Here again the degree of each \( X_i \) (\( 1 \leq i \leq 2g \)) is 1 and that of each \( Z_j \) (\( 1 \leq j \leq r - 1 \)) is 2. Let \( I_m \) be the ideal of \( A \) consisting of all power series whose lowest degree is greater than or equal to \( m \). Then \( G(m) \) is mapped into \( 1 + I_m \) (\( m \geq 1 \)) and we can associate to each element of \( gr^*G \) a homogeneous polynomial of degree \( m \) in \( X_1, \ldots, X_{2g}, Z_1, \ldots, Z_{r-1} \) (\( \deg(X_i) = 1, \deg(Z_j) = 2 \)). This gives the desired representation of \( grG \). Calculation of the rank is also carried out in the similar manner as that in [7].
2. Filtration of "braid type" automorphism group.

Put

\[ \tilde{\Gamma} = \Gamma_{\sigma, r} = \{ \sigma \in \text{Aut} G_{\sigma, r} \mid z_j^* \sim z_j^\sigma, \exists x_j \in Z_i^r, 1 \leq j \leq r \}, \]

where \( \sim \) denotes conjugacy in \( G = G_{\sigma, r} \). Since each element in \( \tilde{\Gamma} \) stabilizes \( N = N_{\sigma, r} \), \( \tilde{\Gamma} \) acts on \( G/G(2) = \mathbb{Z}_2^r \). Taking the class of \( x_i \) \((1 \leq i \leq 2g)\) in \( G/G(2) \) as coordinates, we get a representation

\[ \tilde{\lambda} : \tilde{\Gamma} \longrightarrow \text{GL}(2g; \mathbb{Z}_i). \]

**Proposition 2.** The representation \( \tilde{\lambda} \) induces an exact sequence

\[ 1 \longrightarrow \tilde{\Gamma}(1) \longrightarrow \tilde{\Gamma} \longrightarrow \text{GS}_p(2g; \mathbb{Z}_i) \longrightarrow 1, \]

where \( \tilde{\Gamma}(1) = \{ \sigma \in \tilde{\Gamma} \mid x_i \cdot x_i^{-1} \in G(2), 1 \leq i \leq 2g \} \) and

\[ \text{GS}_p(2g; \mathbb{Z}_i) = \left\{ A \in \text{GL}(2g; \mathbb{Z}_i) \left| tA_j^*A = \mu(A)J, \mu(A) \in Z_i^r, J = \begin{pmatrix} 0 & -1 \n 1 & 0 \end{pmatrix} \right\} \right\}. \]

Moreover, for \( \sigma \in \tilde{\Gamma} \), we have \( z_j^\sigma \sim z_j^{(x_1^\sigma)} (1 \leq j \leq r) \).

**Proof.** The fact that the image of \( \tilde{\lambda} \) is contained in \( \text{GS}_p(2g; \mathbb{Z}_i) \) and the relation \( z_j^\sigma \sim z_j^{(x_1^\sigma)} \) are easily seen by a calculation modulo \( G(3) \) of the effect of \( \sigma \) on the relation

\[ [x_1, x_{r+1}][x_2, x_{2r+2}] \cdots [x_r, x_{2^r}]z_1 \cdots z_r = 1. \]

The crucial part is to show that the image of \( \tilde{\lambda} \) coincides with \( \text{GS}_p(2g; \mathbb{Z}_i) \). As in the proof of Proposition 1 in [2], the essential tool for that is the "successive approximation lemma" presented below. Once established the lemma, the proof of Proposition 2 is totally the same as that of Proposition 1 in [2].

For \( A \in \text{GS}_p(2g; \mathbb{Z}_i) \), let \( a_i \) denote the \( i \)-th column vector of \( A \) \((1 \leq i \leq 2g)\) and \( x^n \) denote \( x_1^n \cdots x_2^n \cdots \), where \( a_i = (a_{i1}, a_{i2}, \ldots, a_{2g}) \in Z_i^{2g} \).

**Lemma 3** (Successive approximation). Let \( m \geq 1 \) and \( A = (a_i)_{1 \leq i \leq 2g} \in \text{GS}_p(2g; \mathbb{Z}_i) \). Suppose the elements \( s_1^{(m)}, \ldots, s_g^{(m)} \in G(2) \) and \( t_1^{(m)}, \ldots, t_r^{(m)} \in G \) satisfy a congruence

\[ ([s_1^{(m)}], [s_2^{(m)}], \ldots, [s_g^{(m)}], [t_1^{(m)}], \ldots, [t_r^{(m)}]) \equiv 1 \mod G(m+2). \]
Then, there exist \( s_i, \cdots, s_{2g} \in G(2) \) and \( t_i, \cdots, t_r \in G \) such that

\[
s_i \equiv s_i^{(m)} \mod G(m+1) \quad (1 \leq i \leq 2g),
\]

\[
t_j \equiv t_j^{(m)} \mod G(m) \quad (1 \leq j \leq r),
\]

and

\[
[s_1x^1, s_{2g}x^{2g+1}] \cdots [s_gx^g, s_{2g}x^{2g+1}] t_1z_1^{-1} \cdots t_rz_r^{-1} = 1.
\]

PROOF. The proof is similar to that of Lemma 1 in [2]. So consult [2] as for the detail of the following calculation. Now, it suffices to prove that there exist \( s_i^{(m+1)} = s_i^{(m)} \mod G(m+1) \) \((1 \leq i \leq 2g)\) and \( t_j^{(m+1)} = t_j^{(m)} \mod G(m) \) \((1 \leq j \leq r)\) satisfying the next higher congruence \((\#_{m+1})\). Put \( s_i^{(m+1)} = S_is_i^{(m)} \) with \( S_i \in G(m+1) \) \((1 \leq i \leq 2g)\) and \( t_j^{(m+1)} = T_jt_j^{(m)} \) with \( T_j \in G(m) \) \((1 \leq j \leq r)\). We shall show that we can choose \( S_i \) and \( T_j \) suitably so that \( s_i^{(m+1)} \) \((1 \leq i \leq 2g)\) and \( t_j^{(m+1)} \) \((1 \leq j \leq r)\) satisfy the congruence \((\#_{m+1})\). By the same calculation as in [2], we obtain

\[
[s_i^{(m+1)}x^i, s_{2g}^{(m+1)}x^{2g+1}] = [x^i, S_i] [S_i, x^{i+1}] [s_i^{(m)}x^i, s_{2g}^{(m)}x^{2g+1}] \mod G(m+3),
\]

\[
t_j^{(m+1)} z_j t_j^{(m+1)} = [T_j, z_j] t_j^{(m)} z_j t_j^{(m)} \mod G(m+3).
\]

Put

\[
\rho = [s_1^{(m)}x^1, s_2^{(m)}x^2+1] \cdots [s_g^{(m)}x^g, s_{2g}^{(m)}x^{2g+1}] \times t_1^{(m)} z_1 t_1^{(m)} \cdots t_r^{(m)} z_r t_r^{(m)} \in G(m+2).
\]

Then the left hand side of \((\#_{m+1})\) is congruent modulo \( G(m+3) \) to

\[
\rho \cdot \prod_{i=1}^g [x^i, S_i] [S_i, x^{i+1}] \cdot \prod_{j=1}^r [T_j, z_j].
\]

Since \( x_i \mod G(2) \) \((1 \leq i \leq 2g)\) and \( z_j \mod G(3) \) \((1 \leq j \leq r-1)\) are the generators of \( grG \) and since \( A \) is invertible, we have

\[
gr^{m+2}G = \sum_{i=1}^g [x^i \mod G(2), gr^{m+1}G] + \sum_{i=1}^g [gr^{m+1}G, x^{i+1} \mod G(2)]
\]

\[
+ \sum_{j=1}^{r-1} [gr^mG, z_j \mod G(3)].
\]

Therefore, we can choose \( S_i, \cdots, S_{2g}, T_1, \cdots, T_r \) such that the congruence

\[
\rho^{-1} = \prod_{i=1}^g [x^i, S_i] [S_i, x^{i+1}] \cdot \prod_{j=1}^r [T_j, z_j] \mod G(m+3)
\]

holds. (Actually we can take \( T_r = 1 \).) Then, \( s_i^{(m+1)} = S_is_i^{(m)} \) \((1 \leq i \leq 2g)\) and \( t_j^{(m+1)} = T_jt_j^{(m)} \) \((1 \leq j \leq r)\) satisfy the congruence \((\#_{m+1})\). ¡
For $m \geq 1$, put

$$\tilde{\Gamma}_r(m) = \tilde{\Gamma}(m) = \{ \sigma \in \tilde{\Gamma} \mid x_i^r \cdot x_i^{-1} \in G(m+1) \ (1 \leq i \leq 2g), \ z_j^m \sim z_j \ (1 \leq j \leq r) \}$$

where \( \sim \) denotes conjugacy by an element in $G(m)$. Let $f_m$ denote the following surjective $Z_r$-linear homomorphism

$$f_m : (gr^{m+G})^2 \times (gr^m)^r \ni (s_{i}, t_{j}) \mapsto \sum_{i=1}^{r} ([x_i^r, s_{i} s_{i+1}^l] + [s_{i}, x_{i} x_{i+1}^l]) + \sum_{j=1}^{r} [t_j, z_j] \in gr^{m+G}$$

where $x_i = x_i \mod G(2)$, $z_j = z_j \mod G(3)$. Assume $m \neq 2$. We can define an injective homomorphism from $\tilde{\Gamma}(m) / \tilde{\Gamma}(m+1)$ to $(gr^{m+G})^2 \times (gr^m)^r$ by

$$\sigma \mapsto (x_i^r \cdot x_i^{-1})_{i=1}^{r} \times (t_j)_{j=1}^{r},$$

where $z_j^r = t_j z_j t_j^{-1}, t_j \in G(m) \ (1 \leq j \leq r)$. At this point we use, to confirm that this is well defined, the fact that the centralizer of $z_j$ in $G$ is $\langle z_j \rangle$, the (topologically) cyclic group generated by $z_j \ (1 \leq j \leq r)$. (See [3, p. 55].)

**Proposition 4.** (1) $[\tilde{\Gamma}(m), \tilde{\Gamma}(n)] \subset \tilde{\Gamma}(m+n), m, n \geq 1$. (2) Assume $m \neq 2$. The $Z_r$-module $\tilde{\Gamma}(m) / \tilde{\Gamma}(m+1)$ is identified with the kernel of $f_m$.

**Corollary.** For $m \geq 1$, $m \neq 2$, $\tilde{\Gamma}(m) / \tilde{\Gamma}(m+1)$ is a finitely generated free $Z_r$-module of rank $2g \rho(m) + \rho(m) - \rho(m+2) \ (\rho(m) = \text{rank}(gr^m))$.

**Proof.** The proof is essentially the same as that of Theorem 1 in [2]. Successive approximation (Lemma 3) will play the crucial role. We omit the details here. \( \square \)

**Remark.** With slight modification, the case of $m=2$ can be described similarly and $\tilde{\Gamma}(2) / \tilde{\Gamma}(3)$ turns out to be a finitely generated free $Z_r$-module whose rank can be given explicitly.

3. Outer automorphism group.

Put

$$\Gamma = \Gamma_{\ast \ast} = \tilde{\Gamma}_{\ast \ast} / \text{Int} G_{\ast \ast}, \quad \Gamma(1) = \Gamma_{\ast \ast}(1) = \tilde{\Gamma}_{\ast \ast}(1) / \text{Int} G_{\ast \ast},$$

where $\text{Int}$ denotes inner automorphism group.

**Lemma 5.** $\text{Int} G \cap \tilde{\Gamma}(m) = \text{Int}_G G(m)$, where $\text{Int}_G G(m) = \{ \sigma \in \text{Int} G \mid \exists g \in G(m), x^r = gxg^{-1}, \forall x \in G \}$. 

Proof. The inclusion $\supset$ is obvious. Conversely, let $g \in G$ satisfy $[g, x_i] \in G(m+1), [g, z_j] \in G(m+2)$. When $g$ belongs to $G(k)$ for $k \leq m-1$, put $g_k = g \mod G(k+1)$. Then in $grG$, $[\bar{x}_i, g_k] = [\bar{z}_j, g_k] = 0$. Since $grG$ is a free Lie algebra generated by $\bar{x}_i$ $(1 \leq i \leq 2g)$ and $\bar{z}_j$ $(1 \leq j \leq r-1)$, we have $g_k = 0$, i.e., $g \in G(k+1)$ ([5, Theorem 5.10]). Hence $g \in G(m)$. \] 

For $m \geq 1$, put 

$$\Gamma(m) = \Gamma_{\nu, r}(m) = (\bar{\Gamma}_{\nu, r}(m) \cdot \text{Int} G)/\text{Int} G.$$ 

According to Lemma 5, we have 

$$\Gamma(m) = \bar{\Gamma}_{\nu, r}(m)/\text{Int}_G G(m).$$ 

There is an exact sequence induced from that in Proposition 2; 

$$1 \longrightarrow \Gamma(1) \longrightarrow \Gamma \longrightarrow G_{S_p}(2g; Z_i) \longrightarrow 1,$$ 

and we have the following proposition analogous to Proposition 4. 

**Proposition 6.** (1) $[\Gamma(m), \Gamma(n)] \subset \Gamma(m+n)$ $(m, n \geq 1).$ (2) Assume $m \neq 2$. The $\mathbb{Z}_r$-module $\Gamma(m)/\Gamma(m+1)$ is identified with the kernel of $f_m$, where $f_m$ is a homomorphism from $\{(gr^{m+1}G)^{\nu} \times (gr^m G)^r\}/gr^m G$ to $gr^{m+1}G$ induced by $\bar{f}_m$. Here, $gr^m G$ is embedded in $(gr^{m+1}G)^{\nu} \times (gr^m G)^r$ by the map 

$$g \mod G(m+1) \longmapsto [(g, x_i)]_{1 \leq i \leq 2g} \times (g, g, \ldots, g).$$ 

**Proof.** This is a consequence of Proposition 4 and Lemma 5. Again the proof is essentially the same as that of Theorem 2 in [2]. Since $G$ is free, present case is a little easier. 

**Corollary.** For $m \geq 1$, $m \neq 2$, $\Gamma(m)/\Gamma(m+1)$ is a finitely generated free $\mathbb{Z}_r$-module of rank $2g\rho(m+1) + (r-1)\rho(m) - \rho(m+2)$ $(\rho(m) = \text{rank}(gr^m G))$. 

**Proof.** It suffices to show that the module $\{(gr^{m+1}G)^{\nu} \times (gr^m G)^r\}/gr^m G$ is $l$-torsion free. Take an element $(s_i)_{1 \leq i \leq 2g} \times (t_j)_{1 \leq j \leq r}$ in $(gr^{m+1}G)^{\nu} \times (gr^m G)^r$ such that 

$$(ls_i)_{1 \leq i \leq 2g} \times (lt_j)_{1 \leq j \leq r} = [(g, x_i)]_{1 \leq i \leq 2g} \times (g, g, \ldots, g)$$

for some $g \in gr^m G$. As $gr^m G$ is torsion free, we must have $t_1 = t_2 = \cdots = t_r$ and $g = lt_j$. Then $ls_i = [lt_j, x_i] = l[t_j, x_i]$. Hence $s_i = [t_j, x_i]$. 

4. **Pro-$l$ version of the theorem of Dehn-Nielsen.**

In this section, we prove the surjectivity of the natural homomorphism
\[ \tilde{I}_{s,r} \longrightarrow \tilde{I}_{s,s} \quad (r > s \geq 0). \]

Here, we understand by $\tilde{I}_{s,0}$ the full automorphism group of the pro-$l$ fundamental group of $R$ Riemann surface of genus $g$, which was studied in [2]. In the classical case, corresponding statement is known as the theorem of Dehn-Nielsen (cf. e.g. [8, 5.6]).

First we define the above homomorphism. Consider the homomorphism $G_{s,r} \rightarrow G_{s,s}$ defined by $x_i \mapsto x_i$ (1 $\leq i \leq 2g$), $z_j \mapsto z_j$ (1 $\leq j \leq s$) and $z_j \mapsto 1$ (s $+ 1 \leq j \leq r$). As each element of $\tilde{I}_{s,r}$ stabilizes the closed normal subgroup generated normally by $z_{s+1}, \ldots, z_r$, the above homomorphism induces a homomorphism $\phi : \tilde{I}_{s,r} \longrightarrow \tilde{I}_{s,s}$.

**THEOREM 7** (Pro-$l$ analogue of the theorem of Dehn-Nielsen). For each $r > s \geq 0$, the homomorphism

\[ \phi : \tilde{I}_{s,r} \longrightarrow \tilde{I}_{s,s} \]

is surjective.

**Proof.** First we prove the theorem in the case where $r > s \geq 1$. Next, by using some results in [2] and [4], we prove the surjectivity of $\tilde{I}_{s,1} \rightarrow \tilde{I}_{s,0}$. We follow the argument in [6] where the corresponding result in the case of $g = 0$ is proved.

So first assume $r > s \geq 1$. In view of the commutative diagram below, it suffices to show that the induced homomorphism $\phi_1 : \tilde{I}_{s,r}(1) \rightarrow \tilde{I}_{s,s}(1)$ is surjective.

\[
\begin{array}{cccc}
1 & \rightarrow & \tilde{I}_{s,r}(1) & \rightarrow & \tilde{I}_{s,r} & \rightarrow & \text{GS}_p(2g; Z_i) & \rightarrow & 1 \\
\downarrow \phi_1 & & \downarrow \phi & & \downarrow \text{id.} & & \\
1 & \rightarrow & \tilde{I}_{s,s}(1) & \rightarrow & \tilde{I}_{s,s} & \rightarrow & \text{GS}_p(2g; Z_i) & \rightarrow & 1.
\end{array}
\]

For that purpose, we only need to check that the induced homomorphisms

\[ gr^m \phi_1 : \tilde{I}_{s,r}(m) / \tilde{I}_{s,r}(m+1) \longrightarrow \tilde{I}_{s,s}(m) / \tilde{I}_{s,s}(m+1) \]

are surjective for all $m \geq 1$. Consider the following commutative diagram;

\[
\begin{array}{cccc}
1 & \rightarrow & \tilde{I}_{s,r}(m) / \tilde{I}_{s,r}(m+1) & \rightarrow & (gr^{m+1}G_{s,r})^p \times (gr^{m}G_{s,r})^r & \rightarrow & \tilde{I}_{s,r} / gr^mG_{s,r} & \rightarrow & 1 \\
\downarrow gr^m \phi_1 & & \downarrow \alpha & & \downarrow \beta & & \\
1 & \rightarrow & \tilde{I}_{s,s}(m) / \tilde{I}_{s,s}(m+1) & \rightarrow & (gr^{m+1}G_{s,s})^p \times (gr^{m}G_{s,s})^r & \rightarrow & \tilde{I}_{s,s} / gr^mG_{s,s} & \rightarrow & 1
\end{array}
\]
where $\alpha$ and $\beta$ are naturally induced homomorphisms. By the snake lemma, the sequence

$$\ker \alpha \longrightarrow \ker \beta \longrightarrow \coker(gr^n \phi_1) \longrightarrow 1$$

is exact. Hence our task is to show that $\ker \alpha \to \ker \beta$ is surjective. We let denote by $\mathcal{A}_m$ the kernel of the surjective homomorphism $gr^n G_{e,r} \to gr^n G_{e,s}$ ($m \geq 1$). Then $\ker \alpha$ (resp. $\ker \beta$) is equal to $\mathcal{A}_{m+1} \oplus \mathcal{A}_m \oplus (gr^n G_{e,r})^{r-1}$ (resp. $\mathcal{A}_{m+2}$). Since the ideal $\mathcal{I} = \oplus_{n \geq 1} \mathcal{A}_m$ is generated by $z_{s+1}, \ldots, z_r$ and $grG$ is generated by $x_i$ ($1 \leq i \leq 2g$) and $z_j$ ($1 \leq j \leq r-1$), we have

$$\mathcal{A}_{m+2} = [\mathcal{A}_{m+1}, gr^i G] + \sum_{j=1}^r [\mathcal{A}_m, z_j] + \sum_{j=s+1}^r [gr^n G, z_j].$$

From this and the definition of the map $f_m$, we conclude that $\ker \alpha \to \ker \beta$ is surjective.

Next we consider the case where $r=1, s=0$. As similar as the case treated above, we need to look at the following diagram and to show that $\ker \alpha \to \ker \beta$ is surjective:

$$
\begin{array}{ccc}
1 & \longrightarrow & \tilde{\Gamma}_{e,1}(m)/\tilde{\Gamma}_{e,1}(m+1) \longrightarrow (gr^{m+1} G_{e,r})^\mathbb{Z} \times gr^m G_{e,s} \xrightarrow{f_m} gr^{m+1} G_{e,s} \longrightarrow 1 \\
& & \downarrow \phi_1 \quad \quad \quad \quad \quad \quad \downarrow \alpha \quad \quad \quad \quad \quad \quad \downarrow \beta \\
1 & \longrightarrow & \tilde{\Gamma}_{e,0}(m)/\tilde{\Gamma}_{e,0}(m+1) \longrightarrow (gr^{m+1} G_{e,0})^\mathbb{Z} \xrightarrow{f_m} gr^{m+1} G_{e,0} \longrightarrow 1.
\end{array}
$$

Let $\mathcal{A}_m$ be the kernel of the homomorphism $gr^n G_{e,1} \to gr^n G_{e,0}$. Also in this case, owing to a theorem of J. Labute [4], the ideal $\mathcal{I} = \oplus_{n \geq 1} \mathcal{A}_m$ is generated by $z_1 = [x_1, x_{s+1}] + \cdots + [x_r, x_{2r}]$. Hence the argument as above also works in this case.

As a direct consequence of the above theorem and Theorem 3 in [2], we get the following

**Corollary.** Suppose $g \geq 3$. Let $A$ be an element of $\text{GS}_p(2g; Z)$ satisfying the following conditions:

$$A \equiv \begin{cases} 1_{2g} \mod l, & l \neq 2 \\ p \mod l, & l = 2, \end{cases}$$

and $C$ be the $\text{GS}_p(2g; Z)$-conjugacy class of $A$. Then $\lambda^{-1}(C)$ contains more than one $\Gamma_{e,r}$-conjugacy class. Here, $\lambda$ is the map induced from the action of $\Gamma_{e,r}$ on $G_{e,r}/G_{e,r}(2) \simeq Z^{2g}$. 

REMARK. By a result of M. Asada [1], the above corollary holds for $g=2$ in a slightly weaker form.

References


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