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Motion of essentially admissible V-shaped polygonal curves governed by generalized crystalline motion with a driving force

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Abstract. In this paper we consider the motion of non-closed planar polygonal curves governed by generalized crystalline curvature flow with a driving force. In the context of "crystalline motion", we usually restrict the curves in the special class of polygonal curves, so-called "admissible class." We here extend the previous results to wider class which is called "essentially admissible class." In such a class, there are no order-preserving structure, thus, controlling the movement of the solution curves becomes more difficult. In this paper we investigate the estimate of the movement of each facet of the solution curves in the essentially admissible class and show the global existence of V-shaped solutions.

Keywords. Motion by crystalline curvature, non-closed polygonal curves, essentially admissible curves, V-shaped interface.

1. Introduction

Around 1990, Angenent and Gurtin [1] (See also [4].) and J. Taylor [12] introduce the motion of polygonal curves in the special class. They establish the notion of crystalline curvature for a polygonal curve in the special class which is called *admissible class*. Now we call these kind of interface motion *crystalline motion* or *crystalline curvature flow*. After their pioneer works, many authors discuss the behavior of the solutions for several types of crystalline motions in the case when the interface is closed (see [3, 6, 2, 13, 7] and their references).

In this paper we consider the motion of non-closed planar polygonal curves governed by generalized crystalline curvature flow with a driving force:

$$\beta(N_j)V_j = U - g(H_j), \tag{1}$$

where V_j, N_j and H_j denote an outward velocity, an outward normal vector and a crystalline curvature of the j-th *facet* of solution curve, respectively. Here, "*facet*" means a lines segment of solution curve $\Gamma(t)$ and $\mathcal{F}_j$ denotes the j-th facet of $\Gamma(t)$. The positive function β and U describe an anisotropy of the mobility and a driving force, respectively. In this paper we consider the case that $U > 0$ and the function $g = g(\lambda)$ is odd, locally Lipschitz continuous, monotone increasing in λ and has linear or superlinear growth for $\lambda \sim \pm\infty$. Marutani *et al.*[11] shown the existence of traveling "V-shaped" solutions for the case $g(\lambda) = \lambda$ as the limit of the smooth solution for weighted curvature flow. In [9], we consider the simple case $\beta \equiv 1$ and $g(\lambda) = \lambda$ and show that the solution with non-V-shaped initial curve eventually becomes V-shaped in finite time under some conditions. We call this phenomena "*eventual monotonicity of the shape*." In [10], we treat the general motion (1) and show the sufficient condition on the anisotropy β for the existence and uniqueness of traveling V-shaped solution and the global existence of V-shaped solution. We also show in [10] eventual monotonicity of the shape for (1) under some additional conditions.

In this paper, we extend the previous results on V-shaped solutions in [10] for wider class of piecewise linear curve which is called *essentially admissible class*. The notion of "essentially admissible class" is proposed for example in [5]. For essentially admissible polygons, the motion of closed interfaces in the case $U = 0$ is discussed in [14] for convex initial curves and [8] for non-convex initial curves. However, for the case $U \neq 0$, we can easily construct the counter example for the comparison theorem. In [10], the comparison argument is very useful to control the movement of solution curves. We constructed some barrier interfaces using traveling V-shape solutions and enclosed the moving region of each facet. However, for essentially admissible curves, we cannot use this argument because the comparison theorem is not valid. Therefore, we need another argument for controlling the movement of each facet to show global existence of V-shaped solutions.

The paper is organized as follows: In next section, we prepare some notation and definitions. In Section 3, the existence of traveling V-shaped solution and the global existence of V-shaped solutions are discussed.

2. Preliminaries

2.1. Notation and definitions

Let $\mathcal{W}_\sigma$ be N_σ-sided convex polygon and denote the Wulff shape. In the physical context, the Wulff shape describes the equilibrium shape of crystal and given by $\mathcal{W}_\sigma = \{\boldsymbol{z} \in \mathbb{R}^2 \,|\, \boldsymbol{z} \cdot \boldsymbol{n} \le \sigma(\boldsymbol{n})$ for all $\boldsymbol{n} \in S^1\}$, where $\sigma = \sigma(\boldsymbol{n})(> 0)$ is an interfacial energy function defined on S^1. The energy σ is called *crystalline* energy if the Wulff shape is polygon.

Let Ω be a simply connected region in the plane which boundary $\Gamma = \partial\Omega$ is a piece-wise linear curve. We denote by $\mathcal{F}_j$ the j-th facet of Γ. Namely, $\mathcal{F}_j$ is a line segment from p_{j-1} to p_j, where $p_j = (x_j, y_j)$ is the j-th vertex of Γ. We denote by d_j the length of $\mathcal{F}_j$, that is, $d_j = |p_j - p_{j-1}|$. Here and hereafter the subscripts are listed in the order which satisfies that the unit tangent and the outward unit normal vectors are given by $\boldsymbol{T}_j = (p_j - p_{j-1})/d_j$ and $\boldsymbol{N}_j = \boldsymbol{T}_j^\perp$, where $(a, b)^\perp = (b, -a)$. Note that $\boldsymbol{N}_j = (\cos\theta_j, \sin\theta_j)$ where θ_j denotes the outward normal angle of $\mathcal{F}_j$.

We here introduce the notion "admissibility" and the definition of crystalline curvature. We denote by $\mathcal{N}_{\mathcal{W}_\sigma}$ the set of all outward unit normal vectors of $\mathcal{W}_\sigma$. The facet $\mathcal{F}_j$ is said to be *admissible* if $\boldsymbol{N}_j \in \mathcal{N}_{\mathcal{W}_\sigma}$ and the vertex p_j is said to be *admissible* if $(s\boldsymbol{N}_j + (1-s)\boldsymbol{N}_{j+1})/|s\boldsymbol{N}_j + (1-s)\boldsymbol{N}_{j+1}| \notin \mathcal{N}_{\mathcal{W}_\sigma}$ for any $0 < s < 1$. If the facet (resp. the vertex) is not admissible, we call it *non-admissible* facet (resp. vertex). The region Ω and its boundary Γ are said to be *essentially admissible* (resp. *admissible*) if all vertices are admissible (resp. all vertices and all facets are admissible). In figure 1, we show examples of admissible, essentially admissible and non essentially admissible polygons when the Wulff shape is a square. We next define the crystalline curvature H_j of $\mathcal{F}_j$ by $H_j := \chi_j l_\sigma(\boldsymbol{N}_j)/d_j$, where $l_\sigma(\boldsymbol{n})$ is the length of the facet of the Wulff shape with normal $\boldsymbol{n}$ if $\boldsymbol{n} \in \mathcal{N}_{\mathcal{W}_\sigma}$, or zero if $\boldsymbol{n} \notin \mathcal{N}_{\mathcal{W}_\sigma}$. The quantity χ_j is called transition number and given by $\chi_j := \big(\mathrm{sgn}(\theta_j - \theta_{j-1}) + \mathrm{sgn}(\theta_{j+1} - \theta_j)\big)/2$. This takes $+1$ (resp. -1) if Γ is convex (resp. concave) at $\mathcal{F}_j$ in the outward normal direction; otherwise it takes zero. Throughout this paper we call a facet with zero transition number "inflection facet."

Remark 1. The crystalline curvature of a non-admissible facet is always zero independently of its transition number χ_j.

The lengths $\{d_j\}$'s satisfy the following system of ordinary differential equations:

$$\dot{d}_j(t) = -(\cot(\theta_{j+1} - \theta_j) + \cot(\theta_j - \theta_{j-1}))V_j + \frac{1}{\sin(\theta_j - \theta_{j-1})}V_{j-1} + \frac{1}{\sin(\theta_{j+1} - \theta_j)}V_{j+1}, \tag{2}$$

where $\{V_j\}$'s satisfy the equation (1). Here and hereafter, we always assume that Γ contains finite facets, that is, the dimension of the above system is finite. Thus, we have the short-time existence and uniqueness results by the standard theory of differential equations. We also have each $d_j(t)$ is positive in a short time interval since the length of each facet at the initial time is positive, from which it deduce that the solution is essentially admissible in short time interval if the initial curve is essentially admissible. However, facet-extinction may occur in finite time and there is a possibility that the solution goes out from the essentially admissible class during a time evolution. In the following sections we treat this issue.

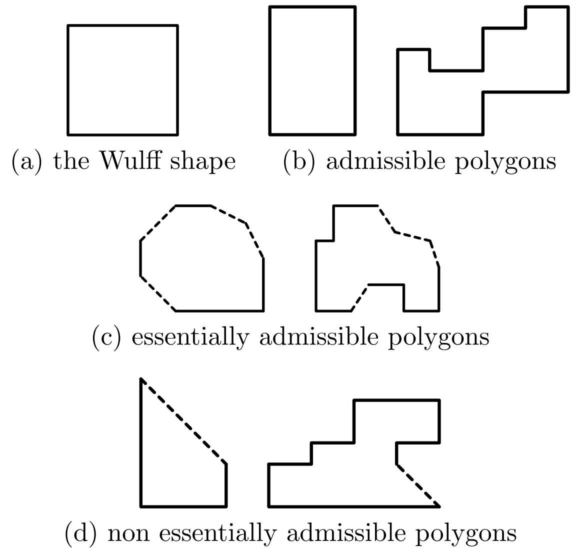

Figure 1: Examples of admissible polygons (b), essentially admissible polygons (c) and non essentially admissible polygons (d) when the Wulff shape is a square (a). In each example, solid (resp. dotted) line segments are *admissible* (resp. *non-admissible*) facets, respectively.

2.2. Setting for V-shaped solutions

We first define "V-shaped" solutions. We say that Ω is V-shaped if Ω satisfies the following: (i) Ω is unbounded simply connected, (ii) Γ has two asymptotic lines, that is, $\partial\Omega = \Gamma = \bigcup_{j=0}^{N+1} \mathcal{F}_j$, where N is the number of facets of Γ except two asymptotic lines and (iii) $\chi_j = -1$ for $j = 1, 2, \cdots, N$. Note that $\mathcal{F}_0$ and $\mathcal{F}_{N+1}$ have infinity length, that is, $d_0 = d_{N+1} = \infty$ and thus $\beta(\boldsymbol{N}_0)V_0 = \beta(\boldsymbol{N}_{N+1})V_{N+1} = U$.

Without loss of generality, we can rotate the plane and the setting of the Wulff shape such that $0 < \theta_{N+1} < \theta_0 < \pi$ and $\beta(\boldsymbol{N}_0)\sin\theta_0 = \beta(\boldsymbol{N}_{N+1})\sin\theta_{N+1}$. Then, the intersection point of two asymptotic lines has a constant velocity vector $(0, c^*)$ where $c^* = U/(\beta(\boldsymbol{N}_0)\sin\theta_0)$ (see the following remark).

Remark 2. Let $q(t) \in \mathbb{R}^2$ be an intersection point of two moving lines which have normal angles θ_a and θ_b and normal velocities V_a and V_b, respectively. Then, the velocity vector of q is given by

$$\dot{q} = \frac{1}{\sin(\theta_a - \theta_b)}\big(V_a(-\sin\theta_b, \cos\theta_b) + V_b(\sin\theta_a, -\cos\theta_a)\big). \tag{3}$$

3. Properties of V-shape solutions

3.1. Traveling V-shaped solutions

In this subsection, we construct a steady traveling V-shaped solution in the (essentially) admissible class. Here, the term "*steady*" means that the solution does not change its shape and has a constant velocity vector.

For steady solutions, the relative positions between each vertex p_j $(j = 0, 1, \cdots, N)$ and the intersection point of two asymptotic lines never change. Thus, we need $\dot{p_j} = (\dot{x}_j, \dot{y}_j) = (0, c^*)$. From (3), we have $\dot{x}_j \sin(\theta_j - \theta_{j+1}) = -V_j \sin\theta_{j+1} + V_{j+1}\sin\theta_j$ and $\dot{y}_j \sin(\theta_j - \theta_{j+1}) = V_j \cos\theta_{j+1} - V_{j+1}\cos\theta_j$. From the first formula, we have $\dot{x}_j = 0$ if and only if $V_j = \mu_j \sin\theta_j$ and $V_{j+1} = \mu_j \sin\theta_{j+1}$ for some $\mu_j > 0$. Then, from the second formula, we have $\dot{y}_j = \mu_j$. From $\dot{y}_j = c^*$, we obtain $\mu_j = c^*$, that is, $V_j = c^* \sin\theta_j$ for each j. By equation (1) and $\chi_j = -1$ for $j = 1, 2, \cdots, N$, we have for admissible facet

$$\begin{aligned} g(l_\sigma(\boldsymbol{N}_j)/d_j) &= \beta(\boldsymbol{N}_j)c^*\sin\theta_j - U \\ &= \Big(\frac{\beta(\boldsymbol{N}_j)\sin\theta_j}{\beta(\boldsymbol{N}_0)\sin\theta_0} - 1\Big)U =: \xi_j \end{aligned}$$

and for non-admissible facet

$$\beta(\boldsymbol{N}_j)c^*\sin\theta_j = U, \text{ that is, } \beta(\boldsymbol{N}_j)\sin\theta_j = \beta(\boldsymbol{N}_0)\sin\theta_0.$$

Since $g(l_\sigma(\boldsymbol{N}_j)/d_j) > 0$ for $d_j > 0$, thus we require $\xi_j > 0$ for admissible facet. Thus, we set the following two conditions:

(B1a): $\beta(\boldsymbol{N}_j)\sin\theta_j > \beta(\boldsymbol{N}_0)\sin\theta_0$,
for admissible facet.
(B1n): $\beta(\boldsymbol{N}_j)\sin\theta_j = \beta(\boldsymbol{N}_0)\sin\theta_0$,
for non-admissible facet.

Under these conditions, we can construct a steady traveling V-shaped solution as follows:

Theorem 1. *There exists a steady traveling V-shaped solution for each g, U and β if and only if (B1a) and (B1n) are satisfied. Moreover, this solution has the following properties:*
(1) $d_j(t) \equiv l_\sigma(\boldsymbol{N}_j)/g^{-1}(\xi_j)$ for each admissible facet.
(2) For non-admissible facet, the length is arbitrary positive.
(3) $\dot{p}_j(t) \equiv (0, c^*)$ for each $j = 0, 1, \cdots, N$.

Remark 3. From (2) in the above theorem, steady traveling V-shaped solution is not unique if the solution curve has a non-admissible facet except $\mathcal{F}_0$ and $\mathcal{F}_{N+1}$.
If the solution is admissible, we only require the condition (B1a) and the solution has the properties (1) and (3) (see [10]).

3.2. Movement of the facet

For V-shaped solution, each facet moves away to infinity as $t \to \infty$ if it exists globally since $V_j \geq U/\beta(\boldsymbol{N}_j)$. In what follows, we see that no facets move away to infinity in finite time.

If $\mathcal{F}_j$ is non-admissible, then $V_j = U/\beta(\boldsymbol{N}_j)$. Thus, $\mathcal{F}_j$ never move away to infinity in finite time, from which it deduce that $d_j(t) < \infty$ for finite $t > 0$ since two intersection points $\mathcal{L}_j(t) \cap \mathcal{L}_0(t)$ and $\mathcal{L}_j(t) \cap \mathcal{L}_{N+1}(t)$ exist as long as $\mathcal{F}_j$ exists. Here $\mathcal{L}_j(t)$ is a line which contains the j-th facet $\mathcal{F}_j(t)$, that is, $\mathcal{L}_j(t) := \{z \in \mathbb{R}^2 | (p_j(t) - z)\cdot \boldsymbol{N}_j = 0\}$.

We next analyze the movement of admissible facet $\mathcal{F}_j$. Note that $V_j = (U + g(l_\sigma(\boldsymbol{N}_j)/d_j))/\beta(\boldsymbol{N}_j) > U/\beta(\boldsymbol{N}_j)$ since $\chi_j = -1$. Assume that $\mathcal{F}_j$ moves away to infinity in finite time $T > 0$, that is,

$$\int_0^T V_j(t)dt = \infty. \tag{4}$$

Then, we have

$$\int_0^T g(1/d_j(t))dt = \infty. \tag{5}$$

Note that we can show $\liminf_{t\to T} d_j(t) = \limsup_{t\to T} d_j(t)$ for $j = 1, 2, \cdots, N$ by the same manner as in [7]. If $\lim_{t\to T} d_j(t) = \infty$, then $d_j(t) > 1$ in $(T-\delta, T)$ for some $\delta > 0$. Thus, we have

$$\int_{T-\delta}^T g(1/d_j(t))dt < \int_{T-\delta}^T g(1)dt < \infty.$$

This leads a contradiction to (5). Hence, we have $\lim_{t\to T} d_j(t) < \infty$ and thus, $w(t) := \sum_{j=1}^N d_j(t) < \infty$ for $0 \leq t \leq T$.

On the other hand, (4) leads

$$\begin{aligned} w_j(t) &:= \operatorname{dist}(\mathcal{L}_j(t)\cap\mathcal{L}_0(t), \mathcal{L}_j(t)\cap\mathcal{L}_{N+1}(t)) \\ &\to \infty \qquad \text{as} \quad t \to \infty \end{aligned}$$

since $V_0, V_{N+1} < \infty$ and $0 < \theta_0 - \theta_{N+1} < \pi$. Since the solution is V-shaped, we also have $w(t) \geq w_j(t)$ and thus we have a contradiction. Therefore, we summarize the above arguments as follows:

Proposition 1. *For any V-shaped solutions in essentially admissible class, all facets remain in bounded region and the length of any facet is bounded in finite time. Moreover, all facets move away to infinity as $t \to \infty$.*

3.3. Global existence of V-shaped solutions

In this subsection, we show that V-shaped solutions exist globally in time in the essentially admissible class.

We first show that any admissible facets never disappear in finite time. Let $T_1 := \sup\{t | \inf_j d_j(t) > 0\}$.

Lemma 1. *No admissible facets of an essentially admissible solution disappear in finite time.*

Proof. Assume that $T_1 < \infty$ and at least one admissible facet disappear at $t = T_1$. Then, there exist j_1 and j_2 such that $0 \leq j_1 < j_2 \leq N+1$, $\bigcup_{k=j_1+1}^{j_2-1}\mathcal{F}_k(t)$ is an extinction part, which contains at least one admissible facet $\mathcal{F}_i$ $(j_1 < i < j_2)$, and $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$ remain at $t = T_1$. Note that $0 < \theta_{j_1} - \theta_{j_2} < \pi$. Set $Q := \{j_1+1, j_1+2, \ldots, j_2-1\}$. Since

$\lim_{t\to T_1} d_j(t) = 0$ for $j \in Q$, we can define the meeting point $p^* := \lim_{t\to T_1} p_i(t)$ for $i \in Q \cup \{j_2\}$. Let $q(t)$ be the intersection point of two lines $\mathcal{L}_{j_1}$ and $\mathcal{L}_{j_2}$ and define two functions $h_1(t)$ and $h_2(t)$ by $h_1(t) := (p^* - q(t)) \cdot \boldsymbol{N}_i$ and $h_2(t) := (p^* - p_i(t)) \cdot \boldsymbol{N}_i$, respectively. Note that $h_1(T_1) = h_2(T_1) = 0$. From $d_i(T_1) = 0$, we have $V_i(t) \to \infty$ as $t \to T_1$. Thus, $V_i(t) > 0$ in $t \in (T_1 - \delta, T_1)$ for some $\delta > 0$. Then, by geometry, $h_1(t) > h_2(t) > 0$ in $t \in (T_1 - \delta, T_1)$. Since d_{j_1} and d_{j_2} remain positive, $\dot{h}_1(t)$ is bounded, while $\dot{h}_2(t) \to -\infty$ as $t \to T_1$. Thus, the function $h_1(t) - h_2(t)$ increases near $t = T_1$ and this is a contradiction. □

From the above lemma, admissible facet never disappear as long as the solution exists. However, there is a possibility of extinction of non-admissible facet. Indeed, we can construct such an example, that is, non-admissible facet disappear in finite time (see Remark 4). On the other hand, for suitable anisotropy β (see (B1a) and (B1n)), there exist steady traveling V-shaped solutions which have non-admissible facets. For these solutions, the facet-extinction of non-admissible facet never occur.

Remark 4. Let $0 = \theta_3 < \theta_2 < \theta_1 < \theta_0 = \pi/2$, $\beta \equiv 1$ and $U = 1$. Set $\{\mathcal{F}_j(0)\}$'s such that $\Gamma(0)$ is circumscribed to a unit circle centered at $(1, 1)$, that is, $\mathcal{L}_0(0) = \{y = 0\}$, $\mathcal{L}_3(0) = \{x = 0\}$. Suppose that $\mathcal{F}_1$ is admissible and $\mathcal{F}_2$ is non-admissible. Assume that all facet exist in $[0, 1]$. Then, $\mathcal{L}_0(1)$, $\mathcal{L}_2(1)$ and $\mathcal{L}_3(1)$ pass through the point $(1, 1)$. More precisely, $\mathcal{L}_0(1) = \{y = 1\}$, $\mathcal{L}_3(1) = \{x = 1\}$ and $\mathcal{L}_2(1)\cap\{x > 1 \text{ and } y > 1\}$ is empty. However, $\mathcal{L}_1(1)\cap\{x > 1 \text{ and } y > 1\}$ is not empty since $V_1 > 1$ and thus this leads a contradiction, from which it deduce that $\mathcal{F}_2$ disappears before $t = 1$.

Remark 5. We can construct steady traveling V-shaped solutions whose all facets are non-admissible. For example, choose two adjacent normals α_1, α_2 of the Wulff shape such that $0 < \alpha_1 < \alpha_2 < \pi$ and set $\{\theta_j\}$'s as $\alpha_1 < \theta_{N+1} < \theta_N < \cdots < \theta_0 < \alpha_2$ and $d_j(0) > 0$ for $j = 1, 2, \cdots, N$, arbitrarily. Then, this V-shaped curve satisfies the condition for essentially admissibility. If β satisfies (B1n), then the solution from this curve as the initial curve is steady traveling V-shaped solution.

Suppose that some non-admissible facets disappear at $t = T_1$. Since there exist the limits $\lim_{t\to T_1} p_j(t)$ for all j, there is the limit shape of the solution: $\Omega(T_1) := \lim_{t\to T_1} \Omega(t)$. Let the curve $\sum_{j=j_1+1}^{j_2-1} \mathcal{F}_j$ be one of the consecutive extinction part of Γ and $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$ remaining facets at $t = T_1$. Since $\boldsymbol{N}_j \notin \mathcal{N}_{\mathcal{W}_\sigma}$ for $j = j_1+1, \cdots, j_2-1$ by Lemma 1, $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$ satisfy

$$\frac{s\boldsymbol{N}_{j_1} + (1-s)\boldsymbol{N}_{j_2}}{|s\boldsymbol{N}_{j_1} + (1-s)\boldsymbol{N}_{j_2}|} \notin \mathcal{N}_{\mathcal{W}_\sigma} \quad \text{for any} \quad 0 < s < 1.$$

Thus, $\Omega(T_1)$ belongs to the essentially admissible class and we obtain the evolution from $\Omega(T_1)$ as the initial data for $t > T_1$. We can repeatedly apply the above argument at the facet-extinction times and obtain the solution for all $t > 0$ since admissible facets and two asymptotic lines $\mathcal{F}_0$ and $\mathcal{F}_{N+1}$ never disappear in finite time. From Proposition 1, we have the following:

Theorem 2. *V-shaped solutions exist globally in time. Moreover, all facets move to infinity as t tends to ∞.*

Acknowledgments

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