A new Euclidean tight 6-design

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Abstract

Euclidean t-designs, which are finite weighted subsets of Euclidean space, were defined by Neumaier-Seidel (1988). A tight t-design is defined as a t-design whose cardinality is equal to the known natural lower bound.

In this paper, we give a new Euclidean tight 6-design in $\mathbb{R}^{22}$. Furthermore, we also show its uniqueness up to similar transformation fixing the origin.

This design has the structure of coherent configuration, which was defined by Higman, and is obtained from the properties of general permutation groups. We also show that the design is obtained by combining two orbits of McLaughlin simple group.

1 Introduction

Euclidean t-designs were defined by Neumaier-Seidel [21] as a two step generalization of spherical designs (cf. [13]). (We note that similar concepts as Euclidean t-designs have existed in numerical analysis as certain cubature formulas, and in statistics as rotatable designs (cf. [8]).

First we give some notation. Let (X, w) be a finite weighted subset in Euclidean space $\mathbb{R}^n$, where X is a finite subset and w is a positive real valued weight function on X. We say X is supported by p concentric spheres. That is, there are distinct nonnegative integers $r_1, \ldots, r_p$, sphere S_i of radius r_i centered at the origin and subset $X_i = X \cap S_i$ for each $1 \le i \le p$, such that $X = X_1 \cup \cdots \cup X_p$. Let $w(X_i) = \sum_{x \in X_i} w(x)$ and $|S_i| = \int_{S_i} d\sigma_i(x)$. We denote by $\mathcal{P}(\mathbb{R}^n)$ the vector space of polynomials in n variables $x_1, \ldots, x_n$ over the fields $\mathbb{R}$ of real numbers. Let $\mathrm{Hom}_l(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ which consists of homogeneous polynomials of degree l, and let $\mathcal{P}_l(\mathbb{R}^n) = \oplus_{i=0}^{l} \mathrm{Hom}_l(\mathbb{R}^n)$.

Definition 1 (Euclidean t-design)**.** (see [21]) *Let t be a positive integer. A weighted finite set (X, w) in $\mathbb{R}^n$ is a Euclidean t-design, if the following equation*

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{\boldsymbol{x} \in S_i} f(\boldsymbol{x})\, d\sigma_i(\boldsymbol{x}) = \sum_{\boldsymbol{x} \in X} w(\boldsymbol{u}) f(\boldsymbol{u}) \qquad (1.1)$$

is satisfied for any polynomial $f \in \mathcal{P}_t(\mathbb{R}^n)$.

There is known a natural lower bound for the cardinalities of Euclidean t-designs (see [20, 14, 21, 9, 6]), and tight t-design is defined as a t-design whose cardinality is equal to this lower bound, (see [14, 21, 4, 6, 9, 8]). This lower bound for t-design for even t is straightforward, but the lower bound for odd t is somewhat delicate. (See [8], or the papers referred there, in particular those by Möller [19, 20], etc., for more details.)

Here, we give only for the case where t is even:

Theorem 2. (see [20, 14]) *Let (X, w) be a Euclidean $2e$-design supported by p concentric spheres S in $\mathbb{R}^n$. Then*

$$|X| \geq \dim(\mathcal{P}_e(S))$$

holds, where $\mathcal{P}_e(S) = \{f|_s : f \in \mathcal{P}_e(\mathbb{R}^n)\}$

Definition 3. (see [14, 4])

(1) *Definitions and notations are the same as above. If the equality* (**??**) *hold, then (X, w) is called a tight $2e$-design on p concentric spheres.*

(2) *Moreover, if $\dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n)) \left(= \binom{n+e}{e}\right)$ holds, then (X, w) is called a Euclidean tight $2e$-design of $\mathbb{R}^n$.*

Many tight Euclidean t-designs have been constructed (see [1, 2, 3, 4, 8, 9, 10, 18]). However, the studies have been so far mostly limited to either (i) for $n = 2$ or (ii) for $t \leq 5$ or $t = 7$ (for $n \geq 3$).

In the previous paper [7], we observed that some of the known tight Euclidean t-designs have the structure of coherent configuration. (In particular, tight t-designs on two concentric spheres always have this property.) Here note that coherent configuration is a concept defined by Higman [15, 16] as a generalization of association schemes. In [7] we tried to classify certain Euclidean t-designs which have the structure of coherent configuration. Furthermore, trying to generalize the work of [7], we started to study Euclidean tight 6-designs on two concentric spheres with one layer (fiber) being a spherical tight 4-design. (Such a tight 6-design on two concentric spheres is automatically a tight 6-design of $\mathbb{R}^n$ as well.) This classification problem is not yet completed, but we were able to show that there is only one feasible parameter set remains if n is small, say $3 \leq n \leq 438$. (This is the parameter set described in Section 2 of this paper.) At first we were a bit surprised and excited at finding this new feasible parameter set. Then we noticed that there actually exists such a Euclidean tight 6-design, by combining two orbits of the McLaughlin simple group acting as orthogonal transformations on the Euclidean space $\mathbb{R}^{22}$. Although these two permutation representations themselves are well known, it seems new to observe that they actually lead to a Euclidean tight 6-design. (Compare this with the well known fact that there exists no spherical tight 6-design for $n \geq 3$.) The main purpose of this short note is to describe this new design. Namely, we obtain:

Theorem 4. *There exists a Euclidean tight 6-design of $\mathbb{R}^{22}$ supported by two concentric spheres of cardinality $\binom{22+3}{3}$ with the ratio $r_2/r_1 = \sqrt{11}$ of the radii r_1 and r_2, and the ratio $w_2/w_1 = 1/729$ of the two weights w_1 and w_2.*

Remark 1. We describe all the parameters of the associated coherent configuration in Section 2. Also, we remark that this Euclidean tight 6-design is unique in $\mathbb{R}^{22}$ supported by two concentric spheres with $|X_1| = 275$, up to similar transformation fixing the origin.

2 Parameters of the Euclidean tight 6-design in $\mathbb{R}^{22}$

Let (X, w) be a tight Euclidean 6-design of $\mathbb{R}^{22}$ supported by 2 concentric spheres of positive radii r_1 and r_2. Let $X = X_1 \cup X_2$, and $X_i \subset S^{21}$ $(i = 1, 2)$. Then Lemma 1.10 in [4] implies that the weight function w is constant on each X_i. Let $w \equiv w_i$ on X_i for $i = 1, 2$. Also Theorem 1.5 in [7] implies that X has the structure of a coherent configuration with 2 fibers. Also Theorem 1.8 in [9] implies that X_1 and X_2 are spherical 4-designs. In the following we assume that X_1 is a tight spherical 4-design, i.e., $|X_1| = 275$.

We define $A(X_i, X_j) = \{\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_i r_j} \mid \boldsymbol{x} \in X_i,\ \boldsymbol{y} \in X_j\ \boldsymbol{x} \neq \boldsymbol{y}\}$ for any $i, j = 1, 2$. Then (Proof of) Lemma 1.10 in [4] implies that $|A(X_i, X_j)| \le 3$. Since we must have $|X_2| = |X| - |X_1| = \binom{22+3}{3} - 275 = 2025$, X_2 must be a 3-distance set, i.e., $|A(X_2, X_2)| = 3$ holds. We can also prove that $|A(X_1, X_2)| = 3$ holds. Also, we can prove that

$$A(X_1, X_1) = \{\alpha_1 = \frac{1}{6},\ \alpha_2 = -\frac{1}{4}\}, \tag{2.1}$$

$$A(X_2, X_2) = \{\beta_1 = \frac{7}{22}, \beta_2 = -\frac{1}{44}, \beta_3 = -\frac{4}{11}\}, \tag{2.2}$$

$$A(X_1, X_2) = \{\gamma_1 = \frac{1}{\sqrt{11}},\ \gamma_2 = -\frac{1}{4\sqrt{11}},\ \gamma_3 = -\frac{3}{2\sqrt{11}}\}, \tag{2.3}$$

$\frac{w_2}{w_1} = \frac{1}{729}$, and $\frac{r_2}{r_1} = \sqrt{11}$ hold.

Define $\alpha_0 = 0$, $\beta_0 = 0$. Then the structure parameters of the coherent algebra (intersection numbers of the coherent configuration) $p^{\alpha_k}_{\alpha_i,\alpha_j}$ $(0 \le i, j, k \le 2)$, $p^{\alpha_k}_{\gamma_i,\gamma_j}$ $(0 \le k \le 2,\ 1 \le i, j \le 3)$, $p^{\beta_k}_{\beta_i,\beta_j}$ $(0 \le i, j, k \le 3)$ $p^{\alpha_k}_{\gamma_i,\gamma_j}$ $(0 \le k \le 3,\ 1 \le i, j \le 3)$, $p^{\gamma_k}_{\alpha_i,\gamma_j} = p^{\gamma_k}_{\gamma_j,\alpha_i}$ $(0 \le i \le 2,\ 1 \le j, k \le 3)$, and $p^{\gamma_k}_{\beta_i,\gamma_j} = p^{\gamma_k}_{\gamma_j,\beta_i}$, $0 \le i \le 3,\ 1 \le j, k \le 3$ are determined uniquely to the values listed below.

Until this stage we just used the values of inner product $\gamma_1, \gamma_2, \gamma_3$ to determine the structure of the coherent configuration. In the following we introduce the following description. $R_{\alpha_i} = \{(\boldsymbol{x}, \boldsymbol{y}) \in X_1 \times X_1 \mid \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_1^2} = \alpha_i\}$ for $i = 0, 1, 2$, $R_{\beta_i} = \{(\boldsymbol{x}, \boldsymbol{y}) \in X_2 \times X_2 \mid \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_2^2} = \beta_i\}$ for $i = 0, 1, 2, 3$, $R_{\gamma_i^{(+)}} = \{(\boldsymbol{x}, \boldsymbol{y}) \in X_1 \times X_2 \mid \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_1 r_2} = \gamma_i\}$ for $i = 1, 2, 3$, and $R_{\gamma_i^{(-)}} = \{(\boldsymbol{x}, \boldsymbol{y}) \in X_2 \times X_1 \mid \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_1 r_2} = \gamma_i\}$ for $i = 1, 2, 3$. Thus $X \times X$ is partitioned into 13 subsets. For $(\boldsymbol{x}, \boldsymbol{y}) \in R_c$ we denote $p^c_{a,b} = |\{\boldsymbol{z} \in X \mid (\boldsymbol{x}, \boldsymbol{z}) \in R_a,\ (\boldsymbol{z}, \boldsymbol{y}) \in R_b\}|$. We define 13×13 matrices B_a, intersection matrix, whose rows and columns are indexed by the set $\{\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \beta_3, \gamma_1^{(+)}, \gamma_2^{(+)}, \gamma_3^{(+)}, \gamma_1^{(-)}, \gamma_2^{(-)}, \gamma_3^{(-)}\}$ with this ordering, where $a \in \{\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \beta_3, \gamma_1^{(+)}, \gamma_2^{(+)}, \gamma_3^{(+)}, \gamma_1^{(-)}, \gamma_2^{(-)}, \gamma_3^{(-)}\}$. The (b, c) entry of B_a is defined by $B_a(b, c) = p^c_{a,b}$ for any $a, b, c \in \{\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \beta_3, \gamma_1^{(+)}, \gamma_2^{(+)}, \gamma_3^{(+)}, \gamma_1^{(-)}, \gamma_2^{(-)}, \gamma_3^{(-)}\}$.

$$B_{\alpha_0} = \left[\begin{array}{c|c|c|c} \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \quad B_{\alpha_1} = \left[\begin{array}{c|c|c|c} \begin{matrix}0&1&0\\162&105&81\\0&56&81\end{matrix} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}60&42&21\\96&105&120\\6&15&21\end{matrix} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

$$B_{\alpha_2} = \left[\begin{array}{c|c|c|c} \begin{matrix}0&0&1\\0&56&81\\112&56&30\end{matrix} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}16&35&56\\80&70&56\\16&7&0\end{matrix} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

$$B_{\beta_0} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \end{array}\right], \quad B_{\beta_1} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}0&1&0&0\\462&185&96&28\\0&256&291&280\\0&20&75&154\end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \begin{matrix}216&105&21\\240&315&336\\6&42&105\end{matrix} \end{array}\right],$$

$$B_{\beta_2} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}0&0&1&0\\0&256&291&280\\1232&776&730&784\\0&200&210&168\end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \begin{matrix}320&357&336\\816&770&840\\96&105&56\end{matrix} \end{array}\right], \quad B_{\beta_3} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}0&0&0&1\\0&20&75&154\\0&200&210&168\\330&110&45&7\end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \begin{matrix}30&105&210\\240&210&120\\60&15&0\end{matrix} \end{array}\right]$$

$$B_{\gamma_1^{(+)}} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}1&0&0\\216&105&21\\320&357&336\\30&105&210\end{matrix} & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \begin{matrix}567&210&81\\0&336&405\\0&21&81\end{matrix} & 0 & 0 \quad 0 \quad 0 & 0 \end{array}\right], \quad B_{\gamma_2^{(+)}} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}0&1&0\\240&315&336\\816&770&840\\240&210&120\end{matrix} & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \begin{matrix}0&336&405\\1296&840&810\\0&120&81\end{matrix} & 0 & 0 & \end{array}\right],$$

$$B_{\gamma_3^{(+)}} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}0&0&1\\6&42&105\\96&105&56\\60&15&0\end{matrix} & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \begin{matrix}0&21&81\\0&120&81\\162&21&0\end{matrix} & 0 & 0 & 0 \end{array}\right], \quad B_{\gamma_1^{(-)}} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \begin{matrix}1&0&0\\60&42&21\\16&35&56\end{matrix} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}77&36&20&7\\0&40&51&56\\0&1&6&14\end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

$$B_{\gamma_2^{(-)}} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \begin{matrix}0&1&0\\96&105&120\\80&70&56\end{matrix} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}0&40&51&56\\176&120&110&112\\0&16&15&8\end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \quad B_{\gamma_3^{(-)}} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \begin{matrix}0&0&1\\6&15&21\\16&7&0\end{matrix} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}0&1&6&14\\0&16&15&8\\22&5&1&0\end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

3 Proof of Theorem 1

As is well know and described in ATLAS (page 100) and Wilson [23], there are two permutation representations of $G = McL$ (McLaughlin simple group) of degree 275 and 2025 in which the one point stabilizers are $H_1 = U_4(3)$ and $H_2 = M_{22}$, respectively. The permutation character χ_1 of the first one is decomposed into irreducible characters as $\chi_1 = 1a + 22a + 252a$ and the second χ_2 as $\chi_2 = 1a + 22a + 252a + 1750a$. This implies that by the method described below, we have a coherent configurations of type $[3, 3; 4]$ in the sense of D. G. Higman [17], by considering the decomposition of the permutation characters χ_1 and χ_2. It is easy to see, that if we take the two nonzero points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in $\mathbb{R}^{22}$ fixed by H_1 and H_2 respectively, then G acts on the union of the two orbits $\boldsymbol{x}_1^G \cup \boldsymbol{x}_2^G$. Then it is easy to show that this intransitive action form a coherent configuration with the same parameters as given in Section 2. (Here the distances of $\boldsymbol{x}_i$ from the origin do not affect the structure of coherent configurations. For example, all the parameters $p_{i,j}^k$ as well as $\gamma_1, \gamma_2, \gamma_3$ do not depend on them. It is easy to perform these calculations, if we use either MAGMA or GAP. (It was actually performed.)

Here we note that this is proved more theoretically, by using the known facts. In ATLAS [11], Conway-Sloane [12], Wilson [23], all the 275 points (corresponding to G/H_1) are explicitly described, and it is shown that the action of $H_2 = M_{22}$ on the 275 points are very visible and divided into three orbits of lengths 22, 77 and 176. (See Wilson page 400.) Then, using the fact that these 275 points form a tight spherical 4-design, using the fundamental equation (cf. Venkov [22], or [3]), we can easily determine $\gamma_1, \gamma_2, \gamma_3$, and they are identical with the parameters given in Section 2. This completes a proof of Theorem 1.

The uniqueness of the Euclidean tight 6-design in $\mathbb{R}^{22}$ is proved as follows. Since X_1 forms a tight 4-design in $S^{21} \subset \mathbb{R}^{22}$, and since the uniqueness of tight 4-design in S^{21} is known, we can fix the 275 points on the unit sphere. We want to determine the points on the sphere $S^{21}(r_2)$, with the angles to any of the 275 points in X_1 are one of $\{\gamma_1, \gamma_2, \gamma_3\}$. (Actually we can fix 22 points of X_1, which are linearly independent, and then we can determine the points which have all the angles one of $\{\gamma_1, \gamma_2, \gamma_3\}$. The calculation shows that there are only 4050 of them. They are divided into two subsets each of 2025 points correspond to the two inequivalent transitive permutation representations of McL/M_{22} of the degree 2025, which are interchanged each other by an outer automorphism of McL. This implies the uniqueness of the Euclidean tight 6-design in $\mathbb{R}^{22}$.

Remark 2. The class 3 association schemes $X_2 = McL/M_{22}$ is Q-polynomial (but not P-polynomial), and so it is in the list of Bill Martin (see the home page: `http://users.wpi.edu/~martin/`). According to Bill Martin, he obtained this information originally from an article (by A. Munemasa on "spherical designs" in the Handbook of Combinatorial Designs, 2nd ed. Chapman and Hall/CRC, pp. 617–622.) It is interesting to note that X_2 is a spherical 4-design. Also, it is interesting to note that the characterization (uniqueness) of the association scheme X_2 itself by parameters seems not yet known at the time of this writing.

Remark 3. In this section, we show that the design is obtained by combining two orbits of McLaughlin simple group, that is $McL/U_4(3)$ and McL/M_{22}. In addition, by the advise of Professor Masaaki Kitazume, we noticed that there is a *relationship* between these two orbits and one orbit of the Conway group Co_2 that is $Co_2/U_6(2)$. It has 4600 points in $\mathbb{R}^{23}$, and we can classify them into 2300 antipodal pairs. The *relationship* is one to one correspondence between these pairs and $275 + 2025$ points of two orbits. Note that Conway group Co_2 acts on these 2300 pairs transitively. We can refer to Remark 4 in detail.

Note that $Co_2/U_6(2)$ with 4600 points is a spherical tight 7-design of $\mathbb{R}^{23}$, which is uniquely determined.

4 Some calculations

Here, we want to explain the method to calculate vectors of X. Again by Wilson [23], we obtain vectors of X from the Leech lattice Λ_{24}.

We must choose the vectors $A, B \in \Lambda_{24}$ of norm 4 such that $(A, B) = -1$. Then, we define the following sets of vectors

$$X_1^0 := \{x \in \Lambda_{24} \mid (x,x) = 6,\ (x,A) = 3,\ (x,B) = -3\}, \tag{4.1}$$
$$X_2^0 := \{x \in \Lambda_{24} \mid (x,x) = 4,\ (x,A) = 2,\ (x,B) = 0\}. \tag{4.2}$$

In conclusion, we obtain just 275 (resp. 2025) vectors for X_1^0 (resp. X_2^0) from the shell of the Leech lattice of norm 6 (resp. 4), which has 16773120 (resp. 196560) vectors. It may not be easy to obtain X_1^0 since the shell of norm 6 has too many vectors, but we performed this calculation.

In addition, we consider the orthogonal projection P from $\mathbb{R}^{24}$ to orthogonal complement of the space spanned by vectors A and B. Finally, with the adjustment on radii, we obtain

$$X_1 = P(X_1^0), \quad X_2 = 3\,P(X_2^0).$$

Then, we obtain the design $X = X_1 \cup X_2$.

In the proof of the uniqueness of the design, we obtain 4050 vectors with the angles to any of the 275 vectors in X_1 are one of $\{\gamma_1, \gamma_2, \gamma_3\}$.

First, we want to explain a method of calculation. We denote

$$\overline{X_2} := \left\{ y \in S^{21} \;\middle|\; \forall x \in \frac{1}{r_1} X_1,\ x \cdot y \in \left\{ \frac{1}{\sqrt{11}}, -\frac{1}{4\sqrt{11}}, -\frac{3}{2\sqrt{11}} \right\} \right\}.$$

The angles between the distinct vectors of X_1 are in $\{\frac{1}{6}, -\frac{1}{4}\}$. Thus, the lattice L generated by all the vectors of $\frac{2\sqrt{3}}{r_1} X_1$ is integral. In addition, by the definition of $\overline{X_2}$, the inner products between the vectors of $\frac{2\sqrt{3}}{r_1} X_1$ and the vectors of $\frac{4\sqrt{11}}{2\sqrt{3}} \overline{X_2}$ are in $\{4, -1, -6\}$. Then, $\frac{4\sqrt{11}}{2\sqrt{3}} \overline{X_2}$ is included in the dual lattice $L^\sharp$. Furthermore, we can choose basis $\{e_1, \ldots, e_{22}\}$ of L from $\frac{2\sqrt{3}}{r_1} X_1$. Then, we can also obtain dual basis $\{e_1', \ldots, e_{22}'\}$ of $L^\sharp$ such that $e_i \cdot e_j' = \delta_{i,j}$ for every $1 \leqslant i, j \leqslant 22$. With the dual basis, we can denote the vectors of

$\frac{4\sqrt{11}}{2\sqrt{3}}\overline{X_2}$ by $\sum_i(5c_i - 1)e_i{}'$ for some $c_i \in \{0, \pm 1\}$. In conclusion, we can choose vectors from the lattice which is generated by the vectors $\{5e_1, \ldots, 5e_{22}, -\sum_i e_i{}'\}$. This lattice is much smaller than the dual lattice $L^\sharp$, and the norm of the every vector is equal to $\frac{44}{3}$. Thus, in this method, we may calculate the vectors of $\overline{X_2}$ more easily.

Now, we can obtain 4050 vectors, half of which are the vectors of $\frac{1}{r_2}X_2$. We denote by $X_2{}' := r_2\overline{X_2} \setminus X_2$. Then, the calculation shows that we have $X_2{}' = P(X_2^{0\prime})$ where

$$X_2^{0\prime} := \{x \in \lambda_{24} \mid (x,x) = 4,\ (x,A) = 0,\ (x,B) = -2\}.$$

Moreover, we can write

$$X_1^0 = \{x \in \lambda_{24} \mid (x,x) = 6,\ (x,(-B)) = 3,\ (x,(-A)) = -3\},$$
$$X_2^{0\prime} = \{x \in \lambda_{24} \mid (x,x) = 4,\ (x,(-B)) = 2,\ (x,(-A)) = 0\}.$$

Comparing with the definitions of X_1^0 and X_2^0, it is clear that $X_1 \cup X_2{}'$ is isometric to $X_1 \cup X_2$.

Remark 4. By Remark 3, we can calculate X_1 by another method. We define

$$\begin{aligned} Y_{+1}^0 &:= \{x \in \Lambda_{24} \mid (x,x) = 4,\ (x,A) = 2,\ (x,B) = 1\} \\ Y_{+2}^0 &:= \{x \in \Lambda_{24} \mid (x,x) = 4,\ (x,A) = 2,\ (x,B) = 0\} \quad (= X_2^0) \\ Y_{-2}^0 &:= \{x \in \Lambda_{24} \mid (x,x) = 4,\ (x,A) = 2,\ (x,B) = -1\} \\ Y_{-1}^0 &:= \{x \in \Lambda_{24} \mid (x,x) = 4,\ (x,A) = 2,\ (x,B) = -2\} \end{aligned}$$

where vectors A, B are same as above. Then, we obtain $X_1 = P(Y_{+1}^0)$ by the same orthogonal projection P.

Moreover, we consider another orthogonal projection P_0 from $\mathbb{R}^{24}$ to orthogonal complement of the space spanned by the only vector A. We denote by $Y_{\pm i} = P_0(Y_{\pm i}^0)$ for $i \in \{1,2\}$. Then, we obtain 4600 vectors in $Y_{+1} \cup Y_{+2} \cup Y_{-1} \cup Y_{-2} = Co_2/U_6(2)$. Since $Y_{+i} = -Y_{-i}$ for $i \in \{1,2\}$, we have 2300 antipodal pairs.

Oppositely, let $X = X_1 \cup X_2$ be the Euclidean tight 6-design, having $|X_1| = 275$ and $|X_2| = 2025$. And let $Z_{\pm 1}$ and $Z_{\pm 2}$ be subsets on $S^{22} \subset \mathbb{R}^{23}$ defined below:

$$\begin{aligned} Z_{+1} &= \{(a_1x_1, b_1) \mid x_1 \in X_1\}, & Z_{-1} &= \{-(a_1x_1, b_1) \mid x_1 \in X_1\}, \\ Z_{+2} &= \{(a_2x_2, b_2) \mid x_2 \in X_2\}, & Z_{-2} &= \{-(a_2x_2, b_2) \mid x_2 \in X_2\}, \end{aligned}$$

where $a_1 = \frac{2}{\sqrt{5}}$, $b_1 = \frac{1}{\sqrt{5}}$, $a_2 = \frac{2}{3\sqrt{5}}$, and $b_2 = \frac{1}{3\sqrt{5}}$. Then, $Z = Z_{+1} \cup Z_{+2} \cup Z_{-1} \cup Z_{-2}$ has 4600 points, and Z is a spherical tight 7-design.

References

[1] B. Bajnok, *On Euclidean designs*, Adv. Geom. 6 (2006), 423–438.

[2] B. Bajnok, *Orbits of the hyperoctahedral group as Euclidean designs*, J. Algebraic Combin. 25 (2007), 375–397.

[3] E. Bannai and E. Bannai, *Algebraic Combinatorics on Spheres* (in Japanese) Springer Tokyo 1999.

[4] E. Bannai and E. Bannai, *On Euclidean tight* 4*-designs*, J. Math. Soc. Japan 58 (2006), 775–804.

[5] E. Bannai and E. Bannai, *Spherical designs and Euclidean designs*, in: Recent Developments in Algebra and Related Areas, ALM 8, Higher Education Press and International Press, Beijing-Boston, 2009, 1–37.

[6] E. Bannai and E. Bannai, *A survey on spherical designs and algebraic combinatorics on spheres*, European J. Combin. 30 (2009), 1392–1425.

[7] E. Bannai and E. Bannai, *Euclidean designs and coherent configurations*, to appear in Contemp. Math. (AMS)

[8] E. Bannai, E. Bannai, M. Hirao and M. Sawa, *Cubature formulas in numerical analysis and Euclidean tight designs*, in a special issue in honor of Michel Deza, European J. Combin. 31 (2010), 423–441.

[9] E. Bannai, *On antipodal Euclidean tight* $(2e + 1)$*-designs*, J. Algebraic Combin. 24 (2006), 391–414.

[10] E. Bannai, *New examples of Euclidean tight* 4*-designs*, European J. Combin. 30 (2009), 655–667.

[11] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson and J. G. Thackaray, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[12] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd edition. Springer-Verlag, New York, 1999.

[13] P. Delsarte, J. M. Goethals and J. J. Seidel, *Spherical codes and designs*, Geom. Dedicata 6 (1977), 363–388.

[14] P. Delsarte and J. J. Seidel, *Fisher type inequalities for Euclidean* t*-designs*, Linear Algebra Appl. 114-115 (1989), 213–230.

[15] D. G. Higman, *Coherent configurations. I*, Rend. Sem. Mat. Univ. Padova 44 (1970), 1–25.

[16] D. G. Higman, *Coherent configurations. I*, Ordinary representation theory, Geom. Dedicata 4 (1975), 1–32.

[17] D. G. Higman, *Coherent algebras*, Linear Algebra Appl. 114/115 (1989), 209–239.

[18] M. Hirao, M. Sawa, and Y. Zhou, *Some remarks on Euclidean tight designs*, J. Combin. Theory Ser. A, in press.

[19] H. M. Möller, *Kubaturformeln mit minimaler Knotenzahl*, Numer. Math. 25 (2) (1975/76), 185–200.

[20] H. M. Möller, *Lower bounds for the number of nodes in cubature formulae*, Numerische Integration (Tagung, Math. Forschungsinst., Oberwolfach, 1978), 221–230, Internat. Ser. Numer. Math., 45, Birkhäuser, Basel-Boston, Mass., 1979.

[21] A. Neumaier and J. J. Seidel, *Discrete measures for spherical designs, eutactic stars and lattices*, Nederl. Akad. Wetensch. Proc. Ser. A 91=Indag. Math. 50 (1988), 321–334.

[22] B.B. Venkov, *Réseaux et "designs" sphériques* (French) [Lattices and spherical designs], in: Réseaux euclidiens, designs sphériques et formes modulaires, 10–86, Monogr. Enseign. Math. 37, Enseignement Math., Geneva, 2001. [Notes by J. Martinet]

[23] R. A. Wilson, *Vector stabilizers and sub groups of Leech lattice groups*, J. Algebra 127 (1989), 387–408.

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