

On Poly-Bernoulli numbers

Arakawa, Tsuneo
Department of Mathematics Rikkyo University

Kaneko, Masanobu
Graduate School of Mathematics, Kyushu University

<https://hdl.handle.net/2324/20483>

出版情報 : Commentarii mathematici Universitatis Sancti Pauli = Rikkyo Daigaku sugaku zasshi.
48 (2), pp.159-167, 1999-11. 立教大学

バージョン :

権利関係 :



On Poly-Bernoulli Numbers

by

Tsuneo ARAKAWA and Masanobu KANEKO

(Received April 8, 1999)

1. Main theorems

In our previous paper [1], we defined and studied “poly-Bernoulli numbers” which generalize the classical Bernoulli numbers. As a continuation, we present here two results, one of which is a further investigation of Clausen-von Staudt type theorem that was treated only in “di-Bernoulli” case in [1], the other being a combinatorial closed formula for negative index poly-Bernoulli numbers.

Poly-Bernoulli numbers $B_n^{(k)}$ ($n=0, 1, 2, \dots$) are defined for each integer k by the generating series

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where $Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$. Table of values of $B_n^{(k)}$ for small k and n will be given at the end of the paper. In [1], we obtained an explicit formula for $B_n^{(k)}$:

$$B_n^{(k)} = (-1)^n \sum_{m=0}^n \frac{(-1)^m m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\}}{(m+1)^k}, \quad (1)$$

where $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ is an integer referred to as the Stirling number of the second kind (“Stirling subset number” in Knuth’s terminology, we adopt his notation [2]).

Let p be a prime number. First of all, it is clear from the above formula (1) that the $B_n^{(k)}$ is p -integral when p is larger than $n+1$. Our first theorem gives an information on the p -part of $B_n^{(k)}$ for $p \leq n+1$.

THEOREM 1 (Clausen-von Staudt type theorem). *Assume $k \geq 2$. Let p be a prime number satisfying $k+2 \leq p \leq n+1$.*

(i) *If $n \equiv 0 \pmod{p-1}$, then $p^k B_n^{(k)}$ is a p -adic integer and satisfies*

$$p^k B_n^{(k)} \equiv -1 \pmod{p\mathbb{Z}_p}.$$

(ii) *If $n \not\equiv 0 \pmod{p-1}$, then $p^{k-1} B_n^{(k)}$ is a p -adic integer. It satisfies*

$$p^{k-1}B_n^{(k)} \equiv \begin{cases} \frac{1}{p} \left\{ \begin{matrix} n \\ p-1 \end{matrix} \right\} - \frac{n}{2^k} \pmod{p\mathbb{Z}_p}, & \text{if } n \equiv 1 \pmod{p-1} \\ \frac{(-1)^{n-1}}{p} \left\{ \begin{matrix} n \\ p-1 \end{matrix} \right\} \pmod{p\mathbb{Z}_p} & \text{otherwise.} \end{cases}$$

REMARK 1. That the $p^{k-1}B_n^{(k)}(p-1 \nmid n)$ and $p^k B_n^{(k)}(p-1 \mid n)$ are p -integral ($k+2 \leq p$) has also been obtained independently by Roberto Sánchez-Peregrino in [3].

2. If $n \not\equiv 0, 1 \pmod{p-1}$, the congruence in (ii) may be written as

$$p^{k-1}B_n^{(k)} \equiv (n-n') \frac{B_n^{(1)}}{n} \pmod{p\mathbb{Z}_p},$$

where n' is a unique integer with $n' \equiv n \pmod{p-1}$ and $1 < n' < p$. Actually, it was shown in [1] the congruence

$$\frac{(-1)^{n-1}}{p} \left\{ \begin{matrix} n \\ p-1 \end{matrix} \right\} \equiv (n-n') \frac{B_n^{(1)}}{n} \pmod{p\mathbb{Z}_p}$$

if $n \not\equiv 0, 1 \pmod{p-1}$ (the assumption made there that n being even can be loosened to the present one).

3. When $p > n+1$, the formula (1) shows that the congruence $B_n^{(k)} \equiv B_n^{(k')} \pmod{p}$ holds for any integers k and k' satisfying $k \equiv k' \pmod{p-1}$.

The number $B_n^{(k)}$ is a positive integer when k is non-positive. Our second theorem is a closed formula (which is completely different from (1)) for this integer.

THEOREM 2 (Closed formula). *For any $n, k \geq 0$, we have*

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\}.$$

REMARK. This formula gives another proof of the symmetry $B_n^{(-k)} = B_k^{(-n)}$ mentioned in [1].

The proofs of Theorems 1 and 2 will be given in §2 and §3 respectively.

2. Proof of Clausen-von Staudt type theorem

Let $k \geq 2$ and p be a prime number satisfying $k+2 \leq p \leq n+1$. To prove Theorem

1, we estimate the p -order of each summand $\frac{(-1)^m m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\}}{(m+1)^k}$ in (1), which we denote hereafter by $b_n^{(k)}(m)$. We prove (i) and (ii) simultaneously. The p -order of an integer a is denoted by $\text{ord}_p(a)$ with the convention $\text{ord}_p(p^t) = t$. Write $m+1 = ap^e$, $(a, p) = 1$, $e \geq 0$. If $e = 0$, then $b_n^{(k)}(m)$ is p -integral. We can ignore this term, because by the assumption $k \geq 2$ we have $p^{k-1}b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$. Since $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ is an integer,

we have

$$\text{ord}_p(b_n^{(k)}(m)) \geq \text{ord}_p\left(\frac{m!}{(m+1)^k}\right).$$

First, assume $e \geq 2$. We show that $p^k b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$ and moreover $p^{k-1} b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$ if $n \not\equiv 0 \pmod{p-1}$. Using $\text{ord}_p(m!) = \sum_{j=1}^{\infty} \left\lfloor \frac{m}{p^j} \right\rfloor$, we have

$$\begin{aligned} \text{ord}_p\left(\frac{m!}{(m+1)^k}\right) &= \sum_{j=1}^{\infty} \left\lfloor \frac{m}{p^j} \right\rfloor - ek \\ &\geq \left\lfloor \frac{m}{p} \right\rfloor - ek = \left\lfloor \frac{ap^e - 1}{p} \right\rfloor - ek = ap^{e-1} - 1 - ek \\ &\geq p^e - 1 - ek = (1+p-1)^{e-1} - 1 - ek \\ &\geq 1 + (e-1)(p-1) - 1 - ek = (e-1)(p-1) - ek \\ &\geq (e-1)(k+1) - ek = -k + e - 1 \\ &\geq -k + 1. \end{aligned}$$

Thus we get $\text{ord}_p\left(\frac{m!}{(m+1)^k}\right) \geq -k + 1$ and so $p^{k-1} b_n^{(k)}(m)$ is p -integral. Hence $p^k b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$. If any one of the above inequalities is strict (i.e. ' $>$ '), then we get $p^{k-1} b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$. Then only case when the equalities hold everywhere is when $e=2$, $m+1=p^2$, and $p=k+2$. In this case, the following lemma ($a=p$) implies $p^{k-1} b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$ if $n \not\equiv 0 \pmod{p-1}$.

LEMMA. *Let n and a be natural numbers. We have the congruence*

$$\left\{ \begin{matrix} n \\ ap-1 \end{matrix} \right\} \equiv \begin{cases} \binom{c-1}{a-1} \pmod{p} & \text{if } n = a-1 + c(p-1) \text{ for some } c \geq a \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

Proof. Use the following formula for a generating function of $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ ([4, (7.47)]):

$$\sum_{n=m}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} x^n = \frac{x^m}{(1-x)(1-2x) \cdots (1-mx)}. \tag{2}$$

If $m = ap - 1$, the right-hand side of this formula is congruent modulo p to

$$\frac{x^{ap-1}}{(1-x^{p-1})^a} = x^{ap-1} \sum_{i=0}^{\infty} \binom{a+i-1}{i} x^{i(p-1)} = \sum_{i=0}^{\infty} \binom{a+i-1}{a-1} x^{a-1+(a+i)(p-1)}$$

(we have used $(1-x)(1-2x) \cdots (1-(p-1)x) \equiv 1 - x^{p-1} \pmod{p}$). Putting $a+i=c$, we obtain the lemma.

Now suppose $e = 1$ ($m = ap - 1$). If $a \geq 3$, then $p^2 \mid (ap - 1)!$. Hence $\text{ord}_p\left(\frac{m!}{(m+1)^k}\right) >$

$-k + 1$, from which follows $p^{k-1}b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$. If $a = 2$, then $\text{ord}_p(m!) = 1$ and $\text{ord}_p\left(\frac{m!}{(m+1)^k}\right) = 1 - k$. Hence $\text{ord}_p(b_n^{(k)}(m)) = 1 - k + \text{ord}_p\left(\left\{\begin{matrix} n \\ m \end{matrix}\right\}\right)$ and so $p^k b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$. If $n \not\equiv 1 \pmod{p-1}$, we see from the above lemma ($a = 2$) that $\left\{\begin{matrix} n \\ 2p-1 \end{matrix}\right\} \equiv 0 \pmod{p}$. From this we have $p^{k-1}b_n^{(k)}(m) \equiv 0 \pmod{p\mathbb{Z}_p}$ if $n \not\equiv 1 \pmod{p-1}$. If $n \equiv 1 \pmod{p-1}$ and $n = 1 + c(p-1)$ with $c \geq 2$ ($c = 1$ cannot occur because $n \geq m$), then we see by the lemma that $\left\{\begin{matrix} n \\ 2p-1 \end{matrix}\right\} \equiv c - 1 \equiv -n \pmod{p}$. From this, we obtain

$$p^{k-1}b_n^{(k)}(m) = p^{k-1} \frac{(-1)^{2p-1}(2p-1)! \left\{\begin{matrix} n \\ 2p-1 \end{matrix}\right\}}{(2p)^k} \\ \equiv \frac{n}{2^k} \pmod{p\mathbb{Z}_p}.$$

Finally, the case $a = 1$ ($m = p - 1$) gives us $b_n^{(k)}(m) = \frac{(p-1)! \left\{\begin{matrix} n \\ p-1 \end{matrix}\right\}}{p^k}$. From the lemma, we have $\left\{\begin{matrix} n \\ p-1 \end{matrix}\right\} \equiv 0 \pmod{p}$ if $n \not\equiv 0 \pmod{p-1}$ and thus $p^{k-1}b_n^{(k)}(m) \equiv -\frac{1}{p} \left\{\begin{matrix} n \\ p-1 \end{matrix}\right\} \pmod{p\mathbb{Z}_p}$. If $n \equiv 0 \pmod{p-1}$, then $\left\{\begin{matrix} n \\ p-1 \end{matrix}\right\} \equiv 1 \pmod{p}$ and $p^k b_n^{(k)}(m) \equiv -1 \pmod{p\mathbb{Z}_p}$. Summing up, and noting the factor $(-1)^n$ before the summation in (1) (also note n is odd if $n \equiv 1 \pmod{p-1}$), we obtain the theorem.

3. Proof of the closed formula for negative index poly-Bernoulli numbers

In this section we prove Theorem 2. In the course of our proof, we obtain

PROPOSITION. For all $n > 0$,

$$\sum_{l=0}^n (-1)^l B_n^{(-l)} = 0.$$

EXAMPLE. $B_2^{(0)} - B_1^{(-1)} + B_0^{(-2)} = 1 - 2 + 1 = 0$, $B_4^{(0)} - B_3^{(-2)} + B_2^{(-2)} - B_1^{(-3)} + B_0^{(-4)} = 1 - 8 + 14 - 8 + 1 = 0$, etc.

This is trivial when n is odd because of the symmetry mentioned in the remark after the theorem.

In order to prove the theorem, we calculate the generating function $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} x^n y^k$ of $B_n^{(-k)}$ in the following form:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} x^n y^k = \sum_{j=0}^{\infty} p_j(x) p_j(y), \tag{3}$$

where

$$p_j(x) = \frac{j! x^j}{(1-x)(1-2x) \cdots (1-(j+1)x)}.$$

Once we establish this, the theorem follows by equating the coefficients of both sides, because we have by the formula (2) in §2

$$p_j(x) = j! \sum_{n=j}^{\infty} \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} x^n. \tag{4}$$

Put the left-hand side of (3) = $B(x, y)$. Using (1) we have

$$\begin{aligned} B(x, y) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left((-1)^n \sum_{m=0}^n (-1)^m m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (m+1)^k \right) x^n y^k \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{m=0}^n (-1)^m m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} x^n \sum_{k=0}^{\infty} (m+1)^k y^k \\ &= \sum_{m=0}^{\infty} (-1)^m m! \sum_{n=m}^{\infty} (-1)^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} x^n \frac{1}{1-(m+1)y}. \end{aligned} \tag{5}$$

Here we use (2) to get

$$B(x, y) = \sum_{m=0}^{\infty} \frac{m! x^m}{(1+x)(1+2x) \cdots (1+mx)(1-(m+1)y)}.$$

The proposition follows from this. Namely, putting $y = -x$ gives

$$\begin{aligned} B(x, -x) &= \sum_{m=0}^{\infty} \frac{m! x^m}{(1+x)(1+2x) \cdots (1+mx)(1+(m+1)x)} \\ &= \sum_{m=1}^{\infty} \frac{(m-1)! x^{m-1}}{(1+x) \cdots (1+mx)} \\ &= \sum_{m=1}^{\infty} (-1)^m (m-1)! \sum_{n=m}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (-1)^n x^{n-1} \quad (\text{by (2)}) \\ &= \sum_{n=1}^{\infty} (-1)^n \left(\sum_{m=1}^n (-1)^m (m-1)! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \right) x^{n-1} \\ &= 1 \quad (\text{by [4, (6.16)]}), \end{aligned}$$

while by definition

$$\begin{aligned} B(x, -x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k B_n^{(-k)} x^{n+k} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n (-1)^l B_{n-l}^{(-l)} \right) x^n \quad (n+k \rightarrow n, k \rightarrow l), \end{aligned}$$

and hence the proposition.

Let us return to the proof of (3). We need the following lemma.

LEMMA. (i) $\frac{1}{1-(m+1)y} = \sum_{j=0}^m \binom{m}{j} p_j(y).$

(ii) $\sum_{m=j}^n (-1)^m m! \binom{m}{j} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = (-1)^n j! \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} \quad (n \geq j \geq 0).$

Proof will be given later. From (5),

$$\begin{aligned} B(x, y) &= \sum_{m=0}^{\infty} (-1)^m m! \sum_{n=m}^{\infty} (-1)^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} x^n \frac{1}{1-(m+1)y} \\ &= \sum_{m=0}^{\infty} \left((-1)^m m! \sum_{n=m}^{\infty} (-1)^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} x^n \right) \sum_{j=0}^m \binom{m}{j} p_j(y) \quad (\text{by Lemma (i)}) \\ &= \sum_{j=0}^{\infty} p_j(y) \left(\sum_{m=j}^{\infty} (-1)^m m! \binom{m}{j} \sum_{n=m}^{\infty} (-1)^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} x^n \right) \\ &= \sum_{j=0}^{\infty} p_j(y) \sum_{n=j}^{\infty} (-1)^n x^n \left(\sum_{m=j}^n (-1)^m m! \binom{m}{j} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \right) \\ &= \sum_{j=0}^{\infty} p_j(y) \sum_{n=j}^{\infty} j! \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} x^n \quad (\text{by Lemma (ii)}) \\ &= \sum_{j=0}^{\infty} p_j(x) p_j(y) \quad (\text{by (4)}). \end{aligned}$$

This is (3) and thus the theorem is proved

Proof of Lemma. (i) The following partial fraction expansion is easily established by residue calculation:

$$\frac{1}{z(z-1)\cdots(z-m)} = \frac{(-1)^m}{m!} \sum_{l=0}^m \frac{(-1)^l \binom{m}{l}}{z-l}.$$

From this we have

$$y p_j(y) = \frac{j! y^{j+1}}{(1-y)(1-2y)\cdots(1-(j+1)y)} = \frac{j!}{\left(\frac{1}{y}-1\right)\left(\frac{1}{y}-2\right)\cdots\left(\frac{1}{y}-(j+1)\right)}$$

$$= \frac{j!(-1)^{j+1}}{y(j+1)!} \sum_{l=0}^{j+1} \frac{(-1)^l \binom{j+1}{l}}{\frac{1}{y} - l} = \frac{(-1)^{j+1}}{j+1} \sum_{l=0}^{j+1} \frac{(-1)^l \binom{j+1}{l}}{1-ly},$$

and therefore,

$$\begin{aligned} y \sum_{j=0}^m \binom{m}{j} p_j(y) &= \sum_{j=0}^m \binom{m}{j} \frac{(-1)^{j+1}}{j+1} \sum_{l=0}^{j+1} \frac{(-1)^l \binom{j+1}{l}}{1-ly} \\ &= \sum_{j=0}^m \binom{m}{j} \frac{(-1)^{j+1}}{j+1} + \sum_{l=1}^{m+1} \frac{(-1)^l}{1-ly} \sum_{j=l-1}^m \frac{(-1)^{j+1}}{j+1} \binom{m}{j} \binom{j+1}{l}. \end{aligned}$$

Now, since

$$\sum_{j=0}^m \binom{m}{j} \frac{(-1)^{j+1}}{j+1} = -\frac{1}{m+1}$$

(take $\int_0^1 dx$ of $(1-x)^m = \sum_{j=0}^m (-1)^j \binom{m}{j} x^j$), and

$$\begin{aligned} \sum_{j=l-1}^m \frac{(-1)^{j+1}}{j+1} \binom{m}{j} \binom{j+1}{l} &= \frac{1}{l} \sum_{j=l-1}^m (-1)^{j+1} \binom{m}{j} \binom{j}{l-1} \\ &= \frac{1}{l} \sum_{j=l-1}^m (-1)^{j+1} \binom{m}{l-1} \binom{m-l+1}{j-l+1} \\ &= \frac{1}{l} \binom{m}{l-1} \sum_{j=l-1}^m (-1)^{j+1} \binom{m-l+1}{j-l+1} \\ &= \begin{cases} \frac{(-1)^{m+1}}{m+1} & l=m+1 \\ 0 & l < m+1 \end{cases} \end{aligned}$$

(the second equality is by $\binom{p}{q} \binom{q}{r} = \binom{p}{r} \binom{p-r}{q-r}$), we have

$$\begin{aligned} y \sum_{j=0}^m \binom{m}{j} p_j(y) &= -\frac{1}{m+1} + \frac{(-1)^{m+1}}{1-(m+1)y} \cdot \frac{(-1)^{m+1}}{m+1} \\ &= \frac{y}{1-(m+1)y}. \end{aligned}$$

This gives Lemma (i).

(ii) First we show

$$\sum_{n=j}^{\infty} (-1)^n j! \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} \frac{t^n}{n!} = (e^{-t} - 1)^j \cdot e^{-t}.$$

For this, we start with

$$\frac{(e^t - 1)^j}{j!} = \sum_{n=j}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \frac{t^n}{n!} \quad (6)$$

(this is [4, (7.49)]). Replacing j by $j+1$,

$$\begin{aligned} \frac{(e^t - 1)^{j+1}}{(j+1)!} &= \sum_{n=j+1}^{\infty} \left\{ \begin{matrix} n \\ j+1 \end{matrix} \right\} \frac{t^n}{n!} \\ &= \sum_{n=j}^{\infty} \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} \frac{t^{n+1}}{(n+1)!} \quad (n \rightarrow n+1). \end{aligned}$$

Differentiation by t gives

$$\frac{(e^t - 1)^j \cdot e^t}{j!} = \sum_{n=j}^{\infty} \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} \frac{t^n}{n!}.$$

From this we have

$$(e^{-t} - 1)^j \cdot e^{-t} = \sum_{n=j}^{\infty} (-1)^n j! \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} \frac{t^n}{n!}.$$

The proof will be finished if we show

$$\sum_{n=j}^{\infty} \left(\sum_{m=j}^n (-1)^m m! \binom{m}{j} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \right) \frac{t^n}{n!} = (e^{-t} - 1)^j \cdot e^{-t}.$$

The left-hand side is equal to

$$\begin{aligned} &\sum_{m=j}^{\infty} (-1)^m m! \binom{m}{j} \sum_{n=m}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{t^n}{n!} \\ &= \sum_{m=j}^{\infty} (-1)^m m! \binom{m}{j} \frac{(e^t - 1)^m}{m!} \quad (\text{by (6)}) \\ &= \sum_{m=j}^{\infty} (-1)^m \binom{m}{j} (e^t - 1)^m. \end{aligned}$$

Since $\sum_{m=j}^{\infty} \binom{m}{j} X^m = \frac{X^j}{(1-X)^{j+1}}$ (replace m by $m-j$ in $\sum_{m=0}^{\infty} \binom{m+j}{j} X^m = (1-X)^{-(j+1)}$)

we obtain

$$\sum_{m=j}^{\infty} (-1)^m \binom{m}{j} (e^t - 1)^m = \sum_{m=j}^{\infty} \binom{m}{j} (1 - e^t)^m = \frac{(1 - e^t)^j}{(1 - (1 - e^t))^{j+1}} = (e^{-t} - 1)^j \cdot e^{-t}.$$

This completes the proof of the lemma and Theorem 2 is thus established.

Table 1. $B_n^{(k)}$ ($-5 \leq k \leq 5, 0 \leq n \leq 7$).

$k \backslash n$	0	1	2	3	4	5	6	7
-5	1	32	454	4718	41506	329462	2441314	17234438
-4	1	16	146	1066	6902	41506	237686	1315666
-3	1	8	46	230	1066	4718	20266	85310
-2	1	4	14	46	146	454	1394	4246
-1	1	2	4	8	16	32	64	128
0	1	1	1	1	1	1	1	1
1	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0
2	1	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{24}$	$\frac{7}{450}$	$\frac{1}{40}$	$-\frac{38}{2205}$	$-\frac{5}{168}$
3	1	$\frac{1}{8}$	$-\frac{11}{216}$	$-\frac{1}{288}$	$\frac{1243}{54000}$	$-\frac{49}{7200}$	$-\frac{75613}{3704400}$	$\frac{599}{35280}$
4	1	$\frac{1}{16}$	$-\frac{49}{1296}$	$\frac{41}{3456}$	$\frac{26291}{3240000}$	$-\frac{1921}{144000}$	$\frac{845233}{1555848000}$	$\frac{1048349}{59270400}$
5	1	$\frac{1}{32}$	$-\frac{179}{7776}$	$\frac{515}{41472}$	$-\frac{216383}{194400000}$	$-\frac{183781}{25920000}$	$\frac{4644828197}{653456160000}$	$\frac{153375307}{49787136000}$

Acknowledgements

The authors are very grateful to Don Zagier and Herbert Gangle for their interests. Their computations stimulated us to obtain Theorem 1 as in the form presented here.

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Tsuneo Arakawa
 Department of Mathematics
 Rikkyo University
 Ikebukuro, Tokyo 171–8501, Japan
 E-mail: arakawa@rkmath.rikkyo.ac.jp

Masanobu Kaneko
 Graduate School of Mathematics
 Kyushu University 33
 Fukuoka 812–8581, Japan
 E-mail: mkaneko@math.kyushu-u.ac.jp