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Asai, Tetsuya
Department of Mathematics, Shizuoka University

Kaneko, Masanobu
Graduate School of Mathematics, Kyushu University

Ninomiya, Hirohito
Department of Mathematics, Kyoto University

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Zeros of Certain Modular Functions and an Application

by

Tetsuya ASAI, Masanobu KANEKO and Hirohito NINOMIYA

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1. Main result

Let $j(\tau)$ be the classical elliptic modular invariant, which is a holomorphic function in the upper half-plane $\boldsymbol{H}$, is invariant under the action of the modular group $SL_2(\boldsymbol{Z})$, and has a simple pole at infinity. Let $\varphi_n(j)$ be the monic polynomial in $j=j(\tau)$ obtained from $j(\tau)-744$ by the action of the n-th Hecke operator (the precise definition of which will be recalled later),

$$\varphi_n(j(\tau))=n(j(\tau)-744)\big|_0 T(n) \qquad (n=1, 2, 3, \cdots). \tag{1}$$

In this paper, we prove the following:

THEOREM 1. *For each n, all the zeros of the polynomial $\varphi_n(j)$ are simple and lie in the interval* (0, 1728).

As has been known since the work of F. K. C. Rankin and H. P. F. Swinnerton-Dyer [8], the values of $j(\tau)$ at the zeros in $\boldsymbol{H}$ of the Eisenstein series $E_k(\tau)$ of any weight k on $SL_2(\boldsymbol{Z})$ always lie in the interval [0, 1728], or equivalently, all the zeros of $E_k(\tau)$ in the standard fundamental domain lie on the unit circle. This result was generalized by R. A. Rankin [7] to certain Poincaré series. Furthermore, the zeros of Atkin's orthogonal polynomials, as well as of certain "hypergeometric modular form", both of which are studied in a joint paper by D. Zagier and Kaneko [5] and have an intimate connection to the j-invariants of supersingular elliptic curves, have the same property. (Here we mention that the Eisenstein series is also related to the supersingular j-invariants [9].) Our Theorem 1 supplies another example with this seemingly peculiar, and not yet fully understood property of zeros.

As an application of this theorem, we can give an interesting proof of the following fact. Let $J(q)$ denote the Laurent series in $q=e^{2\pi i\tau}$ of the Fourier expanison of $j(\tau)$,

$$J(q)=\frac{1}{q}+744+196884q+21493760q^2+864299970q^3+\cdots,$$

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and r_n the coefficient of q^n of the reciprocal of $J(q)$,

$$\frac{1}{J(q)} = \sum_{n \geq 1} r_n q^n = q - 744q^2 + 356652q^3 - 140361152q^4 + 49336682190q^5 - \cdots .$$

COROLLARY 2. *The signs of the coefficients of* $1/J(q)$ *are strictly alternating. In other words,* $(-1)^{n-1} r_n$ *is always a positive integer.*

We derive this from Theorem 1 using the following expansion formula of the reciprocal of $J(q)-j$, where j represents a variable:

$$\frac{1}{J(q)-j} = \sum_{n=1}^{\infty} \varphi_n'(j) \frac{q^n}{n}, \tag{2}$$

where the symbol $'$ denotes differentiation with respect to j. The formula (2) is obtained as a corollary of a special case of an expansion formula for certain Green's kernel functions, which, as will be recalled in §3, has appeared in several places, notably in R. Borcherds's work on the Moonshine Conjecture in an equivalent form as a product formula.

From Theorem 1 it is obvious that the sign of $\varphi_n'(0)$ is plus for odd n and minus for even n, which readily proves Corollary 2 by virtue of formula (2). We note that, as Borcherds and Zagier pointed out to us, the corollary can also be proven directly by investigating asymptotic behaviour of r_n through residue caluculation.

In the next section, we prove Theorem 1. In §3, we discuss the expansion formula mentioned above, and also discuss briefly some results similar to Corollary 2.

2. The location of zeros of $\varphi_n(j)$

By definition, the Hecke operator $T(n)$ $(n=1, 2, 3, \cdots)$ acts on a modular form $f(\tau)$ of weight k (on $SL_2(\boldsymbol{Z})$) as

$$(f|_k T(n))(\tau) = n^{k-1} \sum_{\substack{ad=n, d>0 \\ 0 \leq b \leq d-1}} d^{-k} f\left(\frac{a\tau+b}{d}\right), \tag{3}$$

or, in terms of Fouier series,

$$(f|_k T(n))(q) = \sum_{m \in \boldsymbol{Z}} \left(\sum_{0 < d | (m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right) \right) q^m, \tag{4}$$

where $f(q) = \sum_{m \in \boldsymbol{Z}} a(m) q^m$ is a Fourier expansion of $f(\tau)$. (See for instance Serre [10].) In this section we prove Theorem 1, which asserts that the zeros of $\varphi_n(j)$ (defined by (1)) are all real, simple, and moreover lie in the interval (0, 1728). To illustrate, we give the first few $\varphi_n(j)$ and their zeros:

$$\varphi_1(j) = j - 744 \qquad ; \ 744.000 ,$$

$$\varphi_2(j) = j^2 - 1428j + 159768 \qquad ; \ 116.491, \ 1371.509 ,$$

$$\varphi_3(j)=j^3-2232j^2+1069956j-36866976\,;\ 37.312,\ 632.482,\ 1562.205\,.$$

For the proof, we may assume $n\geq 2$. Let D denote the standard fundamental domain in the upper half-plane $\boldsymbol{H}$ under the action of the modular group:

$$D=\{\tau\in\boldsymbol{H}:|\tau|\geq 1,\ -1/2<\mathrm{Re}(\tau)\leq 1/2,\ \text{and}\ |\tau|>1\ \text{if}\ -1/2<\mathrm{Re}(\tau)<0\}\,.$$

Then let C be a part of the boundary of D defined by

$$C=\{\tau\in\boldsymbol{H}:|\tau|=1,\ 0\leq\mathrm{Re}(\tau)\leq 1/2\}\,.$$

In the following, we consider the function

$$F_n(\tau)=\varphi_n(j(\tau))$$

on the arc C. Recall that the map $\tau\mapsto j(\tau)$ gives a 1-1 correspondence between C and the interval [0, 1728]; in particular, the function $F_n(\tau)$ takes real values on C because the polynomial $\varphi_n(j)$ has rational integer coefficients. Since the degree of $\varphi_n(j)$ is n, it is sufficient to show that the function $F_n(\tau)$ has at least n distinct zeros on the arc C.

The essential point in the proof is the following estimate, which we refer to as the Key Lemma.

KEY LEMMA. *Let $\tau_0=x_0+iy_0\in C$. Then we have*

$$|F_n(\tau_0)e^{-2\pi n y_0}-2\cos(2\pi n x_0)|<2\,.$$

This lemma implies that the function $F_n(\tau)$ changes sign at least once in each part of the arc C with $\dfrac{\nu-1}{2n}<\mathrm{Re}(\tau)<\dfrac{\nu}{2n}$ for $\nu=1, 2, \cdots, n$, and hence $F_n(\tau)$ has at least n distinct zeros on C, as desired.

Proof of Key Lemma. By the definition (3) of the Hecke operators, we have

$$F_n(\tau)=\sum_{\substack{ad=n,d>0\\0\leq b\leq d-1}}\left(j\left(\frac{a\tau+b}{d}\right)-744\right). \tag{5}$$

Let M be the maximum of $|j(\tau)-744-e^{-2\pi i\tau}|$ in the closure of D:

$$M=\max_{\tau\in\bar{D}}|j(\tau)-744-e^{-2\pi i\tau}|\,.$$

The following estimate based on the positivity of the Fourier coefficients c_n of $j(\tau)$ provides the inequality $M<1335$ (note that $\mathrm{Im}(\tau)\geq\dfrac{\sqrt{3}}{2}$ when $\tau\in\bar{D}$):

$$\begin{aligned}|j(\tau)-744-e^{-2\pi i\tau}|&\leq\sum_{n\geq 1}c_n|q|^n\\&=\sum_{n\geq 1}c_n e^{-2\pi\mathrm{Im}(\tau)n}\end{aligned}$$

$$\leq \sum_{n\geq 1} c_n e^{-2\pi(\sqrt{3}/2)n}$$

$$= \left| j\left(\frac{\sqrt{-3}}{2}\right) - 744 - e^{2\pi(\sqrt{3}/2)} \right| = 1334.813\cdots .$$

For any $\tau \in \boldsymbol{H}$, let τ^* denote the unique point in D which is equivalent to τ under the action of the modular group. We claim that the following estimates concerning the values of each summand in (5) hold for any $\tau_0 = x_0 + iy_0 \in C$ and any $n \geq 2$.

(i) $|j(n\tau_0) - 744 - e^{-2\pi i n \tau_0}| \leq M$.

(ii) $\left| j\left(\frac{\tau_0}{n}\right) - 744 - e^{2\pi i n \bar{\tau}_0} \right| \leq M$.

(iii) Assume $\frac{a\tau_0 + b}{d}$ is distinct from $n\tau_0$, $\frac{\tau_0}{n}$, and $\frac{\tau_0 + n - 1}{n}$. Then

$$\left| j\left(\frac{a\tau_0 + b}{d}\right) - 744 \right| \leq e^{\pi n y_0} + M .$$

The inequality (i) is easily demonstrated because $n\tau_0 - (n\tau_0)^* \in \boldsymbol{Z}$, and thus

$$|j(n\tau_0) - 744 - e^{-2\pi i n \tau_0}| = |j((n\tau_0)^*) - 744 - e^{-2\pi i (n\tau_0)^*}| \leq M ,$$

the last inequality following directly from the definition of M, since $(n\tau_0)^* \in D$. The inequality (ii) is similarly derived from $-\frac{n}{\tau_0} - \left(\frac{\tau_0}{n}\right)^* \in \boldsymbol{Z}$ and $n\bar{\tau}_0 = \frac{n}{\tau_0}$:

$$\left| j\left(\frac{\tau_0}{n}\right) - 744 - e^{2\pi i n \bar{\tau}_0} \right| = \left| j\left(-\frac{n}{\tau_0}\right) - 744 - e^{-2\pi i(-n/\tau_0)} \right|$$

$$= \left| j\left(\left(\frac{\tau_0}{n}\right)^*\right) - 744 - e^{-2\pi i(\tau_0/n)^*} \right| \leq M .$$

As for (iii), we proceed as follows. Put $z = \frac{a\tau_0 + b}{d}$ and $z^* = \frac{\alpha z + \beta}{\gamma z + \delta}$. We then have

$$\mathrm{Im}(z^*) = \frac{n y_0}{|\gamma a \tau_0 + \gamma b + \delta d|^2} .$$

In order to prove (iii), it is sufficient to show that $\mathrm{Im}(z^*) \leq \frac{n y_0}{2}$ if (a, b, d) satisfies the condition in (iii), because we have $|j(z) - 744| \leq e^{2\pi \mathrm{Im}(z^*)} + M$ by the definition of M and the triangle inequality. Put $L = |\gamma a \tau_0 + \gamma b + \delta d|$; we now show that $L \geq \sqrt{2}$. We may assume $\gamma \geq 0$. If $\gamma = 0$, then $\delta = \pm 1$ and $L = |d| \geq 2$. If $\gamma \geq 2$ or $a \geq 2$, then we easily see that $L \geq \sqrt{3}$. Suppose $\gamma = a = 1$. Then we have $d = n$ and $L = |\tau_0 + b + n\delta|$. In this case, noting that $b + n\delta$ is a non-zero integer because $1 \leq b \leq n - 1$, we have $L \geq \sqrt{2}$ unless $b + n\delta = -1$, which is possible only when $b = n - 1$, the case being

excluded by the assumption. This proves (iii). From (i), (ii), (iii) and the trivial estimate

$$\left| j\left(\frac{\tau_0+n-1}{n}\right)-744\right| \leq e^{2\pi n y_0}+M$$

for the excluded case, we obtain

$$|F_n(\tau_0)-(e^{-2\pi i n\tau_0}+e^{2\pi i n\bar{\tau}_0})| \leq \sigma_1(n)M+(\sigma_1(n)-3)e^{\pi n y_0}+e^{2\pi n y_0},$$

where $\sigma_1(n)$ is the sum of positive divisors of n. Multiplying both sides by $e^{-2\pi n y_0}$, we have

$$|F_n(\tau_0)e^{-2\pi n y_0}-2\cos(2\pi n x_0)| \leq \sigma_1(n)Me^{-2\pi n y_0}+(\sigma_1(n)-3)e^{-\pi n y_0}+1.$$

Using the bound $M \leq 1335$ and the trivial estimate $\sigma_1(n) \leq n^2$ (and so $\sigma_1(n)-3 \leq n^2$), as well as $y_0 \geq \frac{\sqrt{3}}{2}$ and the fact that $n^2e^{-\pi n\sqrt{3}}$ and $n^2e^{-\pi n\sqrt{3}/2}$ are monotonically decreasing for $n \geq 2$, we finally obtain

$$\begin{aligned} |F_n(\tau_0)e^{-2\pi n y_0}-2\cos(2\pi n x_0)| &\leq n^2(Me^{-\pi n\sqrt{3}}+e^{-\pi n\sqrt{3}/2})+1 \qquad (6)\\ &\leq 4(1335e^{-2\pi\sqrt{3}}+e^{-\pi\sqrt{3}})+1 \\ &= 1.1176\cdots<2, \end{aligned}$$

which completes our proof of Theorem 1. ∎

REMARK. Theorem 1 is valid when we replace $j(\tau)-744$ in the definition (1) of $\varphi_n(j)$ by any $j(\tau)-a$ with $0<a<1728$; the proof is completely analogous. Moreover, our method of proof implies that when we take any real number a, the zeros of the resulting polynomials $n(j(\tau)-a)|_0 T(n)$ have the property stated in Theorem 1 for all sufficiently large n. This is because $n^2(Me^{-\pi n\sqrt{3}}+e^{-\pi n\sqrt{3}/2})$ in (6) tends to zero as n becomes large.

3. Expansion formula for certain Green's kernel functions

For each integer k in the set $S:=\{0, 4, 6, 8, 10, 14\}$ and each positive integer n, let $f_n^{(k)}(q)$ and $g_n^{(k)}(q)$ be the Fourier series of the unique meromorphic modular forms on $SL_2(\boldsymbol{Z})$ of weight k and $2-k$, respectively, characterized by the following properties:

(i) They are holomorphic in τ ($q=e^{2\pi i\tau}$) in the upper half-plane.

(ii) $f_n^{(k)}(q)-q^{-n} \in q\boldsymbol{Z}[[q]]$, whereas $g_n^{(k)}(q)-q^{-n} \in \boldsymbol{Z}[[q]]$.

The uniqueness of $f_n^{(k)}(q)$ (resp., $g_n^{(k)}(q)$) follows from the fact that no holomorphic cusp (resp., modular) forms of weight k (resp., $2-k$) exist when k is in the set S. As for existence, we can construct the forms $f_n^{(k)}(q)$ and $g_n^{(k)}(q)$ in the manner described below; the verification of the properties (i) and (ii) will then be straightforward.

For $n=1$, put

$$f_1^{(k)}(q)=\left(J(q)-744+\frac{2k}{B_k}\right)\cdot E_k(q)\,,\quad g_1^{(k)}(q)=\frac{E_{14-k}(q)}{\Delta(q)}\,,$$

where $E_k(q)$ is the Eisenstein series of weight k ($E_0(q)=1$),

$$E_k(q)=1-\frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n\quad\left(B_k=k\text{-th Bernoulli number},\ \sigma_{k-1}(n)=\sum_{d\mid n}d^{k-1}\right)$$

(note that the number $\frac{2k}{B_k}$ is an integer when $k\in S$), and $\Delta(q)$ is the discriminant function of weight 12 defined by

$$\Delta(q)=\frac{E_4(q)^3-E_6(q)^2}{1728}\,.$$

For general n, we put

$$f_n^{(k)}(q)=n^{1-k}\cdot f_1^{(k)}(q)\big|_k T(n)\,,\quad g_n^{(k)}(q)=n^{k-1}\cdot g_1^{(k)}(q)\big|_{2-k}T(n)\,,$$

where $T(n)$ is the Hecke operator. Further, we put $f_0^{(k)}(q)=E_k(q)$. Note in particular that $f_n^{(0)}(q)=\varphi_n(J(q))$.

Now we have

THEOREM 3. *Let $k\in S$. Then*

$$\frac{E_k(p)E_{14-k}(q)\Delta(q)^{-1}}{J(q)-J(p)}=\sum_{n=0}^{\infty}f_n^{(k)}(p)q^n=-\sum_{m=1}^{\infty}g_m^{(k)}(q)p^m\,,$$

where p and q are independent formal variables.

This theorem is fairly well-known. When $k\geq 4$, the function on the left-hand side is essentially the Green's function in the sense of Eichler [3], $f_n^{(k)}(q)$ is a Poincaré series and is the $k-1$-st derivative of $g_n^{(k)}(q)$ up to a constant. The case of $k=0$ constitutes a restatement of a product formula for the j-function (equivalently, the denominator formula for the Monster Lie algebra) stated in the introduction of Borcherds [2] and also appearing in Norton [6] and Alexander-Cummins-Mckay-Simons [1]. Furthermore, the polynomials $\varphi_n(j)$ $(=f_n^{(0)}(q))$ are viewed as Faber polynomials, the subject of vast study since the original work of G. Faber [4], mainly from analytical points of view. We also mention that Zagier (in preparation) obtained a similar formula which involves meromorphic modular forms of *half-integral* weight. Here, for the reader's convenience, we give a simple and elmentary unified proof for all k in question.

Proof. First we note that any meromorphic modular form on $SL_2(\boldsymbol{Z})$ of weight 2 which is holomorphic in $\boldsymbol{H}$ is a derivative (with respect to τ) of a polynomial in $j(\tau)$. In particular, the constant term of the Fourier series of such a form always vanishes (recall: $\frac{1}{2\pi i}\frac{d}{d\tau}=q\frac{d}{dq}$). Now let us put

$$J(q)=q^{-1}+\sum_{n=0}^{\infty} c_n q^n, \quad f_1^{(k)}(q)=q^{-1}+\sum_{n=1}^{\infty} a_n^{(k)} q^n, \quad g_1^{(k)}(q)=q^{-1}+\sum_{n=0}^{\infty} b_n^{(k)} q^n.$$

From (4), we have

$$f_n^{(k)}(q)=q^{-n}+n^{1-k}a_n^{(k)}q+\cdots \qquad \forall n\geq 1.$$

As a consequence of the preceding remark, looking at the constant terms of $f_n^{(k)}(q)g_1^{(k)}(q)$ and $g_1^{(k)}(q)E_k(q)$, we obtain

$$b_n^{(k)}=-n^{1-k}a_n^{(k)} \qquad \forall n\geq 1, \tag{7}$$

and

$$b_0^{(k)}=\frac{2k}{B_k}. \tag{8}$$

Since the forms $J(p)f_n^{(k)}(p)$ and $J(q)g_m^{(k)}(q)$ are uniquely determined by their principal parts (terms with non-positive exponents), we obtain, by comparing the coefficients and using relations (7) and (8) as well as the fact that the constant terms of $g_m^{(k)}(q)$ is $b_0^{(k)}\sigma_{k-1}(m)$, the recurrence relations

$$J(p)f_n^{(k)}(p)=f_{n+1}^{(k)}(p)+\sum_{l=0}^{n} c_{n-l}f_l^{(k)}(p)-b_n^{(k)}f_0^{(k)}(p) \qquad \forall n\geq 0, \tag{9}$$

and

$$J(q)g_m^{(k)}(q)=g_{m+1}^{(k)}(q)+\sum_{l=1}^{m} c_{m-l}g_l^{(k)}(q)+\frac{2k}{B_k}\sigma_{k-1}(m)g_1^{(k)}(q) \qquad \forall m\geq 1. \tag{10}$$

Multiplying both sides of (9) (resp., (10)) by q^n (resp., p^m) and summing, we have

$$J(p)F(p,q)=\frac{1}{q}(F(p,q)-f_0^{(k)}(p))+\left(J(q)-\frac{1}{q}\right)F(p,q)+f_0^{(k)}(p)\left(-g_1^{(k)}(q)+\frac{1}{q}\right)$$

(resp.,

$$J(q)G(p,q)=\frac{1}{p}(G(p,q)-g_1^{(k)}(q)p)+\left(J(p)-\frac{1}{p}\right)G(p,q)+g_1^{(k)}(q)(1-E_k(p)),$$

where $F(p,q)=\sum_{n=0}^{\infty} f_n^{(k)}(p)q^n$ (resp., $G(p,q)=\sum_{m=1}^{\infty} g_m^{(k)}(q)p^m$). This can easily be transformed into the formula in Theorem 3. ∎

COROLLARY 4. *Let $k\in S$. Then*

$$\frac{E_{14-k}(q)\Delta(q)^{-1}}{J(q)-j}=\sum_{n=0}^{\infty}\varphi_n^{(k)}(j)q^n, \quad \textit{and} \quad \frac{E_k(q)}{J(q)-j}=\sum_{n=1}^{\infty}\psi_n^{(k)}(j)q^n,$$

where $\varphi_n^{(k)}(j)$ and $\psi_n^{(k)}(j)$ are monic polynomials of respective degrees n and $n-1$ which are determined by

$$f_n^{(k)}(q)=\varphi_n^{(k)}(J(q))E_k(q)$$

and

$$g_n^{(k)}(q)=\psi_n^{(k)}(J(q))E_{14-k}(q)\Delta(q)^{-1}\,.$$

In fact, the polynomials $\varphi_n^{(k)}(j)$ and $\psi_n^{(k)}(j)$ are determined inductively using the relations (9) and (10). In particular, $\varphi_n^{(0)}(j)$ is identical to $\varphi_n(j)$ in §1, and $\psi_n^{(0)}(j)=\frac{\varphi_n'(j)}{n}$. The latter is because we have

$$-q\frac{d}{dq}\left(\frac{\varphi_n(J(q))}{n}\right)=\frac{\varphi_n'(J(q))}{n}\frac{E_{14}(q)}{\Delta(q)}\,,$$

and the left-hand side of this clearly satisfies the conditions that characterize $g_n^{(0)}(q)$. We therefore obtain the formula (2) in §1.

If we substitute $j=1728$ in the formula (2), we can derive a result for the coefficients of $\frac{1}{J(q)-1728}=\frac{\Delta}{E_6^2}$ demonstrating that they are all positive. Similarly, substituting $j=0$ or $j=1728$ into the formula

$$\frac{E_{14}(q)\Delta(q)^{-1}}{J(q)-j}=\sum_{n=0}^{\infty}\varphi_n(j)q^n$$

which represents the case $k=0$ of the first formula in the Corollary 4, we conclude by Theorem 1 that the sign of the coefficient in the Fourier expansion of $\frac{E_6}{E_4}$ is strictly alternating, while that of $\frac{E_4^2}{E_6}$ is always positive; the latter case is, however, obvious from the expansions of E_4 and E_6.

We note that the statement of Theorem 1 also holds for all $\varphi_n^{(k)}(j)$ and $\psi_n^{(k)}(j)$ with $k\in S$, which is proved along the same line, though each evaluation becomes rather complicated. We can therefore obtain similar results for the signs of the Fourier expansions of various meromorphic modular forms, such as $\frac{1}{E_4}$, $\frac{\Delta}{E_4}$, $\frac{E_6\Delta}{E_4}$, $\cdots$ (alternating type) and $\frac{\Delta}{E_6}$, $\frac{E_4\Delta}{E_6}$, $\frac{E_4\Delta}{E_6^2}$, $\cdots$ (always positive).

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Tetsuya Asai:
Department of Mathematics
Shizuoka University
Shizuoka 422, Japan
E-mail: smtasai@ipcs.shizuoka.ac.jp

Masanobu Kaneko:
Graduate School of Mathematics
Kyushu University 33
Fukuoka 812, Japan
E-mail: mkaneko@math.kyushu-u.ac.jp

Hirohito Ninomiya:
Department of Mathematics
Kyoto University
Kyoto 606, Japan
E-mail: ninomiya@kusm.kyoto-u.ac.jp