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WHEN IS A POLYGONAL PYRAMID NUMBER AGAIN POLYGONAL?

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ABSTRACT. We consider a Diophantine equation arising from a generalization of the classical Lucas problem of the square pyramid: when is the sum of the first m-gonal numbers n-gonal? We use the theory of elliptic surfaces to deduce several families of parametric solutions of the problem.

1. Introduction. In 1875 Lucas proposed the problem of proving that $1^2 + 2^2 + \cdots + 24^2 = 70^2$ is the only nontrivial solution to the problem referred to as "the cannonball problem" or "the problem of the square pyramid": When is the sum of the first n squares a perfect square? This problem was settled finally by G.N. Watson in 1918 (see [**1**] for the history and an elementary proof of the problem).

In the present paper, by regarding a perfect square as 4-gonal number, we consider the following generalization of the cannonball problem: When is the sum of the first m-gonal numbers once again an m-gonal number, or more generally, a polygonal number of possibly different order n? Here, for positive integers $m \geq 3$ and $i \geq 1$, the ith *polygonal number of order* m or the ith m*-gonal number*, is given by

$$G_m(i) = \frac{m-2}{2}i^2 - \frac{m-4}{2}i. \tag{1}$$

We call the sum of the first i m-gonal numbers the ith polygonal pyramid number of order m, or the ith m*-gonal pyramid number* and denote it by $\mathrm{Pyr}_m(i)$: $\mathrm{Pyr}_m(i) = \sum_{j=1}^{i} G_m(j)$. Then our problem of the polygonal pyramid is to find (positive) integer solutions (x, y) to the equation

$$G_n(y) = \mathrm{Pyr}_m(x) \tag{2}$$

for fixed integers $m, n \geq 3$. By (1), we can write this equation explicitly as

$$3(n-2)y^2 - 3(n-4)y = (m-2)x^3 + 3x^2 - (m-5)x. \tag{3}$$

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For each pair (m, n), this defines an elliptic curve over $\mathbf{Q}$ and, according to Siegel's theorem, there are only finitely many integral points on the curve (3). However, it is in general very hard to find them all. Instead, we view m or n as independent variable and consider a $\mathbf{Q}[m]$ or $\mathbf{Q}[n]$ point on the curve. This makes it possible to find several parametric solutions to (2) (with varying m or n).

Our main results are as follows.

Theorem 1.1. (a) *For each integer $m \geq 3$, there are infinitely many m-gonal pyramid numbers expressible as a polygonal number. Specifically, we have the following identities*:

$$\begin{aligned}\mathrm{Pyr}_m(3(m-2)k-2) &= G_{9k+2}((m-2)^2k-m+3),\\ \mathrm{Pyr}_m(3k-1) &= G_{(m-2)k+3}(3k-1),\\ \mathrm{Pyr}_m(6k-3) &= G_{4(m-2)(2k-1)+6}(3k-1)\end{aligned}$$

for any integer $k \geq 1$.

(b) *Moreover, if $m \equiv 2 \bmod 3$, then*

$$\begin{aligned}\mathrm{Pyr}_m\Big(\frac{1}{3}(m-2)k-2\Big) &= G_{k+2}\Big(\frac{1}{9}(m-2)^2k-m+3\Big),\\ \mathrm{Pyr}_m(k) &= G_{(1/3)(m-2)(k+1)+3}(k),\\ \mathrm{Pyr}_m(2k-1) &= G_{(4/3)(m-2)(2k-1)+6}(k)\end{aligned} \tag{4}$$

for any integer $k \geq 1$ (for which the expression has meaning).

Theorem 1.2. (a) *For each integer $n \geq 3$, there are infinitely many n-gonal numbers expressible as a polygonal pyramid number. Specifically, we have*

$$G_n((n-2)k^2-3k+1) = \mathrm{Pyr}_{3k+2}((n-2)k-2), \tag{5}$$

$$G_n(8k^2-6k+1) = \mathrm{Pyr}_{3(n-2)k+2}(4k-2). \tag{6}$$

Moreover, if $n \equiv 2 \bmod 9$, we have

$$G_n\Big(\frac{n-2}{9}k^2-k+1\Big) = \mathrm{Pyr}_{k+2}\Big(\frac{n-2}{3}k-2\Big).$$

(b) *For $n = 3, 5$ and 8, there are other families of n-gonal numbers that can be expressed as a polygonal pyramid number. Namely, we have*

$$\begin{aligned}
G_3(2(12k^2+13k+3)) &= \mathrm{Pyr}_{27k+20}(4k+2),\\
G_5(6k^2+k) &= \mathrm{Pyr}_{12k+6}(3k),\\
G_5(6k^2+5k+1) &= \mathrm{Pyr}_{12k+10}(3k+1),\\
G_5(2(12k^2-25k+13)) &= \mathrm{Pyr}_{3k}(12k-14),\\
G_5(2(12k^2-17k+6)) &= \mathrm{Pyr}_{3k+1}(12k-10),\\
G_8(30k^2+k) &= \mathrm{Pyr}_{75k+7}(6k),\\
G_8(30k^2+41k+14) &= \mathrm{Pyr}_{75k+57}(6k+4).
\end{aligned}$$

Here $k \geq 0$ is any integer for each equation (for which the sum has meaning).

Theorem 1.3. *When both m and n are viewed as independent variables, one and only one $\mathbf{Q}[m,n]$ point exists on the elliptic curve (3), namely,*

$$(x,y) = \Big(\frac{1}{3}(mn-2m-2n-2), \frac{1}{9}(m^2n-2m^3-4mn-m+4n+19)\Big).$$

The equation (5), or equivalently (4), since these two represent essentially the same solution, when specialized to $n = 3k+2$, gives us a parametric solution $(x,y,m) = (3k^2-2, 3k^3-3k+1, 3k+2)$ to the special case $m = n$ of (2) (when $m \equiv 2 \bmod 3$). We state this as

Corollary 1.4. *Suppose $m \equiv 2 \bmod 3$ and write $m = 3k+2$ with $k \geq 1$. The equation*

$$G_m(y) = \mathrm{Pyr}_m(x)$$

has a parametric solution

$$(x,y) = (3k^2-2, 3k^3-3k+1).$$

Notice that one can easily verify the identities in Theorems 1.1 and 1.2. But we show here how we can obtain these identities systematically from the polynomial points on the elliptic curve (3). In order to obtain every polynomial point, we use an improved version of the results of Hindy-Silverman [**4**] on an upper bound of the height pairing (the canonical height) for polynomial points on an elliptic curve over a rational function field (Proposition 4.1 in Section 4).

Various other variations to the original square pyramid problem have been considered. Of them, we mention that Kuwata and Top [**5**] treated the equation $y^2 = \sum_{i=0}^{n-1}(x+i)^2$ for a positive integer $n \geq 2$ also from a viewpoint of the theory of elliptic surfaces. We also note in passing that Beukers and Top [**2**] considered the problem that is equivalent to finding integer solutions to the cubic equation $\mathrm{Pyr}_3(m) = \mathrm{Pyr}_4(n)$ in our notation.

Theorems will be proved in Section 3 after the determination in Section 2 of the structure of the Mordell-Weil group of the curve (3), viewed as defined over the rational function field $\mathbf{Q}(m)$ (with fixed n and varying m). The last Section 4 will be devoted to giving an upper bound of heights of polynomial points, which is used in Section 3, as mentioned.

2. Mordell-Weil groups and generators. Fixing an integer $n \geq 3$ and viewing m as a variable, we regard the elliptic curve (3) as being defined over $\mathbf{Q}(m)$. Now let us make a change of variables:

$$t = m-2, \quad x = \frac{3}{(n-2)t}X - \frac{1}{t}, \quad y = \frac{3}{(n-2)^2 t}Y + \frac{n-4}{2(n-2)}. \tag{7}$$

This changes the equation (3) into the following elliptic curve over the rational function field $\mathbf{Q}(t)$ defined by a (minimal) Weierstrass form

$$E_n : Y^2 = X^3 + p(t)X + q(t), \tag{8}$$

where

$$\begin{aligned} p(t) &= -\frac{1}{9}(n-2)^2(t^2-3t+3), \\ q(t) &= \frac{1}{108}(n-2)^2((3n^2-20n+40)t^2 \\ &\qquad - 12(n-2)t + 8n - 16). \end{aligned}$$

In the present section we will determine the structure of the Mordell-Weil group $E_n(\bar{\mathbf{Q}}(t))$ for each $n \geq 3$. And we will also find generators of each Mordell-Weil group by calculating the height pairing. Notations here follow [**7**]. We note that the minimal elliptic surface corresponding to our elliptic curve (8) is rational, and hence the structure theorem of the Mordell-Weil lattice as developed in Theorem 10.3 in [**7**] is applicable. Namely, the structure of the Mordell-Weil group is determined by the data of singular fibers which is described in the following lemma.

Lemma 2.1. *For each $n \geq 3$, reducible singular fibers of the minimal elliptic surface over $\mathbf{P}^1$ corresponding to the elliptic curve* (8) *are fully described as follows*:

(a) *Both E_3 and E_6 have an I_3-fiber at $t = 0$ and an I_0^*-fiber at $t = \infty$.*

(b) *E_4 has an I_2-fiber at each $t = 0, 3$ and $3/2$ and an I_0^*-fiber at $t = \infty$.*

(c) *Both E_5 and E_n for $n \geq 7$ have an I_2-fiber at $t = 0$ and an I_0^*-fiber at $t = \infty$.*

Proof. It is known as Tate's algorithm [**9**] that the type of singular fiber over $t = t_0$ is completely determined and calculable by the orders at t_0 of the discriminant $\Delta = 4p(t)^3 + 27q(t)^2$, $q(t)$ and the j-invariant of the elliptic curve (8). The discriminant is given by

$$\begin{aligned}\Delta = &-\frac{1}{11664}(n-2)^4t^2\big(64(n-2)^2t^4 - 576(n-2)^2t^3\\ &- 9(27n^4 - 360n^3 + 1664n^2 - 3776n + 3776)t^2\\ &+ 648(n-2)(3n^2 - 28n + 56)t - 1296(n-2)(n-3)(n-6)\big).\end{aligned}$$

The order of Δ at $t = 0$ is 2 for all n except for 3 and 6, in which cases the order is 3. And we can verify by computing the discriminant that the second factor, the polynomial of degree 4, has double zeros only when $n = 4$, for integer $n \geq 3$, and the roots are $t = 3$ and $3/2$. The order of Δ at $t = \infty$ is equal to that of Δ at $s = 0$ in the equation obtained from the change of variables $s = 1/t$, $X' = s^2X$ and $Y' = s^3Y$. From these the lemma is established using Tate's algorithm.

Remark. Of course, there are also irreducible singular fibers on each elliptic surface. However, those make no effect on the structure of the Mordell-Weil group and thus we do not describe such fibers here.

For each $n \geq 3$, let

$$P_1^{(n)} = \Big(\frac{1}{3}(n-2)(t+1), \frac{1}{6}\, n(n-2)t\Big),$$
$$P_2^{(n)} = \Big(\frac{1}{3}(n-2), -\frac{1}{6}(n-2)(n-4)t\Big)$$

and

$$P_3^{(n)} = \Big(\frac{1}{3}(n-2)(-t+1), -\frac{1}{6}(n-2)(n-4)t\Big)$$

be points on the elliptic curve (8) corresponding to trivial points $(1,1), (0,0)$ and $(-1,0)$ on the original elliptic curve (3). And, for $n = 4$, we add one more point: Let $P_4^{(4)} = (2(t-2)/3, 0)$ be the point corresponding to another trivial point $((m-5)/(m-2), 0)$. The following proposition gives the structure and generators of the Mordell-Weil group of $E_n(\bar{\mathbf{Q}}(t))$.

Proposition 2.2. *The structure and generators of the Mordell-Weil group of* $E_n(\bar{\mathbf{Q}}(t))$ *are as follows*:

(a) $E_3(\bar{\mathbf{Q}}(t)) \cong E_6(\bar{\mathbf{Q}}(t)) \cong \mathbf{Z} \oplus \mathbf{Z}$ *and generators are* $P_1^{(3)}, P_2^{(3)}$ *and* $P_1^{(6)}, P_2^{(6)}$*, respectively.*

(b) $E_4(\bar{\mathbf{Q}}(t)) \cong \mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})^2$ *and* $P_1^{(4)}$ *is a free generator. Torsion points of order* 2 *are* $P_2^{(4)}, P_3^{(4)}$ *and* $P_4^{(4)}$.

(c) *For* $n = 5$ *and* $n \geq 7$*,* $E_n(\bar{\mathbf{Q}}(t)) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ *and generators are* $P_1^{(n)}, P_2^{(n)}$ *and* $P_3^{(n)}$.

Proof. (a) There is essentially no difference between two cases $n = 3$ and 6, and so we treat the first here. From the types of two reducible fibers in Lemma 2.1 (a), the lattice of irreducible components of those fibers is $\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$. Considering the dual lattice of the orthogonal of this lattice in E_8, cf. [**3**], the Mordell-Weil group $E_3(\bar{\mathbf{Q}}(t))$ is isomorphic to $(1/6)\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ as a lattice. (The lattice is obtained also from the table of [**6**], and so are the other ones below.) It follows that the rank of the group is 2 and the height pairing of each generator with itself has to be $2/3$.

By the explicit formula of the height pairing [**7**], we have (under the notation in [**7**])

$$\langle P_1^{(3)}, P_1^{(3)}\rangle = 2 - \operatorname{contr}_0(P_1^{(3)}) - \operatorname{contr}_\infty(P_1^{(3)}).$$

Since $\operatorname{contr}_0(P_1^{(3)}) = 2/3$ and $\operatorname{contr}_\infty(P_1^{(3)}) = 2/3$, we have $\langle P_1^{(3)}, P_1^{(3)}\rangle = 2/3$. Hence, $P_1^{(3)}$ is a generator of $E_3(\bar{\mathbf{Q}}(t))$, and by the same calculation, so is $P_2^{(3)}$.

(b) From the previous lemma, the lattice of irreducible components is $D_4 \oplus A_1^{\oplus 3}$ and therefore, $E_4(\bar{\mathbf{Q}}(t))$ is isomorphic to $A_1^* \oplus (\mathbf{Z}/2\mathbf{Z})^2$. Torsion points correspond to the roots of the righthand side of the equation (8), i.e., $P_2^{(4)}, P_3^{(4)}$ and $P_4^{(4)}$. Since the free part is isomorphic to A_1^*, the height pairing of a generator is $1/2$. Observing singular points on each singular fiber, we have $\operatorname{contr}_0(P_1^{(4)}) = 1/2$, $\operatorname{contr}_\infty(P_1^{(3)}) = 1$ and $\operatorname{contr}_3(P_1^{(4)}) = \operatorname{contr}_{3/2}(P_1^{(4)}) = 0$, and hence $\langle P_1^{(4)}, P_1^{(4)}\rangle = 1/2$ from the formula. Therefore, $P_1^{(4)}$ is a generator of $E_4(\bar{\mathbf{Q}}(t))$.

(c) In every case, the lattice of irreducible components is $D_4 \oplus A_1$ and hence $E_n(\bar{\mathbf{Q}}(t))$ is isomorphic to $A_1^{*\oplus 3}$. From singular points on each singular fiber, $\operatorname{contr}_0(P_i^{(n)}) = 1/2$ and $\operatorname{contr}_\infty(P_i^{(n)}) = 1$, and it follows that $\langle P_i^{(n)}, P_i^{(n)}\rangle = 1/2$, $i = 1, 2$ and 3, from the formula, as is required.

Corollary 2.3. *$E_n(\bar{\mathbf{Q}}(t)) = E_n(\mathbf{Q}(t))$ for every $n \geq 3$.*

Proof. This is clear because all the generators of each $E_n(\bar{\mathbf{Q}}(t))$ given in the proposition are defined over $\mathbf{Q}(t)$.

3. Proof of theorems. By (7), the point (x, y) on (3) corresponds to the point (X, Y) on the elliptic curve (8) where

$$(9)\qquad X = \frac{1}{3}(n-2)(tx+1), \qquad Y = \frac{1}{6}(n-2)(2(n-2)y - n + 4)t.$$

In order to find integral points on the curve (3) over $\mathbf{Q}$ (integers m and n being fixed), we search for $\mathbf{Q}[m]$-integral points (x, y), i.e., x and $y \in \mathbf{Q}[m]$, on (3) viewed as over $\mathbf{Q}(m)$, with an integer n fixed. Then, by (9), corresponding points on (8) must be $\mathbf{Q}[t]$-integral (polynomial points). We can find all the polynomial points from the generators of $E_n(\mathbf{Q}(t))$ because heights of polynomial points are bounded. We need a practical bound for actual computation and this will be given by Corollary 4.4 to Proposition 4.1 in the next section.

Proposition 3.1. *For each value of $n \geq 3$, the following table lists the parametric integer solutions (x, y, m) coming from $\mathbf{Q}[t]$-integral points on the elliptic curve* (8)*.*

TABLE 1. Parametric solutions in case of fixing n.

n		(x, y, m)
$n = 3$	i)	$(k-2, k^2-3k+1, 3k+2)$
	ii)	$(4k-2, (2k-1)(4k-1), 3k+2)$
	iii)	$(4k+2, 2(3k+1)(4k+3), 27k+20)$
$n = 4$	i)	$(2k-2, (k-1)(2k-1), 3k+2)$
$n = 5$	i)	$(3k-2, 3k^2-3k+1, 3k+2)$
	ii)	$(4k-2, (2k-1)(4k-1), 9k+2)$
	iii)	$(3k, k(6k+1), 12k+6)$
	iv)	$(3k+1, (2k+1)(3k+1), 12k+10)$
	v)	$(12k-14, 2(k-1)(12k-13), 3k)$
	vi)	$(12k-10, 2(3k-2)(4k-3), 3k+1)$
$n = 6$	i)	$(4k-2, 4k^2-3k+1, 3k+2)$
	ii)	$(k-2, (1/2)(k-1)(k-2), 3k+2)$
$n \geq 7$	i)	$((n-2)k-2, (n-2)k^2-3k+1, 3k+2)$
		$(n-2)k/3-2, (n-2)k^2/9-k+1, k+2)$
		if $n \equiv 2 \bmod 9$
	ii)	$(4k-2, (2k-1)(4k-1), 3(n-2)k+2)$
$n = 8$	iii)	$(6k, k(30k+1), 75k+7)$
	iv)	$(6k+4, (3k+2)(10k+7), 75k+57)$

Proof. We find all $\mathbf{Q}[t]$-integral points on the elliptic curve (8) using the estimate of heights given in Corollary 4.4 in the next section, and check if they correspond to $\mathbf{Q}[m]$-integral points on the curve (3). We shall simply write P_i for $P_i^{(n)}$.

When $n = 3$, computation shows that only $2P_2, -2P_1 - 2P_2$ and $-2P_1$ give $\mathbf{Q}[m]$-integral points on (3) and they are respectively $((m-8)/3, (m^2-13m+31)/9)$, $(2(2m-7)/3, (1/9)(2m-7)(4m-11))$ and $(2(2m-13)/27, 2(m-11)(4m+1)/243)$. The first and the second points become ($\mathbf{Z}$- and positive) integral points if and only if m is an integer of the form $3k+2$ where $k \geq 3$ for the first and $k \geq 1$ for the second. For the third point the condition is $m = 27k+20$ for $k \geq 0$.

When $n = 4$, $-P_1 + P_4$ is the unique point to look at and it gives $(2(m-5)/3, (m-5)(2m-7)/9)$ on the curve (3). This will become a positive integer point if and only if $m = 3k+2$ for $k \geq 2$.

For the case $n = 5$, four points occur: $-P_1 + P_2 + P_3$, $-P_1 - P_2 - P_3$, $-P_1 - P_2 + P_3$ and $-P_1 + P_2 - P_3$ on (3). These correspond respectively to $(m-4, (m^2-7m+13)/3))$, $(2(2m-13)/9, (2m-13)(4m-17)/81)$, $((m-6)/4, (m-4)(m-6)/24))$ and $(4m-14, 2(m-3)(4m-13)/3)$. The first and the second are integer points only in the case $m = 3k+2$ for $k \geq 1$ and $m = 9k+2$ for $k \geq 1$, respectively. For the third, $m = 12k+6$ for $k \geq 1$ or $m = 12k+10$ for $k \geq 0$ is the condition. And $m = 3k$ for $k \geq 2$ or $m = 3k+1$ for $k \geq 1$ is for the last.

When $n = 6$, $-2P_1 + 2P_2$ and $-2P_1$ are required points. (Here we note that $2P_2$ is also $\mathbf{Q}[m]$-integral but this point gives negative y for all m.) These two points become $(2(4m-14)/3, (4m^2-25m+43)/9)$ and $((m-8)/3, (m-5)(m-8)/18)$ on (3), and then $m = 3k+2$ is the condition to be positive integral points for both cases where $k \geq 1$ and $k \geq 3$, respectively.

For every $n \geq 7$, $-P_1 + P_2 + P_3$, $-P_1 - P_2 - P_3$ and $-P_1 - P_2 + P_3$ satisfy the condition. On the curve (3) these three points correspond to

$$\left(\frac{1}{3}(mn-2m-2n-2), \frac{1}{9}(m^2n-2m^2-4mn-m+4n+19)\right) \tag{10}$$

$$= \left(\frac{1}{3}(m-2)(n-2) - 2, \frac{1}{9}(m-2)^2(n-2) - m + 3\right), \tag{11}$$

$$\Big(\frac{4(m-2)}{3(n-2)}-2, \Big(\frac{2(m-2)}{3(n-2)}-1\Big)\Big(\frac{4(m-2)}{3(n-2)}-1\Big)\Big),$$

and

$$\Big(\frac{(mn-2m-8n+22)}{3(n-3)^2}, \frac{(mn-2m-8n+22)(mn-2m-5n+13)}{9(n-2)(n-3)^3}\Big).$$

The first is positive integral if $m = 3k+2$ for $k \geq 1$. When $n \equiv 2 \bmod 9$, the first is always integral for any value of m. The second is integral if $m = 3(n-2)k + 2$ for $k \geq 1$. (When $n-2$ is even, the condition can be weakened. However, we do not want to make our case distinction too complicated to describe.) The third requires a little more work and will be treated as

Lemma 3.2. *The point*

$$(x, y) = \Big(\frac{(mn-2m-8n+22)}{3(n-3)^2}, \frac{(mn-2m-8n+22)(mn-2m-5n+13)}{9(n-2)(n-3)^3}\Big)$$

on the curve (3) *becomes an integer point only when $n = 8$. In that case the condition on m for (x, y) to be a positive integer point is that $m = 75k + 7$ for $k \geq 1$ or $m = 75k + 57$ for $k \geq 0$.*

Proof of Lemma 3.2. Put $n - 2 = q$. In order that y is an integer, q must divide $(mn - 2m - 8n + 22)(mn - 2m - 5n + 13)$. Since $(mn-2m-8n+22)(mn-2m-5n+13) \equiv 6\cdot 3 = 18 \bmod q$, q must be a divisor of 18 and hence, because $n \geq 7$, possible values of $n(= q+2)$ are 8, 11, 20. It is easy to see that $n = 11$ and $n = 20$ are impossible. When $n = 8$, the point (x, y) becomes $(2(m-7)/25, (m-7)(2m-9)/375)$. For x to be integral, m must be of the form $25k+7$. Then $y = k(10k+1)/3$ and thus k must be congruent to 0 or $2 \bmod 3$. Therefore (x, y) is positive integral if and only if $m = 75k+7$ for $k \geq 1$ or $m = 75k+57$ for $k \geq 0$. This proves the lemma and thus the proposition is established.

In the course of our computation of points on (3) for general $n \geq 7$, we found points other than (10) whose coordinates are polynomials in n; namely,

$$-P_1 - P_2 = \Big(\frac{3n-m-7}{m-2}, \frac{3n-m-7}{m-2}\Big)$$

and

$$-P_1 - P_3 = \Big(\frac{3(n-6)}{4(m-2)},\ \frac{3n+4m-26}{8(m-2)}\Big).$$

For a fixed value of m, we shall investigate the condition on n in order for these points to be integral. First, if $m = 3$, the first point is always integral and positive provided that n is an integer ≥ 4 whereas the second is a positive intgral when and only when $n = 8k+2$ with $k \geq 1$. Assume $m > 3$. The point

$$\Big(\frac{3n-m-7}{m-2}, \frac{3n-m-7}{m-2}\Big) = \Big(\frac{3(n-3)}{m-2} - 1, \frac{3(n-3)}{m-2} - 1\Big)$$

is integral if $n-3$ is of the form $(m-2)k$. If $m-2$ is divisible by 3, then the condition can be weakened to $n-3 = (m-2)k/3$. The point

$$\Big(\frac{3(n-6)}{4(m-2)}, \frac{3n+4m-26}{8(m-2)}\Big) = \Big(\frac{3(n-6)}{4(m-2)}, \frac{1}{2}\cdot\frac{3(n-6)}{4(m-2)} + \frac{1}{2}\Big)$$

is integral if and only if $3(n-6)/4(m-2)$ is an odd integer. This is so if $n-6$ is of the form $4(m-2)(2k+1)$. If $m-3$ is divisible by 3, then it is enough for $n-6$ to be of the form $4(m-2)(2k+1)/3$. Summing up, we obtain the following

Proposition 3.3. *Suppose an integer $m \geq 3$ is fixed. We have the following parametric solutions (x, y, n) where x, y and n are polynomials in an integer $k \geq 1$:*

$$\begin{aligned}(x,y,n) &= (3(m-2)k-2, (m-2)^2k - m + 3, 9k+2),\\ &= (3k-1, 3k-1, (m-2)k+3),\\ &= (6k-3, 3k-1, 4(m-2)(2k-1)+6).\end{aligned}$$

Moreover, if $m \equiv 2 \bmod 3$, then we have

$$\begin{aligned}(x,y,n) &= \frac{1}{3}(m-2)k-2, \frac{1}{9}(m-2)^2k - m + 3, k+2,\\ &= \Big(k, k, \frac{1}{3}(m-2)(k+1)+3\Big),\\ &= \Big(2k-1, k, \frac{4}{3}(m-2)(2k-1)+6\Big).\end{aligned}$$

Propositions 3.1 and 3.3 provide the identities of Theorems 1.1 and 1.2. Theorem 1.3 is also deduced from the above consideration, since a $\mathbf{Q}[m,n]$ point must be a priori a $\mathbf{Q}[m]$ point (when n is fixed).

Remark. If we regard the elliptic curve (8) as being defined over $\mathbf{Q}(n)$ (we denote it by E'_m here), the Mordell-Weil group $E'_m(\mathbf{Q}(n))$ is a little complicated. In this case there are two types of the Mordell-Weil group according as $m = 5$ or $m \neq 5$. But we have to pay more attention to their generators. First, in case of $m = 5$ ($t = 3$), the Mordell-Weil group $E'_5(\bar{\mathbf{Q}}(n))$ is isomorphic to $\mathbf{Z}^3$. We can easily verify that $P_1^{(n)}$ and $P_2^{(n)}$ are again generators in this case, too. But a remaining generator is completely different from any generators of $E_n(\mathbf{Q}(t))$. Indeed,

$$\Big(\frac{1}{3}(3\sqrt{-1}-2)(n-2),\ \frac{1}{2}n(n-2)\Big)$$

is a third generator, which does not correspond to a trivial point on the elliptic curve (3). Thus the Mordell-Weil group $E'_5(\bar{\mathbf{Q}}(n))$ is actually $E'_5(\mathbf{Q}\sqrt{-1})(n))$. Anyway, we can search for $\mathbf{Q}[n]$-integral points from these generators. But there is no new $\mathbf{Q}[n]$-point other than those already obtained before.

In case of $m \neq 5$, the Mordell-Weil group $E'_m(\bar{\mathbf{Q}}(n))$ is isomorphic to $\mathbf{Z}^4$. Generators are $P_1^{(n)}, P_2^{(n)}, P_3^{(n)}$ and a point with the following X-coordinate:

$$X = -\frac{1}{6}\left(m-2 \pm \sqrt{m^2-22m+49}\right)(n-2).$$

Thus, the field of definition becomes strictly larger than $\mathbf{Q}$ (except for $m = 20$, 22 and 30). It would, therefore, be hard to investigate the whole $\mathbf{Q}(n)$-rational points and presumably no new $\mathbf{Q}[n]$ points will be found.

4. Upper bound for heights of polynomial points. In this section we shall prove the following

Proposition 4.1. *Let k be an algebraically closed field of characteristic other than 2 or 3, E an elliptic curve given by a minimal Weierstrass form over $k(t)$, and r the number of distinct zeros of the discriminant*

Δ. *Then for a polynomial point* $P = (x(t), y(t)) \in E(k(t))$, *we have the following properties*:

(i) *When* $r \geq 2$,

$$\langle P, P \rangle \leq \begin{cases} 2\chi + 4r - 6 & \text{if } 12 \nmid \deg(\Delta) \\ 2\chi + 4r - 4 & \text{if } 12 \mid \deg(\Delta). \end{cases}$$

(ii) *When* $r = 1$, P *is a torsion point, and hence*

$$\langle P, P \rangle = 0.$$

Here $\langle P, P \rangle$ *is the height pairing in the sense of* [**7**] *and* χ *is the arithmetic genus of the minimal model.*

We note that the height pairing here is twice the canonical height. Suppose E is given by a minimal Weierstrass form;

$$E : y^2 = x^3 + p(t)x + q(t), \tag{12}$$

where $p(t)$ and $q(t)$ are in $k[t]$. By minimal we mean the degree of the discriminant Δ is minimal. Then a positive integer $\mu = \min\{m \in \mathbf{Z} \mid \deg(p(t)) \leq 4m \text{ and } \deg(q(t)) \leq 6m\}$ is uniquely determined. For a point $P = (x(t), y(t))$ on the elliptic curve E, if $\deg(x(t)) \leq 2\mu$ and $\deg(y(t)) \leq 3\mu$, then we can easily show $\langle P, P \rangle \leq 2\chi$ [**7**]. Thus we treat a polynomial point $P = (x(t), y(t))$ on the elliptic curve (12) such that $\deg(x(t)) \geq 2\mu$ and $\deg(y(t)) \geq 3\mu$.

In order to prove the theorem above, we need the following two lemmas.

Lemma 4.2. *For a polynomial point* $P = (x(t), y(t)) \in E(k(t))$,

$$(PO) = \frac{1}{2}\deg(x(t)) - \mu,$$

where (PO) *is the intersection number of the section* (P) *and the zero section* (O).

Proof. Since $P = (x(t), y(t))$ is a polynomial point, the section (P) does not intersect the zero section (O) except on the fiber at $t = \infty$.

It follows that $(PO) = (PO)_{t=\infty}$, where $(PO)_{t=\infty}$ is the local index at the intersecting point of two sections on the fiber at $t = \infty$. Now we change the coordinate of the elliptic curve (12). Let $s = 1/t$, $X = s^{2\mu}x$ and $Y = s^{3\mu}y$, and moreover, $W = X/Y$ and $Z = 1/X$. Then we have the elliptic curve

$$\tilde{E} : Z = W^3 + \tilde{p}(s)WZ^2 + \tilde{q}(x)Z^3,$$

where $\tilde{p}(s) = s^{4\mu}p(1/s)$ and $\tilde{q}(s) = s^{6\mu}q(1/s)$ are in $k[s]$. By this coordinate change, the point P becomes $\tilde{P} = (w(s), z(s)) = (X(s)/Y(s), 1/Y(s))$ and the point at infinity O is the origin $\tilde{O} = (0, 0)$. So we may calculate $(\tilde{P}\tilde{O})_{s=0}$, i.e., the local intersection number of two sections $(\tilde{P})$ and $(\tilde{O})$ at the point $p = (0, 0, 0)$ on $\{(W, Z, s)\}$, which is the affine expression of the minimal elliptic surface S of $\tilde{E}$.

The local ring of this surface at the point p is

$$\mathcal{O}_{S,p} = k[W, Z, s]_p/(Z - W^3 - \tilde{p}(s)WZ^2 - \tilde{q}(s)Z^3)$$

and the local equation of the section $(\tilde{P})$, respectively $(\tilde{O})$, at p is $W - w(s) = 0$, respectively $W = 0$. Since $w(s) = us^{(1/2)\deg(x(t))-\mu}$ for some u in $k[s]_0^*$, it follows from the definition of an intersection number that

$$\begin{aligned}(PO) &= \dim_k \mathcal{O}_{S,p}/(W - w(s), W)\\ &= \dim_k k[s]_0/(s^{(1/2)\deg(x(t))-\mu})\\ &= \frac{1}{2}\deg(x(t)) - \mu,\end{aligned}$$

where the second equality is obtained from the isomorphism of k-vector spaces

$$\mathcal{O}_{S,p}/(W - w(s), W) \cong k[s]_0/(w(s)).$$

Lemma 4.3. *For a polynomial point $P = (x(t), y(t)) \in E(k(t))$,*

$$\langle P, P\rangle \leq 2\chi - 2\mu + \deg(x(t)).$$

Proof. From the formula, Theorem 8.6 in [**7**], of the height pairing, we have

$$\langle P, P\rangle \leq 2\chi + 2(PO).$$

This combined with Lemma 4.2 gives the desired inequality.

Proof of Proposition 4.1. Let $h : K \to \mathbf{Z}$ be the height function (K is the function field) i.e., $h(f)$ is either the total number of zeros of f or the total number of poles [**8**]. Choosing zeros of Δ and ∞ as the set S in the inequality of [**4**, Proposition 8.2] or [**8**, Theorem 12.3], we have

$$h\Big(\frac{y^4}{\Delta}\Big) \leq 24(r-1). \tag{13}$$

Case (i). As we have noticed, we are treating a polynomial point $P = (x(t), y(t))$ such that $\deg(x(t)) \geq 2\mu$ and $\deg(y(t)) \geq 3\mu$. Since the discriminant Δ is also a polynomial and $\deg(\Delta) \leq 12m$ by the assumption, we have

$$h\Big(\frac{y^4}{\Delta}\Big) \geq 4h(y) - h(\Delta) = 4\deg(y(t)) - \deg(\Delta).$$

Thus from the inequality (13),

$$\begin{aligned} \deg(y(t)) &\leq 6(r-1) + \frac{1}{4}\deg(\Delta) \\ &\leq 6(r-1) + \begin{cases} 3(\mu-1) & \text{if } 12 \nmid \deg(\Delta) \\ 3\mu & \text{if } 12 \mid \deg(\Delta) \end{cases} \\ &= \begin{cases} 6r+3\mu-9 & \text{if } 12 \nmid \deg(\Delta) \\ 6r+3\mu-6 & \text{if } 12 \mid \deg(\Delta). \end{cases} \end{aligned}$$

Since $\deg(x(t)) = 2/3\deg(y(t))$, we have

$$\deg(x(t)) \leq \begin{cases} 4r+2\mu-6 & \text{if } 12 \nmid \deg(\Delta) \\ 4r+2\mu-4 & \text{if } 12 \mid \deg(\Delta). \end{cases}$$

Hence we obtain the desired result by Lemma 4.3.

Case (ii). Since $h(y^4/\Delta) = 0$ from the inequality (13), $y^4 = \alpha\Delta$ for some α in k. By the assumption that $y(t)$ is a polynomial, if $4 \nmid \deg(\Delta)$, then $\alpha = 0$, and, therefore, $P = (x(t), 0)$, i.e., a torsion point of order 2. If $4 \mid \deg(\Delta)$, then a minimal Weierstrass form is as follows:

$$y^2 = x^3 + \beta(t+\gamma)^{2l}, \quad l = 1 \text{ or } 2,$$

for some β and γ in k. But both of the elliptic curves above have two singular fibers of the types IV and IV*, and hence the Mordell-Weil groups are isomorphic to $\mathbf{Z}/2\mathbf{Z}$. Indeed, for $P = (0, \beta^{1/2}(t+\gamma)^l)$, we have $E(k(t)) = \{O, P, 2P\}$. The fact that $\langle P, P\rangle = 0$ is immediately obtained from the definition of the height pairing.

Corollary 4.4. *In addition to the assumption in Proposition* 4.1, *if E is a rational elliptic surface, then*

$$\langle P, P\rangle \le \begin{cases} 4r-4 & \text{if } \deg(\Delta) < 12, \\ 4r-2 & \text{if } \deg(\Delta) = 12. \end{cases}$$

Proof. This follows immediately from the fact that $\chi = \mu = 1$ when E is rational.

Corollary 4.5. *In general, for a polynomial point P,*

$$\langle P, P\rangle \le 2\chi + 48\mu - 4.$$

In particular, $\langle P, P\rangle \le 46$ *if E is rational.*

Proof. Immediately from the fact that $r \le \deg(\Delta) \le 12\mu$.

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