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# Quasimodular solutions of a differential equation of hypergeometric type

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## §1. INTRODUCTION AND MAIN THEOREM

In our previous paper [2], we studied further the solutions of the following differential equation in the upper half-plane  $\mathfrak{H}$  which was originally found and studied in [4] in connection with the arithmetic of supersingular elliptic curves;

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0.$$

Here,  $k$  is an integer or half an integer, the symbol  $'$  denotes the differentiation  $(2\pi i)^{-1}d/d\tau = q \cdot d/dq$  ( $q = e^{2\pi i\tau}$ ), and  $E_2(\tau)$  is the “quasimodular” Eisenstein series of weight 2 for the full modular group  $\mathrm{SL}_2(\mathbb{Z})$ :

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) q^n.$$

Let  $p \geq 5$  be a prime number and  $F_{p-1}(\tau)$  be the solution of the above differential equation for  $k = p - 1$  which is modular on  $\mathrm{SL}_2(\mathbb{Z})$  (such a solution exists and is unique up to a scalar multiple). For any zero  $\tau_0$  in  $\mathfrak{H}$  of the form  $F_{p-1}(\tau)$ , the value of the  $j$ -function at  $\tau_0$  is algebraic and its reduction modulo (an extension of)  $p$  is a supersingular  $j$ -invariant of characteristic  $p$ , and conversely, all the supersingular  $j$ -invariants are obtained in this way from the single solution  $F_{p-1}(\tau)$  with suitable choices of  $\tau_0$ . This is the arithmetic connection that motivated our study of the differential equation.

Various modular forms on  $\mathrm{SL}_2(\mathbb{Z})$  and its subgroups were obtained in [2] as solutions to this differential equation, the groups depending on the choice of  $k$ . Every modular solution is expressed in terms of a hypergeometric polynomial in a suitable modular function (hence the “hypergeometric type” in the title of the paper), also depending on the choice of  $k$ . For instance, if  $k \equiv 0, 4 \pmod{12}$ , we have a modular solution

$$E_4(\tau)^{\frac{k}{4}} F\left(-\frac{k}{12}, -\frac{k-4}{12}, -\frac{k-5}{6}; \frac{1728}{j(\tau)}\right),$$

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where

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1)$$

is the Gauss hypergeometric series (which becomes a polynomial when  $a$  or  $b$  is a negative integer, which is the case here),  $E_4(\tau)$  the Eisenstein series of weight 4 on  $\mathrm{SL}_2(\mathbb{Z})$ , and  $j(\tau)$  the elliptic modular invariant.

In addition to the modular solutions, quite remarkable was an occurrence of a *quasi-modular form*, not of weight  $k$  as in the modular case but of weight  $k+1$ . In the present paper, we give another supply of examples of quasimodular forms as solutions to an analogous differential equation attached to the group  $\Gamma_0^*(2)$ , which is *not* contained in  $\mathrm{SL}_2(\mathbb{Z})$ ;

$$\Gamma_0^*(2) = \left\langle \Gamma_0(2), \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \right\rangle$$

where

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{2} \right\}.$$

( $\Gamma_0^*(2)$  is the triangular group “2A” in the notation of Conway-Norton [1].)

Let

$$E_{2A}(\tau) := (E_2(\tau) + 2E_2(2\tau))/3 = 1 - 8q - 40q^2 - 32q^3 - \cdots$$

be the quasimodular form of weight 2 on  $\Gamma_0^*(2)$  which is the logarithmic derivative of the form

$$\Delta_{2A}(\tau) := \eta(\tau)^8 \eta(2\tau)^8 = q - 8q^2 + 12q^3 + 64q^4 - \cdots$$

of weight 8 on  $\Gamma_0^*(2)$ ;  $E_{2A}(\tau) = \Delta'_{2A}(\tau)/\Delta_{2A}(\tau)$ , an analogous situation in the  $\mathrm{SL}_2(\mathbb{Z})$  case where  $E_2(\tau)$  is the logarithmic derivative of the Ramanujan  $\Delta(\tau)$ . Consider the following differential equation;

$$(\#)_k \quad f''(\tau) - \frac{k+1}{4} E_{2A}(\tau) f'(\tau) + \frac{k(k+1)}{8} E'_{2A}(\tau) f(\tau) = 0.$$

Solutions which are modular on the group  $\Gamma_0^*(2)$  and its subgroups were studied in [6, 7]. In particular, when  $k$  is a non-negative integer congruent to 0 or 6 modulo 8, the equation  $(\#)_k$  has a one dimensional space of solutions which are modular on the group  $\Gamma_0^*(2)$  itself. We note here that the equation  $(\#)_k$  has a characterization by the invariance of the space of solutions under the action of  $\Gamma_0^*(2)$ , similar to the previous case for  $\mathrm{SL}_2(\mathbb{Z})$ , owing to the fact that there is no holomorphic modular form of weight 2 on  $\Gamma_0^*(2)$  (see [5] and [2, §5]). By a general theory of ordinary differential equations, we see that the equation  $(\#)_k$  has a quasimodular solution (which, since its transformation under  $\tau \rightarrow -1/2\tau$  is also a solution, inevitably gives a solution having  $\log q$  term in the expansion at  $q = 0$ ) only when  $k$  is a positive integer congruent to 3 modulo 4.

In the following, we show there indeed exists a quasimodular solution in this case and describe explicitly the solution in terms of a certain orthogonal polynomials. First we need to develop some notations. Put

$$\begin{aligned} C(\tau) &:= 2E_2(2\tau) - E_2(\tau) \\ &= 1 + 24 \sum_{n=1}^{\infty} \left( \sum_{\substack{d|n \\ d:\text{odd}}} d \right) q^n = 1 + 24q + 24q^2 + 96q^3 + \cdots, \\ D(\tau) &:= \frac{\eta(2\tau)^{16}}{\eta(\tau)^8} = \sum_{n=1}^{\infty} \left( \sum_{\substack{d|n \\ d:\text{odd}}} (n/d)^3 \right) q^n = q + 8q^2 + 28q^3 + 64q^4 + \cdots, \end{aligned}$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} - q^{\frac{25}{24}} - q^{\frac{49}{24}} + q^{\frac{121}{24}} + \cdots$$

is the Dedekind eta function. The functions  $C(\tau)$  and  $D(\tau)$  are modular forms of respective weights 2 and 4 on the group  $\Gamma_0(2)$  ( $= "2B"$ ) and the graded ring of modular forms of integral weights on  $\Gamma_0(2)$  is generated by these  $C(\tau)$  and  $D(\tau)$ . Recall that (see [3]) an element of degree  $k$  in the graded ring  $\mathbb{C}[E_2(\tau), C(\tau), D(\tau)]$ , where the generators  $E_2(\tau), C(\tau), D(\tau)$  have degrees 2, 2, and 4 respectively, is referred to as a quasimodular form of weight  $k$  (on  $\Gamma_0(2)$ ). Incidentally, the graded ring of modular forms of integral weights on  $\Gamma_0^*(2)$  is generated by three elements  $C(\tau)^2 = (E_4(\tau) + 4E_4(2\tau))/5$ ,  $C(\tau)^3 - 128C(\tau)D(\tau) = (E_6(\tau) + 8E_6(2\tau))/9$ , and  $\Delta_{2A}(\tau)$  of respective weights 4, 6, 8, of which  $C(\tau)^2$  and  $\Delta_{2A}(\tau)$  generate freely the subring consisting forms of weight being multiple of 4, and the whole space as a graded module is generated over this ring by  $C(\tau)^3 - 128C(\tau)D(\tau)$ .

Now define a sequence of polynomials  $P_n(x)$  ( $n = 0, 1, 2, \dots$ ) by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_{n+1}(x) = xP_n(x) + \lambda_n P_{n-1}(x) \quad (n = 1, 2, \dots)$$

where

$$\lambda_n = 4 \frac{(4n+1)(4n+3)}{n(n+1)}.$$

First few examples are

$$P_2(x) = x^2 + 70, \quad P_3(x) = x^3 + 136x, \quad P_4(x) = x^4 + 201x^2 + 4550, \dots$$

The  $P_n(x)$  is even or odd polynomial according as  $n$  is even or odd. We also define a second series of polynomials  $Q_n(x)$  by the same recursion (with different initial values):

$$Q_0(x) = 0, \quad Q_1(x) = 1, \quad Q_{n+1}(x) = xQ_n(x) + \lambda_n Q_{n-1}(x) \quad (n = 1, 2, \dots),$$

a couple of examples being

$$Q_2(x) = x, \quad Q_3(x) = x^2 + 66, \quad Q_4(x) = x^3 + 131x, \dots$$

The  $Q_n(x)$  has opposite parity: It is even if  $n$  is odd and odd if  $n$  is even.

Put  $G(\tau) = C(\tau)^2 - 128D(\tau)$  ( $= (4E_4(2\tau) - E_4(\tau))/3$ ).

**Theorem.** *Let  $k = 4n+3$  ( $n = 0, 1, 2, \dots$ ). The following quasimodular form of weight  $k+1$  on  $\Gamma_0(2)$  is a solution of  $(\#)_k$ :*

$$\sqrt{\Delta_{2A}(\tau)}^n P_n\left(\frac{G(\tau)}{\sqrt{\Delta_{2A}(\tau)}}\right) \frac{C'(\tau)}{24} - \sqrt{\Delta_{2A}(\tau)}^{n+1} Q_n\left(\frac{G(\tau)}{\sqrt{\Delta_{2A}(\tau)}}\right).$$

**Remark.** The appearance of the square root  $\sqrt{\Delta_{2A}(\tau)}$  in the formula is superficial because of the parities of  $P_n(x)$  and  $Q_n(x)$ , that is, the form is actually an element in  $\mathbb{Q}[E_2(\tau), C(\tau), D(\tau)]$ , by noting  $\Delta_{2A}(\tau) = D(\tau)(C(\tau)^2 - 64D(\tau))$  and  $C'(\tau) = (E_2(\tau)C(\tau) - C(\tau)^2)/6 + 32D(\tau)$ . The form does not belong to  $\Gamma_0^*(2)$ .

## §2. PROOF OF THEOREM

It is convenient to introduce the operator  $\vartheta_k$  defined by

$$\vartheta_k(f)(\tau) = f'(\tau) - \frac{k}{8} E_{2A}(\tau) f(\tau).$$

By the quasimodular property of  $E_2(\tau)$  or the fact that  $E_{2A}(\tau)$  is the logarithmic derivative of  $\Delta_{2A}(\tau)$ , we have the transformation formulas

$$E_{2A}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_{2A}(\tau) + \frac{4}{\pi i} c(c\tau+d) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)\right)$$

and

$$E_{2A}\left(-\frac{1}{2\tau}\right) = 2\tau^2 E_{2A}(\tau) + \frac{8}{\pi i} \tau.$$

From these we see that if  $f$  is modular of weight  $k$  on a subgroup of  $\Gamma_0^*(2)$ , then  $\vartheta_k(f)$  is modular of weight  $k+2$  on the same group. If  $f$  and  $g$  have weights  $k$  and  $l$ , the Leibniz rule

$$\vartheta_{k+l}(fg) = \vartheta_k(f)g + f\vartheta_l(g)$$

holds. We sometimes drop the suffix of the operator  $\vartheta_k$  when the weights of modular forms we consider are clear. With this operator, the equation  $(\#)_k$  can be rewritten as

$$(\#')_k \quad \vartheta_{k+2}\vartheta_k(f)(\tau) = \frac{k(k+2)}{64} C(\tau)^2 f(\tau),$$

(use  $E'_{2A}(\tau) = (E_{2A}(\tau)^2 - C(\tau)^2)/8$ ).

Denote the form in the theorem by  $F_k(\tau)$ . We first establish the recurrence relation (note  $n = (k-3)/4$ ):

$$(1) \quad F_{k+4}(\tau) = G(\tau)F_k(\tau) + \lambda_n \Delta_{2A}(\tau)F_{k-4}(\tau).$$

This is a consequence of the recursion of  $P_n$  and  $Q_n$ , namely (we often omit the variable  $\tau$  hereafter)

$$\begin{aligned}
& GF_k + \lambda_n \Delta_{2A} F_{k-4} \\
&= G \left( \sqrt{\Delta_{2A}}^n P_n \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) \frac{C'}{24} - \sqrt{\Delta_{2A}}^{n+1} Q_n \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) \right) \\
&+ \lambda_n \Delta_{2A} \left( \sqrt{\Delta_{2A}}^{n-1} P_{n-1} \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) \frac{C'}{24} - \sqrt{\Delta_{2A}}^n Q_{n-1} \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) \right) \\
&= \sqrt{\Delta_{2A}}^{n+1} \left( \frac{G}{\sqrt{\Delta_{2A}}} P_n \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) + \lambda_n P_{n-1} \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) \right) \frac{C'}{24} \\
&- \sqrt{\Delta_{2A}}^{n+2} \left( \frac{G}{\sqrt{\Delta_{2A}}} Q_n \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) + \lambda_n Q_{n-1} \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) \right) \\
&= \sqrt{\Delta_{2A}}^{n+1} P_{n+1} \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) \frac{C'}{24} - \sqrt{\Delta_{2A}}^{n+2} Q_{n+1} \left( \frac{G}{\sqrt{\Delta_{2A}}} \right) \\
&= F_{k+4}.
\end{aligned}$$

Now we prove by induction that the  $F_k(\tau)$  satisfies the equation  $(\#')_k$ . We can check the cases  $k = 3$  and  $7$  directly. Assume  $F_{k-4}$  and  $F_k$  satisfy  $(\#')_{k-4}$  and  $(\#')_k$  respectively. Then by using (1) and the formulas

$$\vartheta(C) = -\frac{1}{4}G, \quad \vartheta(G) = -\frac{1}{2}C^3, \quad \vartheta(\Delta_{2A}) = 0$$

we have

$$\begin{aligned}
\vartheta^2(F_k) &= \vartheta(\vartheta(F_k)G - \frac{1}{2}C^3F_k) + \lambda_n \Delta_{2A} \vartheta^2(F_{k-4}) \\
&= \vartheta^2(F_k)G - \frac{1}{2}\vartheta(F_k)C^3 + \frac{3}{8}C^2GF_k - \frac{1}{2}C^3\vartheta(F_k) + \lambda_n \Delta_{2A} \vartheta^2(F_{k-4}) \\
&= \frac{k(k+2)}{64}C^2GF_k - C^3\vartheta(F_k) + \frac{3}{8}C^2GF_k + \frac{(k-4)(k-2)}{64}\lambda_n \Delta_{2A} C^2F_{k-4} \\
&= \frac{k^2 + 2k + 24}{64}C^2GF_k + \frac{(k-4)(k-2)}{64}\lambda_n \Delta_{2A} C^2F_{k-4} - C^3\vartheta(F_k).
\end{aligned}$$

Hence we find

$$\begin{aligned}
&\vartheta^2(F_{k+4}) - \frac{(k+4)(k+6)}{64}C^2F_{k+4} \\
&= \left( \frac{k^2 + 2k + 24}{64} - \frac{(k+4)(k+6)}{64} \right) C^2GF_k \\
&+ \left( \frac{(k-4)(k-2)}{64} - \frac{(k+4)(k+6)}{64} \right) \lambda_n \Delta_{2A} C^2F_{k-4} \\
&= -C^2 \left( \frac{k}{8}GF_k + C\vartheta(F_k) + \frac{k+1}{4}\lambda_n \Delta_{2A} F_{k-4} \right).
\end{aligned}$$

The proof of the theorem therefore boils down to show the equation

$$\frac{k}{8}GF_k + C\vartheta(F_k) = -\frac{k+1}{4}\lambda_n\Delta_{2A}F_{k-4}.$$

For this we also proceed by induction. For  $k = 7$  the equation is checked directly. Assuming that this is valid for  $k$ , we have

$$F_{k+4} = GF_k + \lambda_n\Delta_{2A}F_{k-4} = \frac{1}{2(k+1)}((k+2)GF_k - 8C\vartheta(F_k))$$

and

$$\begin{aligned} & \frac{k+4}{8}GF_{k+4} + C\vartheta(F_{k+4}) \\ &= \frac{k+4}{16(k+1)}G((k+2)GF_k - 8C\vartheta(F_k)) \\ &+ \frac{1}{2(k+1)}C\left(-\frac{1}{2}(k+2)C^3F_k + (k+2)G\vartheta(F_k) + 2G\vartheta(F_k) - 8C\vartheta^2(F_k)\right) \\ &= \frac{(k+2)(k+4)}{16(k+1)}(G^2 - C^4)F_k \\ &= -\frac{k+5}{4}\lambda_{n+1}\Delta_{2A}F_k. \end{aligned}$$

Here we have used the (previous) induction assumption that  $F_k$  satisfies  $(\#')_k$  and the relation  $G^2 - C^4 = -256\Delta_{2A}$ . This completes our proof.

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