On a kind of duality of multiple zeta-star values

Kaneko, Masanobu
Faculty of Mathematics, Kyushu University

Ohno, Yasuo
Department of Mathematics, Kinki University

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On a kind of duality of multiple zeta-star values

Masanobu Kaneko and Yasuo Ohno

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1 Main result

In this note, we prove a certain duality-type result for height 1 *multiple zeta-star values* and discuss its possible generalization.

For an index set $(k_1, k_2, \dots, k_n)$ of positive integers with $k_1 > 1$, the multiple zeta-star value $\zeta^*(k_1, k_2, \dots, k_n)$ is defined by

$$\zeta^*(k_1, k_2, \dots, k_n) := \sum_{m_1 \geq m_2 \geq \dots \geq m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}.$$

If we remove the equality signs in the summation, we obtain the usual *multiple zeta value*:

$$\zeta(k_1, k_2, \dots, k_n) := \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}.$$

The *height* of the multiple zeta or zeta-star value is the number of k_i in the index set which is greater than 1. The following theorem can be regarded as a kind of duality for multiple zeta-star values of height 1.

Theorem 1 *For any integers $k, n \geq 1$, we have*

$$(-1)^k \zeta^*(k+1, \underbrace{1, \dots, 1}_n) - (-1)^n \zeta^*(n+1, \underbrace{1, \dots, 1}_k) \in \mathbf{Q}[\zeta(2), \zeta(3), \zeta(5), \dots],$$

the right-hand side being the algebra over $\mathbf{Q}$ generated by the values of the Riemann zeta function at positive integer arguments (> 1).

Remark For multiple zeta values, there is a well-known duality formula [9], and the height 1 case of the formula reads as

$$\zeta(k+1, \underbrace{1, \dots, 1}_{n-1}) = \zeta(n+1, \underbrace{1, \dots, 1}_{k-1})$$

for $k, n \geq 1$. No such simple formula has been known for multiple zeta-star values. It should be noted that the pair of indices

$$(k+1, \underbrace{1, \dots, 1}_n) \longleftrightarrow (n+1, \underbrace{1, \dots, 1}_k)$$

in Theorem 1 is different from that in the duality formula for multiple zeta values above.

We can also compute the generating function of the quantity

$$(-1)^k \zeta^\star(k+1, \underbrace{1, \dots, 1}_n) - (-1)^n \zeta^\star(n+1, \underbrace{1, \dots, 1}_k)$$

in Theorem 1.

Theorem 2 *We have*

$$\begin{aligned} \sum_{k, n \geq 1} ((-1)^k \zeta^\star(k+1, \underbrace{1, \dots, 1}_n) - (-1)^n \zeta^\star(n+1, \underbrace{1, \dots, 1}_k)) x^k y^n \\ = \psi(x) - \psi(y) + \pi (\cot(\pi x) - \cot(\pi y)) \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}. \end{aligned}$$

Here, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, the logarithmic derivative of the gamma function.

2 Proof of Theorems

We prove the following basic identity, from which follow both Theorem 1 and Theorem 2.¹

Proposition *For $k, n \geq 1$, we have*

$$\begin{aligned} & (-1)^k \zeta^\star(k+1, \underbrace{1, \dots, 1}_n) - (-1)^n \zeta^\star(n+1, \underbrace{1, \dots, 1}_k) \\ = & k\zeta(k+2, \underbrace{1, \dots, 1}_{n-1}) - n\zeta(n+2, \underbrace{1, \dots, 1}_{k-1}) \\ & + (-1)^k \sum_{j=0}^{k-2} (-1)^j \zeta(k-j) \zeta(n+1, \underbrace{1, \dots, 1}_j) \\ & - (-1)^n \sum_{j=0}^{n-2} (-1)^j \zeta(n-j) \zeta(k+1, \underbrace{1, \dots, 1}_j), \end{aligned}$$

where we understand an empty sum to be 0.

Proof. We use two formulas for the special value of the function $\xi_k(s)$ defined for $k \geq 1$ by

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t}) dt. \quad (1)$$

¹Recently, C. Yamazaki ([8]) gave another proof of them. It uses a generating function of certain sums of multiple zeta-star values which was introduced in [1].

In [3], we studied this function and obtained among others the formula

$$\begin{aligned}
\xi_k(n+1) &= (-1)^{k-1} [\zeta(n+1, \underbrace{2, 1, \dots, 1}_{k-1}) + \zeta(n+1, \underbrace{1, 2, 1, \dots, 1}_{k-1}) + \dots \\
&\quad \dots + \zeta(n+1, \underbrace{1, \dots, 1, 2}_{k-1}) + (n+1) \cdot \zeta(n+2, \underbrace{1, \dots, 1}_{k-1})] \\
&\quad + \sum_{j=0}^{k-2} (-1)^j \zeta(k-j) \cdot \zeta(n+1, \underbrace{1, \dots, 1}_j),
\end{aligned} \tag{2}$$

where k, n are integers ≥ 1 .

On the other hand, we showed in [6] that the value $\xi_k(n)$ is nothing but the multiple zeta-star value of height 1, i.e., we have the formula

$$\xi_k(n+1) = \zeta^*(k+1, \underbrace{1, \dots, 1}_n). \tag{3}$$

Since the index sets $(k+1, \underbrace{1, \dots, 1}_{n-1})$ and $(n+1, \underbrace{1, \dots, 1}_{k-1})$ are dual (in the context of multiple zeta values) with each other, the main theorem in [6] applied to these index sets with $l = 1$ gives the identity

$$\begin{aligned}
&\zeta(k+2, \underbrace{1, \dots, 1}_{n-1}) + \zeta(k+1, \underbrace{2, 1, \dots, 1}_{n-1}) + \zeta(k+1, \underbrace{1, 2, 1, \dots, 1}_{n-1}) + \dots \\
&\quad \dots + \zeta(k+1, \underbrace{1, \dots, 1, 2}_{n-1}) \\
&= \zeta(n+2, \underbrace{1, \dots, 1}_{k-1}) + \zeta(n+1, \underbrace{2, 1, \dots, 1}_{k-1}) + \zeta(n+1, \underbrace{1, 2, 1, \dots, 1}_{k-1}) + \dots \\
&\quad \dots + \zeta(n+1, \underbrace{1, \dots, 1, 2}_{k-1}).
\end{aligned} \tag{4}$$

Combining (2), (3) and (4), we obtain the proposition. $\square$

Proof of Theorems 1 and 2. Recall the formula of Aomoto [2] and Drinfeld [4]

$$\sum_{k, n \geq 1} \zeta(k+1, \underbrace{1, \dots, 1}_{n-1}) x^k y^n = 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}. \tag{5}$$

This together with the standard Taylor expansion of the (logarithm of) gamma function

$$\Gamma(1+x) = \exp\left(-\gamma x + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} x^n\right) \quad (|x| < 1, \gamma : \text{Euler's constant}) \tag{6}$$

shows that all multiple zeta values of height 1 (= of type $\zeta(m, 1, \dots, 1)$) can be expressed as polynomials over $\mathbf{Q}$ in the Riemann zeta values. Theorem 1 therefore follows from the formula in Proposition.

As for the generating series, we start with the formula (5). Replace k with $k + 1$ in (5) and divide the both-hand sides out by xy , and then differentiate with respect to x and multiply xy . Then we obtain

$$\begin{aligned} & \sum_{k,n \geq 1} k \zeta(k+2, \underbrace{1, \dots, 1}_{n-1}) x^k y^n \\ &= -\frac{1}{x} + \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \left(\frac{1}{x} + \psi(1-x) - \psi(1-x-y) \right), \end{aligned}$$

and hence by interchanging x and y and subtracting, we have

$$\begin{aligned} & \sum_{k,n \geq 1} \left(k \zeta(k+2, \underbrace{1, \dots, 1}_{n-1}) - n \zeta(n+2, \underbrace{1, \dots, 1}_{k-1}) \right) x^k y^n \\ &= -\frac{1}{x} + \frac{1}{y} + \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \left(\frac{1}{x} + \psi(1-x) - \frac{1}{y} - \psi(1-y) \right). \quad (7) \end{aligned}$$

Next, by the formula

$$\sum_{i=2}^{\infty} (-1)^i \zeta(i) x^{i-1} = \psi(1+x) + \gamma$$

(take the logarithmic derivative of (6)) and by (5), we have

$$\begin{aligned} & \sum_{k,n \geq 1} (-1)^k \sum_{j=0}^{k-2} (-1)^j \zeta(k-j) \zeta(n+1, \underbrace{1, \dots, 1}_j) x^k y^n \\ &= \sum_{i \geq 2, j, n \geq 1} (-1)^i \zeta(i) \zeta(n+1, \underbrace{1, \dots, 1}_{j-1}) x^{i+j-1} y^n \\ &= \left(\sum_{i \geq 2} (-1)^i \zeta(i) x^{i-1} \right) \left(\sum_{j,n \geq 1} \zeta(n+1, \underbrace{1, \dots, 1}_{j-1}) x^j y^n \right) \\ &= (\psi(1+x) + \gamma) \left(1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \right), \end{aligned}$$

and thus we obtain

$$\begin{aligned} & \sum_{k,n \geq 1} \left((-1)^k \sum_{j=0}^{k-2} (-1)^j \zeta(k-j) \zeta(n+1, \underbrace{1, \dots, 1}_j) \right. \\ & \quad \left. - (-1)^n \sum_{j=0}^{n-2} (-1)^j \zeta(n-j) \zeta(k+1, \underbrace{1, \dots, 1}_j) \right) x^k y^n \\ &= \left(1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \right) (\psi(1+x) - \psi(1+y)). \quad (8) \end{aligned}$$

By Proposition, Theorem 2 follows from (7), (8), and the standard identities

$$\psi(1+x) = \frac{1}{x} + \psi(x) \quad \text{and} \quad \pi \cot(\pi x) = \frac{1}{x} + \psi(1-x) - \psi(1+x).$$

3 Possible generalization

In this section, we propose a possible generalization of Theorem 1 for arbitrary heights.

First, we recall a few notations which are used in [1]. The *weight* and the *depth* of multiple zeta-star values $\zeta^*(k_1, k_2, \dots, k_n)$ are the sum $k_1 + k_2 + \dots + k_n$ and the length n of its index, respectively. We denote by $X_0(k, n, s)$ the sum of all multiple zeta-star values of weight k , depth n and height s , for $k \geq n + s$ and $n \geq s \geq 1$.

Based on the numerical experiments up to weight 11, we conjecture the following.

Conjecture For any integers $k, n \geq s \geq 1$, we have

$$(-1)^k X_0(k+n+1, n+1, s) - (-1)^n X_0(k+n+1, k+1, s) \in \mathbf{Q}[\zeta(2), \zeta(3), \zeta(5), \dots].$$

Remark Theorem 1 is nothing but the case when $s = 1$ of the above conjecture.

Examples When the weight is 8 and the height is 2 or 3, we can show (using the double shuffle relations of multiple zeta values) the following identities, which are in favor of the conjecture.

$$\begin{aligned} X_0(8, 3, 2) + X_0(8, 6, 2) &= \frac{876}{175} \zeta(2)^4 - \zeta(2) \zeta(3)^2 - 3 \zeta(3) \zeta(5) \\ X_0(8, 4, 2) + X_0(8, 5, 2) &= \frac{1083}{280} \zeta(2)^4 + \zeta(2) \zeta(3)^2 + 2 \zeta(3) \zeta(5) \\ X_0(8, 4, 3) + X_0(8, 5, 3) &= \frac{1349}{280} \zeta(2)^4 - \frac{1}{2} \zeta(2) \zeta(3)^2 - \zeta(3) \zeta(5) \end{aligned}$$

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Masanobu Kaneko
 Faculty of Mathematics,
 Kyushu University,
 Motoooka, Nishi-ku, Fukuoka 819-0395, Japan.
 E-mail: mkaneko@math.kyushu-u.ac.jp

Yasuo Ohno
 Department of Mathematics,
 Kinki University
 Higashi-Osaka, Osaka 577-8502, Japan.
 E-mail: ohno@math.kindai.ac.jp