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ON ORDINARY PRIMES FOR MODULAR FORMS
AND THE THETA OPERATOR

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Abstract. We give a criterion for a prime being ordinary for a modular form, by using the theta operator of Ramanujan.

1. Introduction and statement of the result

A normalized Hecke eigenform is said to be ordinary at a prime $p$ if $p$ does not divide its $p$-th Fourier coefficient. In the theory of $p$-adic modular forms and Galois representations attached to modular forms, this notion has fundamental importance, and there is extensive literature on the subject.

In the present paper, we shall give a criterion for ordinarity in terms of certain polynomials attached to derivatives of given modular forms. Throughout the paper, the modular forms considered are those on the full modular group $SL_2(\mathbb{Z})$.

For any $f = f(z) = \sum_{n=0}^{\infty} a(n)q^n$ ($q = e^{2\pi i z}$), we define

$$\theta f := q \frac{d}{dq} f = \sum_{n=0}^{\infty} n a(n) q^n.$$ 

This is the derivative with respect to $2\pi iz$, and is often referred to as the “theta operator” of Ramanujan. The derivative of a modular form is no longer modular but “quasimodular”, which means, in the case of $SL_2(\mathbb{Z})$, that it is an isobaric element of the ring $\mathbb{C}[E_2, E_4, E_6]$. Here, $E_k = E_k(z)$ for even $k$ is the standard Eisenstein series

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \left( \sum_{d|n} d^{k-1} \right) q^n,$$

$B_k$ being the $k$-th Bernoulli number. For $k \geq 4$, the function $E_k(z)$ is modular of weight $k$, but $E_2(z)$ is not quite modular. The operator $\theta$ preserves the ring $\mathbb{C}[E_2, E_4, E_6]$ (as is seen by Ramanujan’s formulæ $\theta E_2 = (E_2^2 - E_4)/12, \theta E_4 = (E_2 E_4 - E_6)/3, \theta E_6 = (E_2 E_6 - E_4)/2$), and hence for any modular form $f$ and non-negative integer $n$, $\theta^n f$ is an element in $\mathbb{C}[E_2, E_4, E_6]$.

To any $g \in \mathbb{C}[E_2, E_4, E_6]$, we attach a polynomial $F(g; X, Y, Z)$ in three variables so that

$$g(z) = F(g; E_2(z), E_4(z), E_6(z))$$

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holds. We also define its “modular part” $F^{(0)}(g; Y, Z)$ by

$$F^{(0)}(g; Y, Z) := F(g; 0, Y, Z).$$

If in particular $g$ is modular (i.e., $g \in \mathbb{C}[E_4, E_6]$), then $F(g; X, Y, Z)$ is free from $X$ and $F(g; X, Y, Z) = F^{(0)}(g; Y, Z)$. If $g$ has $p$-integral Fourier coefficients, the polynomial $F$ (and hence $F^{(0)}$) also has $p$-integral coefficients.

For a prime $p > 3$, set $H_p(Y, Z) = F^{(0)}(E_{p-1}; Y, Z)(= F(E_{p-1}; X, Y, Z))$. The polynomial $H_p(Y, Z)$ has $p$-integral coefficients, and $H_p(Y, Z) \mod p$ is known as the “Hasse invariant” ([3], [4]).

Now we can state our main theorem.

**Theorem 1.1.** Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized eigencusp form of weight $k$ and $p$ a prime number greater than $k$. Then the following conditions are equivalent:

1. $a(p) \not\equiv 0 \mod p$.
2. $H_p(Y, Z) \equiv F^{(0)}(\theta^{p-k+1} f; Y, Z) \mod p$.

2. **Proof of the theorem and a corollary**

In order to prove the theorem, we use the theory of filtration of modular forms modulo $p$ developed by Swinnerton-Dyer [4], the theory of theta cycles by Tate [1], and a formula for the derivative $\theta^m f$. We first recall the definition of the filtration and then review theorems of Tate and Swinnerton-Dyer.

Let $M_k(\mathbb{Z}(p))$ be the set of modular forms of weight $k$ (on $SL_2(\mathbb{Z})$) whose Fourier coefficients belong to $\mathbb{Z}(p)$, the local ring of $\mathbb{Q}$ at $p$. Following [4], let $\widehat{M}_k$ be the $\mathbb{F}_p$-vector space (in $\mathbb{F}_p[[g]]$) obtained from $M_k(\mathbb{Z}(p))$ by reducing Fourier coefficients modulo $p$. We note that, since we have $E_{p-1} \equiv 1 \mod p$ and $E_2 \equiv E_{p+1} \mod p$ by the Kummer congruences of Bernoulli numbers, any quasimodular form having $p$-integral Fourier coefficients is congruent modulo $p$ to a modular form of suitable weight.

**Definition 2.1.** For $f \in \widehat{M}_k$, we define the filtration $w(f)$ of $f$ to be the least $\ell$ such that $f$ belongs to $\widehat{M}_\ell$. For a modular or quasimodular form $f$ whose Fourier coefficients are $p$-integral, we shall write $w(f)$ instead of $w(f \mod p)$.

We call an element in $\widehat{M}_k$ an eigenform if it is congruent modulo $p$ to a Hecke-eigencusp form. Tate’s theory of theta cycles connects the ordinarity of an eigenform $f$ to the filtration of the derivative of $f$.

**Proposition 2.2** (Tate [1]). Let $f = \sum_{n=1}^{\infty} a(n)q^n \in \widehat{M}_k$ be an eigenform. We assume $k < p$ and $w(f) = k$. Then we have

$$w(\theta^{p-k+1} f) = \begin{cases} 
2p - k + 2 & \text{if } a(p) \not\equiv 0 \mod p, \\
p - k + 3 & \text{if } a(p) \equiv 0 \mod p.
\end{cases}$$

(In [1] the assumption is weaker (that $f$ is in the kernel of the “$U$-operator”), but for our purpose it is enough to restrict to the case of eigenform.)

On the other hand, the filtration of a modular form $g$ is related to the divisibility of $F^{(0)}(g; Y, Z) \mod p$ by the Hasse invariant.
Proposition 2.3 (Swinnerton-Dyer [4, Lemma 5]). For \( g \in M_k(\mathbb{Z}_{(p)}) \), the following hold:

1. If \( w(g) = k' \), then \( H_p(Y, Z) \not\equiv F^{(0)}(g; Y, Z) \mod p \).
2. If \( w(g) = k' - p + 1 \), then \( H_p(Y, Z) \mid F^{(0)}(g; Y, Z) \mod p \).

Now assume that \( f \) is a normalized eigenform of weight \( k \). The derivative \( \theta^{p-k+1} f \) is quasimodular of weight \( 2p - k + 2 \). If \( \theta^{p-k+1} f \) is congruent modulo \( p \) to a (true) modular form \( g \) of weight \( 2p - k + 2 \), then, combining Proposition 2.2 and Proposition 2.3 (with \( k' = 2p - k + 2 \)), the condition \( a(p) \not\equiv 0 \mod p \) is equivalent to the polynomial \( F^{(0)}(g; X, Y, Z) \mod p \) not being divisible by \( H_p(Y, Z) \mod p \). Our theorem is therefore a consequence of the following observation that we can indeed take \( g \) to be the modular part of \( \theta^{p-k+1} f \). Here, for a quasimodular form \( g = \sum_{i=0}^m g_i E_{4i} \), \( g_i \in \mathbb{C}[E_4, E_6] \), we call \( g_0 \) its modular part.

Lemma 2.4. Let \( p > 3 \) be a prime and \( f \) a modular form of weight \( k < p \) with \( p \)-integral Fourier coefficients. Then we have

\[ \theta^{p-k+1} f \equiv (\theta^{p-k+1} f)_0 \mod p. \]

This is a consequence of a general formula for \( \theta^n f \) given in [4]. Recall that, if \( f \) is modular of weight \( k \), then

\[ \partial f := \theta f - \frac{k}{12} E_4 f \]

is modular of weight \( k + 2 \). For a modular form \( f \) of weight \( k \), define a sequence of modular forms \( f_r \) of weight \( k + 2r \) recursively by

\[ f_{r+1} = \partial f_r - \frac{r(r + k - 1)}{144} E_4 f_{r-1} \quad (r \geq 0) \]

with initial condition \( f_0 = f \). Then the formula (37) in [5] is equivalent to the following closed formula.

Proposition 2.5. Let \( f \) be a modular form of weight \( k \). Then for any \( n \geq 0 \) we have

\[ \frac{\theta^n f}{n!} = \sum_{i=0}^n \frac{[k + n - 1]_{k+n-1}}{(n-i)!} \left( \frac{E_2}{12} \right)^i. \]

When \( n = p - k + 1 \), the binomial coefficients \( \binom{k+n-1}{i} \) are divisible by \( p \) for all \( i > 0 \), and hence Lemma 2.4 follows \( (f_n = (\theta^n f)_0) \). This completes the proof of the theorem.

Here we give a corollary to the theorem. As in the theorem, assume that \( f(z) = \sum_{n=1}^\infty a(n)q^n \) is a normalized eigenform of weight \( k \) and \( p \) is a prime number greater than \( k \). We denote by \( b(l, m, n) \) the coefficient of \( X^l Y^m Z^n \) in \( F(\theta^{p-k+1} f; X, Y, Z) \):

\[ F(\theta^{p-k+1} f; X, Y, Z) = \sum_{2l + 4m + 6n = 2p - k + 2} b(l, m, n) X^l Y^m Z^n. \]

Corollary 2.6. (1) Assume that \( k \equiv 0 \mod 6 \) and \( p \equiv 2 \mod 3 \).

If \( b(0, 0, \frac{2p - k + 2}{6}) \not\equiv 0 \mod p \), then \( a(p) \not\equiv 0 \mod p \).

(2) Assume that \( k \equiv 0 \mod 4 \) and \( p \equiv 3 \mod 4 \).

If \( b(0, \frac{2p - k + 2}{4}, 0) \not\equiv 0 \mod p \), then \( a(p) \not\equiv 0 \mod p \).
Proof. We only prove (1), the proof of (2) being similar. Write
\[ H_p(Y, Z) = \sum_{4m+6n=p-1} c(m, n) Y^m Z^n. \]

By the assumption, \( p - 1 \) is not divisible by 6, and hence the term with \( m = 0 \) does not occur on the right. Therefore, if \( b(0, 0, \frac{2p-k+2}{6}) \not\equiv 0 \mod p \), the polynomial \( F(\theta^{p-k+1} f; X, Y, Z) \mod p \) is not a multiple of \( H_p(Y, Z) \mod p \), and thus \( a(p) \not\equiv 0 \mod p \) by Theorem 1.1.

\[ \square \]

3. Relation to supersingular \( j \)-invariants of elliptic curves

We may rephrase the theorem in terms of the supersingular \( j \)-polynomial. Let \( f \) be a modular form of weight \( k \). Write \( k = 12m + 4\delta + 6\varepsilon \) with \( m \geq 0 \), \( \delta \in \{0, 1, 2\} \), \( \varepsilon \in \{0, 1\} \). Then there exists a unique polynomial \( G(f; x) \) such that
\[ f(z) = \Delta(z)^m E_4(z)^6 E_6(z)^\delta G(f; j(z)), \]
where \( \Delta(z) = (E_4(z)^3 - E_6(z)^2)/1728 \) is the discriminant function and \( j(z) = E_4(z)^3/\Delta(z) \) is the modular invariant. Moreover we put
\[ \tilde{G}(f; x) := x^{\delta} (x - 1728)^\varepsilon G(f; x). \]

For a prime number \( p \), we define the supersingular \( j \)-polynomial \( S_p(x) \) by
\[ S_p(x) := \prod_{E/F_p: \text{supersingular}} (x - j(E)) \in \mathbb{F}_p[x], \]
where the product runs over the isomorphism classes of supersingular elliptic curves in characteristic \( p \) and \( j(E) \) is the \( j \)-invariant of \( E \). Assume \( p > 3 \). A theorem of Deligne (cf. [3], [2]) then asserts that
\[ \tilde{G}(E_{p-1}; x) \equiv S_p(x) \mod p. \]

By this and Theorem 1.1, we have the following.

**Theorem 3.1.** The assumption being the same as in Theorem 1.1, the following conditions are equivalent:

1. \( a(p) \not\equiv 0 \mod p \).
2. \( S_p(x) \not| \tilde{G}((\theta^{p-k+1} f)_0; x) \mod p \).

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**References**


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