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A generalization of restricted isometry property and applications to compressed sensing

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Abstract

This paper introduces a new general theory of compressed sensing. We give a natural generalization of restricted isometry property called weak RIP. Though our approach is based on the Candès ideas, we consider that the proposed results are more useful for real data anaysis. In this note, we establish our new results about the accuracy of the reconstruction from undersampling measurements where are possible to improve estimation in various situations that we have knowledge.

Key Words and Phrases: Compressed sensing, Weak restricted isometry property.

1 Introduction

1.1 RIP and the results

We supose that we observe

$$\boldsymbol{y} = A\boldsymbol{x}, \quad \boldsymbol{x} \in \boldsymbol{R}^n, \tag{1}$$

where A is $m \times n$ matrix. Our goal is to reconstruct $\boldsymbol{x} \in \boldsymbol{R}^n$ with good accuracy. We are interested in $m < n$ case. It occurs the problem is of course ill-posed, but we know important results when we suppose $\boldsymbol{x}$ is known to be sparse or nearly sparse and A obeys restricted isometry property(RIP) introduced below. Then we can reconstruct $\boldsymbol{x} \in \boldsymbol{R}^n$ with good accuracy. In detail, this premise changes the problem, making the seach for solutions feasible. In fact, we show that the solution $\boldsymbol{x}^\star$ to the following optimazation problem

$$\min_{\tilde{\boldsymbol{x}} \in \boldsymbol{R}^n} \| \tilde{\boldsymbol{x}} \|_1 \quad \textit{subject to} \quad \boldsymbol{y} = A\tilde{\boldsymbol{x}} \tag{2}$$

recovers $\boldsymbol{x}$ exactly, where $\|\cdot\|_1$ is l_1 norm. Furthermore, we extend the results for noisy recovery. We observe

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{z}, \tag{3}$$

where $\boldsymbol{z}$ is an unknown noise term. In this context, we consider reconstructing $\boldsymbol{x}$ as the solution $\boldsymbol{x}^\star$ to the optimization problem

$$\min_{\tilde{\boldsymbol{x}}\in\boldsymbol{R}^n} \| \tilde{\boldsymbol{x}} \|_1 \quad \textit{subject to} \quad \| \boldsymbol{y} - A\tilde{\boldsymbol{x}} \|_2 \leq \varepsilon, \tag{4}$$

where ε is an upper bounded on the size of the noisy contribution and $\|\cdot\|_2$ is l_2-norm.

Definition 1.1. A matrix A satisfies the restricted isometry property(RIP) of order s if there exists a constant δ with $0 < \delta < 1$ such that

$$(1-\delta) \| \boldsymbol{a} \|_2^2 \leq \| A\boldsymbol{a} \|_2^2 \leq (1+\delta) \| \boldsymbol{a} \|_2^2 \tag{5}$$

for all s-sparse vectors $\boldsymbol{a}$. A vector is said to be s-sparse if it has at most s nonzero entries. The minimum of above constants δ is said to be the isometry constant of A and it is denoted by δ_s.

The condition (5) is equivalent to requiring that the matrix $A_S^T A_S$ has all of its eigenvalues in $[1-\delta_s, 1+\delta_s]$, where A_S is the $m \times |S|$ matrix composed of these columns for any subset S of $\{1, 2, \cdots, n\}$. It is well-known in [1, 4, 5] that RIP is very useful to study the general robustness of CS. In particular, Candès [2] has obtained the following results: Assume that $\delta_{2s} < \sqrt{2} - 1$ and $\| \boldsymbol{z} \|_2 \leq \varepsilon$. Then the solution $\boldsymbol{x}^\star$ to (4) obeys

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_2 \leq C_0 \frac{1}{\sqrt{s}} \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + C_1\varepsilon, \tag{6}$$

where $\boldsymbol{x}_s$ is the vector $\boldsymbol{x}$ with all but the largest s components set to zero and C_0, C_1 are explicitly given constants. In a previous paper (Inoue [6]), we have improved this result:

Theorem 1.1. Let $s, s' \in \boldsymbol{N}$ with $s < n$ and $s' < n - s$. Assume that A satisfies the RIP of order $(s + 2s')$ and

$$\delta_{s+s'} + \sqrt{\frac{s}{s'}}\delta_{s+2s'} < 1. \tag{7}$$

Then, the solution $\boldsymbol{x}^\star$ to (4) obeys

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_1 \leq D_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + D_1\varepsilon \tag{8}$$

and

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_2 \leq \frac{2}{\sqrt{s}} D_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + \frac{1}{\sqrt{s'}} D_1 \varepsilon, \tag{9}$$

where,

$$\begin{aligned} \gamma &= \frac{\delta_{s+2s'}}{1-\delta_{s+s'}}, \\ D_0 &= \left\{ \begin{array}{ll} \left(\frac{1+\gamma}{1-\gamma}\right)\sqrt{\frac{s}{s'}} & , s' \leq s \\ \left(\frac{1+\gamma}{1-\gamma}\right)\sqrt{\frac{s'}{s}} & , s' \geq s \end{array} \right. , \quad D_1 = \left\{ \begin{array}{ll} \frac{\alpha}{1-\gamma} 2\sqrt{s} & , s' \leq s \\ \frac{\alpha}{1-\gamma} 2\sqrt{s'} & , s' \geq s. \end{array} \right. \end{aligned} \tag{10}$$

1.2 Main theorem

The RIP requires bounded condition number for all submatrices built by selecting s arbitrary columns and the spectral norm of a matrix is not generally easy to commute. In order to solve this problem, Candès and Plan introduce the notion of the weak RIP [3]. But the notion is not explicitly defined. And so, we shall define the notion weakened the RIP. And we introduce new results obtained by using weak RIP.

Throughout this section, let A be a $m \times n$ matrix.

Definition 1.2. (Weak RIP) Let T be a fixed subset of $\{1, 2, \cdots, n\}$ with $|T| = s$ and $1 < r < s$. A obeys the weak RIP with respect to T of order r if there exists $0 < \delta < 1$ such that for any subset $R \subset T^c$ with $|R| \leq r$,

$$(1-\delta) \| \boldsymbol{x} \|_2^2 \leq \| A\boldsymbol{x} \|_2^2 \leq (1+\delta) \| \boldsymbol{x} \|_2^2 \tag{11}$$

for all $\boldsymbol{x} \in \boldsymbol{R}^n$ with *supp* $\boldsymbol{x} \subset T \cup R$. The minimum of such constants δ is denoted by $\delta_{T,r}$.

Roughly speaking, we consider an arbitrary submatrix formed by concatenating columns in T with r other columns from A selected in any way you like. Then we would like this submatrix to be well conditioned. Because T is fixed, we prove the good condition as following:

Theorem 1.2. Let T_0 be the location of the s-largest coefficients of $\boldsymbol{x}$. Assume that A obeys the weak RIP with respect to T_0 of order r, and $\delta_{T_0.r} < \frac{1}{1+\sqrt{\frac{2s}{[\frac{r}{2}]}}}$, where $[\cdot]$ is the floor function. Then the solution $\boldsymbol{x}^\star$ to (4) obeys

$$\| \boldsymbol{x} - \boldsymbol{x}^\star \|_2 \leq C_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + C_1 \varepsilon, \tag{12}$$

where C_0, C_1 are explicitly given constants.

In Section 2, we prove the main theorem. The proofs are based on those of Theorem1.1.

2 Proof

In this section we prove Theorem1.2. Here we simply put $\delta = \delta_{T_0,r}$. We first show that for any $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^n$ with *supp* $\boldsymbol{a} \subset T_0$ and *supp* $\boldsymbol{b} \subset R \subset T_0^c$,

$$|< A\boldsymbol{a}, A\boldsymbol{b} >| \leq \delta \|\boldsymbol{a}\|_2 \|\boldsymbol{b}\|_2, \tag{13}$$

where,$< \cdot, \cdot >$ denotes the inner product.

Indeed, it suffices to show in case that $\|\boldsymbol{a}\|_2 = \|\boldsymbol{b}\|_2 = 1$. Since *supp* $(\boldsymbol{a}+\boldsymbol{b})$, *supp* $(\boldsymbol{a}-\boldsymbol{b}) \subset (T_0 \cup R)$ and $\boldsymbol{a} \perp \boldsymbol{b}$, we have

$$\begin{aligned} |< A\boldsymbol{a}, A\boldsymbol{b} >| &= \frac{1}{4}\left\{\|A(\boldsymbol{a}+\boldsymbol{b})\|_2^2 - \|A(\boldsymbol{a}-\boldsymbol{b})\|_2^2\right\} \\ &\leq \frac{1}{4}\left\{(1+\delta)\|\boldsymbol{a}+\boldsymbol{b}\|_2^2 - (1-\delta)\|\boldsymbol{a}-\boldsymbol{b}\|_2^2\right\} \\ &= \frac{1}{4}\left\{\left(\|\boldsymbol{a}+\boldsymbol{b}\|_2^2 - \|\boldsymbol{a}-\boldsymbol{b}\|_2^2\right) + \delta\left(\|\boldsymbol{a}+\boldsymbol{b}\|_2^2 + \|\boldsymbol{a}-\boldsymbol{b}\|_2^2\right)\right\} \\ &= \delta. \end{aligned}$$

We put

$$\boldsymbol{h} \equiv \boldsymbol{x}^\star - \boldsymbol{x}.$$

By the linearity of A and the triangle equality, we have

$$\|A\boldsymbol{h}\|_2 \leq 2\varepsilon. \tag{14}$$

For any $\boldsymbol{a} \in \boldsymbol{R}^n$ and $T \subset \{1, 2, , \cdots, n\}$, we put

$$\boldsymbol{a}_T = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix}, \; a_i^T = \begin{cases} a_i & , \; i \in T \\ 0 & , \; i \in T^c. \end{cases}$$

Let T_0 be the location of the s-largest coefficients of $\boldsymbol{x}$. Then,$\boldsymbol{x}_{T_0} = \boldsymbol{x}_s$. Let T_1 be the location of $[\frac{r}{2}]$-largest coefficients of $\boldsymbol{h}_{T_0^c}$. Repeating this method, $\{1, 2, , \cdots, n\}$ is decomposed into $\{1, 2, \cdots, n\} = T_0 \cup T_1 \cup \cdots \cup T_{l-1} \cup T_l$, $|T_l| \leq [\frac{r}{2}]$. Then, since

$$\left|h_k^{T_{j-1}}\right| \geq \max_{k \in T_j}\left|h_k^{T_j}\right|, \quad 2 \leq j \leq l, \; 1 \leq k \leq \left[\frac{r}{2}\right]$$

it follows that

$$\begin{aligned}
\| \boldsymbol{h}_{T_j} \|_2 &= \left(\sum_{k=1}^{\left[\frac{r}{2}\right]} \left| h_k^{T_j} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{\left[\frac{r}{2}\right]} \, \| \boldsymbol{h}_{T_j} \|_\infty \\
&\equiv \sqrt{\left[\frac{r}{2}\right]} \max_{k \in T_j} \left| h_k^{T_j} \right| \\
&\leq \frac{1}{\sqrt{\left[\frac{r}{2}\right]}} \|\boldsymbol{h}_{T_{j-1}}\|_1, \quad 2 \leq j \leq l,
\end{aligned} \tag{15}$$

which implies that

$$\begin{aligned}
\| \boldsymbol{h}_{(T_0 \cup T_1)^c} \|_2 &= \| \sum_{j \geq 2} h_{T_j} \|_2 \\
&\leq \sum_{j \geq 2} \| \boldsymbol{h}_{T_j} \|_2 \\
&\leq \frac{1}{\sqrt{\left[\frac{r}{2}\right]}} \sum_{j \geq 1} \| \boldsymbol{h}_{T_j} \|_1 \\
&= \frac{1}{\sqrt{\left[\frac{r}{2}\right]}} \| \sum_{j \geq 1} \boldsymbol{h}_{T_j} \|_1 \\
&= \frac{1}{\sqrt{\left[\frac{r}{2}\right]}} \| \boldsymbol{h}_{T_0^c} \|_1 \, .
\end{aligned} \tag{16}$$

Since

$$\begin{aligned}
\| \boldsymbol{x} \|_1 &\geq \| \boldsymbol{x}^\star \|_1 \\
&= \| \boldsymbol{x} + \boldsymbol{h} \|_1 \\
&= \| \boldsymbol{x}_{T_0} + \boldsymbol{h}_{T_0} + \boldsymbol{x}_{T_0^c} + \boldsymbol{h}_{T_0^c} \|_1 \\
&= \| \boldsymbol{x}_{T_0} + \boldsymbol{h}_{T_0} \|_1 + \| \boldsymbol{x}_{T_0^c} + \boldsymbol{h}_{T_0^c} \|_1 \\
&\geq \| \boldsymbol{x}_{T_0} \|_1 - \| \boldsymbol{h}_{T_0} \|_1 + \| \boldsymbol{h}_{T_0^c} \|_1 - \| \boldsymbol{x}_{T_0^c} \|_1,
\end{aligned}$$

it follows that

$$\begin{aligned}
\| \boldsymbol{h}_{T_0^c} \|_1 &\leq \| \boldsymbol{x} \|_1 - \| \boldsymbol{x}_{T_0} \|_1 + \| \boldsymbol{x}_{T_0^c} \|_1 + \| \boldsymbol{h}_{T_0} \|_1 \\
&= \| \boldsymbol{x}_{T_0^c} \|_1 + \| \boldsymbol{x}_{T_0^c} \|_1 + \| \boldsymbol{h}_{T_0} \|_1 \\
&= 2 \| \boldsymbol{x}_{T_0^c} \|_1 + \| \boldsymbol{h}_{T_0} \|_1 \\
&= 2 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + \| \boldsymbol{h}_{T_0} \|_1,
\end{aligned} \tag{17}$$

which implies by (15) that

$$\begin{aligned}\| \boldsymbol{h}_{(T_0\cup T_1)^c} \|_2 &\le \frac{1}{\sqrt{[\frac{r}{2}]}} \| \boldsymbol{h}_{T_0^c} \|_1 \\ &\le \frac{1}{\sqrt{[\frac{r}{2}]}} (\| \boldsymbol{h}_{T_0} \|_1 + 2 \| \boldsymbol{x}-\boldsymbol{x}_s \|_1) \\ &\le \sqrt{\frac{s}{[\frac{r}{2}]}} \| \boldsymbol{h}_{T_0} \|_2 + \frac{2}{\sqrt{[\frac{r}{2}]}} \| \boldsymbol{x}-\boldsymbol{x}_s \|_1 \\ &\le \sqrt{\frac{s}{[\frac{r}{2}]}} \| \boldsymbol{h}_{T_0\cup T_1} \|_2 + \frac{2}{\sqrt{[\frac{r}{2}]}} \| \boldsymbol{x}-\boldsymbol{x}_s \|_1 . \qquad (18)\end{aligned}$$

Hence it follows from (13) that for any $j \ge 2$

$$\begin{aligned}\left|\langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}_{T_j}\rangle\right| &\le \left|\langle A\boldsymbol{h}_{T_0}, A\boldsymbol{h}_{T_j}\rangle\right| + \left|\langle A\boldsymbol{h}_{T_1}, A\boldsymbol{h}_{T_j}\rangle\right| \\ &\le \delta \| \boldsymbol{h}_{T_0} \|_2 \| \boldsymbol{h}_{T_j} \|_2 + \delta \| \boldsymbol{h}_{T_1} \|_2 \| \boldsymbol{h}_{T_j} \|_2 \\ &\le \delta \| \boldsymbol{h}_{T_j} \|_2 (\| \boldsymbol{h}_{T_0} \|_2 + \| \boldsymbol{h}_{T_1} \|_2) \\ &\le \delta \| \boldsymbol{h}_{T_j} \|_2 \sqrt{2} \left(\| \boldsymbol{h}_{T_0} \|_2^2 + \| \boldsymbol{h}_{T_1} \|_2^2\right)^{\frac{1}{2}} \quad (by\ \boldsymbol{h}_{T_0} \perp \boldsymbol{h}_{T_1}) \\ &= \sqrt{2}\delta \| \boldsymbol{h}_{T_j} \|_2 \| \boldsymbol{h}_{T_0} + \boldsymbol{h}_{T_1} \|_2 \\ &= \sqrt{2}\delta \| \boldsymbol{h}_{T_j} \|_2 \| \boldsymbol{h}_{T_0\cup T_1} \|_2,\end{aligned}$$

which implies by (14) and (15) that

$$\begin{aligned}\| A\boldsymbol{h}_{T_0\cup T_1} \|_2^2 &= \langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h} - \sum_{j\ge 2} A\boldsymbol{h}_{T_j}\rangle \\ &= \langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}\rangle - \langle A\boldsymbol{h}_{T_0\cup T_1}, \sum_{j\ge 2} A\boldsymbol{h}_{T_j}\rangle \\ &\le \| A\boldsymbol{h}_{T_0\cup T_1} \|_2 \| A\boldsymbol{h} \|_2 + \sum_{j\ge 2} \left|\langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}_{T_j}\rangle\right| \qquad (19) \\ &\le \sqrt{1+\delta} \| \boldsymbol{h}_{T_0\cup T_1} \|_2 \, 2\varepsilon \\ &+ \sqrt{2}\delta \left(\sum_{j\ge 2} \| \boldsymbol{h}_{T_j} \|_2\right) \| \boldsymbol{h}_{T_0\cup T_1} \|_2 \\ &= \| \boldsymbol{h}_{T_0\cup T_1} \|_2 \left(2\varepsilon\sqrt{1+\delta} + \sqrt{2}\delta \sum_{j\ge 2} \| \boldsymbol{h}_{T_j} \|_2\right) \\ &\le \| \boldsymbol{h}_{T_0\cup T_1} \|_2 \left(2\varepsilon\sqrt{1+\delta} + \sqrt{2}\delta \frac{1}{\sqrt{[\frac{r}{2}]}} \sum_{j\ge 1} \| \boldsymbol{h}_{T_j} \|_1\right) \\ &= \| \boldsymbol{h}_{T_0\cup T_1} \|_2 \left(2\varepsilon\sqrt{1+\delta} + \sqrt{2}\delta \frac{1}{\sqrt{[\frac{r}{2}]}} \| \boldsymbol{h}_{T_0^c} \|_1\right)\end{aligned}$$

Hence we have

$$\begin{aligned}(1-\delta)\parallel \boldsymbol{h}_{T_0\cup T_1}\parallel_2 &\leq 2\varepsilon\sqrt{1+\delta}+\sqrt{2}\delta\frac{1}{\sqrt{[\frac{r}{2}]}}\parallel \boldsymbol{h}_{T_0^c}\parallel_1,\\ &\leq 2\varepsilon\sqrt{1+\delta}+\frac{\sqrt{2}\delta}{\sqrt{[\frac{r}{2}]}}\left(2\|\boldsymbol{x}-\boldsymbol{x}_s\|_1+\|\boldsymbol{h}_{T_0}\|_1\right)\\ &\leq 2\varepsilon\sqrt{1+\delta}+\frac{\sqrt{2}\delta}{\sqrt{[\frac{r}{2}]}}\left(2\|\boldsymbol{x}-\boldsymbol{x}_s\|_1+\sqrt{s}\|\boldsymbol{h}_{T_0\cup T_1}\|_2\right),\end{aligned}$$

and so

$$\left(1-\left(1+\sqrt{\frac{2s}{[\frac{r}{2}]}}\right)\delta\right)\|\boldsymbol{h}_{T_0\cup T_1}\|_2 \leq 2\varepsilon\sqrt{1+\delta}+\frac{2\sqrt{2}\delta}{\sqrt{[\frac{r}{2}]}}\|\boldsymbol{x}-\boldsymbol{x}_s\|_1. \tag{20}$$

By the assumption; $\delta<\frac{1}{1+\sqrt{\frac{2s}{[\frac{r}{2}]}}}$ (if and only if, $1-\left(1+\sqrt{\frac{2s}{[\frac{r}{2}]}}\right)\delta>0$), we have

$$\|\boldsymbol{h}_{T_0\cup T_1}\|_2\leq\frac{2\sqrt{1+\delta}}{1-\left(1+\sqrt{\frac{2s}{[\frac{r}{2}]}}\right)\delta}\varepsilon+\frac{2\sqrt{2}\delta}{\left(1-\left(1+\sqrt{\frac{2s}{[\frac{r}{2}]}}\right)\delta\right)\sqrt{[\frac{r}{2}]}}\|\boldsymbol{x}-\boldsymbol{x}_s\|_1. \tag{21}$$

Thus we have by (18) and (21),

$$\begin{aligned}\parallel \boldsymbol{x}-\boldsymbol{x}^\star\parallel_2=\parallel \boldsymbol{h}\parallel_2 &\leq \parallel \boldsymbol{h}_{T_0\cup T_1}\parallel_2+\parallel \boldsymbol{h}_{(T_0\cup T_1)^c}\parallel_2\\ &\leq \parallel \boldsymbol{h}_{T_0\cup T_1}\parallel_2+\sqrt{\frac{s}{[\frac{r}{2}]}}\parallel \boldsymbol{h}_{T_0\cup T_1}\parallel_2+\frac{2}{\sqrt{[\frac{r}{2}]}}\parallel \boldsymbol{x}-\boldsymbol{x}_s\parallel_1\\ &= \left(1+\sqrt{\frac{s}{[\frac{r}{2}]}}\right)\|\boldsymbol{h}_{T_0\cup T_1}\|_2+\frac{2}{\sqrt{[\frac{r}{2}]}}\parallel \boldsymbol{x}-\boldsymbol{x}_s\parallel_1\\ &\leq \left(1+\sqrt{\frac{s}{[\frac{r}{2}]}}\right)\frac{2\sqrt{1+\delta}}{\left(1-\left(1+\sqrt{\frac{s}{[\frac{r}{2}]}}\right)\delta\right)}\varepsilon\\ &+\frac{2\left(1+(\sqrt{2}-1)\left(1+\sqrt{\frac{s}{[\frac{r}{2}]}}\right)\delta\right)}{\sqrt{[\frac{r}{2}]}\left(1-\left(1+\sqrt{\frac{s}{[\frac{r}{2}]}}\right)\delta\right)}\|\boldsymbol{x}-\boldsymbol{x}_s\|_1.\end{aligned} \tag{22}$$

This complete the proof.

3 Examples

In this section we give simple examples of $m\times n$ matrices obeying weak RIP.

Example 3.1. Suppose that a subset $T=\{n_1,n_2,\cdots,n_s\}$ of $\{1,2,\cdots,n\}$ satisfies the following conditions (3.1.1), (3.1.2) and (3.1.3).

(3.1.1) A_T obeys the RIP of order s. We denote by δ_T the isometry constant of A_T.

(3.1.2) A_{T^c} obeys the RIP of order r.

(3.1.3) $\left|< A_{\{i\}}, A_{\{j\}} >\right| < \varepsilon$, for any $i \in T$ and $j \in T^c$, where $0 < \varepsilon < \frac{1-\max(\delta_T,\delta_r)}{sr}$, where $A_{\{i\}}$ is column vector of A with index i.

Then A obeys the weak RIP with respect to T of order r. Indeed, take arbitrary $R = \{n'_1, n'_2, \cdots, n'_r\} \subset T^c$ and $\boldsymbol{x} \in \boldsymbol{R}^n$ with $supp\ \boldsymbol{x} \subset T \cup R$.

Since

$$
\begin{aligned}
|\langle A_T^* A_R \boldsymbol{x}_R, \boldsymbol{x}_T \rangle| &\leq \sum_{j=1}^{r} \sum_{i=1}^{s} \left| \left\langle A_{\{n_j\}}, A_{\{n'_j\}} \right\rangle \right| \left| x_{n_i} x_{n'_j} \right| \\
&\leq \varepsilon \sum_{j=1}^{r} \sum_{i=1}^{s} \left| x_{n_i} x_{n'_j} \right| \\
&\leq sr\varepsilon \|\boldsymbol{x}_T\|_2 \|\boldsymbol{x}_R\|_2,
\end{aligned}
$$

it follow that

$$
\begin{aligned}
|\langle (A_{T\cup R}^* A_{T\cup R} - I)\, \boldsymbol{x}, \boldsymbol{x} \rangle| &= |\langle (A_T^* A_T - I)\, \boldsymbol{x}_T, \boldsymbol{x}_T \rangle| + 2\, |\langle A_T^* A_R \boldsymbol{x}_R, \boldsymbol{x}_T \rangle| \\
&+ |\langle (A_R^* A_R - I)\, \boldsymbol{x}_R, \boldsymbol{x}_R \rangle| \\
&\leq \delta_T \|\boldsymbol{x}_T\|_2^2 + \delta_r \|\boldsymbol{x}_R\|_2^2 + 2\, |\langle A_T^* A_R \boldsymbol{x}_R, \boldsymbol{x}_T \rangle| \\
&\leq \delta_T \|\boldsymbol{x}_T\|_2^2 + \delta_r \|\boldsymbol{x}_R\|_2^2 + 2sr\varepsilon \|\boldsymbol{x}_T\|_2 \|\boldsymbol{x}_R\|_2 \\
&\leq \delta_T \|\boldsymbol{x}_T\|_2^2 + \delta_r \|\boldsymbol{x}_R\|_2^2 + sr\varepsilon \left(\|\boldsymbol{x}_T\|_2^2 + \|\boldsymbol{x}_R\|_2^2 \right) \\
&\leq (\max(\delta_T, \delta_r) + sr\varepsilon) \left(\|\boldsymbol{x}_T\|_2^2 + \|\boldsymbol{x}_R\|_2^2 \right) \\
&= \delta \|\boldsymbol{x}\|_2^2,
\end{aligned}
$$

where $\delta \equiv \max(\delta_T, \delta_r) + sr\varepsilon$. By (3.1.2) we have $0 < \delta < 1$. Thus A obeys the weak RIP with respect to T of order r.

Example 3.2. Suppose that a subset $T = \{n_1, n_2, \cdots, n_s\}$ of $\{1, 2, \cdots, n\}$ satisfies the following conditions (3.2.1), (3.2.2) and (3.2.3).

(3.2.1) $A_T = (A_{\{n_1\}}, A_{\{n_2\}}, \cdots, A_{\{n_s\}})$ is an orthonormal system in $\boldsymbol{R}^n$.

(3.2.2) $\{B_{\{n_1\}}, B_{\{n_2\}}, \cdots, B_{\{n_s\}}, B_{\{j\}};\ j \in T^c\}$ is an orthonormal base in $\boldsymbol{R}^n$, where

$$B_{\{n_i\}} = \begin{pmatrix} A_{\{n_1\}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad 0 \leq i \leq s,$$

$$B_{\{j\}} = \begin{pmatrix} A_{\{j\}} \\ a_{m+1j} \\ \vdots \\ a_{nj} \end{pmatrix}, \quad j \in T^c.$$

(3.2.3) A_{T^c} obeys the RIP of order r.

Then A obeys the weak RIP with respect to T of order r. Indeed, this is easily shown by using Example3.1.2.

4 Discussions

In case that we analyze the data by compressed sensing, we often random matrices without structure of data. When we use matrices with structure of data, it is very difficult to investigate whether matrices obey the RIP. And so, it seems useful to weaken the conditions of RIP. In this paper, we have defined the notion of weak RIP which is a generalization of RIP. In case that we have the knowledges of data, that is, we know good location T_0, it seems better to analyze data using the weak RIP and to have more possibilities of applications to statistics and the other fields.

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