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Abstract
The initial boundary value problem for the compressible Navier-Stokes equation is considered in an infinite layer of $\mathbb{R}^2$. It is proved that if the Reynolds and Mach numbers are sufficiently small, then strong solutions to the compressible Navier-Stokes equation around parallel flows exist globally in time for sufficiently small initial perturbations. The large time behavior of the solution is described by a solution of a 1-dimensional viscous Burgers equation. The proof is given by a combination of spectral analysis of the linearized operator and a variant of the Matsumura-Nishida energy method.

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1. Introduction

This paper is concerned with the stability of parallel flows of the compressible Navier-Stokes equation

\begin{align}
\partial_t \rho + \text{div} (\rho \mathbf{v}) &= 0, \\
\rho (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \mu \Delta \mathbf{v} - (\mu + \mu') \nabla \text{div} \mathbf{v} + \nabla P(\rho) &= \rho \mathbf{g}
\end{align}

in an $n$-dimensional infinite layer $\Omega_\ell = \mathbb{R}^{n-1} \times (0, \ell)$:

$$\Omega_\ell = \{ \tilde{x} = (\tilde{x}', \tilde{x}_n) ; \tilde{x}' = (\tilde{x}_1, \cdots, \tilde{x}_{n-1}) \in \mathbb{R}^{n-1}, 0 < \tilde{x}_n < 1 \} \ (n \geq 2).$$

Here $\rho = \rho(\tilde{x}, \tilde{t})$ and $\mathbf{v} = \nabla (\tilde{v}^1(\tilde{x}, \tilde{t}), \cdots, \tilde{v}^n(\tilde{x}, \tilde{t}))$ denote the unknown density and velocity at time $\tilde{t} \geq 0$ and position $\tilde{x} \in \Omega_\ell$, respectively; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of $\rho$ satisfying

$$P'(\rho_*) > 0$$

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for a given constant \( \rho_* > 0 \); \( \mu \) and \( \mu' \) are viscosity coefficients that are assumed to be constants and satisfy
\[
\mu > 0, \quad \frac{2}{n} \mu + \mu' \geq 0;
\]
div, \( \nabla \) and \( \Delta \) denote the usual divergence, gradient and Laplacian with respect to \( \tilde{x} \); and \( \tilde{g} \) is an external force which is a function of \( \tilde{x}_n \) only. Here and in what follows \( \top \cdot \) stands for the transposition.

Under a suitable assumption on \( \tilde{g} \), say, \( \tilde{g} \) is in the form
\[
\tilde{g} = \top \left( \tilde{g}^1(\tilde{x}_n), 0, \cdots, 0, \tilde{g}^n(\tilde{x}_n) \right),
\]
with bounded smooth functions \( \tilde{g}^j(\tilde{x}_n) \) \( (j = 1, n) \), problem (1.1)–(1.2) has a smooth stationary solution \( \tilde{u}_s = \top(\tilde{\rho}_s, \tilde{v}_s) \) of parallel flow satisfying
\[
\tilde{\rho}_s = \tilde{\rho}_s(\tilde{x}_n), \quad \frac{1}{\ell} \int_0^\ell \tilde{\rho}_s(\tilde{x}_n) d\tilde{x}_n = \rho_*.
\]

In this paper we are interested in the stability of parallel flows. Typical examples of parallel flows are the plane Couette flow:
\[
\tilde{\rho}_s = \rho_*, \quad \tilde{v}^1_s = \frac{V^1}{\ell} \tilde{x}_n
\]
for \( \tilde{g} = 0 \) with a constant \( V^1 \neq 0 \); and the Poiseuille flow
\[
\tilde{\rho}_s = \rho_*, \quad \tilde{v}^1_s = \frac{\rho_* \tilde{g}^1}{2\mu} \tilde{x}_n(\ell - \tilde{x}_n)
\]
for \( \tilde{g} = \top(\tilde{g}^1, 0, \cdots, 0) \) with a constant \( \tilde{g}^1 \neq 0 \).

As for the stability of parallel flows, Iooss and Padula [3] studied the linearized stability of parallel flows in a cylindrical domain. It was shown in [3] that if the Reynolds number is small in some sense, then the parallel flow is linearly stable under perturbations periodic in the unbounded direction of the domain. Furthermore, the solution of the linearized problem decays exponentially as \( t \to \infty \) if the density component of initial perturbation has a vanishing mean value in the periodicity cell. In [12] the linearized stability of \( \tilde{u}_s \) under perturbations in \( L^2(\Omega_\ell) \) was studied; and it was shown that \( \tilde{u}_s \) is linearly stable if the Reynolds and Mach numbers are sufficiently small and \( \tilde{\rho}_s \) is sufficiently close to \( \rho_* \). Furthermore, the asymptotic behavior of solutions of the linearized problem is described by an \( n - 1 \) dimensional
linear heat equation. The nonlinear stability of $\tilde{u}_s$ was then studied in [8] in the case $n \geq 3$. It was shown in [8] that $\tilde{u}_s$ is asymptotically stable under sufficiently small initial perturbations in the Sobolev space $H^m(\Omega_\ell)$ with $m \geq \lfloor n/2 \rfloor + 1$ if $n \geq 3$, provided that the Reynolds number $Re = \frac{\rho_* \ell^2 V}{\mu}$ and Mach number $Ma = \frac{V}{\sqrt{P'(\rho_*)}}$ are sufficiently small and $\tilde{\rho}_s$ is sufficiently close to $\rho_*$ in $C^{m+1}[0, \ell]$. Here $V$ is a non-dimensional number satisfying $V \sim \|\tilde{v}_s^1\|_{C^{m+1}[0,\ell]}$. Furthermore, the asymptotic behavior of perturbations is described by an $n - 1$ dimensional linear heat equation. The proof in [8] is based on the Matsumura-Nishida energy method ([15]). But the argument in [8] does not work well for the case $n = 2$ due to a quadratic nonlinearity which cannot be controlled by the standard energy method. A similar aspect appears in the case of 1-dimensional viscous conservation laws, where the asymptotic behavior is described by a solution of a 1-dimensional viscous Burgers equation (cf., [13, 16]).

The purpose of this paper is to prove the nonlinear stability of $\tilde{u}_s$ in the case $n = 2$ for small Reynolds and Mach numbers. Furthermore, we will show that the large time behavior of the perturbation is described by a solution of a 1-dimensional viscous Burgers equation.

To state the results of this paper more precisely, we introduce the following non-dimensional variables:

$$\tilde{x} = \ell x, \quad \tilde{t} = \frac{\ell}{V} t, \quad \tilde{v} = V v, \quad \tilde{\rho} = \rho_* \rho, \quad \tilde{P} = \rho_* V^2 P$$

with $V = \|\tilde{v}_s^1\|_{C^{m+1}[0,\ell]}$ for an integer $m \geq \lfloor n/2 \rfloor + 1$. Here

$$\|\tilde{v}_s^1\|_{C^{m+1}[0,\ell]} = \sum_{k=0}^{m+1} \sup_{0 \leq \tilde{x}_n \leq \ell} |\ell^k \partial_{\tilde{x}_n}^k \tilde{v}_s^1(\tilde{x}_n)|.$$ 

Under this non-dimensionalization the domain $\Omega_\ell$ is transformed into $\Omega \equiv \Omega_1$ and the parallel flow $\tilde{u}_s$ is transformed into $u_s = \frac{\gamma}{\gamma} (\rho_s, v_s)$ with

$$\rho_s = \rho_s(x_n) > 0, \quad \int_0^1 \rho_s(x_n) dx_n = 1,$$
$$v_s = \frac{\gamma}{\gamma} (v_s^1(x_n), 0, \cdots, 0), \quad \|v_s^1\|_{C^{m+1}[0,1]} = 1.$$ 

The perturbation $u(t) = \frac{\gamma}{\gamma} (\phi(t), w(t)) \equiv \frac{\gamma}{\gamma} (\gamma^2 (\rho(t) - \rho_*) - v(t) - v_s)$ is governed by the system of equations

$$(1.3) \quad \partial_t \phi + v_s \cdot \nabla \phi + \gamma^2 \text{div} (\rho_s w) = f^0,$$
\[
\partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \text{div} w + \nabla \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) \\
+ \frac{\nu_2}{\gamma^2 \rho_s^2} v_s \phi e_1 + v_s \cdot \nabla w + w \cdot \nabla v_s = f.
\]

Here \( \text{div} \), \( \nabla \) and \( \Delta \) denote the divergence, gradient and Laplacian with respect to \( x \); \( e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n \); \( \nu \), \( \tilde{\nu} \) and \( \gamma \) are non-dimensional parameters defined by

\[
\nu = \frac{\mu}{\rho_s \ell V}, \quad \tilde{\nu} = \frac{\mu + \mu'}{\rho_s \ell V}, \quad \gamma = \sqrt{\frac{P''(\rho_s)}{V}};
\]

and \( f^0 \) and \( f = (f', f^n) \), \( f' = (f^1, \cdots, f^{n-1}) \), denote the nonlinearities:

\[
f^0 = -\text{div}(\phi w),
\]

\[
f = -w \cdot \nabla w + \frac{\nu \phi}{(\phi + \gamma^2 \rho_s)^2} \left( -\Delta w + \frac{\partial^2 v^1}{\gamma^2 \rho_s} \phi e_1 \right) \\
- \frac{\nu \phi}{\rho_s^2 (\phi + \gamma^2 \rho_s)^2} \nabla \text{div} w \\
+ \frac{\phi}{\gamma^2 \rho_s} \nabla \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) - \frac{1}{2\gamma^2 \rho_s} \nabla (P''(\rho_s)\phi^2) \\
+ \tilde{P}_3(\rho_s, \phi, \partial_x \phi),
\]

where

\[
\tilde{P}_3 = \frac{\phi^3}{\gamma^2 (\phi + \gamma^2 \rho_s)^2} \nabla P(\rho_s) - \frac{1}{2\gamma^2 \rho_s} \nabla (\phi^3 P_3(\rho_s, \phi)) \\
+ \frac{\phi}{2\gamma^2 \rho_s^2} \nabla \left( P''(\rho_s)\phi^2 + \frac{1}{\gamma^2} \phi^3 P_3(\rho_s, \phi) \right) \\
- \frac{\phi^2}{\gamma^2 (\phi + \gamma^2 \rho_s)^2} \nabla \left( \frac{P'(\rho_s)}{\gamma^3} \phi + \frac{1}{2\gamma^3} P''(\rho_s)\phi^2 + \frac{1}{2\gamma^3} \phi^3 P_3(\rho_s, \phi) \right)
\]

with

\[
P_3(\rho_s, \phi) = \int_0^1 (1 - \theta)^2 P''(\theta \gamma^{-2} \phi + \rho_s) d\theta.
\]

We consider (1.3)–(1.4) under the boundary condition

\[
w|_{\partial \Omega} = 0
\]

and the initial condition

\[
u|_{t=0} = u_0 = (\phi_0, w_0).
\]

We will show that the following assertion holds when \( n = 2 \); if \( u_0 \) is sufficiently small in \( H^m(\Omega) \cap L^1(\Omega) \) for an integer \( m \geq 2(= \lfloor n/2 \rfloor + 1) \) and
$u_0$ satisfies a suitable compatibility condition, then there exists a unique solution $u(t)$ of (1.3)–(1.6) in $C([0, \infty); H^m(\Omega))$, provided that $\nu \gg 1$, $\gamma \gg 1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \ll 1$. Furthermore, $u(t)$ satisfies
\[
\|u(t)\|_{L^2} = O(t^{-\frac{1}{2}}),
\]
\[
\|u(t) - (\sigma u^{(0)})(t)\|_{L^2} = O(t^{-\frac{3}{4} + \delta}) \quad (\forall \delta > 0)
\]
as $t \to \infty$. Here $u^{(0)}$ is a function of $x_2$ only that satisfies $u^{(0)}(x_2) \sim t^{-\frac{1}{2}}(1, O(\gamma^{-2}), 0)$; and $\sigma$ is a function of $(x_1, t)$ that satisfies
\[
(1.7) \quad \partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma + a_0 \partial_{x_2} \sigma + a_1 \partial_{x_1} (\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2) dx_2
\]
with some constants $\kappa_0 > 0$, $a_0, a_1 \in \mathbb{R}$. We note that the result also holds for the case of the motionless state $\tilde{u}_s = (\tilde{\rho}_s, 0)$ with $a_0 = 0$ and some constants $\kappa_0 > 0$ and $a_1 \in \mathbb{R}$ possibly different from the ones in (1.7). The coefficient $a_1$ in (1.7) vanishes in the case of the motionless state $\tilde{u}_s = (\tilde{\rho}_s, 0)$ with constant density $\tilde{\rho}_s \equiv \rho_s$ ([6]) and in the case of the plane Couette flow ([7]). These are the special cases where (1.7) becomes a linear equation. In general, one can see that $a_1 \neq 0$. Even in the case of the motionless state $\tilde{u}_s = (\tilde{\rho}_s, 0)$, if the density $\tilde{\rho}_s$ is not a constant, then $a_1 \neq 0$ in general.

We also note that when $n \geq 3$, as was remarked in [8], the large time behavior of $u(t)$ is described by a linear heat equation in such a way that
\[
\|u(t) - (\sigma u^{(0)})(t)\|_{L^2} = O(t^{-\frac{n+1}{4} - \frac{1}{2} \eta_n(t)}).
\]
Here $\sigma$ is a function of $(x', t)$ satisfying
\[
\partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + a_0 \partial_{x_2} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n) dx_n
\]
with some constants $\kappa_0, \kappa'' > 0$, $a_0 \in \mathbb{R}$, where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$, and $\eta_n(t) = \log(1 + t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \geq 4$.

The proof of the main results is given by a combination of the spectral analysis of the linearized problem and a variant of the Matsumura-Nishida energy method. The nonlinearity of the Burgers equation (1.7) stems from the terms $-\text{div}(\phi w)$ in (1.3) and $-\frac{1}{2\kappa_0} \partial_{x_2} (P''(\rho_s) \phi')$ in the $n$ th equation of (1.4) which cannot be controlled by the standard Matsumura-Nishida energy method when $n = 2$. To deal with this term we will introduce a decomposition of the solution according to spectral properties of the linearized operator. Roughly speaking we decompose the solution into a low frequency
part and a high frequency part. For the low frequency part, we will apply the decay estimates on the linearized problem. On the other hand, for the high frequency part, we will apply a variant of the Matsumura-Nishida energy method similar to the analysis for the case $n \geq 3$ in [8]. The symmetric property of the system (1.1)–(1.2) is a little bit disturbed by introducing the decomposition. However, using the fact that spatial differentiation of any order acts on the low frequency part as a bounded operator, we can obtain the estimates on the nonlinearities necessary to obtain the a priori estimate for the global existence.

This paper is organized as follows. In section 2 we introduce notations used in this paper. In section 3 we state the main results of this paper. Section 4 is devoted to spectral properties of the linearized problem, most of which were essentially obtained in [12]. In section 5 we introduce a decomposition of the solution and reformulate the problem. In section 6 we investigate the low frequency part, the slowly decaying part, by using the decay estimates on the linearized problem; and in section 7 we estimate the high frequency part, the fast decaying part, by the energy method. In section 8 we give necessary estimates on the nonlinearities. In section 9 we show that the asymptotic behavior of $u(t)$ is described by the Burgers equation (1.7).

2. Preliminaries

In this section we first state assumptions on the parallel flow $u_s$ and then introduce some notations used in this paper.

In this paper we consider the two-dimensional problem ($n = 2$). Throughout this paper we assume that the parallel flow $u_s = (\rho_s(x_2), v_s(x_2))$ is bounded smooth and satisfies

\begin{equation}
0 < \rho_1 \leq \rho_s(x_2) \leq \rho_2, \quad \int_0^1 \rho_s(x_2) \, dx_2 = 1, \quad v_s(x_2) = (v_s^1(x_2), 0)
\end{equation}

and

\begin{equation}
P'(\rho) > 0 \quad \text{for} \quad \rho_1 \leq \rho \leq \rho_2
\end{equation}

with some constants $0 < \rho_1 < 1 < \rho_2$. Existence of such a parallel flow was shown, e.g., in [8] when $\tilde{g}$ is sufficiently small. Note also that, due to the non-dimensionalization, we have

$$\|v_s\|_{C^{m+1}[0,1]} = 1.$$

In the remaining of this section, we introduce some notations which will be used throughout the paper. For a domain $D$ and $1 \leq p \leq \infty$ we denote by
Let $m$ be a nonnegative integer. The symbol $H^m(D)$ denotes the $m$th order Sobolev space on $D$ with norm $\| \cdot \|_{H^m(D)}$. $C^m_0(D)$ stands for the set of all $C^m$ functions which have compact support in $D$. We denote by $H^1_0(D)$ the completion of $C^1_0(D)$ in $H^1(D)$.

We simply denote by $L^p(D)$ (resp., $H^m(D)$) the set of all vector fields $w = \tau(w^1, w^2)$ on $D$ with $w^j \in L^p(D)$ (resp., $H^m(D)$), $j = 1, 2$, and its norm is also denoted by $\| \cdot \|_{L^p(D)}$ (resp., $\| \cdot \|_{H^m(D)}$). For $u = \tau(\phi, w)$ with $\phi \in H^k(D)$ and $w = \tau(w^1, w^2) \in H^m(D)$, we define $\| u \|_{H^k(D) \times H^m(D)}$ by

$$
\| u \|_{H^k(D) \times H^m(D)} = \| \phi \|_{H^k(D)} + \| w \|_{H^m(D)}.
$$

When $k = m$, we simply write $\| u \|_{H^k(D) \times H^m(D)} = \| u \|_{H^k(D)}$.

In case $D = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^m(\Omega)$) as $L^p$ (resp., $H^m$). In particular, the norm $\| \cdot \|_{L^p(\Omega)} = \| \cdot \|_{L^p}$ is denoted by $\| \cdot \|_p$.

The inner product of $L^2(\Omega)$ is denoted by

$$
(f, g) = \int_{\Omega} f(x)g(x) \, dx, \quad f, g \in L^2(\Omega).
$$

We also denote the inner product of $L^2(0, 1)$ by

$$
(f, g) = \int_0^1 f(x_2)g(x_2) \, dx_2, \quad f, g \in L^2(0, 1),
$$

if no confusion occurs. We further introduce a weighted inner product $\langle \langle \cdot, \cdot \rangle \rangle$ defined by

$$
\langle \langle u_1, u_2 \rangle \rangle = \int_{\Omega} \phi_1 \phi_2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} \, dx + \int_{\Omega} w_1 \cdot w_2 \rho_s \, dx
$$

for $u_j = \tau(\phi_j, w_j) \in L^2(\Omega)$ ($j = 1, 2$); and, also, $\langle \cdot, \cdot \rangle$ defined by

$$
\langle u_1, u_2 \rangle = \int_0^1 \phi_1 \phi_2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} \, dx_2 + \int_0^1 w_1 \cdot w_2 \rho_s \, dx_2
$$

for $u_j = \tau(\phi_j, w_j) \in L^2(0, 1)$ ($j = 1, 2$). Here $\rho_s = \rho_s(x_2)$ denotes the density of the parallel flow $u_s$. We note that $\langle \langle \cdot, \cdot \rangle \rangle$ and $\langle \cdot, \cdot \rangle$ define inner products of $L^2(\Omega)$ and $L^2(0, 1)$, respectively, due to (2.1) and (2.2).

We denote the mean value of $f \in L^1(0, 1)$ by $\langle \cdot \rangle$:

$$
\langle f \rangle = \int_0^1 f(x_2) \, dx_2.
$$
For \( u = \top(\phi, w) \in L^1(0,1) \) with \( w = \top(w^1, w^2) \) we define \( \langle u \rangle \) by
\[
\langle u \rangle = \langle \phi \rangle + \langle w^1 \rangle + \langle w^2 \rangle.
\]

We denote the \( k \times k \) identity matrix by \( I_k \). We also define \( 3 \times 3 \) diagonal matrices \( Q_0 \) and \( e^Q \) by
\[
Q_0 = \text{diag}(1, 0, 0), \quad e^Q = \text{diag}(0, 1, 1).
\]
Note that
\[
\langle Q_0 u \rangle = \langle \phi \rangle \quad \text{for} \quad u = \top(\phi, w).
\]

Partial derivatives of a function \( u \) in \( x, x_j \) and \( t \) are denoted by \( \partial x u \), \( \partial x_j u \) and \( \partial t u \), respectively. We also write higher order partial derivatives of \( u \) as \( \partial^k x u \) where
\[
\partial^k x u = (\partial^{\alpha} x u; |\alpha| = k).
\]

For a function \( f = f(x_1) (x_1 \in \mathbb{R}) \), we denote its Fourier transform by \( \hat{f} \) or \( \mathcal{F}f \):
\[
\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}} f(x_1) e^{-i\xi x_1} dx_1 \quad (\xi \in \mathbb{R}).
\]
The inverse Fourier transform is denoted by \( \mathcal{F}^{-1} \):
\[
(\mathcal{F}^{-1}f)(x_1) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\xi) e^{i\xi x_1} d\xi.
\]

We will denote the resolvent set of a closed operator \( A \) by \( \rho(A) \) and the spectrum of \( A \) by \( \sigma(A) \). For \( \Lambda \in \mathbb{R} \) and \( \theta \in (\frac{\pi}{2}, \pi) \) we denote the set \( \{ \lambda \in \mathbb{C}; |\arg(\lambda - \Lambda)| \leq \theta \} \) by \( \Sigma(\Lambda, \theta) \):
\[
\Sigma(\Lambda, \theta) = \{ \lambda \in \mathbb{C}; |\arg(\lambda - \Lambda)| \leq \theta \}.
\]

3. Main Results

In this section we state the main results of this paper.

We first mention the compatibility condition for \( u_0 = \top(\phi_0, w_0) \). We will look for a solution \( u = \top(\phi, w) \) of (1.3)–(1.6) with \( n = 2 \) in \( \cap_{j=0}^{\frac{n}{2}} C^j([0, \infty); H^{m-2j}) \) satisfying
\[
\int_0^t \|\partial_x w\|^2_{H^m} d\tau < \infty
\]
for all \( t \geq 0 \) with \( m \geq 2 \). Therefore, we need to require the compatibility condition for the initial value \( u_0 = \top(\phi_0, w_0) \), which is formulated as follows.

Let \( u = \top(\phi, w) \) be a smooth solution of (1.3)–(1.6) with \( n = 2 \). Then
\[
\partial_t^j u = \top(\partial_t^j \phi, \partial_t^j w) \quad (j \geq 1)
\]
is inductively determined by
\[
\partial_t^j \phi = -v_s \cdot \nabla \partial_t^{j-1} \phi - \gamma^2 \text{div}(\rho_s \partial_t^{j-1} w) + \partial_t^{j-1} f^0
\]
and
\[
\partial_t^j w = \frac{\nu}{\rho_s} \Delta \partial_t^{j-1} w + \frac{\nu}{\rho_s} \nabla \text{div} \partial_t^{j-1} w - \nabla \left( \frac{P'(\rho_s)}{\gamma \rho_s} \partial_t^{j-1} \phi \right) \\
- \frac{\nu \partial_t^j v_i}{\gamma \rho_s^2} \partial_t^{j-1} \phi e_i - v_s \cdot \nabla \partial_t^{j-1} w - \partial_t^{j-1} w \cdot \nabla v_s + \partial_t^{j-1} f.
\]

From these relations we see that \( \partial_t^j u|_{t=0} = \top (\partial_t^j \phi, \partial_t^j w)|_{t=0} \) is inductively given by \( u_0 = \top (\phi_0, w_0) \) in the following way:
\[
\partial_t^j u|_{t=0} = \top (\partial_t^j \phi, \partial_t^j w)|_{t=0} = \top (\phi_j, w_j) = u_j,
\]
where
\[
\phi_j = -v_s \cdot \nabla \phi_{j-1} - \gamma^2 \text{div} (\rho_s w_{j-1}) \\
+ f_{j-1}^0 (u_0, \ldots, u_{j-1}, \partial_x u_0, \ldots, \partial_x u_{j-1}),
\]
\[
w_j = \frac{\nu}{\rho_s} \Delta w_{j-1} + \frac{\nu}{\rho_s} \nabla \text{div} w_{j-1} - \nabla \left( \frac{P'(\rho_s)}{\gamma \rho_s} \phi_{j-1} \right) \\
- \frac{\nu \partial_t^j v_i}{\gamma \rho_s^2} \phi_{j-1} e_i - v_s \cdot \nabla w_{j-1} - w_{j-1} \cdot \nabla v_s \\
+ f_{j-1} (u_0, \ldots, u_{j-1}, \partial_x u_0, \ldots, \partial_x u_{j-1}).
\]
Here \( f_{j}^0 (u_0, \ldots, u_j, \ldots) \) is a certain polynomial in \( u_0, \ldots, u_j, \ldots \); \( \ldots \), and so on.

By the boundary condition \( w|_{\partial \Omega} = 0 \) in (1.5), we necessarily have \( \partial_t^j w|_{\partial \Omega} = 0 \), and hence,
\[
w_j|_{\partial \Omega} = 0.
\]

Assume that \( u = \top (\phi, w) \) is a solution of (1.3)–(1.6) with \( n = 2 \) in \( \cap_{j=0}^{[n/2]} C^1([0, T_0]; H^{m-2j}) \) for some \( T_0 > 0 \). Then, from the above observation, we need the regularity \( u_j = \top (\phi_j, w_j) \in H^{m-2j} \) for \( j = 0, \ldots, \lfloor m/2 \rfloor \), which, indeed, follows from the fact that \( u_0 = \top (\phi_0, w_0) \in H^m \) with \( m \geq 2(= \lceil n/2 \rceil + 1) \). Furthermore, it is necessary to require that \( u_0 = \top (\phi_0, w_0) \) satisfies the \( m \)th order compatibility condition:
\[
w_j \in H^1_0 \quad \text{for} \quad j = 0, 1, \ldots, \hat{m} = \left\lfloor \frac{m-1}{2} \right\rfloor.
\]

We are now in a position to state our main results of this paper.

**Theorem 3.1.** Suppose that \( n = 2 \). Let \( m \) be an integer satisfying \( m \geq 2(= \lceil n/2 \rceil + 1) \). Assume that \( u_0 = \top (\rho_0, v_0) \) satisfies (2.1) and (2.2). Then there are positive numbers \( \nu_0, \gamma_0 \) and \( \omega_0 \) such that if \( \nu \geq \nu_0, \gamma^2/(\nu + \gamma) \geq \gamma_0^2 \) and \( \|\rho_0 - 1\|_{C^{m+1}[0, 1]} \leq \omega_0 \), then the following assertion holds. There is a positive
number $\varepsilon_0$ such that if $u_0 = \top(\phi_0, w_0) \in H^m \cap L^1$ satisfies $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_0$ and the $m$ th compatibility condition, then there exists a unique global solution $u(t) = \top(\phi(t), w(t))$ of (1.3)–(1.6) with $n = 2$ in $\cap_{j=0}^{\infty} C^j([0, \infty); H^{m-2j})$ which satisfies

(3.1) $\|\partial_t^k u(t)\|_2 = O(t^{-\frac{n}{4} - \frac{k}{2}})$ \hspace{1em} (k = 0, 1),

(3.2) $\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{n}{4} + \delta})$ \hspace{1em} ($\forall \delta > 0$)

as $t \to \infty$. Here $u^{(0)} = u^{(0)}(x_2)$ is a function given in Lemma 4.1 below; and $\sigma = \sigma(x_1, t)$ is a function satisfying

$$\partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma + a_0 \partial_{x_1} \sigma + a_1 \partial_{x_1}(\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2) \, dx_2$$

with some constants $\kappa_0 > 0$, $a_0, a_1 \in \mathbb{R}$.

**Remark.** As was remarked in [8], in case $n \geq 3$, one can establish the estimates

$$\|\partial_t^k u(t)\|_2 = O(t^{-\frac{n}{4} - \frac{k}{2}}) \hspace{1em} (k = 0, 1),$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{n}{4} + \delta})$$

as $t \to \infty$, provided that $u_0 = (\phi_0, w_0) \in H^m \cap L^1$ with $\|u_0\|_{H^m \cap L^1} \ll 1$, $m \geq [n/2] + 1$. Here $\sigma$ is a function of $(x', t)$ satisfying

$$\partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + a_0 \partial_{x_1} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n) \, dx_n$$

with some constants $\kappa_0, \kappa'' > 0$, $a_0 \in \mathbb{R}$, where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$; and $\eta_n(t) = \log(1 + t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \geq 4$.

As in [15, 10], Theorem 3.1 is proved by showing the local existence and the a priori estimates. The local existence is proved by applying the local solvability result in [9]. In fact, we can show the following assertion. We introduce notations:

$$[f(t)]_k = \left( \sum_{j=0}^{[\frac{k}{2}]} \|\partial_t^j f(t)\|_{H^{k-2j}}^2 \right)^{\frac{1}{2}},$$

$$|||Df(t)|||_k = \begin{cases} \|\partial_x f(t)\|_2 & \text{for } k = 0, \\ (\|\partial_x f(t)\|_k^2 + \|\partial_t f(t)\|_{k-1}^2)^{\frac{1}{2}} & \text{for } k \geq 1. \end{cases}$$

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For $T > 0$ we define a function space $Z^m(T)$ by

$$Z^m(T) = \{ u \in \cap_{j=0}^{[m/2]} C^j([0,T]; H^{m-2j}); \| u \|_{Z^m(T)} < \infty \},$$

where

$$\| u \|_{Z^m(T)} = \sup_{0 \leq t \leq T} \| u(t) \|_m + \left( \int_0^T \| D^2 w(t) \|_m^2 \, dt \right)^{1/2}.$$ 

**Proposition 3.2.** Let $m \geq 2 (= \lfloor n/2 \rfloor + 1)$. Assume that $u_0 = \top (\phi_0, w_0) \in H^m$ satisfies the following conditions.

(a) $u_0 \in H^m$ satisfies the $\hat{m}$-th compatibility condition.

(b) $-\frac{\gamma^2}{2} \rho_1 \leq \phi_0$.

Then there exists a positive number $T_0$ depending on $\| u_0 \|_{H^m}$ and $\rho_1$ such that problem (1.3)–(1.6) has a unique solution $u(t) \in Z^m(T_0)$ satisfying

$$\phi(x, t) \geq -\frac{\gamma^2}{2} \rho_1 \text{ for } \forall (x, t) \in \Omega \times [0, T_0].$$

Furthermore, the inequality

$$\| u \|_{Z^m(T_0)}^2 \leq C_0 \{ 1 + \| u_0 \|_{H^m}^2 \}^a \| u_0 \|_{H^m}^2$$

holds for some constants $C_0 > 0$ and $a > 0$ depending on $m$.

The global existence of the solution $u(t)$ follows from Proposition 3.2 and the following a priori estimate in a standard manner.

**Proposition 3.3.** Let $m$ be an integer satisfying $m \geq 2 (= \lfloor n/2 \rfloor + 1)$. Then there are positive numbers $\nu_0$, $\gamma_0$ and $\omega_0$ such that if $\nu \geq \nu_0$, $\gamma_0^2 / (\nu + \nu_0) \geq \gamma_0^2$ and $\| \rho_0 - 1 \|_{L^{m+1}[0,1]} \leq \omega_0$, then the following assertion holds.

There exists a number $\varepsilon_1 > 0$ such that if a solution $u(t)$ of (1.3)–(1.6) in $Z^m(T)$ satisfies $\| u_0 \|_{H^m \cap L^1} \leq \varepsilon_1$, then there holds the estimate

$$\| u(t) \|_m^2 \leq C_1 \| u_0 \|_{H^m \cap L^1}^2$$

for some constant $C_1 > 0$ independent of $T$.

Proposition 3.3, together with $L^2$ decay estimate (3.1), will be proved in sections 4–8. The asymptotic behavior (3.2) will be proved in section 9.

4. Spectral properties of the linearized operator
In this section we consider the spectral properties of the linearized problem which will be employed in the analysis of the nonlinear problem.

Problem (1.3)--(1.6) with \( n = 2 \) is written in the form

(4.1) \[ \partial_t u + Lu = F, \quad w|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \]

Here \( u = \top(\phi, w); \quad F = \top(f^0, f) \) with \( f = \top(f^1, f^2) \) is the nonlinearity; and \( L \) is the operator of the form

\[
L = A + B + C_0,
\]

where

\[
A = \begin{pmatrix}
0 & 0 \\
0 & -\frac{\nu}{\rho_s} \Delta I_2 - \frac{2\nu}{\rho_s} \nabla \text{div}
\end{pmatrix}, \quad B = \begin{pmatrix}
v^1_\phi \partial_x & \gamma^2 \text{div}(\rho_s \cdot) \\
\nabla\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot\right) & v^1 \partial_x I_2
\end{pmatrix},
\]

\[
C_0 = \begin{pmatrix}
0 & 0 \\
\frac{\nu \partial^2 \rho_s}{\gamma^2 \rho_s^2} e_1 & (\partial_{x_2} v^1_\phi) e_1 \top e_2
\end{pmatrix}
\]

with \( e_1 = \top(1,0) \) and \( e_2 = \top(0,1) \).

We here consider the operator \( L \) as an operator on \( H^1 \times \tilde{H}^1 \) with domain \( D(L) = \{ u = \top(\phi, w) \in H^1 \times \tilde{H}^1; \ w \in H^2 \cap \tilde{H}^1_0, \ Lu \in H^1 \times \tilde{H}^1 \} \), where \( \tilde{H}^1 = \{ w \in L^2; \ \partial_{x_1} w \in L^2 \} \) with norm \( \| w \|_{\tilde{H}^1} = (\| w \|^2_2 + \| \partial_{x_1} w \|^2_2)^{\frac{1}{2}} \). One can see that there exists a \( \Lambda \gg 1 \) such that \( \{ \lambda \in \mathbb{C}; \ \text{Re} \lambda \geq \Lambda \} \subset \rho(-L) \).

Let \( \tilde{Z}(T) \) be a function space defined by

\[
\tilde{Z}(T) = \{ u = \top(\phi, w); \ u \in C([0,T]; H^1 \times \tilde{H}^1) \}
\]

\[
\partial_{x_1}^k w \in L^2(0,T; H^1_0), \ k = 0,1,
\]

\[
w \in C((0,T]; H^1_0).\]

Then one can show that for any \( u_0 \in H^1 \times \tilde{H}^1 \) there exists a unique solution \( u(t) \) in \( \tilde{Z}(T) \) (\( \forall T > 0 \)) of the linear problem

(4.2) \[ \partial_t u + Lu = 0, \quad u|_{t=0} = u_0; \]

and \( u(t) \) satisfies

\[
\| u(t) \|_{H^1 \times \tilde{H}^1}^2 + t \| \partial_{x_2} w(t) \|_2^2 + \sum_{k=0}^{1} \int_0^t \| \partial_{x_1}^k w(\tau) \|_{\tilde{H}^1}^2 \ d\tau \leq C \| u_0 \|_{H^1 \times \tilde{H}^1}^2.
\]
Furthermore, if \( u_0 \in D(L) \), then \( u \in C^1([0, T]; H^1 \times \dot{H}^1) \). It then follows that 
\(-L\) generates a \( C_0\)-semigroup \( U(t) \) that is defined by \( U(t)u_0 = u(t) \), where \( u(t) \) is a solution of (4.2) with \( u_0 \in H^1 \times H^1 \). See also [2, 3] for a generation of a \( C_0\)-semigroup. We will employ some spectral properties of \( U(t) \).

In the analysis of (4.1) we will decompose the solution \( u(t) \) of (4.1) by a projection operator associated with \( U(t) \) which is obtained through the Fourier transform in \( x_1 \).

Let us consider the Fourier transform of (1.3)–(1.6) in \( x_1 \in \mathbb{R} \):

\[
\partial_t \hat{\phi} + i \xi_1 v_1^1 \hat{\phi} + i \gamma \xi (\rho_s \hat{w}^1) + \gamma^2 \partial_{x_2} (\rho_s \hat{w}^2) = \hat{f}_0, \tag{4.3}
\]

\[
\partial_t \hat{w}^1 + \frac{\nu}{\rho_s} (|\xi|^2 - \partial^2_{x_2}) \hat{w}^1 - i \frac{\tilde{\nu}}{\rho_s} \xi (i \xi \hat{w}^1 + \partial_{x_2} \hat{w}^2) + i \xi \left( \frac{\partial (\rho_s)}{\gamma \rho_s} \right) \hat{\phi} \\
+ \frac{\nu^2 \gamma^2 v_1^1}{\gamma^2 \rho_s^2} \hat{\phi} + i \xi v_1^1 \hat{w}^1 + (\partial_{x_2} v_1^1) \hat{w}^2 = \hat{f}_1, \tag{4.4}
\]

\[
\partial_t \hat{w}^2 + \frac{\nu}{\rho_s} (|\xi|^2 - \partial^2_{x_2}) \hat{w}^2 - \frac{\tilde{\nu}}{\rho_s} \partial_{x_2} (i \xi \hat{w}^1 + \partial_{x_2} \hat{w}^2) \\
+ \partial_{x_2} \left( \frac{\partial (\rho_s)}{\gamma \rho_s} \right) \hat{\phi} + i \xi v_1^1 \hat{w}^2 = \hat{f}_2, \tag{4.5}
\]

\[
\hat{w}|_{x_2=0,1} = 0, \tag{4.6}
\]

\[
\hat{u}|_{t=0} = \hat{u}_0 = \hat{F}(\hat{\phi}_0, \hat{w}_0). \tag{4.7}
\]

Here \( \hat{\phi} = \hat{\phi}(\xi, x_2, t) \) and \( \hat{w} = \hat{w}(\xi, x_2, t) \) are the Fourier transform of \( \phi = \phi(x_1, x_2, t) \) and \( w = w(x_1, x_2, t) \) in \( x_1 \in \mathbb{R} \) with \( \xi \in \mathbb{R} \) being the dual variable. We thus arrive at the following problem

\[
\partial_t \hat{u} + \hat{L}_\xi \hat{u} = \hat{F}, \quad \hat{u}|_{t=0} = \hat{u}_0 \tag{4.8}
\]

with a parameter \( \xi \in \mathbb{R} \). Here \( \hat{L}_\xi \) is the operator on \( H^1(0,1) \times L^2(0,1) \) of the form

\[
\hat{L}_\xi = \hat{A}_\xi + \hat{B}_\xi + \hat{C}_0,
\]

where

\[
\hat{A}_\xi = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{\nu}{\rho_s} (|\xi|^2 - \partial^2_{x_2}) + \frac{\tilde{\nu}}{\rho_s} |\xi|^2 & -i \frac{\tilde{\nu}}{\rho_s} \xi \partial_{x_2} \\
0 & -i \frac{\tilde{\nu}}{\rho_s} \xi \partial_{x_2} & \frac{\nu}{\rho_s} (|\xi|^2 - \partial^2_{x_2}) - \frac{\tilde{\nu}}{\rho_s} \partial^2_{x_2}
\end{pmatrix},
\]

\[
\hat{B}_\xi = \begin{pmatrix}
0 & \nu \partial_{x_2} \\
\nu \partial_{x_2} & 0 \\
0 & 0
\end{pmatrix},
\]

\[
\hat{C}_0 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]
\[
\hat{B}_\xi = \begin{pmatrix}
i \xi v_s^1 & i \gamma^2 \rho_s \xi & \gamma^2 \partial x_2 (\rho_s \cdot ) \\
i \xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} & i \xi v_s^1 & 0 \\
\partial x_2 \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) & 0 & i \xi v_s^1
\end{pmatrix},
\]

with domain
\[
D(\hat{L}_\xi) = H^1(0,1) \times (H^2(0,1) \cap H^1_0(0,1)).
\]

In [12] some spectral properties of \(\hat{L}_\xi\) were investigated. (In [12], the linearized operator at a Poiseuille type flow was studied, but one can see that the arguments in [12] are valid for our parallel flows since only the properties of parallel flow (2.1), (2.2) and \(\| \rho_s - 1 \|_{C^{m+1}[0,1]} \ll 1\) were used in [12] to show the spectral properties in Lemmas 4.1–4.9 below.) We will employ the following properties of \(\hat{L}_\xi\) for \(|\xi| \ll 1\).

We begin with the case \(\xi = 0\). Let us introduce a formal adjoint operator \(\hat{L}_\xi^*\) of \(\hat{L}_\xi\) with respect to the inner product \(\langle \cdot, \cdot \rangle\):
\[
\hat{L}_\xi^* = \hat{A}_\xi - \hat{B}_\xi + \hat{C}_0^*,
\]

with domain \(D(\hat{L}_\xi^*) = D(\hat{L}_\xi)\), where
\[
\hat{C}_0^* = \begin{pmatrix}
0 & \frac{\gamma^2 \nu \partial x_2 v_s^1}{\rho_s P'(\rho_s)} & 0 \\
0 & 0 & 0 \\
0 & \partial x_2 v_s^1 & 0
\end{pmatrix}.
\]

**Lemma 4.1.** ([12]) Let \(u_s = \top(\rho_s, v_s)\) be a smooth stationary solution satisfying (2.1) and (2.2). Then there are positive constants \(\nu_0, \gamma_0\) and \(\omega_0\) such that if \(\nu \geq \nu_0, \gamma^2 / (\nu + \bar{\nu}) \geq \gamma_0^2\) and \(\| \rho_s - 1 \|_{C^{m+1}[0,1]} \leq \omega_0\), then the following assertions hold.

(i) There are positive numbers \(\eta_0\) and \(\theta_0 \in \left(\frac{\pi}{2}, \pi\right)\) such that \(\Sigma(-\eta_0, \theta_0) \setminus \{0\} \subset \rho(-\hat{L}_0)\). Furthermore, the following estimates hold uniformly for \(\lambda \in \rho(-\hat{L}_0) \cap \Sigma(-\eta_0, \theta_0) \setminus \{0\}\):
\[
| (\lambda + \hat{L}_0)^{-1} f |_{H^1 \times L^2} \leq \frac{C}{|\lambda|} | f |_{H^1 \times L^2},
\]
\[
\left| \partial_{x_2}^l Q(\lambda + \hat{L}_0)^{-1} f \right|_2 \leq C \left( \frac{1}{|\lambda|} + \frac{1}{(|\lambda| + 1)^{1 - \frac{l}{2}}} \right) |f|_{H^1 \times L^2}
\]
for \( l = 1, 2, \)
\[
\left| \partial_{x_2}^2 Q_0(\lambda + \hat{L}_0)^{-1} f \right|_2 \leq C \left( \frac{1}{|\lambda|} + \frac{1}{(|\lambda| + 1)^{\frac{1}{2}}} \right) |f|_{H^2 \times H^1}.
\]

The same assertion holds for \(-\hat{L}_0^*\).

(ii) \( \lambda = 0 \) is a simple eigenvalue of \( \hat{L}_0 \) and \( \hat{L}_0^* \).

(iii) The eigenspaces for \( \lambda = 0 \) of \( \hat{L}_0 \) and \( \hat{L}_0^* \) are spanned by \( u^{(0)} \) and \( u^{(0)*} \), respectively, where

\[
u^{(0)} = \top(\phi^{(0)}, w^{(0)}), \quad w^{(0)} = \top(w^{(0)}, 1, 0)
\]

and

\[
u^{(0)*} = \top(\phi^{(0)*}, 0)
\]

with

\[
\phi^{(0)}(x_2) = \alpha_0 \frac{\gamma^2 \rho_s(x_2)}{P'(\rho_s(x_2))}, \quad \alpha_0 = \left( \int_0^1 \frac{\gamma^2 \rho_s}{P'(\rho_s)} \, dx_2 \right)^{-1},
\]

\[
w^{(0)}(x_2) = -\frac{1}{\gamma^2} \int_0^1 G(x_2, y_2) \frac{\partial^2 v^1(y_2)}{\rho_s(y_2)} \phi^{(0)}(y_2) \, dy_2,
\]

\[
\phi^{(0)*}(x_2) = \frac{\gamma^2}{\alpha_0} \phi^{(0)}(x_2).
\]

Here

\[
G(x_2, y_2) = \begin{cases} 
(1 - x_2)y_2 & (0 < y_2 < x_2) \\
 x_2(1 - y_2) & (x_2 < y_2 < 1).
\end{cases}
\]

(iv) The eigenprojections \( \hat{\Pi}^{(0)} \) and \( \hat{\Pi}^{(0)*} \) for \( \lambda = 0 \) of \( \hat{L}_0 \) and \( \hat{L}_0^* \) are given by

\[
\hat{\Pi}^{(0)} u = \langle u, u^{(0)*} \rangle u^{(0)} = \langle \phi \rangle u^{(0)},
\]

and

\[
\hat{\Pi}^{(0)*} u = \langle u, u^{(0)} \rangle u^{(0)*}
\]

for \( u = \top(\phi, w) \), respectively.

(v) Let \( u^{(0)} \) be written as \( u^{(0)} = u_0^{(0)} + u_1^{(0)} \), where

\[
u_0^{(0)} = \top(\phi^{(0)}, 0), \quad u_1^{(0)} = \top(0, w^{(0)}), \quad w^{(0)} = \top(w^{(0)}, 1, 0).
\]

Then \( u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)} \) and

\[
\langle u, u^{(0)} \rangle = \frac{\alpha_0}{\gamma^2} \langle \phi \rangle + (w^1, w^{(0)}, \rho_s)\]

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for \( u = \hat{v} (\phi, w), \) \( w = \hat{v} (w^1, w^2) \).

As for the spectrum of \(-\hat{L}_\xi\), we have the following result. Let \( \hat{L}_\xi \) be denoted by

\[
\hat{L}_\xi = \hat{L}_0 + \xi \hat{L}^{(1)} + \xi^2 \hat{L}^{(2)},
\]

where

\[
\hat{L}^{(1)} = \begin{pmatrix}
i v_1^1 & i v_2^1 & 0 \\
i v_1^1 & \frac{iv'}{\gamma \rho_x} & -i \frac{v}{\rho_x} \partial x_2 \\
c & -i \frac{v}{\rho_x} \partial x_2 & 0
\end{pmatrix}, \quad \hat{L}^{(2)} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{\nu + \xi \rho_x}{\rho_x} & 0 \\
0 & 0 & \frac{\nu}{\rho_x}
\end{pmatrix}.
\]

Likewise, we denote \( \hat{L}_\xi^* \) by \( \hat{L}_0^* + \xi \hat{L}^{(1)*} + \xi^2 \hat{L}^{(2)*} \).

**Lemma 4.2.** ([12]) If \( \nu \geq \nu_0, \gamma^2 / (\nu + \nu) \geq \gamma_0^2 \) and \( \| \rho_x - 1 \|_{C^{m+1}[0,1]} \leq \omega_0 \), then there exists a positive number \( r_0 = r_0 (\nu, \hat{v}, \gamma, \eta_0, \theta_0) \) such that the following assertions hold.

(i) \( \Sigma (\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{n}{2} \} \subset \rho (\hat{L}_\xi) \) for \( |\xi| \leq r_0 \). Furthermore, the following estimates hold uniformly in \( \lambda \in \Sigma (\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{n}{2} \} \) and \( \xi \) with \( |\xi| \leq r_0 \):

\[
| (\lambda + \hat{L}_\xi)^{-1} f |_{H^1 \times L^2} \leq \frac{C}{|\lambda|} |f|_{H^1 \times L^2},
\]

\[
| \partial x_2 Q (\lambda + \hat{L}_\xi)^{-1} f |_{L^2} \leq C \left( \frac{1}{|\lambda|} + \frac{1}{(|\lambda| + 1)^{1/2}} \right) |f|_{H^1 \times L^2}
\]

for \( l = 1, 2 \),

\[
| \partial x_2 Q_0 (\lambda + \hat{L}_\xi)^{-1} f |_{L^2} \leq C \left( \frac{1}{|\lambda|} + \frac{1}{(|\lambda| + 1)^{1/2}} \right) |f|_{H^2 \times H^2}.
\]

The same assertion also holds for \(-\hat{L}_\xi^*\).

(ii) There holds

\[
\sigma (\hat{L}_\xi) \cap \{ \lambda; |\lambda| < \frac{n}{2} \} = \{ \hat{\lambda}_0 (\xi) \} \quad \text{for} \quad |\xi| \leq r_0,
\]

where \( \hat{\lambda}_0 (\xi) \) is a simple eigenvalue of \(-\hat{L}_\xi\) that has the form

\[
\hat{\lambda}_0 (\xi) = -i a_0 \xi - \kappa_0 \xi^2 + O (|\xi|^3)
\]

as \( \xi \to 0 \). Here \( a_0 \in \mathbb{R} \) and \( \kappa_0 > 0 \) are the numbers given by

\[
a_0 = - \langle v_1^1 \phi (0) + \gamma^2 \rho_x w (0), 1 \rangle,
\]

\[
\kappa_0 = \frac{a_0^2}{\nu} \left( - \partial x_2 \right)^{-1/2} \rho_x |^2 + O (\frac{\nu + \nu}{\gamma^2}) + O (\frac{1}{\nu}) (1 + O (\omega_0))
\]

\[
= \left( \frac{\gamma^2}{12 \nu} + O (\frac{\nu + \nu}{\gamma^2}) + O (\frac{1}{\nu}) \right) (1 + O (\omega_0)) > 0.
\]
(iii) The eigenprojection $\hat{\Pi}(\xi)$ for the eigenvalue $\hat{\lambda}_0(\xi)$ is expanded as

$$\hat{\Pi}(\xi) = \hat{\Pi}^{(0)} + \xi \hat{\Pi}^{(1)} + \xi^2 \hat{\Pi}^{(2)}(\xi),$$

where

$$\hat{\Pi}^{(1)} = -\hat{\Pi}^{(0)} \hat{L}^{(1)} \hat{S} - \hat{S} \hat{L}^{(1)} \hat{\Pi}^{(0)}$$

with $\hat{S} = ((I - \hat{\Pi}^{(0)}) \hat{L}_0 (I - \hat{\Pi}^{(0)}))^{-1}$; and $\hat{\Pi}^{(2)}(\xi)$ is a bounded operator on $H^1(0, 1) \times L^2(0, 1)$ satisfying $|\hat{\Pi}^{(2)}(\xi)u|_{H^1 \times L^2} \leq C|u|_{H^1 \times L^2}$. Furthermore, it holds that $\tilde{Q} \hat{\Pi}(\xi)u|_{x_2=0,1} = 0$.

Concerning the eigenprojection $\hat{\Pi}(\xi)$, the following estimates hold true.

**Lemma 4.3.** (i) The eigenprojection $\hat{\Pi}(\xi)$ is written in the form

$$(\hat{\Pi}(\xi)u)(x_2) = \int_0^1 \hat{\Pi}(\xi, x_2, y_2)u(y_2) \, dy_2$$

with

$$\hat{\Pi}(\xi, x_2, y_2) = \hat{\Pi}^{(0)}(x_2) + \xi \hat{\Pi}^{(1)}(x_2, y_2) + \xi^2 \hat{\Pi}^{(2)}(\xi, x_2, y_2).$$

Here $\hat{\Pi}^{(0)}(x_2) = u^{(0)}(x_2)^T e_0$, $e_0 = \tau (1, 0, 0)$; and $\hat{\Pi}^{(1)}(x_2, y_2)$ and $\hat{\Pi}^{(2)}(\xi, x_2, y_2)$ satisfy the estimates

$$\left| \partial_{x_2}^k \partial_{y_2}^l \hat{\Pi}^{(1)}(\xi, \cdot, \cdot) \right|_{L^\infty((0,1) \times (0,1))} \leq C_{k,l},$$

$$\left| \partial_{x_2}^k \partial_{y_2}^l \hat{\Pi}^{(2)}(\xi, \cdot, \cdot) \right|_{L^\infty((0,1) \times (0,1))} \leq C_{k,l}$$

uniformly in $\xi$ with $|\xi| \leq r_0$ for any $k, l \geq 0$.

As a consequence, there hold, for $0 \leq \forall k \leq m + 2$, $1 \leq p \leq 2$,

$$|\partial_{x_2}^k \hat{\Pi}(\xi)u|_2 \leq C_k |u|_p,$$

$$|\partial_{x_2}^k \hat{\Pi}^{(j)} u|_2 \leq C_k |u|_p \quad (j = 0, 1),$$

$$|\partial_{x_2}^k \hat{\Pi}^{(2)}(\xi)u|_2 \leq C_k |u|_p.$$

(ii) Let $\hat{\Pi}^*(\xi)$ be defined by

$$(\hat{\Pi}^*(\xi)u)(y_2) = \int_0^1 \hat{\Pi}^*(\xi, x_2, y_2)u(x_2) \, dx_2,$$

where $\hat{\Pi}^*(\xi, x_2, y_2)$ is given by

$$\hat{\Pi}^*(\xi, x_2, y_2) = W(y_2)^{-1T} \left( \frac{\alpha_s(x_2)}{\gamma_s(x_2)} \right) \left( \frac{\rho_s(x_2)}{\gamma_s(x_2)} \right) \frac{W(x_2)}{\gamma_s(x_2)} \right) W(x_2)$$

with $W(x_2) = \text{diag} \left( \frac{\rho_s(x_2)}{\gamma_s(x_2)} \right)$. Then $\hat{\Pi}^*(\xi)$ is the eigenprojection for the eigenvalue $\hat{\lambda}_0(\xi)$ of $-\hat{L}_\xi$; and there hold $\langle \hat{\Pi}(\xi)u_1, u_2 \rangle = \langle u_1, \hat{\Pi}^*(\xi)u_2 \rangle$.
and \( \tilde{Q}\hat{H}^*(\xi)u|_{x_2=0,1} = 0 \). Furthermore, estimates similar to those for \( \hat{H}(\xi) \) given in (i) hold for \( \hat{H}^*(\xi) \).

**Proof.** The integral expression of \( \hat{H}(\xi) \) and estimates for the case \( k, l \leq 1 \) are given in [12] following the argument in [5]. Estimates for the case \( k, l \geq 2 \) can be obtained as follows. We follow the argument in [5, Proof of Theorem 3.3].

Let \( \tilde{u}(\xi) \) and \( \tilde{u}^*(\xi) \) be defined by

\[
\tilde{u}(\xi) = \frac{1}{2\pi i} \int_{\{\lambda = \frac{\eta_0}{2}\}} (\lambda + \hat{L}_\xi)^{-1} u(0) d\lambda,
\]

\[
\tilde{u}^*(\xi) = \frac{1}{2\pi i} \int_{\{\lambda = \frac{\eta_0}{2}\}} (\lambda + \hat{L}_\xi^*)^{-1} u(0)^* d\lambda.
\]

Then \( \hat{H}(\xi) \) is given by

\[
(\hat{H}(\xi)u)(x_2) = \frac{\langle u, \tilde{u}^*(\xi) \rangle}{\langle \tilde{u}(\xi), \tilde{u}^*(\xi) \rangle} \tilde{u}(\xi; x_2)
\]

\[
= \int_0^1 \tilde{u}(\xi; x_2) \otimes W(y_2)\tilde{u}^*(\xi; y_2)u(y_2) dy_2.
\]

(4.9)

Here for \( a = \top (a_1, a_2) \) and \( b = \top (b_1, b_2) \), \( a \otimes b \) denotes the matrix \((a_1b_2, a_2b_1)\).

Since

\[
|\hat{L}^{(1)}(\lambda + \hat{L}_0)^{-1}f|_{H^{k+1} \times H^k} \leq C_k |(\lambda + \hat{L}_0)^{-1}f|_{H^{k+1}},
\]

(4.10)

\[
|\hat{L}^{(2)}(\lambda + \hat{L}_0)^{-1}f|_{H^{k+1} \times H^k} \leq C_k |(\lambda + \hat{L}_0)^{-1}f|_{H^{k}},
\]

(4.11)

for \( k \geq 0 \), by using Lemma 4.1, we have the Neumann series expansion

\[
(\lambda + \hat{L}_\xi)^{-1} = (\lambda + \hat{L}_0)^{-1} \sum_{N=0}^{\infty} (-1)^N \{ (\xi \hat{L}^{(1)} + \xi^2 \hat{L}^{(2)})(\lambda + \hat{L}_0)^{-1} \}^N
\]

for \( |\xi| \ll 1 \). One can see that the same expansion also holds for \( (\lambda + \hat{L}_0^*)^{-1} \)

\[
(\lambda + \hat{L}_\xi^*)^{-1} = (\lambda + \hat{L}_0^*)^{-1} \sum_{N=0}^{\infty} (-1)^N \{ (\xi \hat{L}^{(1)*} + \xi^2 \hat{L}^{(2)*})(\lambda + \hat{L}_0^*)^{-1} \}^N.
\]

It then follows that

\[
\tilde{u}(\xi) = u^{(0)} + \xi \tilde{u}^{(1)} + \xi^2 \tilde{u}^{(2)}(\xi),
\]

(4.12)
\( \tilde{u}^*(\xi) = u^{(0)*} + \xi \tilde{u}^{(1)*} + \xi^2 \tilde{u}^{(2)*}(\xi), \)

\( \langle \tilde{u}(\xi), \tilde{u}^*(\xi) \rangle = 1 + O(\xi) \geq \frac{1}{2}, \)

for \( |\xi| \ll 1. \) Here

\( \tilde{u}^{(1)} = -\frac{1}{2\pi i} \int_{|\lambda| = \frac{\eta_0}{2}} (\lambda + \hat{L}_0)^{-1} \hat{L}^{(1)}(\lambda + \hat{L}_0)^{-1} u^{(0)} \, d\lambda, \)

\( \tilde{u}^{(2)}(\xi) = \frac{1}{2\pi i} \int_{|\lambda| = \frac{\eta_0}{2}} R^{(2)}(\lambda, \xi) u^{(0)} \, d\lambda \)

with

\[ R^{(2)}(\lambda, \xi) = -(\lambda + \hat{L}_0)^{-1} \hat{L}^{(2)}(\lambda + \hat{L}_0)^{-1} + (\lambda + \hat{L}_0)^{-1} \sum_{N=2}^{\infty} (-1)^N \xi^{N-2} \{(\hat{L}^{(1)} + \xi \hat{L}^{(2)})(\lambda + \hat{L}_0)^{-1}\}^N \]

and, likewise, \( \tilde{u}^{(1)*} \) and \( \tilde{u}^{(2)*}(\xi) \) are given by formulas similar to (4.15) and (4.16) with \( \hat{L}_0 \) replaced by \( \hat{L}_0^* \), etc. The integral expression now follows from (4.9), (4.12)–(4.14). Furthermore, estimates for kernel functions would follow from regularity estimates on the resolvents \( (\lambda + \hat{L}_0)^{-1} \) and \( (\lambda + \hat{L}_0^*)^{-1} \). In fact, we deduce from Lemma 4.1, (4.10) and (4.11) that for \( |\lambda| = \frac{\eta_0}{2} \),

\[ |\hat{L}^{(j)}(\lambda + \hat{L}_0)^{-1} f|_{H^2 \times L^2} \leq C |f|_{H^2 \times L^2} \quad (j = 1, 2), \]

and hence,

\[ |\tilde{u}^{(1)}|_{H^2 \times H^2} \leq C \int \left| \hat{L}^{(1)}(\lambda + \hat{L}_0)^{-1} u^{(0)} \right|_{H^2 \times L^2} |d\lambda| \]

\[ \leq |u^{(0)}|_{H^2 \times L^2} \leq C. \]

Similarly,

\[ |\tilde{u}^{(2)}(\xi)|_{H^2 \times H^2} \leq C \int \left( 1 + \sum_{N=2}^{\infty} |N-2| \right) \left| u^{(0)} \right|_{H^2 \times L^2} |d\lambda| \]

\[ \leq |u^{(0)}|_{H^2 \times L^2} \leq C. \]
provided that $|\xi|$ is sufficiently small. Note also that $\partial_\xi (\lambda + \hat{L}_\xi)^{-1} u^{(0)}|_{\xi = 0} = - (\lambda + \hat{L}_0)^{-1} L^{(1)} (\lambda + \hat{L}_0)^{-1} u^{(0)}$ in $H^1(0,1) \times H^2(0,1)$.

To estimate higher order derivatives, we first show that

\begin{equation}
(4.17) \quad \left| (\lambda + \hat{L}_\xi)^{-1} f \right|_{H^{k+1} \times H^{k+2}} \leq C_k |f|_{H^{k+1} \times H^k}
\end{equation}

for $|\lambda| = \frac{m_0}{2}, \ k = 1, 2, \ldots$. The case $k = 0$ follows from Lemma 4.2. As for $k \geq 1$, we set $u = \tau(\phi, w^1, w^2) = (\lambda + \hat{L}_\xi)^{-1} f$. Then

\begin{equation}
(4.18) \quad \lambda \phi + i \xi v_1^1 \phi + i \gamma^2 \xi \rho_s w^1 + \gamma^2 \partial_{x_2}(\rho_s w^2) = f^0,
\end{equation}

\begin{equation}
(4.19) \quad \lambda w^1 + \frac{\nu}{\rho_s} (|\xi|^2 - \partial_{x_2}^2) w^1 - i \frac{\nu}{\rho_s} \xi (i \xi w^1 + \partial_{x_2} w^2) + i \xi \frac{P(\rho_s)}{\gamma^2 \rho_s} \phi + \frac{\nu}{\gamma^2 \rho_s} v_1^1 \phi + i \xi v_1^1 w^1 + \partial_{x_2} v_1^1 w^2 = f^1,
\end{equation}

\begin{equation}
(4.20) \quad \lambda w^2 + \frac{\nu}{\rho_s} (|\xi|^2 - \partial_{x_2}^2) w^2 - \frac{\nu}{\rho_s} \partial_{x_2} (i \xi w^1 + \partial_{x_2} w^2) + \partial_{x_2} \left( \frac{P(\rho_s)}{\gamma^2 \rho_s} \phi \right) + i \xi v_1^1 w^2 = f^2,
\end{equation}

where $f = \tau (f^0, f^1, f^2)$. By adding $\frac{\gamma^2 \rho_s^2}{\nu + \rho_s} \times (4.20)$ to $\partial_{x_2} (4.18)$, we have

\[ \lambda \partial_{x_2} \phi + \frac{\nu}{\rho_s} P(\rho_s) \partial_{x_2} \phi + i \xi v_1^1 \partial_{x_2} \phi = H, \]

where

\[ H = \partial_{x_2} f^0 - \left\{ i \gamma^2 \xi \partial_{x_2}(\rho_s w^1) + \gamma^2 \partial_{x_2} \rho_s \partial_{x_2} w^2 + \gamma^2 \partial_{x_2}(w^2 \partial_{x_2} \rho_s) \right\} + \frac{\gamma^2 \rho_s^2}{\nu + \rho_s} \left\{ f^2 - (\lambda w^2 + i \xi v_1^1 w^2 + \frac{\nu}{\rho_s} |\xi|^2 w^2 - \frac{\nu}{\rho_s} \partial_{x_2} i \xi w^1 + \phi \partial_{x_2} \left( \frac{P(\rho_s)}{\gamma^2 \rho_s} \right) \right\}, \]

It then follows that

\[ \lambda \partial_{x_2}^{k+1} \phi + \frac{\nu}{\rho_s} P(\rho_s) \partial_{x_2}^{k+1} \phi + i \xi v_1^1 \partial_{x_2}^{k+1} \phi = \partial_{x_2}^{k+1} H - i \xi \partial_{x_2}^{k+1} v_1^{(1)} \partial_{x_2} \phi - \frac{\nu}{\rho_s} \partial_{x_2}^{k+1} \left[ \partial_{x_2}^{k+1} \phi, \frac{P(\rho_s)}{\gamma^2 \rho_s} \right] \partial_{x_2} \phi. \]

We thus obtain

\begin{align*}
\text{Re} \lambda |\partial_{x_2}^{k+1} \phi|_2^2 + |\sqrt{\frac{\nu}{\rho_s} P(\rho_s)} \partial_{x_2}^{k+1} \phi|^2 & \leq C \{ |\partial_{x_2}^{k+1} f^0|_2 + |\partial_{x_2} \phi|_{H^{k-1}} + |\phi|_{H^k} + |w|_{H^{k+1}} + |f^2|_{H^k} + |\lambda| w^2_{H^k} \} |\partial_{x_2}^{k+1} \phi|_2.
\end{align*}

Therefore, if $|\rho_s - 1|_\infty \ll 1$, then

\begin{equation}
(4.21) \quad \text{Re} \lambda + \frac{\nu \gamma^2}{\nu + \rho_s} |\partial_{x_2}^{k+1} \phi|_2^2 \leq C \{ |\partial_{x_2}^{k+1} f^0|_2^2 + |\partial_{x_2} \phi|_{H^{k-1}}^2 + |\phi|_{H^k}^2 + |w|_{H^{k+1}}^2 + |f^2|_{H^k}^2 + |\lambda|^2 |w^2|_{H^k}^2 \}.
\end{equation}
By (4.19) and (4.20), we have
\[ \partial^2_{x_2} w^1 = \frac{\rho_0}{\nu + \tilde{\nu}} \left\{ \lambda w^1 + \frac{\nu}{\rho_s} |\xi|^2 w^1 - i \frac{\tilde{\rho}}{\rho_s} \xi (i \xi w^1 + \partial_{x_2} w^2) + i \xi \frac{\rho'(\rho_s)}{\gamma \rho_s} \phi 
+ \frac{\nu \partial^2_x v^1}{\gamma \rho_s} \phi + i \xi v_1^1 w^1 + \partial_{x_2} v_1^1 w^2 - f^1 \right\}, \]
\[ \partial^2_{x_2} w^1 = \frac{\rho_0}{\nu + \tilde{\nu}} \left\{ \lambda w^2 + \frac{\nu}{\rho_s} |\xi|^2 w^2 - i \frac{\tilde{\rho}}{\rho_s} \xi \partial_{x_2} w^1 + \partial_{x_2} \left( \frac{\rho'(\rho_s)}{\gamma \rho_s} \phi \right) 
+ i \xi v_1^1 w^2 - f^2 \right\}, \]
and hence,
\[ |\partial^k_{x_2} w|^2 \leq C \left\{ |\lambda|^2 |w|^2_{H^k} + |\phi|^2 |_{H^{k+1}} + |w|^2_{H^{k+1}} + |f^1|^2_{H^k} + |f^2|^2_{H^k} \right\} \]
for $|\lambda| = \frac{\rho_0}{\tilde{\nu}}$. This proves (4.17).

We next estimate integrands $z^{(1)} = -(\lambda + \tilde{L}_0)^{-1} \hat{L}^{(1)} (\lambda + \tilde{L}_0)^{-1} u^{(0)}$ and $z^{(2)}_\xi = R^{(2)} (\lambda, \xi) u^{(0)}$ in the right-hand side of (4.15) and (4.16), respectively. Set $z_\xi = (\lambda + \tilde{L}_0)^{-1} u^{(0)}$. Then, $z_\xi \in H^1 (0, 1) \times (H^2 (0, 1) \cap H^1_0 (0, 1))$, and
\[ (\lambda + \tilde{L}_0) z_\xi + \xi \hat{L}^{(1)} z_\xi + \xi^2 \hat{L}^{(2)} z_\xi = u^{(0)}. \]
By (4.17), we have, for $k = 0, 1, 2, \cdots$,
\[ |z_\xi|^2_{H^{k+1} \times H^{k+2}} \leq C_k |u^{(0)}|^2_{H^{k+1} \times H^k} \leq C_k. \]
We next set $z^{(0)} = \frac{1}{\lambda} u^{(0)}$ and $z^{(1)}_\xi = \frac{1}{\xi} (z_\xi - z^{(0)})$. Then, $z^{(1)}_\xi \to z^{(1)}$ in $H^1 (0, 1) \times H^2 (0, 1)$ as $\xi \to 0$, and
\[ |z^{(1)}|^2_{H^{k+1}} \leq C_k |u^{(0)}|^2_{H^{k+1}} \leq C_k. \]
Furthermore, since
\[ (\lambda + \tilde{L}_0) z^{(0)} = u^{(0)}, \]
we see from (4.23) and (4.26) that
\[ (\lambda + \tilde{L}_0) z^{(1)}_\xi + \hat{L}^{(1)} z_\xi + \xi \hat{L}^{(2)} z_\xi = 0. \]
Therefore, we have
\[ z^{(1)}(\xi) = -(\lambda + \hat{L}_0)^{-1}(\hat{L}^{(1)} z_\xi + \xi \hat{L}^{(2)} z_\xi). \]

It then follows from (4.17) and (4.24) that
\[ |z^{(1)}_{\xi}|_{H^{k+1} \times H^{k+2}} \leq C_k |\hat{L}^{(1)} z_\xi + \xi \hat{L}^{(2)} z_\xi|_{H^{k+1} \times H^k} \]
(4.28)
\[ \leq C_k |z_\xi|_{H^{k+1}} \]
\[ \leq C_k. \]

We next consider \( z^{(2)}_{\xi} = R^{(2)}(\lambda, \xi)u^{(0)}. \) We have \( z^{(2)}_{\xi} = \frac{1}{\xi}(z_\xi - z^{(0)} - \xi z^{(1)}) \in H^1(0, 1) \times (H^2(0, 1) \cap H^1_0(0, 1)) \) and, by (4.23), (4.26) and (4.27),
\[ (\lambda + \hat{L}_0)z^{(2)}_{\xi} + \hat{L}^{(1)}z^{(1)}_{\xi} + \hat{L}^{(2)}z_\xi = 0. \]

We thus obtain
\[ z^{(2)}_{\xi} = -(\lambda + \hat{L}_0)^{-1}(\hat{L}^{(1)} z^{(1)}_{\xi} + \hat{L}^{(2)} z_{\xi}). \]

It then follows from (4.17), (4.24) and (4.28) that
\[ |z^{(2)}_{\xi}|_{H^{k+1} \times H^{k+2}} \leq C_k |\hat{L}^{(1)} z^{(1)}_{\xi} + \hat{L}^{(2)} z_{\xi}|_{H^{k+1} \times H^k} \]
(4.29)
\[ \leq C_k \{ |z^{(1)}_{\xi}|_{H^{k+1}} + |z_{\xi}|_{H^k} \} \]
\[ \leq C_k. \]

We see from (4.24), (4.25) and (4.29) that
\[ |\tilde{u}^{(1)}|_{H^{k+1} \times H^{k+2}} + |\tilde{u}^{(2)}(\xi)|_{H^{k+1} \times H^{k+2}} \leq C_k \]
for \( k = 1, 2, \ldots. \)

Similarly, one can show
\[ |\tilde{u}^{(1)}(\xi)|_{H^{k+1} \times H^{k+2}} + |\tilde{u}^{(2)}(\xi)|_{H^{k+1} \times H^{k+2}} \leq C_k |u^{(0)}(\xi)|_{H^{k+1} \times H^k} \leq C_k \]
for \( k = 1, 2, \ldots. \)

The desired estimates for integrands \( \tilde{H}^{(1)}(x_2, y_2) \) and \( \tilde{H}^{(2)}(\xi, x_2, y_2) \) are now obtained by (4.9), (4.30), (4.31) and the Sobolev embedding. This completes the proof. \( \square \)
We now introduce a projection based on $\hat{\Pi}(\xi)$. For an interval $J \subset \mathbb{R}$, we denote the characteristic function of $J$ by $1_J$. We define $\hat{\chi}_1$ by

$$\hat{\chi}_1(\xi) = 1_{[0,r_1]}(|\xi|) = \begin{cases} 1 & (|\xi| < r_1) \\ 0 & (|\xi| \geq r_1). \end{cases}$$

Here and in what follows we take a number $r_1 > 0$ satisfying $r_1 \leq \min\{r_0, 1\}$.

We define $P_1$ and $P_1^*$ by

$$P_1 u = \mathcal{F}^{-1}(\hat{\chi}_1(\xi) \hat{\Pi}(\xi) \hat{u})$$

and

$$P_1^* u = \mathcal{F}^{-1}(\hat{\chi}_1(\xi) \hat{\Pi}^*(\xi) \hat{u}).$$

It then follows that $P_1^2 = P_1$, $P_1^* = P_1^*$ and

$$\langle \langle P_1 u_1, u_2 \rangle \rangle = \langle \langle u_1, P_1^* u_2 \rangle \rangle.$$

Based on Lemma 4.2 (iii), we expand $P_1$ as

$$P_1 = P_1^{(0)} + \partial_{x_1} P_1^{(1)} + \partial_{x_1}^2 P_1^{(2)},$$

where $P_1^{(j)} u = \mathcal{F}^{-1}(\hat{P}_1^{(j)}(\xi) \hat{u})$ $(j = 0, 1, 2)$,

$$\hat{P}_1^{(0)} = \hat{\chi}_1 \hat{\Pi}^{(0)}; \quad \hat{P}_1^{(1)} = -i \hat{\chi}_1 \hat{\Pi}^{(1)}; \quad \hat{\Pi}^{(1)} = -(\hat{\Pi}^{(0)} \hat{L}^{(1)} \hat{S} + \hat{S} \hat{L}^{(1)} \hat{\Pi}^{(0)}),$$

$$\hat{P}_1^{(2)} = -\hat{\chi}_1 \hat{\Pi}^{(2)}(\xi).$$

In what follows we will also denote $\hat{\Pi}^{(0)}$ by $\Pi^{(0)}$.

The following properties of $\Pi^{(0)}$ will be used in the subsequent analysis.

**Lemma 4.4.** (i) For any $k = 1, 2, \ldots$, $\partial_{x_1}^k \Pi^{(0)} = \Pi^{(0)} \partial_{x_1}^k$.

(ii) If $Q_0 u|_{x_2=0,1} = 0$, then $\Pi^{(0)}(\partial_{x_2} u) = 0$.

(iii) For any $l = 0, 1, \ldots$, there holds $\|\partial_{x_2}^l \Pi^{(0)} u\|_2 \leq C l \|\Pi^{(0)} u\|_2$.

(iv) If $\tilde{Q} u|_{x_2=0,1} = 0$, then $\|(I - \Pi^{(0)}) u\|_2 \leq C \|\partial_{x_2} (I - \Pi^{(0)}) u\|_2$.

**Proof.** The assertions (i)–(iii) easily follow from the definition of $\Pi^{(0)}$:

$$\Pi^{(0)} u = \langle u, u^{(0)*} \rangle u^{(0)} = \langle Q_0 u \rangle u^{(0)}.$$
As for (iv), since \( \tilde{Q}u^{(0)}|_{x_2=0,1} = \top (0, w^{(0)})|_{x_2=0,1} = 0 \), if \( \tilde{Q}u|_{x_2=0,1} = 0 \), then we have \( \tilde{Q}(I - II^{(0)})u|_{x_2=0,1} = 0 \). By the Poincaré inequality, \( \|\tilde{Q}(I - II^{(0)})u\|_2 \leq C\|\partial_{x_2}\tilde{Q}(I - II^{(0)})u\|_2 \). We also have \( \langle Q_0(I - II^{(0)})u \rangle = 0 \). Therefore, by the Poincaré inequality, \( \|Q_0(I - II^{(0)})u\|_2 \leq C\|\partial_{x_2}Q_0(I - II^{(0)})u\|_2 \). This completes the proof. \( \square \)

We next state boundedness properties of \( P_1 \) and \( P_1^* \).

Lemma 4.5. (i) For any \( k = 1, 2, \cdots, x, \partial_{x_1}^k P_1 = P_1 \partial_{x_1}^k \) and \( \partial_{x_1}^k P_{1}^{(j)} = P_{1}^{(j)} \partial_{x_1}^k \) \( (j = 0, 1, 2) \).
(ii) There holds \( \tilde{Q} P_1 u|_{x_2=0,1} = \tilde{Q} P_{1}^{*} u|_{x_2=0,1} = 0 \).
(iii) For any \( k, l = 0, 1, \cdots \), there hold
\[
\|\partial_{x_1}^k \partial_{x_2}^l P_1 u\|_2 \leq C_l \|u\|_2,
\|\partial_{x_1}^k \partial_{x_2}^l P_{1}^{*} u\|_2 \leq C_l \|u\|_2,
\|\partial_{x_1}^k \partial_{x_2}^l P_{1}^{(j)} u\|_2 \leq C_l \|u\|_2 \quad (j = 0, 1, 2).
\]

Proof. It is clear that (i) holds true; (ii) follows from Lemma 4.2 (iii) and Lemma 4.3 (ii); and (iii) follows from Lemma 4.3. This completes the proof. \( \square \)

We next consider \( P_1 \)-part of \( U(t) \). We define \( U_1(t) \) by
\[
U_1(t) = P_1 U(t) = U(t)P_1 = P_1 U(t)P_1.
\]
Then we have \( \widehat{U_1(t)}u = \hat{x}_1 e^{\hat{\lambda}_0(t)\hat{H}(\xi)} \hat{u} \). We also define \( U_1^{(j)}(t) \) \( (j = 0, 1, 2) \) by
\[
U_1^{(j)}(t) = \mathcal{F}^{-1}(\hat{x}_1(\xi)e^{\hat{\lambda}_0(t)\hat{P}_{1}^{(j)} \hat{u}}) \quad (j = 0, 1, 2).
\]
Then
\[
U_1(t) = U_1^{(0)}(t) + \partial_{x_1} U_1^{(1)}(t) + \partial_{x_1}^2 U_1^{(2)}(t).
\]
Furthermore, we have the following properties.

Lemma 4.6. (i) There hold \( \partial_{x_1}^k U_1(t) = U_1(t)\partial_{x_1}^k \); \( \partial_{x_1}^k U_1^{(j)}(t) = U_1^{(j)}(t)\partial_{x_1}^k \) \( (j = 0, 1, 2) \) for \( k = 0, 1, 2, \cdots \), and
\[
\|\partial_{x_1}^k \partial_{x_2}^l U_1(t)u\|_2 \leq C_l \|u\|_2, \quad \|\partial_{x_1}^k \partial_{x_2}^l U_1^{(j)}(t)u\|_2 \leq C_l \|u\|_2 \quad (j = 0, 1, 2)
\]
for \( t \geq 0 \) and \( k, l = 0, 1, 2, \cdots \).
(ii) There hold

\[ \Pi(0)U_1^{(0)}(t) = U_1^{(0)}(t)\Pi(0) = U_1^{(0)}(t), \]
\[ \Pi(0)U_1^{(1)}(t) = \Pi(0)U_1^{(1)}(t)(I - \Pi(0)), \]

and

\[ (I - \Pi(0))U_1^{(1)}(t) = (I - \Pi(0))U_1^{(1)}(t)\Pi(0). \]

**Proof.** Properties in (i) and the first relation in (ii) are easy to verify. The last two relations in (ii) follow from the fact that \( \hat{S}\Pi(0) = \Pi(0)\hat{S} = O \), which yields relations

\[ \Pi(0)\hat{\Pi}(1) = -\Pi(0)(\Pi(0)\hat{L}(1)\hat{S} + \hat{S}\hat{L}(1)\Pi(0)) \]
\[ = -(\Pi(0)\hat{L}(1)\hat{S} + \hat{S}\hat{L}(1)\Pi(0))\Pi(0) = \hat{\Pi}(1)(I - \Pi(0)), \]

and likewise, \( (I - \Pi(0))\hat{\Pi}(1) = \hat{\Pi}(1)\Pi(0) \). This completes the proof. \( \square \)

We next consider the decay estimates on \( U_1(t) \) and \( U_1^{(j)}(t) \) \( (j = 0, 1, 2) \).

**Lemma 4.7.** There hold the following estimates uniformly for \( t \geq 0 \).

(i) \[ \| \partial_{x_1}^k \partial_{x_2}^l U_1(t)u \|_2 \leq C_l(1 + t)^{-\frac{1}{4} - \frac{k}{2}} \| u \|_1 \quad (k, l = 0, 1, 2, \cdots), \]

(ii) \[ \| \partial_{x_1}^k \partial_{x_2}^l \Pi(0)U_1(t)u \|_2 \leq C_l(1 + t)^{-\frac{1}{4} - \frac{k}{2}} \| u \|_1 \quad (k, l = 0, 1, 2, \cdots), \]

(iii) \[ \| \partial_{x_1}^k \partial_{x_2}^l (I - \Pi(0))U_1(t)u \|_2 \leq C_l(1 + t)^{-\frac{1}{4} - \frac{k + 1}{2}} \| u \|_1 \quad (k, l = 0, 1, 2, \cdots). \]

(iv) \[ \| \partial_{x_1}^k \partial_{x_2}^l U_1^{(j)}(t)u \|_2 \leq C_l(1 + t)^{-\frac{1}{4} - \frac{1}{2} \frac{k}{2}} \| u \|_1 \quad (j = 0, 1, 2; \ k, l = 0, 1, 2, \cdots). \]

**Proof.** Since \( \hat{\lambda}_0(\xi) = -ia_0\xi - \kappa_0\xi^2 + O(|\xi|^3) \), we see from Lemma 4.3 that

\[ \| \partial_{x_1}^k \partial_{x_2}^l U_1(t)u \|_2 \leq C_l \left( \int_{\mathbb{R}} \hat{\chi}_1(\xi)^2|\xi|^{2k} e^{-\kappa_0|\xi|^2 t} |\partial_{x_2}^l \hat{\Pi}(\xi)\hat{u}|^2 d\xi \right)^{\frac{1}{2}} \]
\[ \leq C_l (\int_{\mathbb{R}} \hat{\chi}_1(\xi)^2|\xi|^{2k} e^{-\kappa_0|\xi|^2 t} |\hat{u}|^2 d\xi)^{\frac{1}{2}} \]
\[ \leq C_l (\int_{\mathbb{R}} \hat{\chi}_1(\xi)^2|\xi|^{2k} e^{-\kappa_0|\xi|^2 t} d\xi)^{\frac{1}{2}} \| u \|_1 \]
\[ \leq C_l \left\{ \begin{array}{l} \| u \|_1, \\
\quad t^{-\frac{1}{4} - \frac{k}{2}} \| u \|_1. \end{array} \right. \]
We thus obtain (i). Estimate (ii) and (iv) can be obtained similarly. As for (iii), one can prove the desired estimate by noting

\[(I - \Pi^{(0)}) \hat{\Pi}(\xi) = (I - \Pi^{(0)})(\xi \hat{\Pi}^{(1)} + \xi^2 \hat{\Pi}^{(2)}(\xi)).\]

This completes the proof. \qed

We finally consider the asymptotic behavior of \(U_1(t)\).

Let \(U_0(t)\) and \(\mathcal{U}(t)\) be defined by

\[U_0(t)u = \mathcal{F}^{-1}(\hat{\chi}_1 e^{\hat{\lambda}_0 t} \langle Q_0 \hat{u} \rangle)\]

and

\[\mathcal{U}(t)\sigma = \mathcal{F}^{-1}(e^{-(i\omega_0 \xi + \kappa_0 \xi^2) t} \hat{\sigma}).\]

It then follows that \(\Pi^{(0)}\)-part of \(U_1(t)\) is given by \(U_0(t)\), i.e., we have

\[(4.32) \quad \Pi^{(0)} U_1(t) u = (U_0(t) P_1 u) u^{(0)}.\]

One can easily see the following properties.

**Lemma 4.8.** There hold the following relations.

(i) \[U_0(t) = U_0(t) \Pi^{(0)} = U_0(t) P_1^{(0)},\]

(ii) \[\partial_{x_1}^k U_0(t) = U_0(t) \partial_{x_1}^k, \quad \partial_{x_1}^k \mathcal{U}(t) = \mathcal{U}(t) \partial_{x_1}^k \quad (k = 1, 2, \cdots),\]

(iii) \[\| \partial_{x_1}^k \mathcal{U}(t) \sigma \|_2 \leq C_k t^{-\frac{1}{4}} \xi^\frac{k}{2} \| \sigma \|_1 \quad (k = 0, 1, 2, \cdots),\]

(iv) \[\| \partial_{x_1}^k U_0(t) u \|_2 \leq C(1 + t)^{-\frac{1}{2}} \xi^\frac{k}{2} \| u \|_1 \quad (k = 0, 1, 2, \cdots),\]

(v) \[\| \partial_{x_1}^k U_0(t) P_1 u \|_2 \leq C(1 + t)^{-\frac{1}{4}} \xi^\frac{k}{2} \| u \|_1 \quad (k = 0, 1, 2, \cdots),\]

\[\| \partial_{x_1}^k U_0(t) P_1^{(j)} u \|_2 \leq C(1 + t)^{-\frac{1}{4}} \xi^\frac{k}{2} \| u \|_1 \quad (j = 1, 2; k = 0, 1, 2, \cdots).\]

We finally show that the leading part of \(U_0(t)\) is given by \(\mathcal{U}(t)\), which, together with Lemma 4.7 (iii) and (4.32), implies that the asymptotic behavior of \(U_1(t)\) is described by \(\mathcal{U}(t)\).

**Lemma 4.9.** Let \(u = \tau(\phi, w) \in (H^1 \times \tilde{H}^1) \cap L^1\) and let \(\sigma = \langle Q_0 u \rangle\). Then

\[\| \partial_{x_1}^k (U_0(t) u - \mathcal{U}(t) \sigma) \|_2 \leq C_k t^{-\frac{1}{4}} \xi^\frac{k}{2} \| u \|_1\]
and

\[ \| \partial_{x_1}^k (U_0(t)P_1 u - \mathcal{U}(t) \sigma) \|_2 \leq C_k t^{-\frac{k+1}{2}} \| u \|_1. \]

**Proof.** We see that

\[ \mathcal{F}(U_0(t)u) = e^{-\langle i a_0 \xi + \kappa_0 \xi^2 \rangle t} \hat{\sigma} + (\hat{\chi}_1 - 1) e^{-\langle i a_0 \xi + \kappa_0 \xi^2 \rangle t} \hat{\sigma} + \hat{\chi}_1 (e^{\hat{\lambda}_0(\xi)t} - e^{-\langle i a_0 \xi + \kappa_0 \xi^2 \rangle t}) \hat{\sigma}. \]

Since \( \hat{\lambda}_0(\xi) + (i a_0 \xi + \kappa_0 \xi^2) = O(\xi^3) \), we have

\[ |e^{\hat{\lambda}_0(\xi)t} - e^{-\langle i a_0 \xi + \kappa_0 \xi^2 \rangle t}| = |e^{-\langle i a_0 \xi + \kappa_0 \xi^2 \rangle t} (e^{\hat{\lambda}_0(\xi)+(i a_0 \xi + \kappa_0 \xi^2))t - 1)| \leq C |\xi| e^{-\frac{\kappa_0}{2} \xi^2 t}. \]

It then follows that

\[ \| \partial_{x_1}^k (U_0(t)u - \mathcal{U}(t) \sigma) \|_2 \leq C_k t^{-\frac{k+1}{2}} \| u \|_1. \]

This shows the first inequality. The second inequality now follows from the first one and Lemma 4.8 (i) and (v) since

\[ U_0(t)P_1 = U_0(t)P_1^{(0)} + \partial_{x_1} U_0(t)P_1^{(1)} + \partial_{x_1}^2 U_0(t)P_1^{(2)}. \]

This completes the proof. \( \square \)

5. Decomposition of the solution

In sections 5–7 we prove the a priori estimate in Proposition 3.3. To do so, we will decompose the solution \( u(t) \) of (4.1) into several parts based on the spectral properties of \( L \).

We first introduce some notation and projection operators. Let \( \hat{\chi}_2 \) and \( \hat{\chi}_3 \) be defined by

\[ \hat{\chi}_2(\xi) = 1_{[\tau_1, 1]}(|\xi|), \quad \hat{\chi}_3(\xi) = 1_{[1, \infty)}(|\xi|). \]

We define \( P_{\infty,j} \) (\( j = 1, 2, 3 \)) by

\[ P_{\infty,1} u = \mathcal{F}^{-1}(\hat{P}_{\infty,1} \hat{u}), \quad \hat{P}_{\infty,1} \hat{u} = \hat{\chi}_1(I - \hat{H}(\xi)) \hat{u}, \]

\[ P_{\infty,j} u = \mathcal{F}^{-1}(\hat{P}_{\infty,j} \hat{u}), \quad \hat{P}_{\infty,j} \hat{u} = \hat{\chi}_j \hat{u} \quad (j = 2, 3). \]

Set

\[ \hat{P}_{\infty} = I - P_1, \quad P_{\infty}^{(0)} = P_{\infty,1} + P_{\infty,2}. \]
We then have
\[ I = P_1 + \tilde{P}_\infty, \quad \tilde{P}_\infty = P_\infty^{(0)} + P_{\infty,3}. \]

We define \( \langle f \rangle_j \) \((j = 1, 2, \infty)\) by
\[ \langle f \rangle_j = \mathcal{F}^{-1}(\hat{\chi}_j(\hat{f})) , \quad j = 1, 2, \]
\[ \langle f \rangle_\infty = \langle f \rangle_1 + \langle f \rangle_2 = \mathcal{F}^{-1}((\hat{\chi}_1 + \hat{\chi}_2)(\hat{f})). \]

We decompose the solution \( u(t) \) into
\[ u(t) = P_1 u(t) + \tilde{P}_\infty u(t), \]
\[ P_1 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t), \]
\[ \tilde{P}_\infty u(t) = (\sigma_\infty u^{(0)})(t) + u_\infty(t), \]
where
\[ (\sigma_1 u^{(0)})(t) = \Pi^{(0)} P_1 u(t), \quad \sigma_1 = \langle Q_0 P_1 u(t) \rangle = \langle Q_0 P_1 u(t) \rangle_1, \]
\[ u_1(t) = (I - \Pi^{(0)}) P_1 u(t), \]
\[ (\sigma_\infty u^{(0)})(t) = \Pi^{(0)} P_\infty^{(0)} u(t), \quad \sigma_\infty = \langle Q_0 P_\infty^{(0)} u(t) \rangle = \langle Q_0 P_\infty^{(0)} u(t) \rangle_\infty, \]
\[ u_\infty(t) = P_\infty u, \]

Here and in what follows, \( P_\infty \) denotes the operator defined by
\[ P_\infty = (I - \Pi^{(0)}) P_\infty^{(0)} + P_{\infty,3}. \]

We now derive the equations for \( \sigma_1, u_1, \sigma_\infty \) and \( u_\infty \). We define \( \tilde{\mathcal{M}} \) by
\[ \tilde{\mathcal{M}} = L - \tilde{L}_0 = \tilde{A} + \tilde{B} \]
with
\[ \tilde{A} = A - \hat{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu + \bar{\nu}}{\rho_s} \partial_{x_1}^2 & -\frac{\bar{\nu}}{\rho_s} \partial_{x_1} \partial_{x_2} \\ 0 & -\frac{\bar{\nu}}{\rho_s} \partial_{x_1} \partial_{x_2} & -\frac{\nu}{\rho_s} \partial_{x_1}^2 \end{pmatrix}, \]
\[ \tilde{B} = B - \hat{B}_0 = \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \rho_s \partial_{x_1} & 0 \\ \partial_{x_1} \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) & v_s^1 \partial_{x_1} & 0 \\ 0 & 0 & v_s^1 \partial_{x_1} \end{pmatrix}. \]
Proposition 5.1. Let $T > 0$ and $u(t)$ be a solution of (4.1) in $Z^m(T)$. Then there hold

$$\sigma_k \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0, T]; H^1(\mathbb{R})) \quad (k = 1, \infty; \forall l = 0, 1, 2, \ldots),$$

$$u_1 \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0, T]; \bar{H}^1(\Omega)) \quad (\forall l = 0, 1, 2, \ldots),$$

$$u_\infty \in Z^m(T).$$

Moreover, $\sigma_1$, $u_1$, $\sigma_\infty$ and $u_\infty$ satisfy

(5.1) $\sigma_1(t) = U_0(t)P_1u_0 + \int_0^t U_0(t - \tau)\Pi^{(0)}P_1F(\tau) d\tau,$

(5.2) $u_1(t) = (I - \Pi^{(0)})U_1(t)u_0 + \int_0^t (I - \Pi^{(0)})U_1(t - \tau)F(\tau) d\tau,$

(5.3) $\partial_t \sigma_\infty + \langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty) \rangle_\infty = \langle Q_0 P_\infty^0 F \rangle_\infty,$

(5.4) $\partial_t u_\infty + Lu_\infty + \tilde{M}(\sigma_\infty u^{(0)}) - \langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty) \rangle_\infty u^{(0)} = F_\infty,$

(5.5) $w_\infty |_{\partial\Omega} = 0,$

(5.6) $\sigma_\infty|_{t=0} = \sigma_{\infty, 0}, \quad u_\infty|_{t=0} = u_{\infty, 0}.$

Here $\sigma_{\infty, 0} = \langle Q_0 P_\infty^{(0)} u_0 \rangle_\infty$, $u_{\infty, 0} = (I - \Pi^{(0)})P_\infty^{(0)}u_0 + P_{\infty, 3}u_0$ and $F_\infty = P_\infty F = (I - \Pi^{(0)})P_\infty^{(0)}F + P_{\infty, 3}F$.

**Proof.** Since $u \in Z^m(T)$, the first assertion for $\sigma_k$ ($k = 1, \infty$) and $u_1$ follows from boundedness properties of $\Pi^{(0)}$ and $P_1$ given in Lemma 4.4 and Lemma 4.5. As for $u_\infty$, it is easy to see that $P_1Z^m(T) \subset Z^m(T)$; and so $P_{\infty, 1}u(t) \in Z^m(T)$. Obviously, $P_{\infty, 2}u(t)$ and $P_{\infty, 3}u(t)$ are in $Z^m(T)$. Since $u^{(0)} = \tilde{\tau}(\phi^{(0)}, w^{(0)})$, $w^{(0)}|_{x_2=0, 1} = 0$, we see that $\Pi^{(0)}Z^m(T) \subset Z^m(T)$, and hence, $\Pi^{(0)}P_\infty^{(0)}u(t) = \Pi^{(0)}(P_{\infty, 1} + P_{\infty, 2})u(t) \in Z^m(T)$. As a consequence, $u_\infty(t) = (I - \Pi^{(0)})P_\infty^{(0)}u(t) + P_{\infty, 3}u(t) \in Z^m(T)$.

We next derive (5.1) and (5.2). Applying $P_1$ and $\tilde{P}_\infty$ to (4.1), we have

(5.7) $\partial_t(P_1u) + LP_1u = P_1F,$

(5.8) $\partial_t(P_\infty u) + LP_\infty u = \tilde{P}_\infty F.$

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It follows from (5.7) that

\[(5.9) \quad P_1 u(t) = U_1(t) u_0 + \int_0^t U_1(t - \tau) F(\tau) d\tau.\]

Applying \(\Pi^{(0)}\) and \(I - \Pi^{(0)}\) to (5.9), we obtain (5.1) and (5.2).

As for (5.3) and (5.4), we apply \(P_\infty^{(0)}\) and \(P_{\infty,3}\) to (5.8) to obtain

\[(5.10) \quad \partial_t (P_\infty^{(0)} u) + L P_\infty^{(0)} u = P_\infty^{(0)} F,\]

\[(5.11) \quad \partial_t (P_{\infty,3} u) + L P_{\infty,3} u = P_{\infty,3} F.\]

Since \(\Pi^{(0)} L = \Pi^{(0)} \tilde{M}\) and \(L \Pi^{(0)} = \tilde{M} \Pi^{(0)}\), applying \(\Pi^{(0)}\) and \(I - \Pi^{(0)}\) to (5.10), we have

\[(5.12) \quad \partial_t (\sigma_\infty u^{(0)}) + \Pi^{(0)} \tilde{M} (P_\infty^{(0)} u) = \Pi^{(0)} P_\infty^{(0)} F,\]

\[(5.13) \quad \partial_t (I - \Pi^{(0)}) P_\infty^{(0)} u + L (I - \Pi^{(0)}) P_\infty^{(0)} u + \tilde{M} (\sigma_\infty u^{(0)}) - \Pi^{(0)} \tilde{M} (P_\infty^{(0)} u)
\]  
\[= (I - \Pi^{(0)}) P_\infty^{(0)} F.\]

Since

\[\Pi^{(0)} \tilde{M} (P_\infty^{(0)} u) = \langle Q_0 \tilde{B} (\sigma_\infty u^{(0)} + (I - \Pi^{(0)}) P_\infty^{(0)} u) \rangle_\infty u^{(0)} \]
\[= \langle Q_0 \tilde{B} (\sigma_\infty u^{(0)} + u_\infty) \rangle_\infty u^{(0)},\]

(5.3) follows from (5.12). Here we used the fact that

\[\langle Q_0 \tilde{B} P_{\infty,3} u \rangle_\infty = \mathcal{F}^{-1} (\langle \hat{\chi}_1 + \hat{\chi}_2 \rangle \langle Q_0 \tilde{B}_\xi \hat{P}_{\infty,3} \hat{u} \rangle) = 0,\]

where \(\tilde{B}_\xi = \tilde{B}_\xi - \tilde{B}_0\). (5.4) now follows by adding (5.11) and (5.13). This completes the proof.

We next state some properties of \(\sigma_\infty\) and \(u_\infty\) parts.

**Lemma 5.2.** There hold the following inequalities.

(i) \[\| \partial_{x_1}^k \langle Q_0 P_\infty^{(0)} u \rangle_\infty \|_2 \leq \|\langle Q_0 P_\infty^{(0)} u \rangle_\infty \|_2 \quad (\forall k = 0, 1, 2, \cdots),\]

(ii) \[\| P_\infty u \|_2 \leq C \| \partial_x P_\infty u \|_2 \quad \text{if} \quad \tilde{Q} u |_{x_2=0,1} = 0.\]
Proof. Inequality (i) is obvious since \( \text{supp} (\hat{\chi}_1 + \hat{\chi}_2) \subset \{ |\xi| \leq 1 \} \). As for (ii), since \( \text{supp} \hat{\chi}_3 \subset \{ |\xi| \geq 1 \} \), we see that

\[
\| P_{\infty, 3} u \|_2 \leq \| \partial_{x_1} P_{\infty, 3} u \|_2.
\]

Since \( \tilde{Q} u \big|_{x_2=0_1} = 0 \), we have \( \tilde{Q} P^{(0)}_{\infty} u \big|_{x_2=0_1} = 0 \), and hence, \( \tilde{Q} (I - \Pi^{(0)}) P^{(0)}_{\infty} u \big|_{x_2=0_1} = 0 \). By the Poincaré inequality we obtain

\[
\| \tilde{Q} (I - \Pi^{(0)}) P^{(0)}_{\infty} u \| \leq \| \partial_{x_2} \tilde{Q} (I - \Pi^{(0)}) P^{(0)}_{\infty} u \|_2.
\]

Furthermore, since \( \langle Q_0 (I - \Pi^{(0)}) P^{(0)}_{\infty} u \rangle = 0 \), we see from the Poincaré inequality that

\[
\| Q_0 (I - \Pi^{(0)}) P^{(0)}_{\infty} u \| \leq C \| \partial_{x_2} Q_0 (I - \Pi^{(0)}) P^{(0)}_{\infty} u \|_2.
\]

It then follows that

\[
\| P_{\infty} u \|_2 \leq C \{ \| \partial_x (I - \Pi^{(0)}) P^{(0)}_{\infty} u \|_2 + \| \partial_x P_{\infty, 3} u \|_2 \} \leq C \| \partial_x P_{\infty} u \|_2.
\]

Here we used \( \langle \partial_x (I - \Pi^{(0)}) P^{(0)}_{\infty} u \rangle \leq 0 \), \( \partial_x P_{\infty, 3} u = 0 \), which follows from the fact \( \hat{\chi}_1 \hat{\chi}_3 = \hat{\chi}_2 \hat{\chi}_3 = 0 \) and the Plancherel theorem. This completes the proof. \( \square \)

To prove the a priori estimate in Proposition 3.3, we will estimate the following quantities.

Let \( u(t) \) be a solution of (4.1) in \( \mathbb{Z}^m (T) \) and let \( u(t) \) be decomposed as above:

\[
u(t) = (\sigma_1 u^{(0)})(t) + u_1(t) + (\sigma_\infty u^{(0)})(t) + u_\infty(t).
\]

We define \( M(t) \geq 0 \) by

\[
M(t)^2 = M_1(t)^2 + \sup_{0 \leq \tau \leq t} (1 + \tau) \frac{3}{2} E_\infty(\tau) \quad (t \in [0, T]).
\]

Here \( M_1(t) \) and \( E_\infty(t) \) are defined by

\[
M_1(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{2}} \| \sigma_1(\tau) \|_2
\]

\[
+ \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}} \left\{ \| \partial_{x_1} \sigma_1(\tau) \|_2 + \| u_1(\tau) \|_2 + \sum_{j=1}^{[m]} (\| \partial_{x_j} \sigma_1(\tau) \|_2 + \| \partial_{x_j} u_1(\tau) \|_2) \right\}.
\]

\[
E_\infty(t) = \left[ u_\infty(t) \right]_m^2 + \left[ \sigma_\infty(t) \right]_m^2.
\]

We also introduce the quantity \( D_\infty(t) \) for \( u_\infty(t) = \top(\phi_\infty(t), w_\infty(t)) \):

\[
D_\infty(t) = \left\| D\phi_\infty(t) \right\|_{m-1}^2 + \left\| Dw_\infty(t) \right\|_{m}^2 + \left\| D\sigma_\infty(t) \right\|_{m-1}^2.
\]
We will show the following estimates for $M_1(t)$ and $E_\infty(t)$.

**Proposition 5.3.** There are positive constants $\nu_0$, $\gamma_0$ and $\omega_0$ such that if $\nu \geq \nu_0$, $\gamma^2/(\nu + \bar{\nu}) \geq \gamma_0^2$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$, then the following assertion holds true.

There exists a number $\varepsilon_2 > 0$ such that if a solution $u(t)$ of (4.1) in $Z^m(T)$ satisfies $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$ for $t \in [0, T]$, then the following estimates hold uniformly for $t \in [0, T]$ with $C > 0$ independent of $T$.

(5.14) \[ M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\}, \]

(5.15) \[ E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau) \, d\tau \]

\[ \leq C\{e^{-at} E_\infty(0) + (1 + t)^{-\frac{3}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)} \tilde{R}(\tau) \, d\tau\}. \]

Here $a = a(\nu, \bar{\nu}, \gamma, \|\rho_s - 1\|_{C^{m+1}[0,1]})$ is a positive constant; and $\tilde{R}(t)$ is a quantity that satisfies

(5.16) \[ \tilde{R}(t) \leq C\{(1 + t)^{-\frac{3}{2}} M(t)^3 + (1 + t)^{-\frac{1}{2}} M(t) D_\infty(t)\} \]

whenever $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$.

The proof of Proposition 5.3 will be given in sections 6–8. We will prove (5.14), (5.15) and (5.16) in sections 6, 7 and 8, respectively.

Assuming Proposition 5.3 to hold true, we can show the following estimate.

**Proposition 5.4.** If $\nu \geq \nu_0$, $\gamma^2/(\nu + \bar{\nu}) \geq \gamma_0^2$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$, then the following assertion holds true.

There exists a number $\varepsilon_3 > 0$ such that if a solution $u(t)$ of (4.1) in $Z^m(T)$ satisfies $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_3$, then there holds the estimate

\[ M(t) \leq C_2 \|u_0\|_{H^m \cap L^1} \]

for some constant $C_2 > 0$ independent of $T$.

The a priori estimate in Proposition 3.3 immediately follows from Proposition 5.4. Furthermore, Proposition 5.4 also provides decay estimates

\[ \|u(t)\|_m \leq C(1 + t)^{-\frac{1}{2}} \|u_0\|_{H^m \cap L^1} \]

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and
\[ \| \partial_{x_1}^k u(t) \|_2 \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{2} t} \| u_0 \|_{H^m \cap L^1} \]
for \( k = 0, 1 \).

**Proof of Proposition 5.4.** If \( \sup_{0 \leq \tau \leq t} \| u(\tau) \|_m \leq \epsilon_2 \) and \( M(t) \leq 1 \), then we see from (5.15) and (5.16) that
\[
E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau \\
\leq C \{ e^{-a t} E_\infty(0) + (1 + t)^{-\frac{3}{2}} M(t)^4 + M(t)^3 \int_0^t e^{-a(t-\tau)} (1 + \tau)^{-\frac{3}{2}} d\tau \\
+ M(t) \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau \} \\
\leq C \{ e^{-a t} E_\infty(0) + (1 + t)^{-\frac{3}{2}} M(t)^3 + M(t) \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau \},
\]
and hence,
\[
(5.17) \quad (1 + t)^{\frac{3}{2}} E_\infty(t) + \mathcal{D}(t) \leq C \{ \| u_0 \|^2_{H^m \cap L^1} + M(t)^3 + M(t) \mathcal{D}(t) \},
\]
where
\[
\mathcal{D}(t) = (1 + t)^{\frac{3}{2}} \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau.
\]
It follows from (5.14) and (5.17) that
\[
(5.18) \quad M(t)^2 + \mathcal{D}(t) \leq C_3 \{ \| u_0 \|^2_{H^m \cap L^1} + M(t)^3 + M(t) \mathcal{D}(t) \}
\]
whenever \( \sup_{0 \leq \tau \leq t} \| u(\tau) \|_m \leq \epsilon_2 \) and \( M(t) \leq 1 \).

We now show that there exists \( \epsilon_3 > 0 \) such that if \( \| u_0 \|_{H^m \cap L^1} < \epsilon_3 \), then \( M(t) < 2C_4 \| u_0 \|_{H^m \cap L^1} \) for all \( t \in [0, T] \) with some \( C_4 > 0 \) independent of \( T \).

We first observe that there is a constant \( C_5 > 0 \) such that \( \sup_{0 \leq \tau \leq t} \| u(\tau) \|_m \leq C_5 M(t) \).

Since \( M(0) \leq C_6 \| u_0 \|_{H^m} \) for some \( C_6 > 0 \) and since \( M(t) \) is continuous in \( t \), we see that there exists \( t_0 > 0 \) such that \( M(t) < 2C_6 \| u_0 \|_{H^m} \) for all \( t \in [0, t_0] \).

We set \( C_4 = \max\{ \sqrt{\frac{C_3}{2}}, C_6 \} \) and \( \epsilon_3 = \min\{ \frac{1}{2C_4}, \frac{1}{4C_3C_4}, \frac{\epsilon_2}{2C_4C_5} \} \). Then
\[
M(t) < 2C_4 \| u_0 \|_{H^m \cap L^1} \quad \text{for} \quad t \in [0, t_0].
\]
Assume that there exists \( t_1 \in (t_0, T) \) such that \( M(t) < 2C_4 \| u_0 \|_{H^m \cap L^1} \) for \( t \in [0, t_1] \) and \( M(t_1) = 2C_4 \| u_0 \|_{H^m \cap L^1} \).

Since \( M(t) \leq 2C_4 \| u_0 \|_{H^m \cap L^1} \leq 1 \) for \( t \in [0, t_1] \), we have \( \sup_{0 \leq \tau \leq t} \| u(\tau) \|_m \leq
\( C_3 M(t) \leq \varepsilon_2 \) for \( t \in [0, t_1] \). We thus see from (5.18) that if \( \| u_0 \|_{H^m \cap L^1} < \varepsilon_3 \), then
\[
M(t)^2 + 2D(t) \leq C_3 \{ \| u_0 \|^2_{H^m \cap L^1} + 2C_4 \| u_0 \|_{H^m \cap L^1} (M(t)^2 + D(t)) \}
\]
and hence,
\[
M(t)^2 + D(t) \leq 2C_3 \| u_0 \|^2_{H^m \cap L^1} \leq 4C_4^2 \| u_0 \|^2_{H^m \cap L^1}
\]
for \( t \in [0, t_1] \). But this contradicts to \( M(t_1) = 2C_4 \| u_0 \|_{H^m \cap L^1} \). We thus conclude that \( M(t) < 2C_4 \| u_0 \|_{H^m \cap L^1} \) for all \( t \in [0, T] \). This completes the proof.

6. Estimates on \( P_1 u(t) \)

In this section we estimate the \( P_1 \)-part of \( u(t) \):
\[
P_1 u(t) = (\sigma_1 u(0))(t) + u_1(t),
\]
where \( \sigma_1(t) = \langle Q_0 P_1 u(t) \rangle \) and \( u_1(t) = (I - P(0))u(t) \).

We will prove (5.14), i.e., the following estimate.

**Proposition 6.1.** There exists a number \( \varepsilon_4 > 0 \) such that if a solution \( u(t) \) of (4.1) in \( Z^m(T) \) satisfies \( \sup_{0 \leq \tau \leq t} \| u(\tau) \|_m \leq \varepsilon_4 \) and \( M(t) \leq 1 \) for all \( t \in [0, T] \), then the estimate
\[
M_1(t) \leq C \{ \| u_0 \|_1 + M(t)^2 \}
\]
holds uniformly for \( t \in [0, T] \) with \( C > 0 \) independent of \( T \).

To prove Proposition 6.1, we employ the results in section 4, together with Lemma 6.2 and Lemma 6.3 below.

Before going further we make one observation. In view of the spectral properties of the linearized operator, the most slowly decaying part of \( u(t) \) is expected to be \( \sigma_1(t) \); and hence, the most slowly decaying part of the nonlinearity \( F(t) \) would be given by the terms involving only \( \sigma_1(t)^2 \) which is, in fact, only one term \( \sigma_1^2 F_1 \) with \( F_1 = F_1(x_2) \):
\[
F_1 = -\sum_{0}^{\gamma_4} \frac{1}{\rho_s(x_2)} \partial_{x_2} (P''(\rho_s(x_2)) \{ \phi^{(0)}(x_2) \}^2).
\]
We thus write \( F \) as
\[
F = \sigma_1^2 F_1 + F_2,
\]
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where \( F_2 = F - \sigma^2 F_1 \) contains terms involving \( u_1, \sigma_\infty, u_\infty \), their derivatives, and terms of order \( O(\sigma_1 \partial_x \sigma_1) \) and \( O(\sigma_1^2) \) but not just \( O(\sigma_1^3) \). In particular, \( \Pi^{(0)} F = \Pi^{(0)} F_2 \).

We make two lemmas on the nonlinearities.

**Lemma 6.2.** There hold the following relations.

(i) \[ \Pi^{(0)} F = -\partial_{x_1} \langle \phi w_1 \rangle u^{(0)}, \]

(ii) \[ \Pi^{(0)} P_1 F = -P_1^{(0)} (\partial_{x_1} \langle \phi w_1 \rangle u^{(0)}) + \Pi^{(0)} \partial_{x_1} P_1^{(1)} (I - \Pi^{(0)}) F + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} F, \]

(iii) \( (I - \Pi^{(0)}) P_1 F = (I - \Pi^{(0)}) \partial_{x_1} P_1^{(1)} \Pi^{(0)} F + (I - \Pi^{(0)}) \partial_{x_1}^2 P_1^{(2)} F. \)

**Proof.** Since \( w|_{x_2=0,1} = 0 \), by integration by parts, we have

\[ \Pi^{(0)} F = \langle Q_0 F \rangle u^{(0)} = -\partial_{x_1} \langle \phi w_1 \rangle u^{(0)}. \]

This shows (i).

We write \( \Pi^{(0)} P_1 F \) as

\[ \Pi^{(0)} P_1 F = \Pi^{(0)} P_1^{(0)} F + \Pi^{(0)} \partial_{x_1} P_1^{(1)} F + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} F \]

\[ = P_1^{(0)} \Pi^{(0)} F + \Pi^{(0)} \partial_{x_1} P_1^{(1)} F + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} F \]

\[ = -P_1^{(0)} (\partial_{x_1} \langle \phi w_1 \rangle u^{(0)}) + \Pi^{(0)} \partial_{x_1} P_1^{(1)} F + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} F. \]

By Lemma 4.2 (iii), \( \hat{\Pi}^{(1)} \) is given by \( - (\Pi^{(0)} \hat{L}^{(1)} \hat{S} + \hat{S} \hat{L}^{(1)} \Pi^{(0)}) \) with \( \hat{S} = (I - \Pi^{(0)}) \hat{L}_0 (I - \Pi^{(0)}) \) and \( \hat{S} = \hat{S} (I - \Pi^{(0)}) \). Since \( (\Pi^{(0)})^2 = \Pi^{(0)}, \Pi^{(0)} \hat{S} = 0 \) and \( \hat{S} \hat{S} = \hat{S} (I - \Pi^{(0)}) \) we see that

\[ \Pi^{(0)} \hat{\Pi}^{(1)} = -\Pi^{(0)} \hat{L}^{(1)} \hat{S} (I - \Pi^{(0)}) = \Pi^{(0)} \hat{\Pi}^{(1)} (I - \Pi^{(0)}), \]

\( (I - \Pi^{(0)}) \hat{\Pi}^{(1)} = -(I - \Pi^{(0)}) \hat{S} \hat{L}^{(1)} \Pi^{(0)} = (I - \Pi^{(0)}) \hat{\Pi}^{(1)} \Pi^{(0)}. \)

It follows that

\[ \Pi^{(0)} \partial_{x_1} P_1^{(1)} F = \mathcal{F}^{-1} (\hat{\chi}_1 \xi \Pi^{(0)} \hat{\Pi}^{(1)} \hat{F} ) \]

\[ = \mathcal{F}^{-1} (\hat{\chi}_1 \xi \Pi^{(0)} \hat{\Pi}^{(1)} (I - \Pi^{(0)}) \hat{F} ) \]

\[ = \Pi^{(0)} \mathcal{F}^{-1} (\hat{\chi}_1 \xi \hat{\Pi}^{(1)} (I - \Pi^{(0)}) \hat{F} ) \]

\[ = \Pi^{(0)} \partial_{x_1} P_1^{(1)} (I - \Pi^{(0)}) F. \]
We thus obtain (ii).

Similarly, we can show

\[(I - \Pi^{(0)})\partial_{x_1} P_{1}^{(1)} F = (I - \Pi^{(0)})\partial_{x_1} P_{1}^{(1)}\Pi^{(0)} F,\]

and obtain (iii). This completes the proof. \qed

It is not difficult to verify the following estimates on the nonlinearities.

**Lemma 6.3.** There exists a number \(\varepsilon_5 > 0\) such that whenever a solution \(u(t)\) of (4.1) in \(Z^m(T)\) satisfies \(\sup_{0 \leq \tau \leq t} \|u(\tau)\|_m \leq \varepsilon_5\) and \(M(t) \leq 1\) for \(t \in [0, T]\), there hold the following estimates for \(t \in [0, T]\) with \(C > 0\) independent of \(T\).

(i) \[\|\partial_{x_1} \langle \phi w^1 \rangle (t)\|_1 \leq C(1 + t)^{-1} M(t)^2,\]

(ii) \[\|\langle \phi w^1 \rangle (t)\|_1 \leq C(1 + t)^{-\frac{1}{2}} M(t)^2,\]

(iii) \[\|\partial_{x_1} \sigma_1^2(t)\|_1 \leq C(1 + t)^{-1} M(t)^2,\]

(iv) \[\|F(t)\|_1 \leq C(1 + t)^{-\frac{1}{2}} M(t)^2,\]

(v) \[\|F_2(t)\|_1 \leq C(1 + t)^{-1} M(t)^2,\]

(vi) \[\|F(t)\|_2 \leq C(1 + t)^{-\frac{3}{2}} M(t)^2.\]

**Proof of Proposition 6.1.** We write (5.1) and (5.2) as

\[\sigma_1(t) = U_0(t)P_1u_0 + I(t),\]

\[u_1(t) = (I - \Pi^{(0)})U_1(t)u_0 + J(t),\]

where

\[I(t) = \int_0^t U_0(t - \tau)\Pi^{(0)}P_1 F(\tau) d\tau,\]

\[J(t) = \int_0^t (I - \Pi^{(0)})U_1(t - \tau) F(\tau) d\tau.\]
Lemma 4.7 and Lemma 4.8 yield
\[
\|\partial_{x_1} U_0(t)P_1u_0\|_2 \leq C(1 + t)^{-\frac{1}{2} - \frac{k}{2}}\|u_0\|_1,
\]
\[
\|(I - \Pi^{(0)})U_1(t)u_0\|_2 \leq C(1 + t)^{-\frac{3}{2}}\|u_0\|_1.
\]

We next estimate \(I(t)\) which we write as
\[
I(t) = I_1(t) + I_2(t),
\]
where
\[
I_1(t) = \int_0^t U_0(t - \tau)\Pi^{(0)}P_1F(\tau) \, d\tau,
\]
\[
I_2(t) = \int_\frac{t}{2}^t U_0(t - \tau)\Pi^{(0)}P_1F(\tau) \, d\tau.
\]

By Lemma 4.8 (i), (ii) and Lemma 6.2 (ii), we have
\[
U_0(t - \tau)\Pi^{(0)}P_1F(\tau) = \partial_{x_1}U_0(t - \tau)\{-P^{(0)}_1(\langle \phi w^1 \rangle u^{(0)}) + P^{(1)}_1(I - \Pi^{(0)})F + \partial_{x_1}P^{(2)}_1F\}(\tau).
\]

It then follows from Lemma 4.8 (ii) and Lemma 6.3 that
\[
\|\partial_{x_1}^k I_1(t)\|_2 \leq C\int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{k}{2}}\{\|\langle \phi w^1 \rangle(\tau)\|_1 + \|F(\tau)\|_1\} \, d\tau
\leq CM(t)^2 \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{k}{2}(1 + \tau)} \, d\tau
\leq C(1 + t)^{-\frac{1}{2} - \frac{k}{2}}M(t)^2
\]
for \(k = 0, 1\).

As for \(I_2(t)\), using Lemma 4.5 (i) and Lemma 6.2 (ii), we write \(\Pi^{(0)}P_1F\) as
\[
\Pi^{(0)}P_1F = -P^{(0)}_1(\langle \partial_{x_1} \langle \phi w^1 \rangle \rangle u^{(0)})
+ \Pi^{(0)}(P^{(1)}_1(I - \Pi^{(0)}) + \partial_{x_1}P^{(2)}_1)_{\sigma_1}(\sigma_1)^2F_1
+ \Pi^{(0)}(\partial_{x_1}P^{(1)}_1(I - \Pi^{(0)}) + \partial_{x_1}^2P^{(2)}_1)F_2.
\]
It then follows from Lemma 4.8 and Lemma 6.3 that
\[ \| \partial_{x_1}^k I_2(t) \|_2 \]
\[ \leq C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2} - \frac{k}{2}} \{ \| \partial_{x_1} \langle \phi w^1 \rangle(\tau) \|_1 + \| \partial_{x_1} (\sigma_1(\tau))^2 \|_1 + \| F_2(\tau) \|_1 \} \, d\tau \]
\[ \leq CM(t)^2 \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2} - \frac{k}{2}} (1 + \tau)^{-1} \, d\tau \]
\[ \leq C(1 + t)^{-\frac{1}{2} - \frac{k}{2}} M(t)^2 \]
for \( k = 0, 1 \).

We next consider \( J(t) \). By Lemma 4.6 (ii), we have
\[
(I - \Pi^{(0)}) U_1(t - \tau) F(\tau) = (I - \Pi^{(0)}) \partial_{x_1} U_1^{(1)}(t - \tau) \Pi^{(0)} F
+ (I - \Pi^{(0)}) \partial_{x_1}^2 U_1^{(2)}(t - \tau) F(\tau).
\]
Furthermore, using Lemma 4.6 (i) and Lemma 6.2 (i), we rewrite this as follows:
\[
(I - \Pi^{(0)}) U_1(t - \tau) F(\tau) = -(I - \Pi^{(0)}) \partial_{x_1}^2 U_1^{(1)}(t - \tau) (\langle \phi w^1 \rangle(\tau) u^{(0)})
+ (I - \Pi^{(0)}) \partial_{x_1}^2 U_1^{(2)}(t - \tau) (\partial_{x_1} (\sigma_1(\tau))^2) F_1
+ (I - \Pi^{(0)}) \partial_{x_1}^2 U_1^{(2)}(t - \tau) F_2(\tau)
\]
for \( 0 \leq \tau \leq \frac{t}{2} \) and
\[
(I - \Pi^{(0)}) U_1(t - \tau) F(\tau) = -(I - \Pi^{(0)}) \partial_{x_1} U_1^{(1)}(t - \tau) (\langle \phi w^1 \rangle(\tau) u^{(0)})
+ (I - \Pi^{(0)}) \partial_{x_1} U_1^{(2)}(t - \tau) (\partial_{x_1} (\sigma_1(\tau))^2) F_1
+ (I - \Pi^{(0)}) \partial_{x_1} U_1^{(2)}(t - \tau) F_2(\tau)
\]
for \( \frac{t}{2} \leq \tau \leq t \). It then follows from Lemma 4.7 and Lemma 6.3 (iv) that
\[ \| J(t) \|_2 \]
\[ \leq C \int_{0}^{\frac{t}{2}} (1 + t - \tau)^{-\frac{1}{2}} \{ \| \langle \phi w^1 \rangle(\tau) \|_1 + \| F(\tau) \|_1 \} \, d\tau \]
\[ + C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2}} \{ \| \partial_{x_1} \langle \phi w^1 \rangle(\tau) \|_1 + \| \partial_{x_1} (\sigma_1(\tau))^2 \|_1 + \| F_2(\tau) \|_1 \} \, d\tau \]
\[ \leq CM(t)^2 \left\{ \int_{0}^{\frac{t}{2}} (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} \, d\tau + \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-1} \, d\tau \right\} \]
\[ \leq C(1 + t)^{-\frac{1}{2}} M(t)^2. \]
We thus obtain
\[(6.2) \sum_{k=0}^{1} (1 + t)^{\frac{3}{2} + \frac{k}{2}} \| \partial_{x_1}^k \sigma_1(t) \|_2 + (1 + t)^{\frac{3}{2}} \| u_1(t) \|_2 \leq C \{ \| u_0 \|_1 + M(t)^2 \} .\]

It remains to estimate time derivatives. Since
\[-LP_1 u = \hat{\chi}_1 \hat{\lambda}_0(\xi) \hat{H}(\xi) \hat{u} = \hat{\chi}_1 \hat{\lambda}_0(\xi)P_1 u ,\]
we see from (5.7) that
\[(6.3) \partial_t P_1 u = \hat{\chi}_1 \hat{\lambda}_0(\xi)P_1 u + \hat{P}_1 \hat{F} \]
with \(\hat{\lambda}_0(\xi) = -(ia_0 \xi + \kappa_0 \xi^2 + O(\xi^3)) = O(\xi)\). It then follows from (6.3) and Lemma 6.2 (vi) that
\[(6.4) \| \partial_t P_1 u(t) \|_2 \leq C \{ \| \partial_{x_1} P_1 u(t) \|_2 + \| P_1 \hat{F}(t) \|_2 \} \leq C(1 + t)^{\frac{3}{4}} \{ \| u_0 \|_1 + M(t)^2 \} .\]

Concerning \(\| \partial_t^{j+1} P_1 u(t) \|_2\) for \(j = 1, \cdots, [m/2] - 1\), we obtain from (6.3)
\[\| \partial_t^{j+1} P_1 u(t) \|_2 \leq C \{ \| \partial_t^j P_1 u(t) \|_2 + \| \partial_t^j P_1 \hat{F} \|_2 \} .\]

Since
\[\| \partial_t^j \hat{F} \|_2 \leq C(1 + t)^{-\frac{3}{4}} M(t)^2 \]
for \(0 \leq j \leq [m/2] - 1\) as we will see in Proposition 8.5 and Proposition 8.6 below, we find by induction on \(j\), together with (6.4), that the estimate
\[(6.5) \| \partial_t^{j+1} P_1 u(t) \|_2 \leq C(1 + t)^{\frac{3}{4}} \{ \| u_0 \|_1 + M(t)^2 \} \]
holds for \(j = 0, 1, \cdots, [m/2] - 1\).

The desired result now follows from (6.2) and (6.5). This completes the proof. \(\square\)

7. Estimates on \(\tilde{P}_\infty u(t)\)

In this section we prove estimate (5.15) for \(\sigma_\infty\) and \(u_\infty\) by a variant of Matsumura-Nishida energy method as in \[8, \text{Section 5}\].

We first show the following inequality.

**Proposition 7.1.** There exist \(\nu_0, \gamma_0\) and \(\omega_0\) such that if \(\nu \geq \nu_0, \gamma^2/(\nu + \nu) \geq \gamma_0^2\) and \(\| \rho_s - 1 \|_{C^{m+1}[0,1]} \leq \omega_0\), then a solution \(u(t)\) of (4.1) in \(Z^m(T)\) satisfies
\[(7.1) \frac{d}{dt} \tilde{E}(t) + 2D(t) \leq \tilde{R}(t) .\]

Here \(\tilde{E}(t), D(t)\) and \(\tilde{R}(t)\) are quantities such that
(i) \( \tilde{E}(t) + [w_\infty(t)]_{m-2}^2 \) is equivalent to \( E_\infty(t) \),
(ii) \( D(t) \) is equivalent to \( D_\infty(t) \),
(iii) \( \tilde{R}(t) \) satisfies estimate \( (5.16) \).

The proof of Proposition 7.1 is similar to that of \( (5.64) \) in [8]. So we here give an outline only.

We introduce some quantities. Let \( E^{(0)}[\tilde{P}_\infty u] \) and \( D^{(0)}[w] \) be defined by

\[
E^{(0)}[\tilde{P}_\infty u] = \frac{\alpha_0}{\gamma^2} \| \sigma_\infty \|^2_2 + \frac{1}{\gamma^2} \sqrt{\frac{P'_{\rho}}{P_{\rho}}} \| \phi_\infty \|^2_2 + \| \sqrt{\rho_{\infty}} w_\infty \|^2_2
\]

for \( \tilde{P}_\infty u = \sigma_\infty u^{(0)} + u_\infty \) with \( u_\infty = \top (\phi_\infty, w_\infty) \); and

\[
D^{(0)}[w] = \nu \| \nabla w \|^2_2 + \tilde{\nu} \| \text{div } w \|^2_2.
\]

Note that

\[
\langle \langle Au(t), u(t) \rangle \rangle = D^{(0)}[w(t)]
\]

for \( u = \top (\phi, w) \in Z^m(T) \).

We denote the tangential derivatives \( \partial^j_t \partial^k_x \) by \( T_{j,k} \):

\[
T_{j,k} u = \partial^j_t \partial^k_x u.
\]

It is easy to see that Lemma 4.4 and Lemma 4.5 of [8] hold with \( \sigma_1 \) and \( \langle \cdot, \cdot \rangle_1 \) in [8] replaced by \( \sigma_\infty \) and \( \langle \cdot, \cdot \rangle_\infty \), respectively. Namely, we have the following lemmas.

**Lemma 7.2 ([8, Lemma 4.4]).** (i) \( \langle \partial_{x_1} f \rangle_\infty = \partial_{x_1} \langle f \rangle_\infty \) and \( \| \partial_{x_1} \langle f \rangle_\infty \|_2 \leq \| \langle f \rangle_\infty \|_2 \).

(ii) Let \( \sigma = \sigma(x_1) \) with \( \text{supp } (\hat{\sigma}) \subset \{ |\xi| \leq 1 \} \). Then

\[
\langle \langle Q_0 \tilde{B} u \rangle \rangle_\infty, \sigma \rangle = \frac{-\gamma^2}{\alpha_0} \langle \langle u, \tilde{B} (\sigma u^{(0)}_0) \rangle \rangle.
\]

(iii) \( \langle \langle \langle f \rangle_\infty u_0^{(0)}, u_\infty \rangle \rangle = 0 \) for \( u_\infty \in \text{Range } (P_\infty) \).

**Lemma 7.3. ([8, Lemma 4.5]).** There hold the following assertions.

(i) \( \| \langle Q_0 \tilde{B} (\sigma_\infty u^{(0)}_0 + u_\infty) \rangle \|_2 \leq C(\| \partial_{x_1} \sigma_\infty \|^2_2 + \| \partial_{x_1} \phi_\infty \|^2_2 + \gamma^4 \| \partial_{x_1} w_\infty \|^2_2) \).
Proposition 7.4. There is a constant $\phi f$ and $\tilde{\omega}$ where $\tilde{\omega}$. Here $\tilde{\omega}$ denotes the 2 $\omega$ energy estimates for tangential derivatives. We set $\sigma_* = \sigma_1 + \sigma_\infty$, $\phi_* = \phi_1 + \phi_\infty$, $w_* = w_1 + w_\infty$, $u_* = \nu (\phi_1, w_*) = (u_1 + u_\infty)$.

We will write $\mathcal{Q}F = \mathcal{T}(0, f)$ in the form $\mathcal{Q}F = \mathcal{F}_0 + \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3$, where $\mathcal{F}_l = \mathcal{T}(0, f_l)$ $(l = 0, 1, 2, 3)$ with

$$f_0 = -w \cdot \nabla w + f_1(\rho_*, \phi)(-\partial_{x_1}^2 \sigma_* w^{(0)} + \frac{\partial_{x_1}^2 w_*}{\gamma^{2p}} \phi_*) + f_2(\rho_*, \phi)(-\partial_{x_1}^2 \sigma_* w^{(0)} - \partial_{x_1} \sigma_* \partial_{x_2} w^{(0)})$$

$$+ f_{01}(x_2, \phi) \phi \sigma_* + f_{02}(x_2, \phi) \phi \partial_{x_1} \sigma_* + f_{03}(x_2, \phi) \phi \phi_*,$$

$$f_1 = -\nabla (f_1(\rho_*, \phi) \nabla w_*) + \nu (\nabla w_*) \nabla (f_1(\rho_*, \phi)),$$

$$f_2 = -\nabla (f_2(\rho_*, \phi) \nabla w_*) + (\nabla w_*) \nabla (f_2(\rho_*, \phi)),$$

$$f_3 = \nabla (f_3(x_2, \phi) \phi \phi_* - \nu \nabla (f_3(x_2, \phi) \phi)).$$

Here $\nabla w_*$ denotes the $2 \times 2$ matrix $(\partial_{x_1} w_*^l)$; $f_1 = \frac{\nu_0 \nu_0}{(\phi + \gamma^{2p})^2}$; $f_2 = \frac{\partial \phi}{(\phi + \gamma^{2p})^2}$; and $f_{0l}(x_2, \phi)$ $(l = 1, 2, 3)$ and $f_3(x_2, \phi)$ are some smooth functions of $x_2$ and $\phi$.

**Proposition 7.4.** There is a constant $\nu_0 > 0$ such that if $\nu \geq \nu_0$ and $\|\rho_* - 1\|_{C^{m+1}[0, 1]} \leq 1$, then the following estimate holds for $0 \leq 2j + k \leq m$:

$$\frac{1}{2} \frac{d}{dt} E^{(0)}[T_{j,k} \tilde{\omega} u] + \frac{3}{4} D^{(0)}[T_{j,k} w_\infty]$$

(7.2)

$$\leq R_{1,j,k}^{(1)} + C \left\{ \left( \frac{1}{\gamma^2} + \frac{\nu_0 \nu_0}{\gamma^2} \right) \| \partial_{x_1}^2 T_{j,k} \sigma_\infty \|_2^2 + \left( \frac{1}{\gamma^2} + \frac{\nu_0 \nu_0}{\gamma^2} \right) \| \tilde{\omega} T_{j,k} \phi_\infty \|_2^2 \right\},$$

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where
\[ \hat{T}_{j,k}\phi_{\infty} = \begin{cases} \partial_x\phi_{\infty} & (j = k = 0), \\ T_{j,k}\phi_{\infty} & (2j + k \geq 1); \end{cases} \]
and \( R^{(1)}_{j,k} \) is given by
\[
R^{(1)}_{j,k} = -\frac{\alpha_0}{\gamma^2} \{ \langle \langle Q_0 T_{j,k} F \rangle_{\infty}, T_{j,k} \sigma_{\infty} \rangle \} - \langle \langle Q_0 T_{j,k} P_{1} F \rangle_{\infty}, T_{j,k} \sigma_{\infty} \rangle 
+ \hat{R}^{(1)}_{j,k} - \langle \langle Q_0 T_{j,k} F \rangle_{\infty}, T_{j,k} u_{\infty} \rangle \}
+ \langle \langle T_{j,k} P_{1} F, T_{j,k} u_{\infty} \rangle \rangle - \langle \langle Q_0 T_{j,k} P_{1} F \rangle_{\infty}, T_{j,k} u_{\infty} \rangle \).

Here
\[
\hat{R}^{(1)}_{j,k} = \langle \langle T_{j,k} F, T_{j,k} u_{\infty} \rangle \rangle
\]
when \( 2j + k \leq m - 1 \); and
\[
\hat{R}^{(1)}_{j,k} = -\langle T_{j,k} (\phi \text{div } w), T_{j,k} \phi_{\infty} \frac{P'(\rho_s)}{\gamma^2 \rho_s} \rangle \]
\[+ \frac{1}{2} (\text{div} \langle \frac{P'(\rho_s)}{\gamma^2 \rho_s} w \rangle, |T_{j,k} \phi_{\infty}|^2) \]
\[- ([T_{j,k}, w_{\infty}][\nabla \phi_{\infty}], T_{j,k} \phi_{\infty} \frac{P'(\rho_s)}{\gamma^2 \rho_s}) \]
\[+ \langle \langle T_{j,k} f_0, T_{j,k} w_{\infty} \rho_s \rangle \rangle + \sum_{l=1}^3 \langle \langle T_{j,k} f_l, T_{j,k} w_{\infty} \rho_s \rangle \rangle \]
\[-1 \]
when \( 2j + k = m \). Here and in what follows, for \( G = g + \partial_x \tilde{g} \) with \( g, \tilde{g} \in L^2 \)
and \( v \in H^1_0 \), \( \langle \langle G, v \rangle \rangle \) denotes
\[
\langle \langle G, v \rangle \rangle \rangle = (g, v) - (\tilde{g}, \partial_x v).
\]

**Outline of Proof.** We apply \( T_{j,k} \) to (5.3) and (5.4). We then take the inner products of the resulting equations with \( T_{j,k} \sigma_{\infty} \) and \( T_{j,k} u_{\infty} \). Using Lemma 7.2 (ii) and integration by parts with the symmetric properties of \( A \) and \( B \):
\[
\langle \langle Au, u \rangle \rangle = D^{(0)}[w],
\]
\[\langle \langle Bu, u \rangle \rangle = -\langle \langle u, Bu \rangle \rangle = 0,\]
we obtain
\[
\frac{d}{dt} \| T_{j,k} \sigma_{\infty} \|_2^2 - \frac{\gamma^2}{\alpha_0} \langle \langle T_{j,k} u_{\infty}, \tilde{B}(T_{j,k} \sigma_{\infty} u_0(0)) \rangle \rangle = \langle \langle Q_0 T_{j,k} P_{(0)} F_{\infty}, T_{j,k} \sigma_{\infty} \rangle \rangle,
\]
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\begin{align}
\frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left\| P_{\rho_0} T_{j,k} \phi_{\infty} \right\|_2^2 + \left\| \sqrt{\gamma} T_{j,k} w_{\infty} \right\|_2^2 \right) + D^{(0)}[T_{j,k} w_{\infty}] + \langle \langle C_0 T_{j,k} u_{\infty}, T_{j,k} u_{\infty} \rangle \rangle + \langle \langle \tilde{M}(T_{j,k} \sigma_{\infty} u^{(0)}), T_{j,k} u_{\infty} \rangle \rangle \\
- \langle \langle Q \tilde{B}(T_{j,k} \sigma_{\infty} u^{(0)} + T_{j,k} u_{\infty})_1 u^{(0)}_1, T_{j,k} u_{\infty} \rangle \rangle = \langle \langle T_{j,k} F_{\infty}, T_{j,k} u_{\infty} \rangle \rangle.
\end{align}

We add $\frac{\alpha_0}{\gamma^2} \times (7.3)$ to (7.4). Then, since
\[\langle \langle \tilde{M}(T_{j,k} \sigma_{\infty} u^{(0)}), T_{j,k} u_{\infty} \rangle \rangle \]
\[= \langle \langle \tilde{A}(T_{j,k} \sigma_{\infty} u^{(0)}), T_{j,k} u_{\infty} \rangle \rangle + \langle \langle \tilde{B}(T_{j,k} \sigma_{\infty} u^{(0)}_0), T_{j,k} u_{\infty} \rangle \rangle \]
\[+ \langle \langle \tilde{B}(T_{j,k} \sigma_{\infty} u^{(0)}_1), T_{j,k} u_{\infty} \rangle \rangle,\]
the term $\langle \langle \tilde{B}(T_{j,k} \sigma_{\infty} u^{(0)}_0), T_{j,k} u_{\infty} \rangle \rangle$ is cancelled. We thus arrive at
\[\frac{1}{2} \frac{d}{dt} E^{(0)}[T_{j,k} \tilde{P}_{\infty} u] + D^{(0)}[T_{j,k} w_{\infty}] + \langle \langle C_0 T_{j,k} u_{\infty}, T_{j,k} u_{\infty} \rangle \rangle + \langle \langle \tilde{A}(T_{j,k} \sigma_{\infty} u^{(0)}), T_{j,k} u_{\infty} \rangle \rangle \]
\[+ \langle \langle \tilde{B}(T_{j,k} \sigma_{\infty} u^{(0)}_0), T_{j,k} u_{\infty} \rangle \rangle \]
\[\langle \langle Q \tilde{B}(T_{j,k} \sigma_{\infty} u^{(0)} + T_{j,k} u_{\infty})_1 u^{(0)}_1, T_{j,k} u_{\infty} \rangle \rangle \}
\[= \frac{\alpha_0}{\gamma^2} \langle \langle Q_0 T_{j,k} P_{\infty} F^{(0)}_{\infty}, T_{j,k} \sigma_{\infty} \rangle \rangle + \langle \langle T_{j,k} F_{\infty}, T_{j,k} u_{\infty} \rangle \rangle.
\]

The desired estimate for the case $2j + k \leq m - 1$ now follows from (7.5) by substituting
\[F_{\infty} = F - \langle \langle Q_0 F \rangle \rangle u^{(0)} - \{ P_{\infty} F - \langle \langle Q_0 P_{\infty} F \rangle \rangle u^{(0)} \}\]
and
\[\langle \langle Q_0 P_{\infty} F \rangle \rangle_{\infty} = \langle \langle Q_0 F \rangle \rangle_{\infty} - \langle \langle Q_0 P_{\infty} F \rangle \rangle_{\infty}\]
into the right-hand side of (7.5) with the aid of Lemma 5.2 (ii) and Lemma 7.3. In case $2j + k = m$, the computation above is formal; and the desired result can formally be obtained by further integration by parts on $\tilde{R}_{j,k}$. This can be justified by an argument of commutator estimate for the transport equation (5.3) and a theory of weak solutions of the parabolic equation (5.4) as in, e.g., [9].

We next derive the $H^1$-parabolic estimates for $w_{\infty}$. We define $J[\tilde{P}_{\infty} u]$ by
\[J[\tilde{P}_{\infty} u] = -2\langle \langle \sigma_{\infty} u^{(0)} + u_{\infty}, B\tilde{Q} u_{\infty} \rangle \rangle \text{ for } \tilde{P}_{\infty} u = \sigma_{\infty} u^{(0)} + u_{\infty}.
\]
A direct computation shows that if $\gamma^2 \geq 1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq 1$, then
\[
|J[\hat{P}_\infty u]| \leq \frac{b_0 \gamma^2}{\nu} E^{(0)}[\hat{P}_\infty u] + \frac{1}{2} D^{(0)}[w]\n
\] for some constant $b_0 > 0$.

Let $b_1$ be a positive constant (to be determined later) and define $E^{(1)}[\hat{P}_\infty u]$ by
\[
E^{(1)}[\hat{P}_\infty u] = \frac{2b_1 \gamma^2}{\nu} E^{(0)}[\hat{P}_\infty u] + D^{(0)}[w] + J[\hat{P}_\infty u].
\]
Note that if $b_1 \geq b_0$, then $E^{(1)}[\hat{P}_\infty u]$ is equivalent to $E^{(0)}[\hat{P}_\infty u] + D^{(0)}[w]$.

**Proposition 7.5.** There exists $b_1 \geq b_0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq 1$, then the following estimate holds for $0 \leq 2k + j \leq m - 1$:
\[
\frac{1}{2} \frac{d}{dt} E^{(1)}[T_{j,k} \hat{P}_\infty u] + \frac{b_1 \gamma^2}{\nu} D^{(0)}[T_{j,k} w] + \frac{1}{2} \|\sqrt{\rho_s} \partial_t T_{j,k} w\|_2^2 
\leq R^{(2)}_{j,k} + C \left\{ \left( \frac{1}{\nu} + \frac{\nu + \gamma^2}{\gamma^2} \right) \left\| \partial_t T_{j,k} \sigma \right\|_2^2 + \left( \frac{1}{\nu} + \frac{1}{\gamma^2} \right) \left\| \partial_x T_{j,k} \phi \right\|_2^2 + \frac{1}{\nu} \|\partial_x T_{j,k} \phi\|_2^2 \right\}.
\]
(7.6)

where
\[
R^{(2)}_{j,k} = \frac{2b_1 \gamma^2}{\nu} R^{(1)}_{j,k} + C \|T_{j,k} F\|_2^2.
\]

**Outline of Proof.** We consider the case $j = k = 0$. We take the inner product of (5.4) with $\partial_t \tilde{Q} u_\infty$ to obtain
\[
\|\sqrt{\rho_s} \partial_t w\|_2^2 + \langle \langle L u_\infty, \partial_t \tilde{Q} u_\infty \rangle \rangle + \langle \langle \tilde{M}(\sigma u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle \rangle
\]
\[
- \langle \langle Q_0 \tilde{B}(\sigma u^{(0)} + u), \partial_t \tilde{Q} u_\infty \rangle \rangle = \langle \langle F_\infty, \partial_t \tilde{Q} u_\infty \rangle \rangle.
\]
(7.7)

By using the symmetric property of $A$ we have
\[
\langle \langle L u_\infty, \partial_t \tilde{Q} u_\infty \rangle \rangle = \frac{1}{4} \frac{d}{dt} D^{(0)}[w] + \langle \langle B u_\infty, \partial_t \tilde{Q} u_\infty \rangle \rangle + \langle \langle C_0 u_\infty, \partial_t \tilde{Q} u_\infty \rangle \rangle.
\]
(7.8)

As for the second term on the right of (7.8), we rewrite it as
\[
\langle \langle B u_\infty, \partial_t \tilde{Q} u_\infty \rangle \rangle = - \frac{d}{dt} \langle \langle u_\infty, B \tilde{Q} u_\infty \rangle \rangle + \langle \langle \partial_t u_\infty, B \tilde{Q} u_\infty \rangle \rangle
\]
\[
- \frac{d}{dt} \langle \langle u_\infty, B \tilde{Q} u_\infty \rangle \rangle + \langle \langle \partial_t Q_0 u_\infty, B \tilde{Q} u_\infty \rangle \rangle
\]
\[
+ \langle \langle \partial_t \tilde{Q} u_\infty, B \tilde{Q} u_\infty \rangle \rangle.
\]
(7.9)
By (5.4), we have
\[
\partial_t \phi_\infty = -\{v_s^1 \partial_x \phi_\infty + \gamma^2 \text{div}(\rho_s w_\infty) + (v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1}) \partial_x \sigma_\infty \\
-\langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty) \rangle_\infty \phi^{(0)} \} + f^0_\infty.
\]
We thus obtain, if \( \| \rho_s - 1 \|_{C^{m+1}[0,1]} \leq 1 \), then
\[
\| \langle \tilde{Q}_0 u_\infty, B\tilde{Q} u_\infty \rangle \| \\
\leq C \left\{ \| \partial_x \phi_\infty \|_2 + \gamma^2 \| \partial_x w_\infty \|_2 + \| \partial_x \sigma_\infty \|_2 \right\} \| \text{div}(\rho_s w_\infty) \|_2 \\
+ C \| Q_0 F_\infty \|_2 \| \text{div}(\rho_s w_\infty) \|_2 \\
\leq C \left\{ \frac{\gamma^2}{\nu} D^{(0)} w_\infty + \frac{1}{\gamma^2} (\| \partial_x \phi_\infty \|_2^2 + \| \partial_x \sigma_\infty \|_2^2 + \| Q_0 F_\infty \|_2^2) \right\}.
\]
It then follows from (7.9) that
\[
\langle Bu_\infty, \partial_t \tilde{Q} u_\infty \rangle
\] (7.10) \[
\geq -\frac{d}{dt} \langle u_\infty, B\tilde{Q} u_\infty \rangle - \| \langle \partial_t \tilde{Q} u_\infty, B\tilde{Q} u_\infty \rangle \| \\
- C \left\{ \frac{\gamma^2}{\nu} D^{(0)} w_\infty + \frac{1}{\gamma^2} (\| \partial_x \phi_\infty \|_2^2 + \| \partial_x \sigma_\infty \|_2^2 + \| Q_0 F_\infty \|_2^2) \right\}.
\]
We deduce from (7.8) and (7.10) that
\[
\langle Lu_\infty, \partial_t \tilde{Q} u_\infty \rangle \geq \frac{1}{2} \frac{d}{dt} \langle D^{(0)} w_\infty - 2 \langle u_\infty, B\tilde{Q} u_\infty \rangle \rangle \\
- \| \langle C_0 u_\infty, \partial_t \tilde{Q} u_\infty \rangle \| - \| \langle \partial_t \tilde{Q} u_\infty, B\tilde{Q} u_\infty \rangle \| \\
- C \left\{ \frac{\gamma^2}{\nu} D^{(0)} w_\infty + \frac{1}{\gamma^2} (\| \partial_x \phi_\infty \|_2^2 + \| \partial_x \sigma_\infty \|_2^2 + \| Q_0 F_\infty \|_2^2) \right\},
\]
provided that \( \gamma^2 \geq 1 \) and \( \| \rho_s - 1 \|_{C^{m+1}[0,1]} \leq 1 \).
We next consider \( \langle \tilde{M}(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle \) on the left-hand side of (7.7):
\[
\langle \tilde{M}(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle = \langle \tilde{A}(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle + \langle \tilde{B}(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle.
\]
By Lemma 5.2 (ii) we have \( \| \partial_{x_1} \sigma_\infty \|_2 \leq \| \partial_{x_1} \sigma_\infty \|_2 \) for \( k \geq 1 \). This, together with the fact \( w^{(0),1} = O(\gamma^{-2}) \), yields
\[
\| \langle \tilde{A}(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle \| \leq C_{\nu, \rho_s} (\| \partial_{x_1} \sigma_1 \|_2 + \| \partial_{x_1} \sigma_1 \|_2) \sqrt{\rho_s} \| \partial_t w_\infty \|_2 \\
\leq \frac{1}{12} \sqrt{\rho_s} \| \partial_t w_\infty \|_2^2 + C_{\nu, \rho_s} \| \partial_{x_1} \sigma_\infty \|_2^2.
\]
The second term on the right of (7.12) is treated in a similar manner to the estimate (7.10). Since $Lu^{(0)} = 0$, we have $\partial_{x_n} \left( \frac{P(\rho_s)}{\sqrt{\rho_s}} \phi^{(0)} \right) = 0$. It then follows that $\tilde{B}(\sigma_\infty u^{(0)}) = B(\sigma_\infty u^{(0)})$, and hence,

$$
\langle \langle \tilde{B}(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle \rangle = \langle \langle B(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle \rangle
$$

$$= -\frac{d}{dt} \langle \langle \sigma_\infty u^{(0)}, B \tilde{Q} u_\infty \rangle \rangle + \langle \langle \partial_t \sigma_\infty u^{(0)}, B \tilde{Q} u_\infty \rangle \rangle.
$$

By (5.3) we have

$$\partial_t \sigma_\infty = -\langle \langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty) \rangle \rangle + \langle \langle Q_0 P_{\infty} F \rangle \rangle_\infty.$$

Substituting this into $\langle \langle \partial_t \sigma_\infty u^{(0)}, B \tilde{Q} u_\infty \rangle \rangle$ we obtain

$$\langle \langle \partial_t \sigma_\infty u^{(0)}, B \tilde{Q} u_\infty \rangle \rangle
$$

$$\leq C \left\{ \frac{2}{\nu} D^{(0)} [w_\infty] + \frac{1}{\gamma^2} \left( \| \partial_{x_1} \sigma_\infty \|_2^2 + \| \partial_{x_2} \sigma_\infty \|_2^2 + \| \partial_{x_3} \sigma_\infty \|_2^2 + \| \langle \langle Q_0 P_{\infty} F \rangle \rangle_\infty \|_2^2 \right) \right\},$$

provided that $\| \rho_s - 1 \|_{C^{m+1}[0,1]} \leq 1$.

We thus find from (7.12)–(7.15) that

$$\langle \langle \mathcal{M}(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle \rangle
$$

$$\geq -\frac{d}{dt} \langle \langle \sigma_\infty u^{(0)}, B \tilde{Q} u_\infty \rangle \rangle - \frac{1}{4} \| \sqrt{\rho_s} \partial_t w_\infty \|_2^2
$$

$$+ C \left\{ \frac{2^2}{\nu} D^{(0)} [w_\infty] + \frac{1}{\gamma^2} \left( \| \partial_{x_1} \sigma_\infty \|_2^2 + \| \partial_{x_2} \sigma_\infty \|_2^2 + \| \partial_{x_3} \sigma_\infty \|_2^2 \right) + \| Q_0 P_{\infty} F \|_\infty^2 \right\}$$

if $\| \rho_s - 1 \|_{C^{m+1}[0,1]} \leq 1$.

It follows from (7.7), (7.11) and (7.16) that

$$\langle \langle C_0 u_\infty, \partial_t \tilde{Q} u_\infty \rangle \rangle + \langle \langle \partial_t \tilde{Q} u_\infty, B \tilde{Q} u_\infty \rangle \rangle$$

$$\langle \langle Q_0 B(\sigma_\infty u^{(0)} + u_\infty), \partial_t \tilde{Q} u_\infty \rangle \rangle + \langle \langle (F_{\infty, \partial_t} \tilde{Q} u_\infty) \rangle \rangle
$$

$$+ C \left\{ \frac{2^2}{\nu} D^{(0)} [w_\infty] + \frac{1}{\gamma^2} \left( \| \partial_{x_1} \phi_\infty \|_2^2 + \| \partial_{x_2} \phi_\infty \|_2^2 + \| Q_0 F \|_2^2 + \| Q_0 P_{\infty} F \|_\infty \right) \right\}$$

$$+ \frac{(\nu + \gamma)^2}{\gamma^2} \| \partial_{x_1} \sigma_\infty \|_2^2 \right\}.$$

We estimate the right-hand side of (7.17) by using the Schwartz inequality
and Lemma 7.3 to obtain
\[ \frac{1}{2} \frac{d}{dt} (D^{(0)}[u]_\infty + J[u]) + \frac{1}{2} \| \sqrt{\rho_s} \partial_t w_\infty \|^2 \leq C' \| F \|^2 + C' \frac{\gamma^2}{\beta} D^{(0)}[w_\infty] \]
(7.18) \[ + C' \left\{ \left( \frac{1}{\gamma^2} + \frac{(\nu + \tilde{\nu})^2}{\gamma^4} \right) \| \partial_x, \sigma_\infty \|^2 + \frac{\gamma^2}{\gamma^2} \| \phi_\infty \|^2 + \frac{1}{\gamma^2} \| \partial_x, \phi_\infty \|^2 \right\}. \]

Taking \( b_1 > 0 \) in such a way that \( b_1 \geq \max\{b_0, 2C\} \) and adding \( \frac{2b_1 \gamma^2}{\nu} \times (7.2) \) to (7.18), we obtain the desired estimate. The case \( 1 \leq 2j + k \leq m - 1 \) can be treated similarly. \( \square \)

As for the dissipative estimates for \( x_2 \)-derivatives of \( \phi_\infty \), we have the following inequality.

**Proposition 7.6.** If \( \| \rho_s - 1 \|_{C^{m+1}[0, 1]} \leq \min\{1, \frac{(\nu + \tilde{\nu})^2}{\gamma^2}, \omega_0\} \), then the following estimate holds for \( 0 \leq j + k \leq m - 1 \):

\[ \frac{1}{2} \frac{d}{dt} \frac{1}{\gamma^2} \left\| \sqrt{\rho_s} \frac{\partial_x^l}{\partial_x^l} \partial_{x_2} \partial_{x_2} \phi_\infty \right\|^2 + \frac{1}{2(\nu + \tilde{\nu})} \left\| \frac{\partial_x^l}{\partial_x^l} \partial_{x_2} \partial_{x_2} \phi_\infty \right\|^2 \leq R^{(3)}_{j,k,l} + C' \frac{\nu + \tilde{\nu}}{\gamma^2} \| K_{j,k,l} \|^2, \]
(7.19)

where

\[ R^{(3)}_{j,k,l} = \left\| \begin{pmatrix} 1 \end{pmatrix} \left( \text{div} \left( \frac{\partial_x^l}{\partial_x^l} \phi_\infty \right), \phi_\infty \right) \right\| + \frac{\nu + \tilde{\nu}}{\gamma^2} \| H_{j,k,l} \|^2 \]

with

\[ \| H_{j,k,l} \|^2 \leq C' \left\{ \| [T_{j,k} \partial_{x_2}^{l+1}, w] \cdot \nabla \phi_\infty \|^2 + \| T_{j,k} \partial_{x_2}^{l+1} f_\infty \|^2 + \| \partial_{x_2} \partial_{x_2}^{l+1} f_\infty \|^2 \right\}, \]

and

\[ f_\infty^0 = -\phi \text{div} w - w \cdot \nabla (\sigma_x \phi_\infty) + \phi_1 - \{ f_1^0 - \langle Q_0 P_1 F \rangle_\infty \} \]

with \( P_1 F = \begin{pmatrix} f_1^0, f_1^{(1)}, f_1^{(2)} \end{pmatrix} \); and \( K_{j,k,l} \) is estimated as

\[ \frac{\nu + \tilde{\nu}}{\gamma^2} \| K_{j,k,l} \|^2 \leq C' \left\{ \| T_{j,k+1} \partial_{x_2}^l \partial_x w_\infty \|^2 + \frac{1}{\nu + \tilde{\nu}} \left\| \sqrt{\rho_s} \partial_{x_2} \partial_{x_2} \partial_x w_\infty \right\|^2 \right\} \]

\[ + \left( \nu + \tilde{\nu} \right) \omega_0^2 \left( \sum_{p=0}^{l-1} \| T_{j,k+1} \partial_{x_2}^l \partial_x w_\infty \| + \| T_{j,k+1} \partial_{x_2}^l \partial_x w_\infty \| \right) \]

\[ + \frac{1}{\nu + \tilde{\nu}} \sum_{p=0}^{l-1} \| T_{j,k+1} \partial_{x_2}^l \partial_x w_\infty \|^2 \]

\[ + \frac{\nu + \tilde{\nu}}{\gamma^2} \left( \sum_{p=0}^{l-1} \| T_{j,k+1} \partial_{x_2}^l \partial_x \phi_\infty \| + \| \partial_x T_{j,k} \sigma_\infty \|^2 \right). \]
Outline of Proof. The first equation of (5.4) is written as
\[
\partial_t \phi_\infty + (v_s + w) \cdot \nabla \phi_\infty + \gamma^2 \rho_s \partial_{x_1} w_\infty + \gamma^2 \partial_{x_2} (\rho_s w_\infty^2) + (v_s^1 \phi_0) + \gamma^2 \rho_s w^{(0),1}) \partial_{x_1} \sigma_\infty - \langle Q_0 \hat{B}(\sigma_\infty u(0) + u_\infty) \rangle_\infty \phi_0
\]
\[= \tilde{j}^0_\infty.
\]
Applying \(T_{j,k} \partial_{x_2}^{l+1}\) to this equation, we have
\[
\partial_t (T_{j,k} \partial_{x_2}^{l+1} \phi_\infty) + (v_s + w) \cdot \nabla (T_{j,k} \partial_{x_2}^{l+1} \phi_\infty) + \gamma^2 \rho_s T_{j,k} \partial_{x_2}^{l+2} w_\infty
\]
\[= -[T_{j,k} \partial_{x_2}^{l+1}, v_s + w] \cdot \nabla \phi_\infty - \gamma^2 \partial_{x_2}^{l+1} (\rho_s \partial_x T_{j,k} w_\infty^1)
\]
\[= -\gamma^2 \partial_{x_2}^{l+1} (v_s^1 \phi_0) + \gamma^2 \rho_s w^{(0),1}) \partial_{x_1} T_{j,k} \sigma_\infty
\]
\[+ \langle Q_0 \hat{B}(T_{j,k} \sigma_\infty u(0) + T_{j,k} u_\infty) \rangle_\infty \partial_{x_2}^{l+1} \phi_0 + T_{j,k} \partial_{x_2}^{l+1} \tilde{j}_\infty^0.
\]
The third equation of (5.4) is written as
\[
-\frac{\nu + \tilde{\nu}}{\rho_s} \partial_{x_2}^2 w_\infty^2 + \partial_{x_2} \left( \frac{P'(\rho_s)}{\gamma' \rho_s} \phi_\infty \right)
\]
\[= -\{ \partial_t w_\infty^2 - \frac{\rho_s}{v_s} \partial_{x_1} w_\infty^2 - \frac{\tilde{\rho}_s}{v_s} \partial_{x_2} \partial_{x_1} w_\infty^1 + v_s^1 \partial_{x_1} w_\infty^1 - \frac{\tilde{\rho}_s}{v_s} (\partial_{x_2} w^{(0),1}) \partial_{x_1} \sigma_\infty \} + f_\infty^2.
\]
Applying \(T_{j,k} \partial_{x_2}^{l}\) to this equation, we have
\[
-\frac{\nu + \tilde{\nu}}{\rho_s} T_{j,k} \partial_{x_2}^{l+2} w_\infty^2 + \frac{P'(\rho_s)}{\gamma' \rho_s} T_{j,k} \partial_{x_2}^{l+1} \phi_\infty
\]
\[= (\nu + \tilde{\nu}) [\partial_{x_2}^{l+1}, \frac{1}{\rho_s}] \partial_{x_2} T_{j,k} w_\infty^2 - [\partial_{x_2}^{l+1}, \frac{P'(\rho_s)}{\gamma' \rho_s}] T_{j,k} \phi_\infty
\]
\[= \{ \partial_{T_{j,k} \partial_{x_2}^{l+1} w_\infty^2 - T_{j,k} \partial_{x_2}^{l+1} (\frac{\rho_s}{v_s} \partial_{x_1} w_\infty^2 + \frac{\tilde{\rho}_s}{v_s} \partial_{x_2} \partial_{x_1} w_\infty^1)
\]
\[+ T_{j,k} \partial_{x_2}^{l+1} (v_s^1 \partial_{x_1} w_\infty^1 - \partial_{x_1} (\frac{\rho_s}{v_s} \partial_{x_2} w^{(0),1}) \partial_{x_1} T_{j,k} \sigma_\infty \} + T_{j,k} \partial_{x_2}^{l+1} f_\infty^2.
\]
Adding \(\frac{\nu + \tilde{\nu}}{\rho_s} \partial_{x_2}^2 \times (7.21) \) to (7.20), we have
\[
\partial_t (T_{j,k} \partial_{x_2}^{l+1} \phi_\infty) + (v_s + w) \cdot \nabla (T_{j,k} \partial_{x_2}^{l+1} \phi_\infty) + \frac{\rho_s P'(\rho_s)}{\nu + \tilde{\nu}} T_{j,k} \partial_{x_2}^{l+1} \phi_\infty
\]
\[= H_{j,k,l} + K_{j,k,l},
\]
where
\[
H_{j,k,l} = -[T_{j,k} \partial_{x_2}^{l+1}, w] \cdot \nabla \phi_\infty + T_{j,k} \partial_{x_2}^{l+1} \tilde{j}_\infty^0 + \frac{\gamma^2 \rho_s^2}{\nu + \tilde{\nu}} T_{j,k} \partial_{x_2}^{l+1} f_\infty^2,
\]
\[48
\]
The following estimate for the material derivative of \( \phi_\infty \) plays an important role to obtain the dissipative estimate for higher order \( x_2 \)-derivatives of \( w_\infty \). We denote the material derivative of \( \phi_\infty \) by \( \dot{\phi}_\infty \):

\[
\dot{\phi}_\infty = \partial_t \phi_\infty + (v_s + w) \cdot \nabla \phi_\infty.
\]

**Proposition 7.7.** (i) If \( \| \rho_s - 1 \|_{C^{m+1}[0,1]} \leq \min \{ 1, (\nu + \bar{\nu})^2, \omega_0 \} \), then the following estimate holds for \( 0 \leq 2j + k + l \leq m - 1 \):

\[
\frac{1}{2} \frac{d}{dt} \frac{1}{\gamma^j} \| T_{j,k} \partial_{x_2}^{j+1} \phi_\infty \|^2 + \frac{1}{4 (\nu + \bar{\nu})} \| \frac{P(\rho_s)}{\gamma^j} T_{j,k} \partial_{x_2}^{j+1} \phi_\infty \|^2 \\
+ c_0 \frac{\nu + \bar{\nu}}{\gamma^j} \| T_{j,k} \partial_{x_2}^{j+1} \phi_\infty \|^2 \\
\leq R_{j,k,l}^{(3)} + C \frac{\nu + \bar{\nu}}{\gamma^j} \| K_{j,k,l} \|^2,
\]

where \( c_0 \) is a positive constant; and \( R_{j,k,l}^{(3)} \) and \( K_{j,k,l} \) satisfy the same estimates as in Proposition 7.6.

(ii) Let \( 0 \leq q \leq k \). Then

\[
\frac{\nu + \bar{\nu}}{\gamma^j} \| T_{j,k} \dot{\phi}_\infty \|^2 \leq C \left\{ R_{j,k}^{(4)} + D^{(0)} [T_{j,k} w_\infty] + (\nu + \bar{\nu}) \omega_0^2 \| T_{j,k} w_\infty \|^2 \\
+ \frac{\nu + \bar{\nu}}{\gamma^j} \| \partial_t T_{j,k} \phi_\infty \|^2 + \frac{\nu + \bar{\nu}}{\gamma^j} \| \partial_{x_1} T_{j,q} \phi_\infty \|^2 \right\},
\]

\( 49 \)
where $R_{j,k}^{(4)} = \frac{\nu + \tilde{\nu}}{\gamma} \|T_{j,k} \tilde{f}_\infty^0\|_2^2$.

**Outline of Proof.** By (7.22), we have

$$T_{j,k} \partial_x^{j+1} \phi_\infty = -\frac{\rho_s \tilde{F}(\rho_s)}{\nu + \tilde{\nu}} T_{j,k} \partial_x^{j+1} \phi_\infty + \tilde{H}_{j,k,l} + \tilde{K}_{j,k,l},$$

where

$$\tilde{H}_{j,k,l} = [T_{j,k} \partial_x^{j+1}, w] \cdot \nabla \phi_\infty + H_{j,k,l},$$

$$\tilde{K}_{j,k,l} = [\partial_x^{j+1}, v_s] \cdot \nabla T_{j,k} \phi_\infty + K_{j,k,l}.$$  

This, together with (7.19), gives (7.23).

The estimate (7.24) follows from the first equation of (5.4) which is written as

$$\partial_t \frac{\nu}{2} \partial_x (T_{j,k} w_\infty) - \gamma^2 \partial_x \partial_x \phi_\infty + \gamma^2 \partial_x \phi_\infty$$

$$+ (Q_0 \tilde{B}(T_{j,k} \sigma_\infty u^{(0)} + T_{j,k} u_\infty)) \phi_\infty + T_{j,k} \tilde{f}_\infty^0.$$  

Applying $T_{j,k}$ to this equation, we can obtain (7.24). \hfill \Box

Let us derive the dissipative estimates for $\sigma_\infty$.

**Proposition 7.8.** There are positive constants $\nu_0$ and $\gamma_0$ such that if $\nu \geq \nu_0$ and $\gamma^2 / (\nu + \tilde{\nu}) \geq \gamma_0^2$, then the following estimate holds for $0 \leq 2j + k \leq m-1$:

$$\frac{1}{2} \frac{d}{dt} \frac{\nu}{\gamma^2} \|T_{j,k} \sigma_\infty\|_2^2 + \frac{\alpha_1}{2}(\nu + \tilde{\nu}) \|\partial_x T_{j,k} \sigma_\infty\|_2^2$$

$$\leq R_{j,k}^{(5)} + C \left\{ \frac{1}{\nu + \tilde{\nu}} \|\partial_x T_{j,k} w_\infty\|_2^2 + D[0][T_{j,k} w_\infty] \right\},$$

where $\alpha_1 > 0$ is a constant; $p$ is any integer satisfying $0 \leq 2j + p + 1 \leq m$ and $0 \leq p \leq k$; and

$$R_{j,k}^{(5)} = \frac{\nu}{\gamma^2} (Q_0 T_{j,k} F^{(0)}(T_{j,k} \sigma_\infty)) - \frac{1}{\nu + \tilde{\nu}} (\rho_s (-\Delta)^{-1} (\rho_s \partial_x T_{j,k} f_\infty^l), T_{j,k} \sigma_\infty).$$

Here $(-\Delta)^{-1}$ is the inverse of $-\Delta$ on $L^2(\Omega)$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$.

**Outline of Proof.** We rewrite (5.3) as

$$\partial_t \sigma_\infty + \gamma^2 \partial_x (\rho_s w_\infty^l) + (Q_0 \tilde{B}(\sigma_\infty u^{(0)}))_\infty$$

$$= (Q_0 F^{(0)}(T_{j,k} \sigma_\infty))_\infty - \partial_{x_1} (v_s^{(0)} \phi_\infty)_\infty.$$  

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Since $Lu(0) = 0$, we have $\partial_{x_2} \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi(0) \right) = 0$. Furthermore, since $1 = \langle u(0), u(0)^* \rangle = \langle \phi(0) \rangle$, we have $\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi(0) = \alpha_0$. Therefore, the second equation of (5.4) is written as

\begin{equation}
(7.27) \quad -\Delta w^1_\infty = -\frac{\alpha_0}{\nu} \rho_s \partial_{x_1} \sigma_\infty + \tilde{f}^1_\infty + \frac{\rho_s}{\nu} f^1_\infty,
\end{equation}

where

\[
\tilde{f}^1_\infty = -\frac{\rho_s}{\nu} \{ \partial_t w^1_\infty - \frac{\nu}{\rho_s} \partial_{x_1} \text{div } w_\infty + \frac{P'(\rho_s)}{\gamma^2 \rho_s} \partial_{x_1} \phi_\infty + v^1_s \partial_{x_1} w^1_\infty \\
+ \frac{\nu_0^2}{\gamma^2 \rho_s^2} \phi_\infty + (\partial_{x_2} v^1_\infty) w^1_\infty - \frac{\nu + \nu_0}{\rho_s} (\partial_{x_1} \sigma_\infty) w^{0(1)}_\infty + v^1_s w^{0(1)} \partial_{x_1} \sigma_\infty \\
- \langle Q_0 \tilde{B}(\sigma_\infty u^{0(1)} + u_\infty) \rangle_\infty w^{0(1)}_\infty \}
\]

It follows from (7.27) that

\begin{equation}
(7.28) \quad w^1_\infty = -\frac{\alpha_0}{\nu} (-\Delta)^{-1}(\rho_s \partial_{x_1} \sigma_\infty) + (-\Delta)^{-1} (\tilde{f}^1_\infty + \frac{\rho_s}{\nu} f^1_\infty).
\end{equation}

Substituting (7.28) into the second term on the left-hand side of (7.26), we have

\begin{equation}
(7.29) \quad \partial_t \sigma_\infty - \frac{\alpha \sigma}{\nu} \{ \rho_s (-\Delta)^{-1}(\rho_s \partial_{x_1}^2 \sigma_\infty) \}_\infty + \langle Q_0 \tilde{B}(\sigma_\infty u^{0(1)}) \rangle_\infty \\
= \langle Q_0 P^{0(1)} F \rangle_\infty - \partial_{x_1} \langle \nu^1_s \phi_\infty \rangle_\infty - \gamma^2 \partial_{x_1} \langle \rho_s (-\Delta)^{-1} (\tilde{f}^1_\infty + \frac{1}{\nu} \rho_s f^1_\infty) \rangle_\infty.
\end{equation}

By the Plancherel theorem,

\[-(\langle \rho_s (-\Delta)^{-1}(\rho_s \partial_{x_1}^2 \sigma_\infty) \rangle_\infty, \sigma_\infty) = (2\pi)^{-1}((\hat{\chi}_1 + \hat{\chi}_2)|\xi|^2 \langle \rho_s (|\xi|^2 - \partial_{x_2}^2)^{-1} \rho_s \rangle \sigma_\hat{\sigma}, \sigma_\hat{\sigma}).\]

Since there exists a constant $c > 0$ such that

\[-(\rho_s (|\xi|^2 - \partial_{x_2}^2)^{-1} \rho_s) = ||\xi|^2 - \partial_{x_2}^2)^{-\frac{1}{2}} \rho_s ||^2_2 \geq c > 0\]

for $|\xi| \leq 1$, we have

\[-(\langle \rho_s (-\Delta)^{-1}(\rho_s \partial_{x_1}^2 \sigma_\infty) \rangle_\infty, \sigma_\infty) \geq (2\pi)^{-1} c |||\xi| \sigma_\hat{\sigma}||^2_2 = c ||\partial_{x_1} \sigma_\infty||^2_2.\]

Furthermore,

\[-(\langle Q_0 \tilde{B}(\sigma_\infty u^{0(1)}) \rangle_\infty, \sigma_\infty) = \langle \nu^1_s \phi(0) + \gamma^2 \rho_s w^{0(1)}, \partial_{x_1} \sigma_\infty, \sigma_\infty \rangle = 0.\]

Therefore, taking the inner product of (7.29) with $\sigma_\infty$, we can obtain the desired estimate. \qed
We next estimate the higher order derivatives.

**Proposition 7.9.** If \( \nu \geq \nu_0, \gamma^2 \geq 1 \) and \( \| \rho_s - 1 \|_{C^{m+1}} \leq \omega_0 \), then there holds the following estimate for \( 0 \leq 2j + k + l \leq m - 1 \):

\[
\begin{align*}
\frac{\nu^2}{\nu + \tilde{\nu}} \| \partial_x^{j+2} T_{j,k} w^\infty \|_2^2 + \frac{1}{\nu + \tilde{\nu}} \| \partial_x^{j+1} T_{j,k} \phi^\infty \|_2^2 \\
\leq C R_{j,k,l}^{(6)} + C \{ \left( \frac{\nu + \tilde{\nu}}{\gamma} \right)^2 \| \partial_x \phi^\infty \|_{H^{l+1}}^2 + \| \partial_x T_{j,k} \rho_s w^\infty \|_{H^l}^2 \\
+ \frac{1}{\nu + \tilde{\nu}} \| \partial_x T_{j,k} \phi^\infty \|_{H^{l+1}}^2 \\
+ \left( \frac{1}{\nu + \tilde{\nu}} + (\nu + \tilde{\nu}) \omega_0^2 \right) \| T_{j,k} w^\infty \|_{H^{l+1}}^2 + D(0) \| T_{j,k} w^\infty \|_{H^{l+1}}^2 \} ,
\end{align*}
\]

where

\[
R_{j,k,l}^{(6)} = \frac{\nu + \tilde{\nu}}{\gamma} \| T_{j,k} f_\infty^0 \|_{H^{l+1}}^2 + \frac{1}{\nu + \tilde{\nu}} \| T_{j,k} f_\infty \|_{H^l}^2 .
\]

**Outline of Proof.** We use the estimates for the Stokes system. Let \( \mathbf{(\mathbf{\tilde{\phi}}, \mathbf{\tilde{w}})} \) be the solution of the Stokes system

\[
\begin{align*}
\text{div} \mathbf{\tilde{w}} &= F \text{ in } \Omega , \\
- \Delta \mathbf{\tilde{w}} + \nabla \mathbf{\tilde{\phi}} &= G \text{ in } \Omega , \\
\mathbf{\tilde{w}} |_{\partial \Omega} &= 0 .
\end{align*}
\]

Then for any \( l \in Z, l \geq 0 \), there exists a constant \( C > 0 \) such that

\[
\| \partial_x^{l+2} \mathbf{\tilde{w}} \|_2^2 + \| \partial_x^{l+1} \mathbf{\tilde{\phi}} \|_2^2 \leq C \{ \| F \|_{H^{l+1}}^2 + \| G \|_{H^l}^2 + \| \partial_x \mathbf{\tilde{w}} \|_2^2 \} ,
\]

(See, e.g., [1], [6, Appendix].)

By (5.4), we have

\[
\begin{align*}
\text{div} \left( T_{j,k} w^\infty \right) &= F_{j,k} \text{ in } \Omega , \\
- \Delta (T_{j,k} w^\infty) + \nabla \left( \frac{F_{j,k}}{\nu^2} T_{j,k} \phi^\infty \right) &= G_{j,k} \text{ in } \Omega , \\
T_{j,k} w^\infty |_{\partial \Omega} &= 0 ,
\end{align*}
\]

where

\[
F_{j,k} = \frac{1}{\gamma^2 \rho_s} T_{j,k} f_\infty^0 - \frac{\partial_x^2 \rho_s}{\gamma^2 \rho_s} T_{j,k} w^\infty \\
- \frac{1}{\gamma^2 \rho_s} \left( T_{j,k} \phi^\infty + (\nu^2 \rho_s w^{(0)}) \partial_x T_{j,k} \sigma^\infty \right) \\
- \left( Q_0 B (T_{j,k} \sigma^\infty u^{(0)} + T_{j,k} u^\infty ) \right) \phi^\infty ,
\]

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Proposition 7.10. (i) If \(0 \leq 2j \leq m - 1\), then there holds the following estimate:

\[
\|\partial^j \sigma\|_2^2 \leq C \{ R^{(7)}_j + \|\partial_x \partial^j \sigma\|_2^2 + \|\partial_x \partial^j \phi\|_2^2 + \gamma^4 \|\partial_x \partial^j w\|_2^2 \}. 
\]

Here \(R^{(7)}_j = \|\langle Q_0 \partial^j_t \rho \rangle \langle P^{(0)}_\infty \rangle \|_2^2\).

(ii) If \(0 \leq 2j \leq m - 1\) and \(\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq 1\), then there holds the following estimate:

\[
\|\partial^j \phi\|_{H^{m-1-2j}}^2 \leq C \{ R^{(8)}_j + \|\partial_x \partial^j \phi\|_{H^{m-1-2j}}^2 + \gamma^4 \|\partial_x \partial^j w\|_{H^{m-1-2j}}^2 + \|\partial_x \partial^j \sigma\|_2^2 \}. 
\]

Here \(R^{(8)}_j = \|\partial^j_t \langle Q_0 \rangle \|_{H^{m-1-2j}}^2\).

Outline of Proof. The estimates (7.32) and (7.33) follows from (5.3) and the first equation of (5.4)

\[
\partial_t \sigma = \langle Q_0 P^{(0)}_\infty \rangle - \langle Q_0 \tilde{B}(\sigma u^{(0)} + u_\infty) \rangle
\]

and

\[
\partial_t \phi = f^{(0)}_\infty - \{ v_s^1 \partial_x \phi + \gamma^2 \text{div}(\rho_s w_\infty) + (v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0,1)}) \partial_x \sigma \}
\]

\[-\langle Q_0 \tilde{B}(\sigma u^{(0)} + u_\infty) \rangle \phi^{(0)} \}.
\]
Proposition 7.1 now follows from combining Propositions 7.4–7.10.

**Outline of Proof of Proposition 7.1.** We proceed as in [8, Section 5.2]. We define

\[
E^{(0)}(t) = \sum_{2j+k \leq m} E^{(0)}[T_{j,k} \check{P}_\infty u(t)], \quad E^{(1)}(t) = \sum_{2j+k \leq m-1} E^{(1)}[T_{j,k} u(t)]
\]

\[
E^{(2)}(t) = \sum_{2j+k \leq m-1} \frac{1}{\tau^2} \| \sqrt{\frac{\rho}{\gamma^2 \rho_x}} \partial_{x_2} T_{j,k} \phi_\infty(t) \|_2^2,
\]

\[
E^{(3)}(t) = \sum_{2j+k \leq m-1} \frac{\nu}{\gamma^2 (\nu + \check{\nu})} \| T_{j,k} \sigma_\infty \|_2^2
\]

and

\[
\dot{D}^{(0)}(t) = \sum_{2j+k \leq m} D^{(0)}[T_{j,k} w_\infty(t)],
\]

\[
D^{(1)}(t) = \sum_{2j+k \leq m-1} \left( \frac{2b_0 \nu^2}{\nu + \check{\nu}} D^{(0)}[T_{j,k} T \nu_\infty(t)] + \frac{1}{\nu + \check{\nu}} \| \sqrt{\rho_x} \partial_t T_{j,k} w_\infty(t) \|_2^2 \right),
\]

\[
D^{(2)}(t) = \sum_{2j+k \leq m-1} \left( \frac{1}{\nu + \check{\nu}} \| \frac{\rho}{\gamma^2} \partial_{x_2} T_{j,k} \phi_\infty(t) \|_2^2 + \frac{\nu + \check{\nu}}{\gamma^2} \| T_{j,k} \phi_\infty \|_H^2 \right),
\]

\[
D^{(3)}(t) = \sum_{2j+k \leq m-1} \frac{a_1}{\nu + \check{\nu}} \| \partial_{x_2} T_{j,k} \sigma_\infty \|_2^2.
\]

Let \( b_j (j = 2, \ldots, 6) \) be positive numbers and consider

\[
\sum_{2j+k \leq m} \{ (7.2) + b_2 \times (7.24) \}
\]

\[
+ \sum_{2j+k \leq m-1} \left\{ \frac{1}{\nu + \check{\nu}} \times (7.6) + b_2 \times (7.23)_{l=0} + b_3 \times (7.25) + b_4 \times (7.30)_{l=0} \right\}
\]

\[
+ \sum_{2j \leq m-2} \frac{\nu}{\gamma^2} b_5 \times \{ (7.32) + (7.33) \} + \frac{b_6 \nu}{\gamma^2} \times (7.2)_{2j=m}.
\]

Taking \( \frac{1}{\nu}, \frac{\nu + \check{\nu}}{\gamma^2}, \omega_0 \) and \( b_j (j = 2, \ldots, 6) \) are suitably small, we can obtain

\[
\frac{d}{dt} E^{(4)}(t) + \frac{1}{2} D^{(4)}(t) + \frac{\omega_0}{\gamma^2} \sum_{2j \leq m-2} \| \partial_{x_2}^{l+1} \sigma_\infty(t) \|_2^2 + \| \partial_{x_2}^{l+1} \phi_\infty(t) \|_2^2
\]

\[
\leq C \sum_{j=1}^8 R^{(j)}(t).
\]

Here

\[
E^{(4)}(t) = \hat{E}^{(0)}(t) + \frac{1}{\nu + \check{\nu}} E^{(1)}(t) + b_2 E^{(2)}(t) + b_3 E^{(3)}(t) + \frac{b_6 \nu}{\gamma^2} E^{(0)}[\partial_{x_2}^{m} \check{P}_\infty u(t)],
\]

\[
D^{(4)}(t) = \frac{1}{2} \left( \dot{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t) \right)
\]

\[
+ b_4 \sum_{2j+k \leq m-1} \left( \frac{\omega^2}{\nu + \check{\nu}} \| \partial_{x_2}^2 T_{j,k} w_\infty(t) \|_2^2 + \frac{1}{\nu + \check{\nu}} \| \partial_{x_2} T_{j,k} \phi_\infty \|_2^2 \right)
\]

\[
+ \frac{b_6 \nu}{\gamma^2} D^{(0)}[\partial_{x_2}^{m} \check{P}_\infty w_\infty(t)]
\]
and

\[ R^{(1)} = \sum_{2j+k \leq m} R_{j,k}^{(1)}, \quad R^{(p)} = \sum_{2j+k \leq m-1} R_{j,k}^{(p)} \quad (p = 2, 5), \quad R^{(4)} = \sum_{2j+k \leq m} R_{j,k}^{(4)} \]

\[ R^{(p)} = \sum_{2j+k+l \leq m-1} R_{j,k,l}^{(p)} \quad (p = 3, 6), \quad R^{(p)} = \sum_{2j \leq m-1} R_{j}^{(8)} \quad (p = 7, 8). \]

To establish the desired estimate (7.1) we need to estimate higher order derivatives in \( x_2 \). For \( 1 \leq l \leq m-1 \), we set

\[ E^{(4)}_l(t) = \sum_{2j+k \leq m-1-l} \frac{1}{\gamma^2} \left\| \frac{P_{\nu}(\rho)}{\psi_x} T_{j,k} \partial^{l+1}_x \phi_\infty(t) \right\|_2 \]

and

\[ D^{(4)}_l(t) = \frac{1}{2} \sum_{2j+k \leq m-1-l} \left( \frac{\nu^2}{\nu+1} \left\| \partial^{l+2}_x T_{j,k} w_\infty(t) \right\|^2_2 + \frac{2\nu}{\nu+1} \left\| \partial^{l+1}_x T_{j,k} \phi_\infty(t) \right\|^2_2 \right) \]

\[ + \frac{b_7}{2} \sum_{2j+k \leq m-1-l} \left( \frac{\nu^2}{\nu+1} \left\| \partial^{l+2}_x T_{j,k} w_\infty(t) \right\|^2_2 + \frac{2\nu}{\nu+1} \left\| \partial^{l+1}_x T_{j,k} \phi_\infty(t) \right\|^2_2 \right). \]

Here \( b_7 \) is a positive number. Using (7.23), (7.30) with \( \nu = 1 \leq l \leq m-1 \) and (7.34), one can find small numbers \( b_7 \) and \( b_8 \) by induction on \( l \) in such a way that the following estimate holds for \( 1 \leq l \leq m-1 \):

\[
\frac{d}{dt} \left( 2^l E^{(4)}_l(t) + \sum_{p=1}^l 2^{l+1-p} b_8 E^{(4)}_p(t) \right) + \sum_{p=0}^l b_8^p D^{(4)}_p(t) \\
\leq (\sum_{p=0}^l 2^p) C_1 \sum_{j=1}^8 R^{(j)}(t),
\]

where \( D^{(4)}_0(t) = D^{(4)}(t) \). See [8, Section 5.2] for the details. The desired estimate (7.1) now follows from (7.36) with \( l = m-1 \); and Proposition 7.1 is proved. \( \square \)

To prove (5.15) we employ the following lemma.

**Lemma 7.11.** There exists a positive number \( \tilde{r}_0 = \tilde{r}_0(\nu, \tilde{\nu}, \gamma) \) such that if \( r_1 \leq \tilde{r}_0 \), then there holds the estimate

\[ \| \Pi^{(0)} P_{\infty,1} u \|_2 \leq C \| \partial_{x_1} (I - \Pi^{(0)}) P_{\infty,1} u \|_2. \]

**Proof.** We set \( \tilde{\Pi}^{(1)}(\xi) = \xi \tilde{\Pi}^{(1)} + \xi^2 \tilde{\Pi}^{(2)}(\xi) \). Since

\[ \Pi^{(0)}(I - \tilde{\Pi}(\xi)) = \Pi^{(0)}(I - \Pi^{(0)} - \tilde{\Pi}^{(1)}(\xi)) = -\Pi^{(0)}(\tilde{\Pi}^{(1)}(\xi)), \]

we see that

\[ \Pi^{(0)} P_{\infty,1} u = \Pi^{(0)} \tilde{\chi}_1(\xi)(I - \tilde{\Pi}(\xi)) \hat{u} = -\tilde{\chi}_1(\xi) \Pi^{(0)} \tilde{\Pi}^{(1)}(\xi) \hat{u}. \]
It then follows that
\[
|\Pi(0)P_{\infty,1}u|_2 \leq C|\xi||\hat{\chi}_1u|_2
\leq C|\xi|\left(\hat{\chi}_1(I - \Pi(0))u + \hat{\chi}_1|\Pi(0)u|_2\right).
\]

Since \((P_{\infty,1})^2 = P_{\infty,1}\), we find that
\[
|\Pi(0)P_{\infty,1}u|_2 \leq C|\xi|\left((I - \Pi(0))P_{\infty,1}u|_2 + |\Pi(0)P_{\infty,1}u|_2\right)
\]
for \(|\xi| \leq r_1\). Therefore, there exists a positive number \(\tilde{r}_0\) such that if \(r_1 \leq \tilde{r}_0\), then
\[
|\Pi(0)P_{\infty,1}u|_2 \leq C|\xi|(I - \Pi(0))P_{\infty,1}u|_2
\]
for \(|\xi| \leq r_1\), from which we obtain
\[
\|\Pi(0)P_{\infty,1}u\|_2 \leq C\|\partial_{x_1}(I - \Pi(0))P_{\infty,1}u\|_2.
\]

This completes the proof.

We now prove (5.15).

**Proof of (5.15).** We fix \(\nu, \tilde{\nu}, \gamma\) and \(\omega_0\) so that Proposition 7.1 holds true; and set \(r_1 = \min\{r_0, \tilde{r}_0, 1\}\). Then, obviously,
\[
\|\Pi(0)P_{\infty,2}u\|_2 \leq \frac{1}{r_1}\|\partial_{x_1}\Pi(0)P_{\infty,2}u\|_2.
\]

By Lemma 7.11 we also have
\[
\|\Pi(0)P_{\infty,1}u\|_2 \leq C\|\partial_{x_1}(I - \Pi(0))P_{\infty,1}u\|_2.
\]

We thus obtain
\[
\|\sigma_\infty\|_2 \leq C\left\{\|\partial_{x_1}(I - \Pi(0))P_{\infty,1}u\|_2 + \frac{1}{r_1}\|\partial_{x_1}\Pi(0)P_{\infty,2}u\|_2\right\}
\leq C(\|\partial_{x_1}\sigma_\infty\|_2 + \|\partial_{x_1}u_\infty\|_2).
\]

By Lemma 5.2 (ii) we have
\[
\|u_\infty\|_2 \leq C\|\partial_{x_1}u_\infty\|_2.
\]

It follows from (7.37) and (7.38) that
\[
D(t) \geq \tilde{a}E(t)
\]

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for a constant $\bar{a} > 0$. We thus deduce from (7.1) and (7.39) that

\begin{equation}
\frac{d}{dt} \tilde{E}(t) + \bar{a} \tilde{E}(t) + D(t) \leq \bar{R}(t).
\end{equation}

This gives

\begin{equation}
\tilde{E}(t) + \int_0^t e^{-\bar{a}(t-\tau)} D(\tau) d\tau \leq e^{-\bar{a}t} \tilde{E}(0) + \int_0^t e^{-\bar{a}(t-\tau)} \bar{R}(\tau) d\tau.
\end{equation}

Writing the second and third equation of (5.4) as

\begin{equation}
-\nu \partial_{x_2}^2 w^1 = \rho_s f^1 - \rho_s \{ \partial_t w^1 - \frac{\nu}{\rho_s} \partial_{x_1}^2 w^1 - \frac{\bar{\nu}}{\rho_s} \partial_{x_1} \text{div} w^\infty + \partial_{x_1} (\frac{P'(\rho_s)}{\gamma^2 \rho_s} \partial_{x_1} \phi) \\
+ v_1^1 \partial_{x_1} w^1 - \frac{\nu}{\gamma^2 \rho_s} v_1^1 \partial_{x_1} \phi - (\partial_{x_2} v_1^1) \partial_{x_1} w^\infty \\
- \frac{\nu + \bar{\nu}}{\rho_s} w^{(0),1} \partial_{x_2} \sigma_\infty + \alpha_0 \partial_{x_1} \sigma_\infty + v_1^1 w^{(0),1} \partial_{x_1} \sigma_\infty \\
- (Q_0 \tilde{B}(\sigma_\infty u^{(0)} + w^\infty), v^{(0),1})
\end{equation}

we can obtain

\begin{equation}
\frac{\nu^2}{\nu + \bar{\nu}} \left[ \partial_{x_2}^2 w^\infty(t) \right]^2_{m-2} \leq C \tilde{E}(t) + C R^{(9)}(t)
\end{equation}

where $R^{(9)}(t) = \left[ Q \tilde{F}_\infty(t) \right]^2_{m-2}$. It then follows from (7.41) and (7.42) that

\begin{equation}
\tilde{E}(t) + \frac{\nu^2}{\nu + \bar{\nu}} \left[ \partial_{x_2}^2 w^\infty(t) \right]^2_{m-2} + \int_0^t e^{-\bar{a}(t-\tau)} D(\tau) d\tau \leq C \left\{ e^{-\bar{a}t} \tilde{E}(0) + R^{(9)}(t) + \int_0^t e^{-\bar{a}(t-\tau)} \bar{R}(\tau) d\tau \right\}
\end{equation}

Since

\begin{equation}
R^{(9)}(t) \leq C(1 + t)^{-\frac{3}{4}} M(t)^4
\end{equation}

as we will see in Proposition 8.6 below, we deduce (5.15) from (7.43) and (7.44). This completes the proof.

\section{8. Estimates on the nonlinearities}

In this section we establish the estimates on the nonlinearities, namely, (5.16) and (7.44)
Proposition 8.1. There exists a number \( \varepsilon_6 > 0 \) such that the following assertion holds true.

Let \( u(t) \) be a solution of (4.1) in \( Z^m(T) \). If \( \sup_{0 \leq \tau \leq T} \| u(\tau) \|_m \leq \varepsilon_6 \) and \( M(t) \leq 1 \) for all \( t \in [0, T] \), then the following estimates hold for all \( t \in [0, T] \) with \( C > 0 \) independent of \( T \).

(i) \[
\| Q^\infty(t) \|_{m-2} \leq C(1+t)^{-\frac{3}{4}} M(t)^2,
\]

(ii) \[
\hat{R}(t) \leq C \{(1+t)^{-\frac{3}{4}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) D^\infty(t)\}.
\]

To prove Proposition 8.1 we employ the following inequalities.

Lemma 8.2. (i) Let \( 2 \leq p \leq \infty \) and let \( j \) and \( k \) be integers satisfying

\[
0 \leq j < k, \quad k > j + n \left( \frac{1}{2} - \frac{1}{p} \right).
\]

Then there exists a constant \( C > 0 \) such that

\[
\| \partial_x^j f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}^{1-a} \| \partial_x^k f \|_{L^2(\mathbb{R}^n)}^a,
\]

where \( a = \frac{1}{k} \left( j + \frac{n}{2} - \frac{n}{p} \right) \).

(ii) Let \( 2 \leq p \leq \infty \) and let \( j \) and \( k \) be integers satisfying

\[
0 \leq j < k, \quad k > j + 2 \left( \frac{1}{2} - \frac{1}{p} \right).
\]

Then there exists a constant \( C > 0 \) such that

\[
\| \partial_x^j f \|_p \leq C \| f \|_{H^s}.
\]

(iii) If \( f \in H^1 \) and \( f = f(x_1) \) is independent of \( x_2 \). Then

\[
\| f \|_{\infty} \leq C \| f \|_2^{\frac{1}{2}} \| \partial_{x_1} f \|_2^{\frac{1}{2}}.
\]

Proof. The inequality in (i) is a special case of the Gagliardo-Nirenberg-Sobolev inequality which can be proved by using Fourier transform. Inequality in (ii) can be obtained by (i) with \( n = 2 \) and the standard extension argument. As for (iii), we note that if \( f = f(x_1) \), then

\[
\| f \|_p = \| f \|_{L^p(\mathbb{R})} \quad (1 \leq p \leq \infty), \quad \| \partial_{x_1} f \|_2 = \| \partial_{x_1} f \|_{L^2(\mathbb{R})}.
\]
Therefore, the inequality in (iii) is a simple consequence of (i) with \( n = 1, \ p = \infty, \ j = 0 \) and \( k = 1 \).

**Lemma 8.3.** (i) Let \( m \) and \( m_k \) \( k = 1, \cdots, \ell \) be nonnegative integers and let \( \alpha_k \) \( k = 1, \cdots, \ell \) be multi-indices. Suppose that

\[
m \geq \left\lceil \frac{n}{2} \right\rceil + 1, \quad 0 \leq |\alpha_k| \leq m \leq |\alpha_k| \quad (k = 1, \cdots, \ell)
\]

and

\[
m_1 + \cdots + m_\ell \geq (\ell - 1)m + |\alpha_1| + \cdots + |\alpha_\ell|.
\]

Then there exists a constant \( C > 0 \) such that

\[
\| \partial_{x_1}^{\alpha_1} \cdots \partial_{x_\ell}^{\alpha_\ell} f \|_2 \leq C \prod_{1 \leq k \leq \ell} \| f_k \|_{H^{m_k}}.
\]

(ii) Let \( 1 \leq k \leq m \). Suppose that \( F(x, t, y) \) is a smooth function on \( \Omega \times [0, \infty) \times I \), where \( I \) is a compact interval in \( \mathbb{R} \). Then for \( |\alpha| + 2j = k \) there hold

\[
\| [\partial_{x}^{\alpha} \partial_{t}^{j}, F(x, t, f_1)] f_2 \|_2 \leq \begin{cases}
C_0(t, f_1(t)) \| f_2 \|_{k-1} + C_1(t, f_1(t)) \{ 1 + \| Df_1 \|_{m-1} \} \| Df_1 \|_{m-1} \| f_2 \|_{k}, \\
C_0(t, f_1(t)) \| f_2 \|_{k-1} + C_1(t, f_1(t)) \{ 1 + \| Df_1 \|_{m-1} \} \| Df_1 \|_{m} \| f_2 \|_{k-1}.
\end{cases}
\]

Here

\[
C_0(t, f_1(t)) = \sum_{(\beta, \ell) \leq (\alpha, j)} \sup_x \| (\partial_{x}^{\beta} \partial_{t}^{j} F)(x, t, f_1(x, t)) \|
\]

and

\[
C_1(t, f_1(t)) = \sum_{1 \leq p \leq j + |\alpha|} \sup_x \| (\partial_{x}^{p} \partial_{t}^{j} \partial_{y}^{p} F)(x, t, f_1(x, t)) \|.
\]

The proof of Lemma 8.3 can be found in [10, 11].

We recall that \( u(t) \) is decomposed into

\[
u(t) = \sigma_1 u(0) + u_1 + \sigma_\infty u(0) + u_\infty;
\]

and we write

\[
\sigma_* = \sigma_1 + \sigma_\infty, \quad \phi_* = \phi_1 + \phi_\infty, \quad w_* = w_1 + w_\infty, \quad u_* = (\phi_*, w_*) = (u_1 + u_\infty)
\]

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as in section 7.

Before estimating the nonlinearities we make a simple observation.

Lemma 8.4. Let \( u(t) = \top(\phi(t), w(t)) = (\sigma_* u^{(0)}(t) + u_*(t) \) be a solution of (4.1) in \( Z^m(T) \). Then there hold the following estimates for all \( t \in [0, T] \) with \( C > 0 \) independent of \( T \).

(i) \[ \| \sigma_*(t) \|_2 \leq C(1 + t)^{-\frac{1}{2}}M(t), \]

(ii) \[ \| D\sigma_*(t) \|_{m-1} + \| u_*(t) \|_m \leq C(1 + t)^{-\frac{3}{2}}M(t), \]

(iii) \[ \| \phi(t) \|_m + \| w(t) \|_m \leq C(1 + t)^{-\frac{1}{2}}M(t), \]

(iv) \[ \| \sigma_*(t) \|_\infty \leq C(1 + t)^{-\frac{1}{2}}M(t), \]

(v) \[ \| \phi(t) \|_\infty + \| w(t) \|_\infty \leq C(1 + t)^{-\frac{1}{2}}M(t). \]

Proof. Estimates (i), (ii) and (iii) immediately follow from the definition of \( M(t) \). As for (iv), we see from Lemma 8.2 (iii) that

\[ \| \sigma_*(t) \|_\infty \leq C\| \sigma_*(t) \|_2 \| \partial_{\pi} \sigma_*(t) \|_2 \leq C(1 + t)^{-\frac{1}{2}}M(t). \]

This shows (iv). Estimate (v) now follows from (iv) and (ii). This completes the proof.

Let us consider estimates on \( Q_0 F \).

Proposition 8.5. Let \( u(t) \) be a solution of (4.1) in \( Z^m(T) \) and assume that \( M(t) \leq 1 \) for all \( t \in [0, T] \). Then the following estimates hold with \( C > 0 \) independent of \( T \).

(i) \[ \| \phi \text{div} w \|_l \leq C \left\{ \begin{array}{ll} (1 + t)^{-1}M(t)^2 + (1 + t)^{-\frac{1}{2}}M(t)\| Dw_\infty(t) \|_m & (l = m), \\ (1 + t)^{-1}M(t)^2 & (l = m - 1), \end{array} \right. \]

(ii) \[ \| w \cdot \nabla (\sigma_* \phi^{(0)} + \phi_1) \|_m \leq C(1 + t)^{-1}M(t)^2, \]
\[ \left[ w \cdot \nabla \phi_\infty \right]_{m-1} \leq C(1 + t)^{-1} M(t)^2, \]

(iii) \[
\left| \left( \text{div} \left( \frac{P(\rho)}{\gamma \rho^s} w \right), \partial_t^j \partial_x^k \phi_\infty \right) \right| \leq C(1 + t)^{-\frac{1}{2}} M(t) D_\infty(t) \]

for \(2j + k = m\),

(iv) \[
\left\| \left[ \partial_t^j \partial_x^k, w \right] \cdot \nabla \phi_\infty \right\|_2 \leq C(1 + t)^{-\frac{1}{2}} M(t) \sqrt{D_\infty(t)} \]

for \(2j + k = m\),

(v) \[
\left\| \partial_t^j (\phi w) \right\|_2 \leq (1 + t)^{-1} M(t)^2 \]

for \(1 \leq j \leq \left[ \frac{m}{2} \right]\).

**Proof.** We write \( \phi \text{div} w \) as

\[
\phi \text{div} w = \sigma_* \phi^{(0)} w^{(0),1} \partial_{x_1} \sigma_* + \sigma_* \phi^{(0)} \text{div} w_* + \phi_* w^{(0),1} \partial_{x_1} \sigma_* + \phi_* \text{div} w_*.
\]

By Lemma 4.5 and Lemma 5.2, we have \( \left[ \partial_{x_1} \sigma_*(t) \right]_m \leq C \left\| \partial_{x_1} \sigma_* \right\|_{m-1} \). So applying Lemma 8.3 to each term on the right-hand side of (8.1) and using Lemma 8.4, we obtain estimate (i). For example, let \(2j + k \leq m\). Then

\[
\left\| \partial_t^j \partial_x^k \sigma_* \phi^{(0)} w^{(0),1} \partial_{x_1} \sigma_* \right\|_2 
\]

\[
\leq \left\| \sigma_* \phi^{(0)} w^{(0),1} \partial_t^j \partial_x^k \partial_{x_1} \sigma_* \right\|_2 + \left\| \left[ \partial_t^j \partial_x^k, \sigma_* \phi^{(0)} w^{(0),1} \right] \partial_{x_1} \sigma_* \right\|_2 
\]

\[
\leq C \left\{ \left\| \sigma_* \phi^{(0)} w^{(0),1} \partial_{x_1} \sigma_* \right\|_2 + \left\| \left[ \partial_t^j \partial_x^k, \sigma_* \phi^{(0)} w^{(0),1} \right] \partial_{x_1} \sigma_* \right\|_2 \right\} 
\]

\[
\leq C \left\{ \left\| \sigma_* \phi^{(0)} \right\|_\infty \left\| \partial_{x_1} \sigma_* \right\|_{m-1} + \left\| \sigma_* \phi^{(0)} w^{(0),1} \partial_{x_1} \sigma_* \right\|_{m-1} \right\} 
\]

\[
\leq C \left\| \sigma_* \phi^{(0)} \right\|_\infty + \left\| \partial_{x_1} \sigma_* \right\|_{m-1} \left\| \partial_{x_1} \sigma_* \right\|_{m-1} 
\]

\[
\leq C(1 + t)^{-\frac{1}{2}} M(t)^2.
\]

As for (ii), we write

\[
w \cdot \nabla (\sigma_* \phi^{(0)} + \phi_1) = \sigma_* \phi^{(0)} w^{(0),1} \partial_{x_1} \sigma_* + w^1_* \partial_{x_1} \sigma_* \phi^{(0)} + w^0_1 \partial_{x_1} \phi_1 + \phi_* \text{div} \phi_1.
\]

The first three terms on the right of (8.2) can be estimated similarly to the right of (8.1). As for the last two terms on the right of (8.2), we employ
the estimate $[\nabla \phi_1]_m \leq C[u_1]_m$, which follows from Lemma 4.5. Using this inequality and Lemma 8.3 and Lemma 8.4, we have, for $2j + k \leq m$,

$$
\| \partial^j_t \partial^k_x (w_\ast \cdot \nabla \phi_1) \|_2 \leq \| w_\ast \partial^j_t \partial^k_x \nabla \phi_1 \|_2 + \| [\partial^j_t \partial^k_x, w_\ast] \cdot \nabla \phi_1 \|_2 \\
\leq C \{ \| w_\ast \|_\infty [\nabla \phi_1]_m + \| D w_\ast \|_{m-1} [\nabla \phi_1]_m \} \\
\leq C \| w_\ast \|_m [\nabla \phi_1]_m \\
\leq C (1 + t)^{-\frac{3}{2}} M(t)^2.
$$

Similarly we can obtain

$$
\| \partial^j_t \partial^k_x (\sigma \ast w^{(0)}, 1 \partial_{x_1} \phi_1) \|_2 \leq C (1 + t)^{-\frac{5}{2}} M(t)^2.
$$

Estimate (iii) can be obtained by applying Lemma 8.3 and Lemma 8.4 to $w \cdot \nabla \phi_\infty = \sigma \ast w^{(0), 1} \partial_{x_1} \phi_\infty + w_\ast \cdot \nabla \phi_\infty$.

As for (iv), since $j \geq 1$ or $k \geq 1$, by the Hölder and Sobolev inequalities, we have

$$
\| [\partial^j_t \partial^k_x, w] \cdot \nabla \phi_\infty \|_2 \leq C \| D w \|_m \| \partial_{x_1} \phi_\infty \|_m - 1 \leq C (1 + t)^{-\frac{1}{2}} M(t) D\infty(t).
$$

Estimate (v) is obtained by Lemma 8.3 and Lemma 8.4:

$$
\| [\partial^j_t \partial^k_x, w] \cdot \nabla \phi_\infty \|_2 \leq C \| D w \|_m \| \partial_{x_1} \phi_\infty \|_m - 1 \leq C (1 + t)^{-\frac{1}{2}} M(t) \sqrt{D\infty(t)}.
$$

This completes the proof. □

We next consider $\hat{Q} F = \overline{(0, f)}$. We first observe that since $m \geq 2(= [n/2] + 1)$ we have the Sobolev inequality

$$
\| f \|_\infty \leq C_S \| f \|_{H^m}.
$$

Let $\varepsilon_6 > 0$ be a number such that $C_S \varepsilon_6 \leq \frac{\gamma^2 \rho_1}{2}$. Then whenever $\| u(t) \|_m \leq \varepsilon_6$, we have

$$
\| \phi(t) \|_\infty \leq C_S \| u(t) \|_m \leq C_S \varepsilon_6 \leq \frac{\gamma^2 \rho_1}{4},
$$

and hence,

$$
\rho(x, t) = \rho_\ast (x_n) + \gamma^{-2} \phi(x, t) \geq \rho_1 - \gamma^{-2} \| \phi(t) \|_\infty \geq \frac{3 \rho_1}{4} (> 0).
$$
We thus see that $\tilde{Q} \mathbf{F}(t)$ is smooth whenever $[u(t)]_m \leq \varepsilon_6$. So, we assume that $[u(t)]_m \leq \varepsilon_6$ for $t \in [0, T]$.

Recall that $\tilde{Q} \mathbf{F}$ is written in the form

$$\tilde{Q} \mathbf{F} = \tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3.$$ 

Here $\tilde{F}_l = \nabla (0, f_l) \ (l = 0, 1, 2, 3)$, where

$$f_0 = -w \cdot \nabla w + f_1(\rho, \phi)(-\partial^2_{x_1} \sigma_* w^{(0)} + \frac{\partial^2_{x_2} v_*}{\gamma \rho_*} \phi_*)$$

$$+ f_2(\rho, \phi)(-\partial^2_{x_1} \sigma_* w^{(0)} - \partial_{x_1} \sigma_* \partial_{x_2} w^{(0)})$$

$$+ f_{01}(x_2, \phi) \phi \sigma_* + f_{02}(x_2, \phi) \phi \partial_{x_1} \sigma_* + f_{03}(x_2, \phi) \phi \phi_*,$$

$$f_1 = -\nabla (f_1(\rho, \phi) \nabla w_*) + \nabla (f_1(\rho, \phi)),$$

$$f_2 = -\nabla (f_2(\rho, \phi) \nabla w_*) + (\nabla w_*) \nabla (f_2(\rho, \phi)),$$

$$f_3 = \nabla (f_3(x_2, \phi) \phi \phi_*) - \phi_* \nabla (f_3(x_2, \phi)).$$

Here $f_1 = \frac{\nu \phi}{(\phi + \gamma \rho_*) \rho_*}$; $f_2 = \frac{\nu \phi}{(\phi + \gamma \rho_*) \rho_*}$; and $f_{0l}(x_2, \phi) \ (l = 1, 2, 3)$ and $f_{3l}(x_2, \phi)$ are smooth functions of $x_2$ and $\phi$.

**Proposition 8.6.** Let $u(t)$ be a solution of (4.1) in $Z^m(T)$ and assume that $\sup_{0 \leq \tau \leq T} [u(\tau)]_m \leq \varepsilon_6$ and $M(t) \leq 1$ for all $t \in [0, T]$. Then the following estimates hold with $C > 0$ independent of $T$.

(i) $$[\tilde{Q} \mathbf{F}]_{m-2} \leq C(1 + t)^{-\frac{3}{2}} M(t)^2,$$

(ii) $$[f_0]_m \leq C\{(1 + t)^{-\frac{3}{2}} M(t)^2 + (1 + t)^{-\frac{3}{2}} M(t) ||Dw_\infty(t)||_m\},$$

(iii) $$\sum_{l=1}^3 [f_l]_{m-1} \leq C\{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{1}{2}} M(t) ||Dw_\infty(t)||_m\},$$

(iv) $$\sum_{l=1}^3 ||T_{j,k} f_l||_{H^{-1}} \leq C\{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{1}{2}} M(t) ||Dw_\infty(t)||_m\}$$

for $2j + k = m$. Here we regard $T_{j,k} f_l$ with $2j + k = m$ as an element in $H^{-1}$ by $(T_{j,k} f_l)[v] = \langle (T_{j,k} f_l), v \rangle_{-1}$ for $v \in H_0^1$.

**Proof.** Estimates (i), (ii), (iii) can be obtained similarly to the proof of Proposition 8.5. So we omit the proof.
Let us prove estimate (iv). Let \(2j + k = m\) and let \(v \in H_0^1\). If \(k \geq 1\), then we see from (iii) that

\[
|\langle (T_{j,k} f_{j_1}, v) \rangle_{-1}| = | - (T_{j,k-1} f_{j_1}, T_{0,1} v)|
\leq \|T_{j,k-1} f_{j_1}\|_2 \|T_{0,1} v\|_2
\leq C \{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{5}{4}} M(t) ||D w_\infty(t)||_{m}\} \|v\|_{H_0^1}.
\]

We thus obtain

\[
\|T_{j,k} f_{j_1}\|_{H^{-1}} \leq C \{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{5}{4}} M(t) ||D w_\infty(t)||_{m}\} \|v\|_{H_0^1}.
\]

If \(k = 0\), i.e., \(m = 2j\), then, by definition,

\[
\langle (\partial_l^j f_{j_1}, v) \rangle_{-1}
= (\partial_l^j (f_1(\rho_s, \phi) \nabla w_s), \nabla v) + (\partial_l^j ([\nabla (\nabla w_s)])_{\rho_s} f_1(\rho_s, \phi) \nabla \rho_s), v)
+ (\partial_l^j (\nabla w_s))_{\rho_s} f_1(\rho_s, \phi) \nabla \phi, v)
\equiv I_1 + I_2 + I_3.
\]

As for \(I_1\), we have

\[
|I_1| \leq \|\partial_l^j (f_1(\rho_s, \phi) \nabla w_s)\|_2 \|\nabla v\|_2.
\]

As in the proof of Proposition 8.5 one can estimate \(\|\partial_l^j (f_1(\rho_s, \phi) \nabla w_s)\|_2\) to obtain

\[
|I_1| \leq C \{(1 + t)^{-\frac{5}{4}} M(t)^2 + (1 + t)^{-\frac{5}{4}} M(t) ||D w_\infty(t)||_{m}\} \|v\|_{H_0^1}.
\]

Similarly we have

\[
|I_2| \leq C \{(1 + t)^{-\frac{5}{4}} M(t)^2 + (1 + t)^{-\frac{5}{4}} M(t) ||D w_\infty(t)||_{m}\} \|v\|_{H_0^1}.
\]

We next consider \(I_3\) which we write as follows:

\[
I_3 = (\partial_{\phi} f_1(\rho_s, \phi) \partial_l^j (\nabla (\nabla w_s) \nabla \phi), v) + ([\partial_l^j, \partial_{\phi} f_1(\rho_s, \phi)] (\nabla (\nabla w_s) \nabla \phi), v)
\equiv J_1 + J_2.
\]

As for \(J_1\), we have

\[
|J_1| \leq |(\partial_{\phi} f_1(\rho_s, \phi) (\nabla \partial_l^{j_1} (\nabla w_s) \nabla \phi), v)| + |(\partial_{\phi} f_1(\rho_s, \phi) (\nabla \partial_l^{j_1} (\nabla w_s) \nabla \phi), v)|
\equiv J_{11} + J_{12}.
\]
Using Lemma 8.2 and Lemma 8.4 we have

\[ J_{11} \leq C \|\nabla \phi\|_4 \|\partial_t^j \partial_x \varphi\|_2 \|v\|_4 \]
\[ \leq C \|\phi\|_{H^2} \{\|\partial_x w_1\|_m + ||Dw_\infty||_m\} \|v\|_{H^1_0} \]
\[ \leq C\{(1 + t)^{-1} \|M(t)^2 + (1 + t)^{-\frac{3}{2}} M(t)||\|Dw_\infty(t)||_m\} \|v\|_{H^1_0}. \]

As for \( J_{12} \), we have

\[ \|\partial_t^j \nabla \phi \|^T (\nabla w_*) \leq C \sum_{i=1}^j \|\partial_t^i \nabla \phi \| \|\partial_t^{j-i} \partial_x \varphi\| \|v\|_{p_3,1}. \]

Since

\[ \frac{1}{2} - \frac{m - 1 - 2l}{2} + \frac{1}{2} - \frac{m - 2(j - l)}{2} = 1 - \frac{m - 1}{2} < 1, \]

we can find \( p_{1,l}, p_{2,l} \geq 2 \) satisfying

\[ \frac{1}{p_{1,l}} > \frac{1}{2} - \frac{m - 1 - 2l}{2}, \quad \frac{1}{p_{2,l}} > \frac{1}{2} - \frac{m - 2(j - l)}{2}, \quad \frac{1}{2} \leq \frac{1}{p_{1,l}} + \frac{1}{p_{2,l}} < 1. \]

We now take a number \( p_{3,l} \geq 2 \) satisfying \( \frac{1}{p_{3,l}} = 1 - \left( \frac{1}{p_{1,l}} + \frac{1}{p_{2,l}} \right) > 0 \). It then follows from Lemma 8.2 that

\[ |J_{12}| \leq C \sum_{i=1}^j \|\partial_t^i \partial_x \phi\|_{p_{1,l}} \|\partial_t^{j-i} \partial_x \varphi\|_{p_{2,l}} \|v\|_{p_{3,l}} \]
\[ \leq C \sum_{i=1}^j \|\partial_t^i \partial_x \phi\|_{H^{m-1-2l}} \|\partial_t^{j-i} \partial_x \varphi\|_{H^{m-2(k-j)}} \|v\|_{H^1_0} \]
\[ \leq C \sum_{i=1}^j \{\|\partial_x w_1\|_m + ||Dw_\infty||_m\} \|v\|_{H^1_0} \]
\[ \leq C\{(1 + t)^{-1} \|M(t)^2 + (1 + t)^{-\frac{3}{2}} M(t)||\|Dw_\infty(t)||_m\} \|v\|_{H^1_0}. \]

It remains to estimate \( J_2 \). By Lemma 8.3 we have

\[ |J_2| \leq C \{\|\partial_x \phi\|_{m-1} \|^T (\nabla w_*) \nabla \phi\}_{m-1} \|v\|_2, \]

and, also,

\[ \|\nabla w_*\|_{H^m} \leq C\{\|\nabla w_*\|_{H^m} + ||\nabla \phi\|_{m-1} \|\nabla \phi\|_{m-1} \} \]
\[ \leq C\{\|\partial_x w_1\|_m + ||Dw_\infty||_m\} \|\partial_x \phi\|_{m-1} \]
\[ \leq C\{(1 + t)^{-1} \|M(t)^2 + (1 + t)^{-\frac{3}{2}} M(t)||\|Dw_\infty(t)||_m\} \].
As for $[\partial_t \phi]_{m-1}$, we see from (6.3) and Proposition 7.10 that
\[
[\partial_t \phi]_{m-1} \leq C\{[\partial_t \sigma_*]_{m-1} + [\partial_t \phi_*]_{m-1}\}
\leq C\{(1 + t)^{-\frac{3}{2}} M(t) + [\partial_x \phi_*]_{m-1} + [\partial_x w_*]_{m-1}
+ [\partial_x \sigma_*]_{m-1} + [P_1 \mathbf{F}]_{m-1} + [Q_0 \mathbf{F}]_{m-1} + [Q_0 \mathbf{F}_\infty]_{m-1}\}
\leq C\{(1 + t)^{-\frac{3}{2}} M(t) + [P_1 \mathbf{F}]_{m-1} + [Q_0 \mathbf{F}]_{m-1}\}.
\]

Since $2j = m$, we have $\frac{m-1}{2} = \frac{m-2}{2}$, and hence, by Lemma 4.5,
\[
[P_1 \mathbf{F}]_{m-1} \leq C[\mathbf{F}]_{m-2}.
\]

It then follows from Proposition 8.5 and Proposition 8.6 (i) that
\[
[P_1 \mathbf{F}]_{m-1} + [Q_0 \mathbf{F}]_{m-1} \leq C\{[Q_0 \mathbf{F}]_{m-1} + [\hat{Q} \mathbf{F}]_{m-2}\} \leq C(1 + t)^{-\frac{3}{2}} M(t)^2,
\]
which implies that
\[
[\partial_t \phi]_{m-1} \leq C(1 + t)^{-\frac{3}{2}} M(t).
\]

We thus obtain
\[
|J_2| \leq C\{(1 + t)^{-\frac{7}{2}} M(t)^2 + (1 + t)^{-1} M(t) ||Dw_\infty(t)||_m\} \|v\|_2.
\]

Consequently,
\[
|I_3| \leq C\{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{5}{2}} M(t) ||Dw_\infty(t)||_m\} \|v\|_{H^1_0}.
\]

Therefore, we arrive at
\[
||\langle \partial_t^j \mathbf{f}_1, v \rangle \rangle_{-1} | \leq C\{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{1}{2}} M(t) ||Dw_\infty(t)||_m\} \|v\|_{H^1_0}.
\]

This gives
\[
||\partial_t^j \mathbf{f}_1||_{H^{-1}} \leq C\{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{1}{2}} M(t) ||Dw_\infty(t)||_m\}.
\]

Clearly, one can obtain the same estimate for $||\partial_t^j \mathbf{f}_2||_{H^{-1}}$. Concerning $||\partial_t^j \mathbf{f}_3||_{H^{-1}}$, one can obtain the desired estimate by replacing $\nabla w_*$ with $\phi_*$ in the argument above for $||\partial_t^j \mathbf{f}_1||_{H^{-1}}$. This completes the proof. \hfill \qed

We are now in a position to prove Proposition 8.1.

Proof of Proposition 8.1. Since $[\hat{Q} \mathbf{F}_\infty]_{m-2} \leq C[\mathbf{F}]_{m-2}$, assertion (i) follows from Proposition 8.5 and Proposition 8.6 (i).
Let us consider $\tilde{R}(t)$. We write $R_{j,k}^{(1)}$ as

$$R_{j,k}^{(1)} = \frac{a_0}{\pi} \langle (Q_0 T_{j,k} F)_{\infty}, T_{j,k} \sigma_\infty \rangle - \frac{a_0}{\pi} \langle (Q_0 T_{j,k} P_1 F)_{\infty}, T_{j,k} \sigma_\infty \rangle + \tilde{R}_{j,k}^{(1)} - \langle \langle (Q_0 T_{j,k} F)_{\infty} u^{(0)}, T_{j,k} u_\infty \rangle \rangle + \langle \langle T_{j,k} P_1 F, T_{j,k} u_\infty \rangle \rangle - \langle \langle (Q_0 T_{j,k} P_1 F)_{\infty} u^{(0)}, T_{j,k} u_\infty \rangle \rangle.$$

$\equiv \sum_{l=1}^{6} I_l.$

We first consider $I_3$. If $2j + k \leq m - 1$, then applying Proposition 8.5 (i)–(iii) and Proposition 8.6 (ii), (iii), we have

$$|I_3| \leq C \{(1 + t)^{-\frac{3}{2}} M(t)^3 + (1 + t)^{-\frac{1}{2}} M(t) |||Dw_\infty(t)|||^2_m \}.$$ 

Here we have used the Poincaré inequality $||w_\infty(t)||_2 \leq ||\partial_x w_\infty(t)||_2$ for $j = k = 0$. In case $2j + k = m$, applying Proposition 8.5 (i), (ii), (iv), (v), and Proposition 8.6 (ii), (iv), we obtain

$$|I_3| \leq C \{(1 + t)^{-\frac{3}{2}} M(t)^3 + (1 + t)^{-\frac{1}{2}} M(t) D_\infty(t) \} \leq C \{(1 + t)^{-\frac{3}{2}} M(t)^3 + (1 + t)^{-\frac{1}{2}} M(t) D_\infty(t) \}$$

if $M(t) \leq 1$.

We next consider $I_5$. We write $I_5$ as

$$I_5 = \langle \langle T_{j,k} F, P_1^* T_{j,k} u_\infty \rangle \rangle.$$

If $2j + k \leq m - 1$, one can estimate $I_5$ as $I_3$. If $2j + k = m$ and $k \geq 1$, then, by Lemma 4.5 (iii),

$$|I_5| \leq | - \langle \langle T_{j,k-1} F, \partial_x, P_1^* T_{j,k} u_\infty \rangle \rangle | \leq C \|T_{j,k-1} F\|_2 \|T_{j,k} u_\infty\|_2 \leq C \{(1 + t)^{-\frac{3}{2}} M(t)^3 + (1 + t)^{-\frac{1}{2}} M(t) |||Dw_\infty(t)|||^2_m \}.$$

In case $2j = m$, let $\Phi$ be defined by $Q_0 P_1^* \partial_t^j u_\infty = \Phi(\Phi, 0)$. Then, by integration by parts, we have

$$\langle \langle \partial_t^j Q_0 F, P_1^* \partial_t^j u_\infty \rangle \rangle = - (\partial_t^j (\phi w), \nabla (\Phi \frac{P_1^*(\rho_s)}{\gamma \rho_s})).$$

Since $\|\nabla (\Phi \frac{P_1^*(\rho_s)}{\gamma \rho_s})\|_2 \leq C \|\partial_t^j u_\infty\|_2$ by Lemma 4.5 (iii), we see from Proposition 8.5 (vi) that

$$|\langle \langle \partial_t^j Q_0 F, P_1^* \partial_t^j u_\infty \rangle \rangle| \leq C (1 + t)^{-\frac{3}{2}} M(t)^3.$$
Since $\hat{Q}P_1^*\partial_t^j u_\infty \in H_0^1$ by Lemma 4.5 (ii), one can estimate $\langle\langle \partial_t^j \hat{Q}F, P_1^*\partial_t^j u_\infty \rangle\rangle$ as in the estimate of $I_3$ and we obtain

$$|I_5| \leq C\{(1 + t)^{-\frac{7}{4}} M(t)^3 + (1 + t)^{-\frac{1}{4}} M(t)D_\infty(t)\}.$$  

As for $I_4$, we have

$$I_4 = -(2\pi)^{-1}\langle\langle \hat{x}_1 + \hat{x}_2)\Pi(0)T_{j,k}F, T_{j,k}u_\infty \rangle\rangle = -(2\pi)^{-1}\langle\langle T_{j,k}F, (\hat{x}_1 + \hat{x}_2)\Pi(0)^*T_{j,k}u_\infty \rangle\rangle = \langle\langle T_{j,k}F, T_{j,k}u_\infty, u(0)^\infty u(0)^* \rangle\rangle.$$  

Here and in what follows we denote $\langle\langle f, u(0)^\infty \rangle\rangle = \mathcal{F}^{-1}(\langle\langle \hat{\chi}_1 + \hat{\chi}_2 \rangle\rangle \langle\langle \hat{f}, u(0)^\infty \rangle\rangle).$

Since $\hat{Q}(\langle\langle T_{j,k}u_\infty, u(0)^\infty u(0)^* \rangle\rangle = 0 (\in H_0^1)$, as in the estimate for $I_5$, we obtain

$$|I_4| \leq C\{(1 + t)^{-\frac{7}{4}} M(t)^3 + (1 + t)^{-\frac{1}{4}} M(t)|Dw_\infty(t)||D_m^2\}.$$  

The same estimate also holds for $I_1$, since

$$I_1 = \frac{\alpha_0}{\gamma^2}(\langle\langle Q_0T_{j,k}F \rangle\rangle T_{j,k}u_\infty) = \langle\langle\langle\langle Q_0T_{j,k}F \rangle\rangle, T_{j,k}u_\infty \rangle\rangle = \langle\langle\langle T_{j,k}F, T_{j,k}u_\infty, u(0)^\infty u(0)^* \rangle\rangle.$$  

Similarly, $I_2$ and $I_6$ can be estimated as $I_5$, since

$$I_2 = \langle\langle T_{j,k}P_1F, T_{j,k}u_\infty, u(0)^\infty u(0)^* \rangle\rangle,$$

$$I_6 = \langle\langle T_{j,k}P_1F, T_{j,k}u_\infty, u(0)^\infty u(0)^* \rangle\rangle.$$  

We thus obtain

$$R^{(1)}(t) \leq C\{(1 + t)^{-\frac{7}{4}} M(t)^3 + (1 + t)^{-\frac{1}{4}} M(t)D_\infty(t)\}.$$  

The remaining terms can be estimated in a similar manner. This completes the proof. \hfill \Box

9. Asymptotic behavior

In this section we prove that the asymptotic behavior of the solution of (4.1) is described by a solution of a 1-dimensional viscous Burgers equation.

Let $\sigma = \sigma(x_1, t)$ be the solution of the following problem:

$$\partial_t\sigma - \kappa_0\partial_{x_1}^2\sigma + a_0\partial_{x_1}\sigma + a_1\partial_{x_1}(\sigma^2) = 0,$$

$$\sigma(x_1, 0) = u_\infty(x_1),$$

$$\sigma(x_1, \infty) = \hat{Q}F = \hat{Q}P_1^*\partial_t^j u_\infty.$$  

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\( \sigma|_{t=0} = \sigma_0, \)

where \( \kappa_0 \) and \( a_0 \) are the numbers given in Lemma 4.2; \( a_1 = \langle \phi^{(0)} \omega^{(0)}, \rangle - i \langle \vec{L}^{(1)} \vec{S} \rangle \); and \( \sigma_0 = \langle Q_0 u_0 \rangle = \langle \phi_0 \rangle. \)

We will show the following estimate.

**Proposition 9.1.** If \( \nu \geq \nu_0, \gamma/\nu + \bar{\nu} \geq \gamma_0^2 \) and \( \|\rho_s - 1\|_{C^{m+1}|\nu,1|} \leq \omega_0, \)
then the following assertion holds. For any \( \delta > 0 \) there exists \( \varepsilon_7 > 0 \) such that if \( \| u_0 \|_{H^m \cap L^1} \leq \varepsilon_7, \)
\[ \| \sigma_1(t) - \sigma(t) \|_2 \leq C(1 + t)^{-\frac{1}{2} - \delta} \| u_0 \|_{H^m \cap L^1}. \]

To prove Proposition 9.1, we will employ the following well-known decay properties of \( \sigma(t) \).

**Lemma 9.2.** Let \( \sigma(t) \) be a solution of (9.1)–(9.2) with \( \sigma_0 \in H^1 \cap L^1. \) Then
\[ \| \partial_{x_1}^k \sigma(t) \|_2 \leq C(1 + t)^{-\frac{k}{2}} \| \sigma_0 \|_{H^1 \cap L^1} \quad (k = 0, 1), \]
\[ \| \sigma(t) \|_\infty \leq C(1 + t)^{-\frac{1}{2}} \| \sigma_0 \|_{H^1 \cap L^1}. \]

We introduce a quantity. Let \( \sigma_1(t) \) and \( \sigma(t) \) be solutions of (5.1) and (9.1)–(9.2), respectively. We define \( N(t) \) by
\[ N(t) = \sup_{0 \leq \tau \leq t} (1 + t)^{\frac{1}{2} - \delta} \| \sigma_1(t) - \sigma(t) \|_{H^1}. \]

Proposition 9.1 would then follow if we could show that \( N(t) \leq C \| u_0 \|_{H^m \cap L^1}. \)

**Proof of Proposition 9.1.** In terms of \( \mathcal{W}(t) \) the solution \( \sigma(t) \) of (9.1)–(9.2) is written as
\[ \sigma(t) = \mathcal{W}(t) \sigma_0 - a_1 \int_0^t \mathcal{W}(t-\tau) \partial_{x_1}(\sigma^2)(\tau) \, d\tau. \]

We next rewrite \( U_0(t-\tau) H^{(0)} \mathcal{P}_1 \mathcal{F}(\tau) \) in (5.1). By Lemma 6.2 (ii), we have
\[ H^{(0)} \mathcal{P}_1 \mathcal{F} = -P_1^{(0)}(\partial_{x_1}(\phi w^{(1)}) u^{(0)}) + \Pi^{(0)} \partial_{x_1} P_1^{(1)} (I - \Pi^{(0)}) \mathcal{F} + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} \mathcal{F} \]
\[ = -P_1^{(0)}(a_1 \partial_{x_1}(\sigma^2) u^{(0)}) - P_1^{(0)}(\partial_{x_1}(\langle \phi w^{(1)} - \langle \phi^{(0)} \omega^{(0)}, \rangle \sigma^2 \rangle) u^{(0)}) \]
\[ + \Pi^{(0)} \partial_{x_1} P_1^{(1)} (I - \Pi^{(0)}) \mathcal{F} + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} \mathcal{F}. \]
Here $a_{11} = \langle \phi^{(0)} w^{(0),1} \rangle$. Since

$$\mathcal{F}(P^{(1)}_1(I - \Pi^{(0)})(\sigma^2_1 F_1)) = i \tilde{\chi}_1 \Pi^{(0)}(\hat{L}^{(1)}(\sigma^2_1) F_1)$$

$$= i \tilde{\chi}_1 \langle \hat{L}^{(1)} \hat{S} F_1 \rangle(\sigma^2_1) u^{(0)} = -a_{12} \mathcal{F}(P^{(0)}_1(\sigma^2_1 u^{(0)})),$$

where $a_{12} = -i \langle \hat{L}^{(1)} \hat{S} F_1 \rangle$, substituting $F = \sigma^2_1 F_1 + F_2$ into $\Pi^{(0)} \partial_{x_1} P^{(1)}_1(I - \Pi^{(0)}) F$, we have

$$\Pi^{(0)} \partial_{x_1} P^{(1)}_1(I - \Pi^{(0)}) F = -a_{12} P^{(0)}_1(\partial_{x_1}(\sigma^2_1) u^{(0)}) + \Pi^{(0)} \partial_{x_1} P^{(1)}_1(I - \Pi^{(0)}) F_2.$$

Using Lemma 4.8 (i), we thus arrive at

$$U_0(t - \tau) \Pi^{(0)} P_1 F(\tau)$$

$$= -a_1 U_0(t - \tau)(\partial_{x_1}(\sigma^2_1(\tau)) u^{(0)})$$

$$- U_0(t - \tau)(\partial_{x_1}(\langle \phi w^{1}(\tau) - \langle \phi^{(0)} w^{(0),1} \rangle(\sigma^2(\tau)) \rangle u^{(0)}))$$

$$+ U_0(t - \tau) h_4(\tau) + U_0(t - \tau) h_5(\tau),$$

where $a_1 = a_{11} + a_{12}$,

$$h_4 = \partial_{x_1} P^{(1)}_1(I - \Pi^{(0)}) F_2 + \sigma^2_{x_1} P^{(2)}_1 F_2,$$

$$h_5 = \sigma^2_{x_1} P^{(2)}_1.$$

It then follows from (5.1) and (9.3) that

$$\sigma_1(t) - \sigma(t) = \sum_{j=0}^{5} I_j(t).$$

Here

$$I_0(t) = U_0(t) u_0 - \mathcal{U}(t) \sigma_0,$$

$$I_1(t) = -a_1 \int_0^t \mathcal{U}(t - \tau)(\partial_{x_1}(\sigma^2_1(\tau)) - \partial_{x_1}(\sigma^2(\tau))) \, d\tau,$$

$$I_2(t) = -a_1 \int_0^t (U_0(t - \tau)(\partial_{x_1}(\sigma^2_1(\tau))u^{(0)})) - \mathcal{U}(t - \tau) \partial_{x_1} \sigma^2_1(\tau) \, d\tau,$$

$$I_3(t) = - \int_0^t U_0(t - \tau)(\partial_{x_1}(\langle \phi w^{1}(\tau) - \langle \phi^{(0)} w^{(0),1} \rangle(\sigma^2(\tau)) \rangle u^{(0)})) \, d\tau,$$

$$I_j(t) = \int_0^t U_0(t - \tau) h_j(\tau) \, d\tau, \quad j = 4, 5.$$
Lemma 4.9 implies that
\[ \|I_0(t)\|_{H^1} \leq C(1 + t)^{-\frac{1}{2}}\|u_0\|_{H^m \cap L^1} . \]

Let us consider \( I_1(t) \). By Lemma 9.1 and the definition of \( M(t) \) and \( N(t) \), we have
\[ \|\sigma_1^2 - \sigma_2^2\|_{1} \leq \|\sigma_1 + \sigma\|_{2} \|\sigma_1 - \sigma_2\|_{2} \]
\[ \leq C(1 + t)^{-1+\delta}\|u_0\|_{H^m \cap L^1} N(t). \]
Furthermore, by Lemma 8.2, we have \( \|\sigma_1 - \sigma\|_{\infty} \leq C(1 + t)^{-\frac{1}{2}+\delta} N(t) \), and hence,
\[ \|\partial_{x_1}(\sigma_1^2 - \sigma_2^2)\|_{2} \leq C\{\|\sigma_1 + \sigma\|_{\infty}\|\partial_{x_1}(\sigma_1 - \sigma_2)\|_{2} \]
\[ + \|\sigma_1 - \sigma\|_{\infty}\|\partial_{x_1}(\sigma_1 + \sigma_2)\|_{2} \} \]
\[ \leq C(1 + t)^{-\frac{5}{2}+\delta}\|u_0\|_{H^m \cap L^1} N(t). \]

It then follows from Lemma 4.8 that for \( k = 0, 1 \),
\[ \|\partial_{x_1}^k I_1(t)\|_{2} \leq C\{ \int_{0}^{t-1} (1 + t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-1+\delta} d\tau \]
\[ + \int_{t-1}^{t} (t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-\frac{1}{2}+\delta} d\tau \} \|u_0\|_{H^m \cap L^1} N(t) \]
\[ \leq C(1 + t)^{-\frac{1}{2}+\delta}\|u_0\|_{H^m \cap L^1} N(t). \]

As for \( I_2(t) \), since \( \partial_{x_1}(\sigma_1^2 u^{(0)}) = \partial_{x_1}(\sigma_2^2) \), we see from Lemma 4.9 that for \( k = 0, 1 \),
\[ \|\partial_{x_1}^k I_2(t)\|_{2} \leq C\{ \int_{0}^{t-1} (1 + t - \tau)^{-\frac{1}{2}}\|\sigma_1^2(\tau)\|_{1} d\tau \]
\[ + \int_{t-1}^{t} (1 + t - \tau)^{-\frac{1}{2}}\|\sigma_1(\partial_{x_1}\sigma_1)(\tau)\|_{1} d\tau \]
\[ + \int_{t-1}^{t} (t - \tau)^{-\frac{1}{2}}\|\sigma_2^2(\tau)\|_{2} d\tau \} \]
\[ \leq C\{ \int_{0}^{t-1} (1 + t - \tau)^{-\frac{3}{2}}(1 + \tau)^{-\frac{1}{2}} d\tau \]
\[ + \int_{t-1}^{t} (1 + t - \tau)^{-\frac{3}{2}}(1 + \tau)^{-1} d\tau \]
\[ + \int_{t-1}^{t} (t - \tau)^{-\frac{3}{2}}(1 + \tau)^{-\frac{1}{2}} d\tau \} M(t)^2 \]
\[ \leq C(1 + t)^{-\frac{5}{2}+\delta}\|u_0\|_{H^m \cap L^1}^2. \]

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As for \( I_3 \), we have
\[
\| \langle \phi w^1 \rangle (\tau) - \langle \phi(0) w^{(0)} \sigma_1^2 \rangle (\tau) \|_1 \\
\leq C \{ \| \sigma_1(\tau) \|_2 \| u(\tau) - (\sigma_1 u^{(0)})(\tau) \|_2 + \| u(\tau) - (\sigma_1 u^{(0)})(\tau) \|_2^2 \} \\
\leq C (1 + \tau)^{-1} M(t)^2.
\]
Lemma 4.8 then gives
\[
\| \partial^k x_1 I_3(\tau) \|_2 \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{3}{4}} (1 + \tau)^{-1} d\tau \\
\leq C (1 + t)^{-\frac{3}{4}} \log(1 + t) \| u_0 \|_{H^m \cap L^1}^2.
\]
To estimate \( I_4(\tau) \), we write \( h_4(\tau) \) as
\[
h_4(\tau) = \partial x_1 (P_1^{(1)} (I - \Pi^{(0)})) F_2 + \partial x_1 P_1^{(2)} F_2(\tau).
\]
Using Lemma 4.5, Lemma 4.8 and Lemma 6.3, we have
\[
\| \partial^k x_1 I_4(\tau) \|_2 \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{3}{4}} (1 + \tau)^{-1} d\tau \\
\leq C (1 + t)^{-\frac{3}{4}} \log(1 + t) \| u_0 \|_{H^m \cap L^1}^2.
\]
As for \( I_5(\tau) \), we write \( h_5(\tau) \) as
\[
h_5(\tau) = \begin{cases} 
\partial^2 x_1 P_1^{(2)} (\sigma_1^2 F_1)(\tau) & \text{for } \tau \in [0, \frac{t}{2}], \\
2 \partial x_1 P_1^{(2)} (\sigma_1 \partial x_1 \sigma_1 F_1)(\tau) & \text{for } \tau \in [\frac{t}{2}, t].
\end{cases}
\]
As in the estimate on \( I_2(\tau) \), we see from Lemma 4.5 and Lemma 4.8 that
\[
\| \partial^k x_1 I_5(\tau) \|_2 \\
\leq C \{ \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{3}{4}} (1 + \tau)^{-\frac{1}{2}} d\tau \\
+ \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\frac{3}{4}} (1 + \tau)^{-1} d\tau \} \\
\leq C (1 + t)^{-\frac{3}{4}} \| u_0 \|_{H^m \cap L^1}^2.
\]
We thus obtain
\[
\| \partial x_1 (\sigma_1 - \sigma)(\tau) \|_2 \leq C (1 + t)^{-\frac{3}{4} + \delta} \| u_0 \|_{H^m \cap L^1} \{ \| u_0 \|_{H^m \cap L^1} + N(t) \},
\]
which yields
\[
N(t) \leq C \| u_0 \|_{H^m \cap L^1} \{ \| u_0 \|_{H^m \cap L^1} + N(t) \}.
\]
The desired result now follows by taking \( \| u_0 \|_{H^m \cap L^1} \) suitably small. This completes the proof. \( \square \)
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