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Abstract

The initial boundary value problem for the compressible Navier-Stokes equation is considered in an infinite layer of $\mathbf{R}^2$. It is proved that if the Reynolds and Mach numbers are sufficiently small, then strong solutions to the compressible Navier-Stokes equation around parallel flows exist globally in time for sufficiently small initial perturbations. The large time behavior of the solution is described by a solution of a 1-dimensional viscous Burgers equation. The proof is given by a combination of spectral analysis of the linearized operator and a variant of the Matsumura-Nishida energy method.

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1. Introduction

This paper is concerned with the stability of parallel flows of the compressible Navier-Stokes equation

(1.1) $$\partial_{\tilde{t}}\tilde{\rho} + \mathrm{div}\,(\tilde{\rho}\tilde{v}) = 0,$$

(1.2) $$\tilde{\rho}(\partial_{\tilde{t}}\tilde{v} + \tilde{v}\cdot\nabla\tilde{v}) - \mu\Delta\tilde{v} - (\mu+\mu')\nabla\mathrm{div}\,\tilde{v} + \nabla\tilde{P}(\tilde{\rho}) = \tilde{\rho}\tilde{\boldsymbol{g}}$$

in an n-dimensional infinite layer $\Omega_\ell = \mathbf{R}^{n-1}\times(0,\ell)$:

$$\Omega_\ell = \{\tilde{x} = (\tilde{x}', \tilde{x}_n)\,;\, \tilde{x}' = (\tilde{x}_1, \cdots, \tilde{x}_{n-1}) \in \mathbf{R}^{n-1},\, 0 < \tilde{x}_n < 1\} \quad (n \geq 2).$$

Here $\tilde{\rho} = \tilde{\rho}(\tilde{x},\tilde{t})$ and $\tilde{v} = {}^\top(\tilde{v}^1(\tilde{x},\tilde{t}), \cdots, \tilde{v}^n(\tilde{x},\tilde{t}))$ denote the unknown density and velocity at time $\tilde{t} \geq 0$ and position $\tilde{x} \in \Omega_\ell$, respectively; $\tilde{P} = \tilde{P}(\tilde{\rho})$ is the pressure that is assumed to be a smooth function of $\tilde{\rho}$ satisfying

$$\tilde{P}'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; μ and μ' are viscosity coefficients that are assumed to be constants and satisfy

$$\mu > 0, \quad \frac{2}{n}\mu + \mu' \geq 0;$$

div, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to $\tilde{x}$; and $\tilde{\boldsymbol{g}}$ is an external force which is a function of $\tilde{x}_n$ only. Here and in what follows ${}^\top\cdot$ stands for the transposition.

Under a suitable assumption on $\tilde{\boldsymbol{g}}$, say, $\tilde{\boldsymbol{g}}$ is in the form

$$\tilde{\boldsymbol{g}} = {}^\top(\tilde{g}^1(\tilde{x}_n), 0, \cdots, 0, \tilde{g}^n(\tilde{x}_n)),$$

with bounded smooth functions $\tilde{g}^j(\tilde{x}_n)$ $(j = 1, n)$, problem (1.1)–(1.2) has a smooth stationary solution $\tilde{u}_s = {}^\top(\tilde{\rho}_s, \tilde{v}_s)$ of parallel flow satisfying

$$\begin{aligned} &\tilde{\rho}_s = \tilde{\rho}_s(\tilde{x}_n) > 0, \quad \frac{1}{\ell}\int_0^\ell \tilde{\rho}_s(\tilde{x}_n)\, d\tilde{x}_n = \rho_*, \\ &\tilde{v}_s = {}^\top(\tilde{v}_s^1(\tilde{x}_n), 0, \cdots, 0). \end{aligned}$$

In this paper we are interested in the stability of parallel flows. Typical examples of parallel flows are the plane Couette flow:

$$\tilde{\rho}_s = \rho_*, \quad \tilde{v}_s^1 = \frac{V^1}{\ell}\tilde{x}_n$$

for $\tilde{\boldsymbol{g}} = \boldsymbol{0}$ with a constant $V^1 \neq 0$; and the Poiseuille flow

$$\tilde{\rho}_s = \rho_*, \quad \tilde{v}_s^1 = \frac{\rho_*\tilde{g}^1}{2\mu}\tilde{x}_n(\ell - \tilde{x}_n)$$

for $\tilde{\boldsymbol{g}} = {}^\top(\tilde{g}^1, 0 \cdots, 0)$ with a constant $\tilde{g}^1 \neq 0$.

As for the stability of parallel flows, Iooss and Padula [3] studied the linearized stability of parallel flows in a cylindrical domain. It was shown in [3] that if the Reynolds number is small in some sense, then the parallel flow is linearly stable under perturbations periodic in the unbounded direction of the domain. Furthermore, the solution of the linearized problem decays exponentially as $t \to \infty$ if the density component of initial perturbation has a vanishing mean value in the periodicity cell. In [12] the linearized stability of $\tilde{u}_s$ under perturbations in $L^2(\Omega_\ell)$ was studied; and it was shown that $\tilde{u}_s$ is linearly stable if the Reynolds and Mach numbers are sufficiently small and $\tilde{\rho}_s$ is sufficiently close to ρ_*. Furthermore, the asymptotic behavior of solutions of the linearized problem is described by an $n - 1$ dimensional linear heat equation. The nonlinear stability of $\tilde{u}_s$ was then studied in [8] in the case $n \geq 3$. It was shown in [8] that $\tilde{u}_s$ is asymptotically stable under sufficiently small initial perturbations in the Sobolev space $H^m(\Omega_\ell)$ with $m \geq [n/2]+1$ if $n \geq 3$, provided that the Reynolds number $Re = \frac{\rho_* \ell V}{\mu}$ and Mach number $Ma = \frac{V}{\sqrt{\tilde{P}'(\rho_*)}}$ are sufficiently small and $\tilde{\rho}_s$ is sufficiently close to ρ_* in $C^{m+1}[0,\ell]$. Here V is a non-dimensional number satisfying $V \sim \|\tilde{v}_s^1\|_{C^{m+1}[0,\ell]}$. Furthermore, the asymptotic behavior of perturbations is described by an $n-1$ dimensional linear heat equation. The proof in [8] is based on the Matsumura-Nishida energy method ([15]). But the argument in [8] does not work well for the case $n = 2$ due to a quadratic nonlinearity which cannot be controlled by the standard energy method. A similar aspect appears in the case of 1-dimensional viscous conservation laws, where the asymptotic behavior is described by a solution of a 1-dimensional viscous Burgers equation (cf., [13, 16]).

The purpose of this paper is to prove the nonlinear stability of $\tilde{u}_s$ in the case $n = 2$ for small Reynolds and Mach numbers. Furthermore, we will show that the large time behavior of the perturbation is described by a solution of a 1-dimensional viscous Burgers equation.

To state the results of this paper more precisely, we introduce the following non-dimensional variables:

$$\tilde{x} = \ell x,\ \tilde{t} = \frac{\ell}{V} t,\ \tilde{v} = V v,\ \tilde{\rho} = \rho_* \rho,\ \tilde{P} = \rho_* V^2 P$$

with $V = \|\tilde{v}_s^1\|_{C_*^{m+1}[0,\ell]}$ for an integer $m \geq [n/2]+1$. Here

$$\|\tilde{v}_s^1\|_{C_*^{m+1}[0,\ell]} = \sum_{k=0}^{m+1} \sup_{0 \leq \tilde{x}_n \leq \ell} \ell^k |\partial_{\tilde{x}_n}^k \tilde{v}_s^1(\tilde{x}_n)|.$$

Under this non-dimensionalization the domain Ω_ℓ is transformed into $\Omega \equiv \Omega_1$ and the parallel flow $\tilde{u}_s$ is transformed into $u_s = {}^\top(\rho_s, v_s)$ with

$$\rho_s = \rho_s(x_n) > 0, \quad \int_0^1 \rho_s(x_n)\, dx_n = 1,$$
$$v_s = {}^\top(v_s^1(x_n), 0, \cdots, 0), \quad \|v_s^1\|_{C^{m+1}[0,1]} = 1.$$

The perturbation $u(t) = {}^\top(\phi(t), w(t)) \equiv {}^\top(\gamma^2(\rho(t) - \rho_s), v(t) - v_s)$ is governed by the system of equations

$$\partial_t \phi + v_s \cdot \nabla \phi + \gamma^2 \mathrm{div}\,(\rho_s w) = f^0, \tag{1.3}$$

$$\begin{aligned}(1.4)\qquad &\partial_t w - \tfrac{\nu}{\rho_s}\Delta w - \tfrac{\tilde{\nu}}{\rho_s}\nabla \mathrm{div}\, w + \nabla\left(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi\right) \\ &+\tfrac{\nu\partial_{x_n}^2 v_s}{\gamma^2\rho_s^2}\phi \boldsymbol{e}_1 + v_s\cdot\nabla w + w\cdot\nabla v_s = \boldsymbol{f}.\end{aligned}$$

Here div , ∇ and Δ denote the divergence, gradient and Laplacian with respect to x; $\boldsymbol{e}_1 = {}^\top(1,0,\cdots,0)\in\mathbf{R}^n$; ν, $\tilde{\nu}$ and γ are non-dimensional parameters defined by

$$\nu = \frac{\mu}{\rho_* \ell V},\ \tilde{\nu} = \frac{\mu+\mu'}{\rho_*\ell V},\ \gamma = \sqrt{P'(1)} = \frac{\sqrt{\tilde{P}'(\rho_*)}}{V};$$

and f^0 and $\boldsymbol{f} = {}^\top(\boldsymbol{f}', f^n)$, $\boldsymbol{f}' = {}^\top(f^1,\cdots,f^{n-1})$, denote the nonlinearities:

$$f^0 = -\mathrm{div}\,(\phi w),$$

$$\begin{aligned}\boldsymbol{f} \;=\; & -w\cdot\nabla w + \tfrac{\nu\phi}{(\phi+\gamma^2\rho_s)\rho_s}\left(-\Delta w + \tfrac{\partial_{x_n}^2 v_s^1}{\gamma^2\rho_s}\phi\boldsymbol{e}_1\right) \\ & -\tfrac{\tilde{\nu}\phi}{(\phi+\gamma^2\rho_s)\rho_s}\nabla\mathrm{div}\, w \\ & +\tfrac{\phi}{\gamma^2\rho_s}\nabla\left(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi\right) - \tfrac{1}{2\gamma^4\rho_s}\nabla\left(P''(\rho_s)\phi^2\right) \\ & +\tilde{P}_3(\rho_s,\phi,\partial_x\phi),\end{aligned}$$

where

$$\begin{aligned}\tilde{P}_3 \;=\; & \tfrac{\phi^3}{\gamma^4(\phi+\gamma^2\rho_s)\rho_s^3}\nabla P(\rho_s) - \tfrac{1}{2\gamma^6\rho_s}\nabla\left(\phi^3 P_3(\rho_s,\phi)\right) \\ & +\tfrac{\phi}{2\gamma^6\rho_s^2}\nabla\left(P''(\rho_s)\phi^2 + \tfrac{1}{\gamma^2}\phi^3 P_3(\rho,\phi)\right) \\ & -\tfrac{\phi^2}{\gamma^2(\phi+\gamma^2\rho_s)\rho_s}\nabla\left(\tfrac{P'(\rho_s)}{\gamma^2}\phi + \tfrac{1}{2\gamma^4}P''(\rho_s)\phi^2 + \tfrac{1}{2\gamma^6}\phi^3 P_3(\rho_s,\phi)\right)\end{aligned}$$

with

$$P_3(\rho_s,\phi) = \int_0^1 (1-\theta)^2 P'''(\theta\gamma^{-2}\phi+\rho_s)\,d\theta.$$

We consider (1.3)–(1.4) under the boundary condition

$$(1.5)\qquad w|_{\partial\Omega} = 0$$

and the initial condition

$$(1.6)\qquad u|_{t=0} = u_0 = {}^\top(\phi_0, w_0).$$

We will show that the following assertion holds when $n = 2$; if u_0 is sufficiently small in $H^m(\Omega)\cap L^1(\Omega)$ for an integer $m \geq 2(=[n/2]+1)$ and

u_0 satisfies a suitable compatibility condition, then there exists a unique solution $u(t)$ of (1.3)–(1.6) in $C([0,\infty); H^m(\Omega))$, provided that $\nu \gg 1$, $\gamma \gg 1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \ll 1$. Furthermore, $u(t)$ satisfies

$$\|u(t)\|_{L^2} = O(t^{-\frac{1}{4}}),$$
$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2} = O(t^{-\frac{3}{4}+\delta}) \quad (\forall \delta > 0)$$

as $t \to \infty$. Here $u^{(0)}$ is a function of x_2 only that satisfies $u^{(0)}(x_2) \sim {}^\top(1, O(\gamma^{-2}), 0)$; and σ is a function of (x_1, t) that satisfies

$$(1.7) \qquad \partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma + a_0 \partial_{x_1} \sigma + a_1 \partial_{x_1}(\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2)\, dx_2$$

with some constants $\kappa_0 > 0$, $a_0, a_1 \in \mathbf{R}$. We note that the result also holds for the case of the motionless state $\tilde{u}_s = (\tilde{\rho}_s, 0)$ with $a_0 = 0$ and some constants $\kappa_0 > 0$ and $a_1 \in \mathbf{R}$ possibly different from the ones in (1.7). The coefficient a_1 in (1.7) vanishes in the case of the motionless state $\tilde{u}_s = (\tilde{\rho}_s, 0)$ with constant density $\tilde{\rho}_s \equiv \rho_*$ ([6]) and in the case of the plane Couette flow ([7]). These are the special cases where (1.7) becomes a linear equation. In general, one can see that $a_1 \neq 0$. Even in the case of the motionless state $\tilde{u}_s = (\tilde{\rho}_s, 0)$, if the density $\tilde{\rho}_s$ is not a constant, then $a_1 \neq 0$ in general.

We also note that when $n \geq 3$, as was remarked in [8], the large time behavior of $u(t)$ is described by a linear heat equation in such a way that

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2} = O(t^{-\frac{n-1}{4}-\frac{1}{2}} \eta_n(t)).$$

Here σ is a function of (x', t) satisfying

$$\partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + a_0 \partial_{x_1} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n$$

with some constants $\kappa_0, \kappa'' > 0$, $a_0 \in \mathbf{R}$, where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$; and $\eta_n(t) = \log(1+t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \geq 4$.

The proof of the main results is given by a combination of the spectral analysis of the linearized problem and a variant of the Matsumura-Nishida energy method. The nonlinearity of the Burgers equation (1.7) stems from the terms $-\mathrm{div}\,(\phi w)$ in (1.3) and $-\frac{1}{2\gamma^4 \rho_s} \partial_{x_2}(P''(\rho_s)\phi^2)$ in the n th equation of (1.4) which cannot be controlled by the standard Matsumura-Nishida energy method when $n = 2$. To deal with this term we will introduce a decomposition of the solution according to spectral properties of the linearized operator. Roughly speaking we decompose the solution into a low frequency part and a high frequency part. For the low frequency part, we will apply the decay estimates on the linearized problem. On the other hand, for the high frequency part, we will apply a variant of the Matsumura-Nishida energy method similar to the analysis for the case $n \geq 3$ in [8]. The symmetric property of the system (1.1)–(1.2) is a little bit disturbed by introducing the decomposition. However, using the fact that spatial differentiation of any order acts on the low frequency part as a bounded operator, we can obtain the estimates on the nonlinearities necessary to obtain the a priori estimate for the global existence.

This paper is organized as follows. In section 2 we introduce notations used in this paper. In section 3 we state the main results of this paper. Section 4 is devoted to spectral properties of the linearized problem, most of which were essentially obtained in [12]. In section 5 we introduce a decomposition of the solution and reformulate the problem. In section 6 we investigate the low frequency part, the slowly decaying part, by using the decay estimates on the linearized problem; and in section 7 we estimate the high frequency part, the fast decaying part, by the energy method. In section 8 we give necessary estimates on the nonlinearities. In section 9 we show that the asymptotic behavior of $u(t)$ is described by the Burgers equation (1.7).

2. Preliminaries

In this section we first state assumptions on the parallel flow u_s and then introduce some notations used in this paper.

In this paper we consider the two-dimensional problem ($n = 2$). Throughout this paper we assume that the parallel flow $u_s = {}^\top(\rho_s(x_2), v_s(x_2))$ is bounded smooth and satisfies

$$(2.1) \qquad 0 < \rho_1 \leq \rho_s(x_2) \leq \rho_2, \quad \int_0^1 \rho_s(x_2)\, dx_2 = 1, \quad v_s(x_2) = {}^\top(v_s^1(x_2), 0)$$

and

$$(2.2) \qquad P'(\rho) > 0 \quad \text{for} \quad \rho_1 \leq \rho \leq \rho_2$$

with some constants $0 < \rho_1 < 1 < \rho_2$. Existence of such a parallel flow was shown, e.g., in [8] when $\tilde{\boldsymbol{g}}$ is sufficiently small. Note also that, due to the non-dimensionalization, we have

$$\|v_s\|_{C^{m+1}[0,1]} = 1.$$

In the remaining of this section, we introduce some notations which will be used throughout the paper. For a domain D and $1 \leq p \leq \infty$ we denote by

$L^p(D)$ the usual Lebesgue space on D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Let m be a nonnegative integer. The symbol $H^m(D)$ denotes the m th order L^2 Sobolev space on D with norm $\|\cdot\|_{H^m(D)}$. $C_0^m(D)$ stands for the set of all C^m functions which have compact support in D. We denote by $H_0^1(D)$ the completion of $C_0^1(D)$ in $H^1(D)$.

We simply denote by $L^p(D)$ (resp., $H^m(D)$) the set of all vector fields $w = {}^\top(w^1, w^2)$ on D with $w^j \in L^p(D)$ (resp., $H^m(D)$), $j = 1, 2$, and its norm is also denoted by $\|\cdot\|_{L^p(D)}$ (resp., $\|\cdot\|_{H^m(D)}$). For $u = {}^\top(\phi, w)$ with $\phi \in H^k(D)$ and $w = {}^\top(w^1, w^2) \in H^m(D)$, we define $\|u\|_{H^k(D)\times H^m(D)}$ by $\|u\|_{H^k(D)\times H^m(D)} = \|\phi\|_{H^k(D)} + \|w\|_{H^m(D)}$. When $k = m$, we simply write $\|u\|_{H^k(D)\times H^k(D)} = \|u\|_{H^k(D)}$.

In case $D = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^m(\Omega)$) as L^p (resp., H^m). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$.

In case D is the interval $(0, 1)$ we denote the norm of $L^p(0, 1)$ by $|\cdot|_p$. The norm of $H^m(0, 1)$ is denoted by $|\cdot|_{H^m}$.

The inner product of $L^2(\Omega)$ is denoted by

$$(f, g) = \int_\Omega f(x)g(x)\,dx, \quad f, g \in L^2(\Omega).$$

We also denote the inner product of $L^2(0, 1)$ by

$$(f, g) = \int_0^1 f(x_2)g(x_2)\,dx_2, \quad f, g \in L^2(0, 1),$$

if no confusion occurs. We further introduce a weighted inner product $\langle\langle\cdot,\cdot\rangle\rangle$ defined by

$$\langle\langle u_1, u_2\rangle\rangle = \int_\Omega \phi_1\phi_2 \frac{P'(\rho_s)}{\gamma^4\rho_s}\,dx + \int_\Omega w_1 \cdot w_2\rho_s\,dx$$

for $u_j = {}^\top(\phi_j, w_j) \in L^2(\Omega)$ $(j = 1, 2)$; and, also, $\langle\cdot,\cdot\rangle$ defined by

$$\langle u_1, u_2\rangle = \int_0^1 \phi_1\phi_2 \frac{P'(\rho_s)}{\gamma^4\rho_s}\,dx_2 + \int_0^1 w_1 \cdot w_2\rho_s\,dx_2$$

for $u_j = {}^\top(\phi_j, w_j) \in L^2(0, 1)$ $(j = 1, 2)$. Here $\rho_s = \rho_s(x_2)$ denotes the density of the parallel flow u_s. We note that $\langle\langle\cdot,\cdot\rangle\rangle$ and $\langle\cdot,\cdot\rangle$ define inner products of $L^2(\Omega)$ and $L^2(0, 1)$, respectively, due to (2.1) and (2.2).

We denote the mean value of $f \in L^1(0, 1)$ by $\langle\,\cdot\,\rangle$:

$$\langle f\rangle = \int_0^1 f(x_2)\,dx_2.$$

For $u = {}^\top(\phi, w) \in L^1(0,1)$ with $w = {}^\top(w^1, w^2)$ we define $\langle u \rangle$ by

$$\langle u \rangle = \langle \phi \rangle + \langle w^1 \rangle + \langle w^2 \rangle.$$

We denote the $k \times k$ identity matrix by I_k. We also define 3×3 diagonal matrices Q_0 and $\widetilde{Q}$ by

$$Q_0 = \mathrm{diag}\,(1,0,0), \quad \widetilde{Q} = \mathrm{diag}\,(0,1,1).$$

Note that

$$\langle Q_0 u \rangle = \langle \phi \rangle \ \text{ for } \ u = {}^\top(\phi, w).$$

Partial derivatives of a function u in x, x_j and t are denoted by $\partial_x u$, $\partial_{x_j} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u;\, |\alpha| = k)$.

For a function $f = f(x_1)$ $(x_1 \in \mathbf{R})$, we denote its Fourier transform by $\hat{f}$ or $\mathscr{F} f$:

$$\hat{f}(\xi) = (\mathscr{F} f)(\xi) = \int_{\mathbf{R}} f(x_1) e^{-i\xi x_1}\, dx_1 \quad (\xi \in \mathbf{R}).$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$(\mathscr{F}^{-1} f)(x_1) = \frac{1}{2\pi} \int_{\mathbf{R}} f(\xi) e^{i\xi \cdot x_1}\, d\xi.$$

We will denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$. For $\Lambda \in \mathbf{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ we denote the set $\{\lambda \in \mathbf{C}; |\arg(\lambda - \Lambda)| \le \theta\}$ by $\Sigma(\Lambda, \theta)$:

$$\Sigma(\Lambda, \theta) = \{\lambda \in \mathbf{C}; |\arg(\lambda - \Lambda)| \le \theta\}.$$

3. Main Results

In this section we state the main results of this paper.

We first mention the compatibility condition for $u_0 = {}^\top(\phi_0, w_0)$. We will look for a solution $u = {}^\top(\phi, w)$ of (1.3)–(1.6) with $n = 2$ in $\cap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j})$ satisfying $\int_0^t \|\partial_x w\|_{H^m}^2\, d\tau < \infty$ for all $t \ge 0$ with $m \ge 2 (= [n/2] + 1)$. Therefore, we need to require the compatibility condition for the initial value $u_0 = {}^\top(\phi_0, w_0)$, which is formulated as follows.

Let $u = {}^\top(\phi, w)$ be a smooth solution of (1.3)–(1.6) with $n = 2$. Then $\partial_t^j u = {}^\top(\partial_t^j \phi, \partial_t^j w)$ $(j \ge 1)$ is inductively determined by

$$\partial_t^j \phi = -v_s \cdot \nabla \partial_t^{j-1} \phi - \gamma^2 \mathrm{div}\,(\rho_s \partial_t^{j-1} w) + \partial_t^{j-1} f^0$$

and

$$\begin{aligned}\partial_t^j w &= \tfrac{\nu}{\rho_s}\Delta \partial_t^{j-1} w + \tfrac{\tilde{\nu}}{\rho_s}\nabla \mathrm{div}\, \partial_t^{j-1} w - \nabla\left(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\partial_t^{j-1}\phi\right)\\ &\quad -\tfrac{\nu\partial_{x_2}^2 v_s^1}{\gamma^2\rho_s^2}\partial_t^{j-1}\phi \boldsymbol{e}_1 - v_s\cdot\nabla\partial_t^{j-1}w - \partial_t^{j-1}w\cdot\nabla v_s + \partial_t^{j-1}\boldsymbol{f}.\end{aligned}$$

From these relations we see that $\partial_t^j u|_{t=0} = {}^\top(\partial_t^j\phi, \partial_t^j w)|_{t=0}$ is inductively given by $u_0 = {}^\top(\phi_0, w_0)$ in the following way:

$$\partial_t^j u|_{t=0} = {}^\top(\partial_t^j\phi, \partial_t^j w)|_{t=0} = {}^\top(\phi_j, w_j) = u_j,$$

where

$$\begin{aligned}\phi_j &= -v_s\cdot\nabla\phi_{j-1} - \gamma^2 \mathrm{div}\,(\rho_s w_{j-1})\\ &\quad + f_{j-1}^0(u_0,\cdots,u_{j-1},\partial_x u_0,\cdots,\partial_x u_{j-1}),\\ w_j &= \tfrac{\nu}{\rho_s}\Delta w_{j-1} + \tfrac{\tilde{\nu}}{\rho_s}\nabla\mathrm{div}\, w_{j-1} - \nabla\left(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi_{j-1}\right)\\ &\quad -\tfrac{\nu\partial_{x_2}^2 v_s^1}{\gamma^2\rho_s^2}\phi_{j-1}\boldsymbol{e}_1 - v_s\cdot\nabla w_{j-1} - w_{j-1}\cdot\nabla v_s\\ &\quad + \boldsymbol{f}_{j-1}(u_0,\cdots,u_{j-1},\cdots,\partial_x u_{j-1},\cdots,\partial_x^2 w_{j-1}).\end{aligned}$$

Here $f_l^0(u_0,\cdots,u_l,\cdots)$ is a certain polynomial in $u_0,\cdots,u_l,\cdots$; $\cdots\cdots$, and so on.

By the boundary condition $w|_{\partial\Omega} = 0$ in (1.5), we necessarily have $\partial_t^j w|_{\partial\Omega} = 0$, and hence,

$$w_j|_{\partial\Omega} = 0.$$

Assume that $u = {}^\top(\phi, w)$ is a solution of (1.3)–(1.6) with $n = 2$ in $\cap_{j=0}^{[\frac{m}{2}]} C^j([0,T_0]; H^{m-2j})$ for some $T_0 > 0$. Then, from the above observation, we need the regularity $u_j = {}^\top(\phi_j, w_j) \in H^{m-2j}$ for $j = 0,\cdots,[m/2]$, which, indeed, follows from the fact that $u_0 = {}^\top(\phi_0, w_0) \in H^m$ with $m \geq 2(= [n/2]+1)$. Furthermore, it is necessary to require that $u_0 = {}^\top(\phi_0, w_0)$ satisfies the $\hat{m}$ *th order compatibility condition*:

$$w_j \in H_0^1 \quad \text{for} \;\; j = 0, 1, \cdots, \hat{m} = \left[\frac{m-1}{2}\right].$$

We are now in a position to state our main results of this paper.

Theorem 3.1. *Suppose that $n = 2$. Let m be an integer satisfying $m \geq 2(= [n/2]+1)$. Assume that $u_s = {}^\top(\rho_s, v_s)$ satisfies* (2.1) *and* (2.2)*. Then there are positive numbers ν_0, γ_0 and ω_0 such that if $\nu \geq \nu_0$, $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$, then the following assertion holds. There is a positive* *number* ε_0 *such that if* $u_0 = {}^\top(\phi_0, w_0) \in H^m \cap L^1$ *satisfies* $\|u_0\|_{H^m\cap L^1} \le \varepsilon_0$ *and the* $\hat{m}$ *th compatibility condition, then there exists a unique global solution* $u(t) = {}^\top(\phi(t), w(t))$ *of* (1.3)–(1.6) *with* $n = 2$ *in* $\cap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j})$ *which satisfies*

$$\|\partial_{x_1}^k u(t)\|_2 = O(t^{-\frac{1}{4}-\frac{k}{2}}) \quad (k = 0, 1), \tag{3.1}$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{3}{4}+\delta}) \quad (\forall \delta > 0) \tag{3.2}$$

as $t \to \infty$*. Here* $u^{(0)} = u^{(0)}(x_2)$ *is a function given in Lemma* 4.1 *below; and* $\sigma = \sigma(x_1, t)$ *is a function satisfying*

$$\partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma + a_0 \partial_{x_1} \sigma + a_1 \partial_{x_1}(\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2)\, dx_2$$

with some constants $\kappa_0 > 0$*,* $a_0, a_1 \in \mathbf{R}$*.*

Remark. As was remarked in [8], in case $n \ge 3$, one can establish the estimates

$$\|\partial_{x'}^k u(t)\|_2 = O(t^{-\frac{n-1}{4}-\frac{k}{2}}) \quad (k = 0, 1),$$
$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t))$$

as $t \to \infty$, provided that $u_0 = (\phi_0, w_0) \in H^m \cap L^1$ with $\|u_0\|_{H^m\cap L^1} \ll 1$, $m \ge [n/2] + 1$. Here σ is a function of (x', t) satisfying

$$\partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + a_0 \partial_{x_1} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n$$

with some constants $\kappa_0, \kappa'' > 0$, $a_0 \in \mathbf{R}$, where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$; and $\eta_n(t) = \log(1 + t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \ge 4$.

As in [15, 10], Theorem 3.1 is proved by showing the local existence and the a priori estimates. The local existence is proved by applying the local solvability result in [9]. In fact, we can show the following assertion. We introduce notations:

$$[\![f(t)]\!]_k = \Big(\textstyle\sum_{j=0}^{[\frac{k}{2}]} \|\partial_t^j f(t)\|_{H^{k-2j}}^2 \Big)^{\frac{1}{2}},$$

$$|||Df(t)|||_k = \begin{cases} \|\partial_x f(t)\|_2 & \text{for } k = 0, \\ \big([\![\partial_x f(t)]\!]_k^2 + [\![\partial_t f(t)]\!]_{k-1}^2 \big)^{\frac{1}{2}} & \text{for } k \ge 1. \end{cases}$$

For $T>0$ we define a function space $Z^m(T)$ by

$$Z^m(T)=\{u\in \cap_{j=0}^{[\frac{m}{2}]}C^j([0,T];H^{m-2j});\|u\|_{Z^m(T)}<\infty\},$$

where

$$\|u\|_{Z^m(T)}=\sup_{0\le t\le T}[\![u(t)]\!]_m+\Big(\int_0^T |||Dw(t)|||_m^2\,dt\Big)^{1/2}.$$

Proposition 3.2. *Let $m\ge 2(=[n/2]+1)$. Assume that $u_0={}^\top(\phi_0,w_0)\in H^m$ satisfies the following conditions.*

(a) *$u_0\in H^m$ satisfies the $\hat{m}$-th compatibility condition.*

(b) $-\frac{\gamma^2}{4}\rho_1\le\phi_0$.

Then there exists a positive number T_0 depending on $\|u_0\|_{H^m}$ and ρ_1 such that problem (1.3)–(1.6) has a unique solution $u(t)\in Z^m(T_0)$ satisfying

$$\phi(x,t)\ge-\frac{\gamma^2}{2}\rho_1\quad\text{for }\forall(x,t)\in\Omega\times[0,T_0].$$

Furthermore, the inequality

$$\|u\|_{Z^m(T_0)}^2\le C_0\{1+\|u_0\|_{H^m}^2\}^a\|u_0\|_{H^m}^2$$

holds for some constants $C_0>0$ and $a>0$ depending on m.

The global existence of the solution $u(t)$ follows from Proposition 3.2 and the following a priori estimate in a standard manner.

Proposition 3.3. *Let m be an integer satisfying $m\ge 2(=[n/2]+1)$. Then there are positive numbers ν_0, γ_0 and ω_0 such that if $\nu\ge\nu_0$, $\gamma^2/(\nu+\tilde{\nu})\ge\gamma_0^2$ and $\|\rho_s-1\|_{C^{m+1}[0,1]}\le\omega_0$, then the following assertion holds.*

There exists a number $\varepsilon_1>0$ such that if a solution $u(t)$ of (1.3)–(1.6) in $Z^m(T)$ satisfies $\|u_0\|_{H^m\cap L^1}\le\varepsilon_1$, then there holds the estimate

$$[\![u(t)]\!]_m^2\le C_1\|u_0\|_{H^m\cap L^1}^2$$

for some constant $C_1>0$ independent of T.

Proposition 3.3, together with L^2 decay estimate (3.1), will be proved in sections 4–8. The asymptotic behavior (3.2) will be proved in section 9.

4. Spectral properties of the linearized operator

In this section we consider the spectral properties of the linearized problem which will be employed in the analysis of the nonlinear problem.

Problem (1.3)–(1.6) with $n = 2$ is written in the form

$$\partial_t u + Lu = \boldsymbol{F}, \quad w|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \tag{4.1}$$

Here $u = {}^\top(\phi, w)$; $\boldsymbol{F} = {}^\top(f^0, \boldsymbol{f})$ with $\boldsymbol{f} = {}^\top(f^1, f^2)$ is the nonlinearity; and L is the operator of the form

$$L = A + B + C_0,$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\nu}{\rho_s}\Delta I_2 - \frac{\tilde{\nu}}{\rho_s}\nabla \mathrm{div} \end{pmatrix}, \quad B = \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \mathrm{div}\,(\rho_s \cdot\,) \\ \nabla\left(\frac{P'(\rho_s)}{\gamma^2\rho_s}\,\cdot\right) & v_s^1\partial_{x_1} I_2 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 0 & 0 \\ \frac{\nu\partial_{x_2}^2 v_s^1}{\gamma^2\rho_s^2}\boldsymbol{e}_1 & (\partial_{x_2} v_s^1)\boldsymbol{e}_1{}^\top\boldsymbol{e}_2 \end{pmatrix}$$

with $\boldsymbol{e}_1 = {}^\top(1,0)$ and $\boldsymbol{e}_2 = {}^\top(0,1)$.

We here consider the operator L as an operator on $H^1 \times \tilde{H}^1$ with domain $D(L) = \{u = {}^\top(\phi, w) \in H^1 \times \tilde{H}^1;\ w \in H^2 \cap H_0^1,\ Lu \in H^1 \times \tilde{H}^1\}$, where $\tilde{H}^1 = \{w \in L^2;\ \partial_{x_1} w \in L^2\}$ with norm $\|w\|_{\tilde{H}^1} = (\|w\|_2^2 + \|\partial_{x_1} w\|_2^2)^{\frac{1}{2}}$. One can see that there exists a $\Lambda \gg 1$ such that $\{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \geq \Lambda\} \subset \rho(-L)$. Let $\tilde{Z}(T)$ be a function space defined by

$$\begin{aligned} \tilde{Z}(T) \;=\; \{u = {}^\top(\phi, w);\ & u \in C([0,T]; H^1 \times \tilde{H}^1) \\ & \partial_{x_1}^k w \in L^2(0,T; H_0^1),\ k = 0, 1, \\ & w \in C((0,T]; H_0^1)\}. \end{aligned}$$

Then one can show that for any $u_0 \in H^1 \times \tilde{H}^1$ there exists a unique solution $u(t)$ in $\tilde{Z}(T)$ $(\forall T > 0)$ of the linear problem

$$\partial_t u + Lu = 0, \quad u|_{t=0} = u_0; \tag{4.2}$$

and $u(t)$ satisfies

$$\|u(t)\|_{H^1\times\tilde{H}^1}^2 + t\|\partial_{x_2} w(t)\|_2^2 + \sum_{k=0}^{1}\int_0^t \|\partial_{x_1}^k w(\tau)\|_{H^1}^2\,d\tau \leq C\|u_0\|_{H^1\times\tilde{H}^1}^2.$$

Furthermore, if $u_0 \in D(L)$, then $u \in C^1([0,T]; H^1 \times \tilde{H}^1)$. It then follows that $-L$ generates a C_0-semigroup $U(t)$ that is defined by $U(t)u_0 = u(t)$, where $u(t)$ is a solution of (4.2) with $u_0 \in H^1 \times \tilde{H}^1$. See also [2, 3] for a generation of a C_0-semigroup. We will employ some spectral properties of $U(t)$.

In the analysis of (4.1) we will decompose the solution $u(t)$ of (4.1) by a projection operator associated with $U(t)$ which is obtained through the Fourier transform in x_1.

Let us consider the Fourier transform of (1.3)–(1.6) in $x_1 \in \mathbf{R}$:

$$\partial_t \hat{\phi} + i\xi_1 v_s^1 \hat{\phi} + i\gamma^2 \xi(\rho_s \hat{w}^1) + \gamma^2 \partial_{x_2}(\rho_s \hat{w}^2) = \hat{f}^0, \tag{4.3}$$

$$\begin{aligned} &\partial_t \hat{w}^1 + \tfrac{\nu}{\rho_s}(|\xi|^2 - \partial_{x_2}^2)\hat{w}^1 - i\tfrac{\tilde{\nu}}{\rho_s}\xi(i\xi\hat{w}^1 + \partial_{x_2}\hat{w}^2) + i\xi(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\hat{\phi}) \\ &+\tfrac{\nu\partial_{x_2}^2 v_s^1}{\gamma^2\rho_s^2}\hat{\phi} + i\xi v_s^1 \hat{w}^1 + (\partial_{x_2} v_s^1)\hat{w}^2 = \hat{f}^1, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\partial_t \hat{w}^2 + \tfrac{\nu}{\rho_s}(|\xi|^2 - \partial_{x_2}^2)\hat{w}^2 - \tfrac{\tilde{\nu}}{\rho_s}\partial_{x_2}(i\xi\hat{w}^1 + \partial_{x_2}\hat{w}^2) \\ &+\partial_{x_2}(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\hat{\phi}) + i\xi v_s^1 \hat{w}^2 = \hat{f}^2, \end{aligned} \tag{4.5}$$

$$\hat{w}|_{x2=0,1} = 0, \tag{4.6}$$

$$\hat{u}|_{t=0} = \hat{u}_0 = {}^\top(\hat{\phi}_0, \hat{w}_0). \tag{4.7}$$

Here $\hat{\phi} = \hat{\phi}(\xi, x_2, t)$ and $\hat{w} = \hat{w}(\xi, x_2, t)$ are the Fourier transform of $\phi = \phi(x_1, x_2, t)$ and $w = w(x_1, x_2, t)$ in $x_1 \in \mathbf{R}$ with $\xi \in \mathbf{R}$ being the dual variable. We thus arrive at the following problem

$$\partial_t \hat{u} + \hat{L}_\xi \hat{u} = \hat{\boldsymbol{F}}, \quad \hat{u}|_{t=0} = \hat{u}_0 \tag{4.8}$$

with a parameter $\xi \in \mathbf{R}$. Here $\hat{L}_\xi$ is the operator on $H^1(0,1) \times L^2(0,1)$ of the form

$$\hat{L}_\xi = \hat{A}_\xi + \hat{B}_\xi + \hat{C}_0,$$

where

$$\hat{A}_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s}(|\xi|^2 - \partial_{x_2}^2) + \frac{\tilde{\nu}}{\rho_s}|\xi|^2 & -i\frac{\tilde{\nu}}{\rho_s}\xi\partial_{x_2} \\ 0 & -i\frac{\tilde{\nu}}{\rho_s}\xi\partial_{x_2} & \frac{\nu}{\rho_s}(|\xi|^2 - \partial_{x_2}^2) - \frac{\tilde{\nu}}{\rho_s}\partial_{x_2}^2 \end{pmatrix},$$

$$\hat{B}_\xi = \begin{pmatrix} i\xi v_s^1 & i\gamma^2\rho_s\xi & \gamma^2\partial_{x_2}(\rho_s\,\cdot) \\ i\xi\frac{P'(\rho_s)}{\gamma^2\rho_s} & i\xi v_s^1 & 0 \\ \partial_{x_2}\left(\frac{P'(\rho_s)}{\gamma^2\rho_s}\,\cdot\right) & 0 & i\xi v_s^1 \end{pmatrix},$$

$$\hat{C}_0 = C_0$$

with domain

$$D(\hat{L}_{\xi'}) = H^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1)).$$

In [12] some spectral properties of $-\hat{L}_\xi$ were investigated. (In [12], the linearized operator at a Poiseuille type flow was studied, but one can see that the arguments in [12] are valid for our parallel flows since only the properties of parallel flow (2.1), (2.2) and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \ll 1$ were used in [12] to show the spectral properties in Lemmas 4.1–4.9 below.) We will employ the following properties of $\hat{L}_\xi$ for $|\xi| \ll 1$.

We begin with the case $\xi = 0$. Let us introduce a formal adjoint operator $\hat{L}_\xi^*$ of $\hat{L}_\xi$ with respect to the inner product $\langle\cdot,\cdot\rangle$:

$$\hat{L}_\xi^* = \hat{A}_\xi - \hat{B}_\xi + \hat{C}_0^*,$$

with domain $D(\hat{L}_\xi^*) = D(\hat{L}_\xi)$, where

$$\hat{C}_0^* = \begin{pmatrix} 0 & \frac{\gamma^2\nu\partial_{x_2}^2 v_s^1}{\rho_s P'(\rho_s)} & 0 \\ 0 & 0 & 0 \\ 0 & \partial_{x_2} v_s^1 & 0 \end{pmatrix}.$$

Lemma 4.1. ([12]) *Let $u_s = {}^\top(\rho_s, v_s)$ be a smooth stationary solution satisfying* (2.1) *and* (2.2). *Then there are positive constants ν_0, γ_0 and ω_0 such that if $\nu \geq \nu_0$, $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$, then the following assertions hold.*

(i) *There are positive numbers η_0 and $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(-\eta_0, \theta_0) \setminus \{0\} \subset \rho(-\hat{L}_0)$. Furthermore, the following estimates hold uniformly for $\lambda \in \rho(-\hat{L}_0) \cap \Sigma(-\eta_0, \theta_0) \setminus \{0\}$:*

$$\left|(\lambda + \hat{L}_0)^{-1} f\right|_{H^1\times L^2} \leq \frac{C}{|\lambda|}|f|_{H^1\times L^2},$$

$$\left|\partial_{x_2}^l \tilde{Q}(\lambda+\hat{L}_0)^{-1}f\right|_2 \le C\Big(\frac{1}{|\lambda|}+\frac{1}{(|\lambda|+1)^{1-\frac{l}{2}}}\Big)|f|_{H^1\times L^2}$$

for $l=1,2$,

$$\left|\partial_{x_2}^2 Q_0(\lambda+\hat{L}_0)^{-1}f\right|_2 \le C\Big(\frac{1}{|\lambda|}+\frac{1}{(|\lambda|+1)^{\frac{1}{2}}}\Big)|f|_{H^2\times H^1}.$$

The same assertion holds for $-\hat{L}_0^*$.

(ii) $\lambda=0$ *is a simple eigenvalue of* $\hat{L}_0$ *and* $\hat{L}_0^*$.

(iii) *The eigenspaces for* $\lambda=0$ *of* $\hat{L}_0$ *and* $\hat{L}_0^*$ *are spanned by* $u^{(0)}$ *and* $u^{(0)*}$ *respectively, where*

$$u^{(0)}={}^\top(\phi^{(0)},w^{(0)}),\quad w^{(0)}={}^\top(w^{(0),1},0)$$

and

$$u^{(0)*}={}^\top(\phi^{(0)*},0)$$

with

$$\phi^{(0)}(x_2)=\alpha_0\tfrac{\gamma^2\rho_s(x_2)}{P'(\rho_s(x_2))},\quad \alpha_0=\Big(\int_0^1\tfrac{\gamma^2\rho_s}{P'(\rho_s)}\,dx_2\Big)^{-1},$$

$$w^{(0),1}(x_2)=-\tfrac{1}{\gamma^2}\int_0^1 G(x_2,y_2)\tfrac{\partial_{y_2}^2 v_s^1(y_2)}{\rho_s(y_2)}\phi^{(0)}(y_2)\,dy_2,$$

$$\phi^{(0)*}(x_2)=\tfrac{\gamma^2}{\alpha_0}\phi^{(0)}(x_2).$$

Here

$$G(x_2,y_2)=\begin{cases}(1-x_2)y_2 & (0<y_2<x_2)\\[2ex] x_2(1-y_2) & (x_2<y_2<1).\end{cases}$$

(iv) *The eigenprojections* $\hat{\Pi}^{(0)}$ *and* $\hat{\Pi}^{(0)*}$ *for* $\lambda=0$ *of* $\hat{L}_0$ *and* $\hat{L}_0^*$ *are given by*

$$\hat{\Pi}^{(0)}u=\langle u,u^{(0)*}\rangle u^{(0)}=\langle\phi\rangle u^{(0)},$$

and

$$\hat{\Pi}^{(0)*}u=\langle u,u^{(0)}\rangle u^{(0)*}$$

for $u={}^\top(\phi,w)$, *respectively.*

(v) *Let* $u^{(0)}$ *be written as* $u^{(0)}=u_0^{(0)}+u_1^{(0)}$, *where*

$$u_0^{(0)}={}^\top(\phi^{(0)},0),\quad u_1^{(0)}={}^\top(0,w^{(0)}),\quad w^{(0)}={}^\top(w^{(0),1},0).$$

Then $u^{(0)*}=\frac{\gamma^2}{\alpha_0}u_0^{(0)}$ *and*

$$\langle u,u^{(0)}\rangle=\frac{\alpha_0}{\gamma^2}\langle\phi\rangle+(w^1,w^{(0),1}\rho_s)$$

for $u = {}^\top(\phi, w)$*,* $w = {}^\top(w^1, w^2)$*.*

As for the spectrum of $-\hat{L}_\xi$, we have the following result. Let $\hat{L}_\xi$ be denoted by

$$\hat{L}_\xi = \hat{L}_0 + \xi \hat{L}^{(1)} + \xi^2 \hat{L}^{(2)},$$

where

$$\hat{L}^{(1)} = \begin{pmatrix} iv_s^1 & i\gamma^2\rho_s & 0 \\ i\frac{P'(\rho_s)}{\gamma^2\rho_s} & iv_s^1 & -i\frac{\tilde{\nu}}{\rho_s}\partial_{x_2} \\ 0 & -i\frac{\tilde{\nu}}{\rho_s}\partial_{x_2} & 0 \end{pmatrix}, \quad \hat{L}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu+\tilde{\nu}}{\rho_s} & 0 \\ 0 & 0 & \frac{\nu}{\rho_s} \end{pmatrix}.$$

Likewise, we denote $\hat{L}_\xi^*$ by $\hat{L}_\xi^* = \hat{L}_0^* + \xi \hat{L}^{(1)*} + \xi^2 L^{(2)*}$.

Lemma 4.2. ([12]) *If* $\nu \geq \nu_0$*,* $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$ *and* $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$*, then there exists a positive number* $r_0 = r_0(\nu, \tilde{\nu}, \gamma, \eta_0, \theta_0)$ *such that the following assertions hold.*

(i) $\Sigma(-\eta_0, \theta_0) \cap \{\lambda;\, |\lambda| \geq \frac{\eta_0}{2}\} \subset \rho(-\hat{L}_\xi)$ *for* $|\xi| \leq r_0$*. Furthermore, the following estimates hold uniformly in* $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \{\lambda;\, |\lambda| \geq \frac{\eta_0}{2}\}$ *and* ξ *with* $|\xi| \leq r_0$*:*

$$\left|(\lambda + \hat{L}_\xi)^{-1} f\right|_{H^1 \times L^2} \leq \tfrac{C}{|\lambda|} |f|_{H^1 \times L^2},$$

$$\left|\partial_{x_2}^l \widetilde{Q}(\lambda + \hat{L}_\xi)^{-1} f\right|_2 \leq C\left(\tfrac{1}{|\lambda|} + \tfrac{1}{(|\lambda|+1)^{1-\frac{l}{2}}}\right) |f|_{H^1 \times L^2}$$

for $l = 1, 2$*,*

$$\left|\partial_{x_2}^2 Q_0(\lambda + \hat{L}_\xi)^{-1} f\right|_2 \leq C\left(\tfrac{1}{|\lambda|} + \tfrac{1}{(|\lambda|+1)^{\frac{1}{2}}}\right) |f|_{H^2 \times H^1}.$$

The same assertion also holds for $-\hat{L}_\xi^*$*.*

(ii) *There holds*

$$\sigma(-\hat{L}_\xi) \cap \{\lambda; |\lambda| < \frac{\eta_0}{2}\} = \{\hat{\lambda}_0(\xi)\} \quad \textit{for } |\xi| \leq r_0,$$

where $\hat{\lambda}_0(\xi)$ *is a simple eigenvalue of* $-\hat{L}_\xi$ *that has the form*

$$\hat{\lambda}_0(\xi) = -ia_0\xi - \kappa_0\xi^2 + O(|\xi|^3)$$

as $\xi \to 0$*. Here* $a_0 \in \mathbf{R}$ *and* $\kappa_0 > 0$ *are the numbers given by*

$$\begin{aligned} a_0 &= -\langle v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1} \rangle, \\ \kappa_0 &= \tfrac{\alpha_0\gamma^2}{\nu}\left|(-\partial_{x_2}^2)^{-\frac{1}{2}}\rho_s\right|_2^2 + \left(O(\tfrac{\nu+\tilde{\nu}}{\gamma^2}) + O(\tfrac{1}{\nu})\right)\left(1 + O(\omega_0)\right) \\ &= \left(\tfrac{\gamma^2}{12\nu} + O(\tfrac{\nu+\tilde{\nu}}{\gamma^2}) + O(\tfrac{1}{\nu})\right)\left(1 + O(\omega_0)\right) > 0. \end{aligned}$$

(iii) *The eigenprojection* $\hat{\Pi}(\xi)$ *for the eigenvalue* $\hat{\lambda}_0(\xi)$ *is expanded as*

$$\hat{\Pi}(\xi) = \hat{\Pi}^{(0)} + \xi \hat{\Pi}^{(1)} + \xi^2 \Pi^{(2)}(\xi),$$

where

$$\hat{\Pi}^{(1)} = -\hat{\Pi}^{(0)} \hat{L}^{(1)} \hat{S} - \hat{S} \hat{L}^{(1)} \hat{\Pi}^{(0)}$$

with $\hat{S} = \big((I - \hat{\Pi}^{(0)}) \hat{L}_0 (I - \hat{\Pi}^{(0)}) \big)^{-1}$*; and* $\Pi^{(2)}(\xi)$ *is a bounded operator on* $H^1(0,1) \times L^2(0,1)$ *satisfying* $|\Pi^{(2)}(\xi) u|_{H^1 \times L^2} \leq C |u|_{H^1 \times L^2}$*. Furthermore, it holds that* $\tilde{Q} \hat{\Pi}(\xi) u|_{x_2 = 0,1} = 0$*.*

Concerning the eigenprojection $\hat{\Pi}(\xi)$, the following estimates hold true.

Lemma 4.3. (i) *The eigenprojection* $\hat{\Pi}(\xi)$ *is written in the form*

$$(\hat{\Pi}(\xi) u)(x_2) = \int_0^1 \hat{\Pi}(\xi, x_2, y_2) u(y_2)\, dy_2$$

with

$$\hat{\Pi}(\xi, x_2, y_2) = \hat{\Pi}^{(0)}(x_2) + \xi \hat{\Pi}^{(1)}(x_2, y_2) + \xi^2 \hat{\Pi}^{(2)}(\xi, x_2, y_2).$$

Here $\hat{\Pi}^{(0)}(x_2) = u^{(0)}(x_2)^\top \boldsymbol{e}_0$, $\boldsymbol{e}_0 = {}^\top(1,0,0)$*; and* $\hat{\Pi}^{(1)}(x_2, y_2)$ *and* $\hat{\Pi}^{(2)}(\xi, x_2, y_2)$ *satisfy the estimates*

$$\left| \partial_{x_2}^k \partial_{y_2}^l \hat{\Pi}^{(1)}(\cdot,\cdot) \right|_{L^\infty((0,1)\times(0,1))} \leq C_{k,l},$$

$$\left| \partial_{x_2}^k \partial_{y_2}^l \hat{\Pi}^{(2)}(\xi,\cdot,\cdot) \right|_{L^\infty((0,1)\times(0,1))} \leq C_{k,l}$$

uniformly in ξ *with* $|\xi| \leq r_0$ *for any* $k, l \geq 0$*.*

As a consequence, there hold, for $0 \leq \forall k \leq m+2$, $1 \leq p \leq 2$,

$$|\partial_{x_2}^k \hat{\Pi}(\xi) u|_2 \leq C_k |u|_p,$$

$$|\partial_{x_2}^k \hat{\Pi}^{(j)} u|_2 \leq C_k |u|_p \quad (j = 0, 1),$$

$$|\partial_{x_2}^k \hat{\Pi}^{(2)}(\xi) u|_2 \leq C_k |u|_p.$$

(ii) *Let* $\hat{\Pi}^*(\xi)$ *be defined by*

$$(\hat{\Pi}^*(\xi) u)(y_2) = \int_0^1 \hat{\Pi}^*(\xi, x_2, y_2) u(x_2)\, dx_2,$$

where $\hat{\Pi}^*(\xi, x_2, y_2)$ *is given by* $\hat{\Pi}^*(\xi, x_2, y_2) = W(y_2)^{-1\top} \big(\overline{\hat{\Pi}(\xi, x_2, y_2)} \big) W(x_2)$ *with* $W(x_2) = \mathrm{diag}\,(\frac{P'(\rho_s(x_2))}{\gamma^4 \rho_s(x_2)}, \rho_s(x_2), \rho_s(x_2))$*. Then* $\hat{\Pi}^*(\xi)$ *is the eigenprojection for the eigenvalue* $\overline{\hat{\lambda}_0(\xi)}$ *of* $-\hat{L}_\xi$*; and there hold* $\langle \hat{\Pi}(\xi) u_1, u_2 \rangle = \langle u_1, \hat{\Pi}^*(\xi) u_2 \rangle$

and $\tilde{Q}\hat{\Pi}^*(\xi)u|_{x_2=0,1} = 0$. *Furthermore, estimates similar to those for* $\hat{\Pi}(\xi)$ *given in* (i) *hold for* $\hat{\Pi}^*(\xi)$.

Proof. The integral expression of $\hat{\Pi}(\xi)$ and estimates for the case $k, l \le 1$ are given in [12] following the argument in [5]. Estimates for the case $k, l \ge 2$ can be obtained as follows. We follow the argument in [5, Proof of Theorem 3.3].

Let $\tilde{u}(\xi)$ and $\tilde{u}^*(\xi)$ be defined by

$$\tilde{u}(\xi) = \frac{1}{2\pi i}\int_{\{|\lambda|=\frac{\eta_0}{2}\}} (\lambda + \hat{L}_\xi)^{-1} u^{(0)}\, d\lambda,$$

$$\tilde{u}^*(\xi) = \frac{1}{2\pi i}\int_{\{|\lambda|=\frac{\eta_0}{2}\}} (\lambda + \hat{L}_\xi^*)^{-1} u^{(0)*}\, d\lambda.$$

Then $\hat{\Pi}(\xi)$ is given by

$$(4.9)\qquad \begin{aligned}(\hat{\Pi}(\xi)u)(x_2) &= \frac{\langle u, \tilde{u}^*(\xi)\rangle}{\langle \tilde{u}(\xi), \tilde{u}^*(\xi)\rangle}\tilde{u}(\xi; x_2) \\ &= \int_0^1 \frac{\tilde{u}(\xi; x_2) \otimes \overline{W(y_2)\tilde{u}^*(\xi; y_2)}u(y_2)}{\langle \tilde{u}(\xi), \tilde{u}^*(\xi)\rangle}\, dy_2.\end{aligned}$$

Here for $a = {}^\top(a_1, a_2)$ and $b = {}^\top(b_1, b_2)$, $a \otimes b$ denotes the matrix $(a_i b_j)$.

Since

$$(4.10)\qquad \big|\hat{L}^{(1)}(\lambda + \hat{L}_0)^{-1} f\big|_{H^{k+1}\times H^k} \le C_k \big|(\lambda + \hat{L}_0)^{-1} f\big|_{H^{k+1}},$$

$$(4.11)\qquad \big|\hat{L}^{(2)}(\lambda + \hat{L}_0)^{-1} f\big|_{H^{k+1}\times H^k} \le C_k \big|(\lambda + \hat{L}_0)^{-1} f\big|_{H^k},$$

for $k \ge 0$, by using Lemma 4.1, we have the Neumann series expansion

$$(\lambda + \hat{L}_\xi)^{-1} = (\lambda + \hat{L}_0)^{-1}\sum_{N=0}^{\infty}(-1)^N\big\{(\xi\hat{L}^{(1)} + \xi^2\hat{L}^{(2)})(\lambda + \hat{L}_0)^{-1}\big\}^N$$

for $|\xi| \ll 1$. One can see that the same expansion also holds for $(\lambda + \hat{L}_\xi^*)^{-1}$

$$(\lambda + \hat{L}_\xi^*)^{-1} = (\lambda + \hat{L}_0^*)^{-1}\sum_{N=0}^{\infty}(-1)^N\big\{(\xi\hat{L}^{(1)*} + \xi^2\hat{L}^{(2)*})(\lambda + \hat{L}_0^*)^{-1}\big\}^N.$$

It then follows that

$$(4.12)\qquad \tilde{u}(\xi) = u^{(0)} + \xi\tilde{u}^{(1)} + \xi^2\tilde{u}^{(2)}(\xi),$$

$$\tilde u^*(\xi) = u^{(0)*} + \xi \tilde u^{(1)*} + \xi^2 \tilde u^{(2)*}(\xi), \tag{4.13}$$

$$\langle \tilde u(\xi), \tilde u^*(\xi)\rangle = 1 + O(\xi) \ge \frac{1}{2}, \tag{4.14}$$

for $|\xi| \ll 1$. Here

$$\tilde u^{(1)} = -\frac{1}{2\pi i}\int_{\{|\lambda|=\frac{\eta_0}{2}\}} (\lambda+\hat L_0)^{-1}\hat L^{(1)}(\lambda+\hat L_0)^{-1}u^{(0)}\,d\lambda, \tag{4.15}$$

$$\tilde u^{(2)}(\xi) = \frac{1}{2\pi i}\int_{\{|\lambda|=\frac{\eta_0}{2}\}} R^{(2)}(\lambda,\xi)u^{(0)}\,d\lambda \tag{4.16}$$

with

$$\begin{aligned} R^{(2)}(\lambda,\xi) &= -(\lambda+\hat L_0)^{-1}\hat L^{(2)}(\lambda+\hat L_0)^{-1} \\ &\quad +(\lambda+\hat L_0)^{-1}\sum_{N=2}^{\infty}(-1)^N\xi^{N-2}\big\{(\hat L^{(1)}+\xi\hat L^{(2)})(\lambda+\hat L_0)^{-1}\big\}^N; \end{aligned}$$

and, likewise, $\tilde u^{(1)*}$ and $\tilde u^{(2)*}(\xi)$ are given by formulas similar to (4.15) and (4.16) with $\hat L_0$ replaced by $\hat L_0^*$, etc. The integral expression now follows from (4.9), (4.12)–(4.14). Furthermore, estimates for kernel functions would follow from regularity estimates on the resolvents $(\lambda+\hat L_0)^{-1}$ and $(\lambda+\hat L_0^*)^{-1}$. In fact, we deduce from Lemma 4.1, (4.10) and (4.11) that for $|\lambda| = \frac{\eta_0}{2}$,

$$\big|\hat L^{(j)}(\lambda+\hat L_0)^{-1}f\big|_{H^1\times L^2} \le C|f|_{H^1\times L^2} \quad (j=1,2),$$

and hence,

$$\begin{aligned} |\tilde u^{(1)}|_{H^1\times H^2} &\le C\int_{\{|\lambda|=\frac{\eta_0}{2}\}} \big|\hat L^{(1)}(\lambda+\hat L_0)^{-1}u^{(0)}\big|_{H^1\times L^2}\,|d\lambda| \\ &\le |u^{(0)}|_{H^1\times L^2} \\ &\le C. \end{aligned}$$

Similarly,

$$\begin{aligned} |\tilde u^{(2)}(\xi)|_{H^1\times H^2} &\le C\int_{\{|\lambda|=\frac{\eta_0}{2}\}} \big(1+\sum_{N=2}^{\infty}|\xi|^{N-2}\big)|u^{(0)}|_{H^1\times L^2}\,|d\lambda| \\ &\le |u^{(0)}|_{H^1\times L^2} \\ &\le C, \end{aligned}$$

provided that $|\xi|$ is sufficiently small. Note also that $\partial_\xi(\lambda+\hat{L}_\xi)^{-1}u^{(0)}|_{\xi=0} = -(\lambda+\hat{L}_0)^{-1}\hat{L}^{(1)}(\lambda+\hat{L}_0)^{-1}u^{(0)}$ in $H^1(0,1)\times H^2(0,1)$.

To estimate higher order derivatives, we first show that

$$\big|(\lambda+\hat{L}_\xi)^{-1}f\big|_{H^{k+1}\times H^{k+2}} \le C_k|f|_{H^{k+1}\times H^k} \tag{4.17}$$

for $|\lambda| = \frac{\eta_0}{2}$, $k=1,2,\cdots$. The case $k=0$ follows from Lemma 4.2. As for $k\ge 1$, we set $u = {}^\top(\phi,w^1,w^2) = (\lambda+\hat{L}_\xi)^{-1}f$. Then

$$\lambda\phi + i\xi v_s^1\phi + i\gamma^2\xi\rho_s w^1 + \gamma^2\partial_{x_2}(\rho_s w^2) = f^0, \tag{4.18}$$

$$\begin{aligned}\lambda w^1 &+ \tfrac{\nu}{\rho_s}(|\xi|^2-\partial_{x_2}^2)w^1 - i\tfrac{\tilde\nu}{\rho_s}\xi(i\xi w^1+\partial_{x_2}w^2) + i\xi\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi\\ &+\tfrac{\nu\partial_{x_2}^2 v_s^1}{\gamma^2\rho_s^2}\phi + i\xi v_s^1 w^1 + \partial_{x_2}v_s^1 w^2 = f^1,\end{aligned} \tag{4.19}$$

$$\begin{aligned}\lambda w^2 &+ \tfrac{\nu}{\rho_s}(|\xi|^2-\partial_{x_2}^2)w^2 - \tfrac{\tilde\nu}{\rho_s}\partial_{x_2}(i\xi w^1+\partial_{x_2}w^2)\\ &+\partial_{x_2}\left(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi\right) + i\xi v_s^1 w^2 = f^2,\end{aligned} \tag{4.20}$$

where $f = {}^\top(f^0,f^1,f^2)$. By adding $\frac{\gamma^2\rho_s^2}{\nu+\tilde\nu}\times$ (4.20) to ∂_{x_2}(4.18), we have

$$\lambda\partial_{x_2}\phi + \tfrac{\rho_s P'(\rho_s)}{\nu+\tilde\nu}\partial_{x_2}\phi + i\xi v_s^1\partial_{x_2}\phi = H,$$

where

$$\begin{aligned} H &= \partial_{x_2}f^0 - \{i\gamma^2\xi\partial_{x_2}(\rho_s w^1) + \gamma^2\partial_{x_2}\rho_s\partial_{x_2}w^2 + \gamma^2\partial_{x_2}(w^2\partial_{x_2}\rho_s)\}\\ &\quad+\tfrac{\gamma^2\rho_s^2}{\nu+\tilde\nu}\big\{f^2 - (\lambda w^2 + i\xi v_s^1 w^2 + \tfrac{\nu}{\rho_s}|\xi|^2 w^2 - \tfrac{\tilde\nu}{\rho_s}\partial_{x_2}i\xi w^1 + \phi\partial_{x_2}\left(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\right))\big\},\end{aligned}$$

It then follows that

$$\lambda\partial_{x_2}^{k+1}\phi + \tfrac{\rho_s P'(\rho_s)}{\nu+\tilde\nu}\partial_{x_2}^{k+1}\phi + i\xi v_s^1\partial_{x_2}^{k+1}\phi = \partial_{x_2}^k H - i\xi[\partial_{x_2}^k, v_s^1]\partial_{x_2}\phi - \big[\partial_{x_2}^k, \tfrac{\rho_s P'(\rho_s)}{\nu+\tilde\nu}\big]\partial_{x_2}\phi.$$

We thus obtain

$$\begin{aligned}&\mathrm{Re}\,\lambda|\partial_{x_2}^{k+1}\phi|_2^2 + \Big|\sqrt{\tfrac{\rho_s P'(\rho_s)}{\nu+\tilde\nu}}\partial_{x_2}^{k+1}\phi\Big|_2^2\\ &\quad\le C\big\{|\partial_{x_2}^{k+1}f^0|_2 + |\partial_{x_2}\phi|_{H^{k-1}} + |\phi|_{H^k} + |w|_{H^{k+1}} + |f^2|_{H^k} + |\lambda||w^2|_{H^k}\big\}|\partial_{x_2}^{k+1}\phi|_2.\end{aligned}$$

Therefore, if $|\rho_s - 1|_\infty \ll 1$, then

$$\begin{aligned}&\big(\mathrm{Re}\,\lambda + \tfrac{c_0\gamma^2}{\nu+\tilde\nu}\big)|\partial_{x_2}^{k+1}\phi|_2^2\\ &\quad\le C\big\{|\partial_{x_2}^{k+1}f^0|_2^2 + |\partial_{x_2}\phi|_{H^{k-1}}^2 + |\phi|_{H^k}^2 + |w|_{H^{k+1}}^2 + |f^2|_{H^k}^2 + |\lambda|^2|w^2|_{H^k}^2\big\}.\end{aligned} \tag{4.21}$$

By (4.19) and (4.20), we have

$$\begin{aligned}\partial_{x_2}^2 w^1 &= \tfrac{\rho_s}{\nu}\big\{\lambda w^1 + \tfrac{\nu}{\rho_s}|\xi|^2 w^1 - i\tfrac{\tilde{\nu}}{\rho_s}\xi(i\xi w^1 + \partial_{x_2} w^2) + i\xi\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi \\ &\quad + \tfrac{\nu\partial_{x_2}^2 v_s^1}{\gamma^2\rho_s^2}\phi + i\xi v_s^1 w^1 + \partial_{x_2} v_s^1 w^2 - f^1\big\}, \\ \partial_{x_2}^2 w^1 &= \tfrac{\rho_s}{\nu+\tilde{\nu}}\big\{\lambda w^2 + \tfrac{\nu}{\rho_s}|\xi|^2 w^2 - i\xi\tfrac{\tilde{\nu}}{\rho_s}\partial_{x_2} w^1 + \partial_{x_2}\left(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi\right) \\ &\quad + i\xi v_s^1 w^2 - f^2\big\},\end{aligned}$$

and hence,

$$|\partial_{x_2}^{k+2} w|_2^2 \le C\{|\lambda|^2|w|_{H^k}^2 + |\phi|_{H^{k+1}}^2 + |w|_{H^{k+1}}^2 + |f^1|_{H^k}^2 + |f^2|_{H^k}^2\} \tag{4.22}$$

for $|\lambda| = \frac{\eta_0}{2}$

Based on (4.21) and (4.22), we obtain, by induction on k,

$$|u|_{H^{k+1}\times H^{k+2}} \le C|f|_{H^{k+1}\times H^k}$$

for $|\lambda| = \frac{\eta_0}{2}$. This proves (4.17).

We next estimate integrands $z^{(1)} = -(\lambda + \hat{L}_0)^{-1}\hat{L}^{(1)}(\lambda + \hat{L}_0)^{-1}u^{(0)}$ and $z_\xi^{(2)} = R^{(2)}(\lambda,\xi)u^{(0)}$ in the right-hand side of (4.15) and (4.16), respectively. Set $z_\xi = (\lambda + \hat{L}_\xi)^{-1}u^{(0)}$. Then $z_\xi \in H^1(0,1)\times(H^2(0,1)\cap H_0^1(0,1))$, and

$$(\lambda + \hat{L}_0)z_\xi + \xi\hat{L}^{(1)}z_\xi + \xi^2\hat{L}^{(2)}z_\xi = u^{(0)}. \tag{4.23}$$

By (4.17), we have, for $k = 0, 1, 2, \cdots$,

$$|z_\xi|_{H^{k+1}\times H^{k+2}} \le C_k|u^{(0)}|_{H^{k+1}\times H^k} \le C_k. \tag{4.24}$$

We next set $z^{(0)} = \frac{1}{\lambda}u^{(0)}$ and $z_\xi^{(1)} = \frac{1}{\xi}(z_\xi - z^{(0)})$. Then, $z_\xi^{(1)} \to z^{(1)}$ in $H^1(0,1)\times H^2(0,1)$ as $\xi \to 0$, and

$$|z^{(1)}|_{H^{k+1}} \le C_k|u^{(0)}|_{H^{k+1}} \le C_k. \tag{4.25}$$

Furthermore, since

$$(\lambda + \hat{L}_0)z^{(0)} = u^{(0)}, \tag{4.26}$$

we see from (4.23) and (4.26) that

$$(\lambda + \hat{L}_0)z_\xi^{(1)} + \hat{L}^{(1)}z_\xi + \xi\hat{L}^{(2)}z_\xi = 0. \tag{4.27}$$

Therefore, we have

$$z_\xi^{(1)} = -(\lambda + \hat{L}_0)^{-1}(\hat{L}^{(1)} z_\xi + \xi \hat{L}^{(2)} z_\xi).$$

It then follows from (4.17) and (4.24) that

$$\begin{aligned} |z_\xi^{(1)}|_{H^{k+1}\times H^{k+2}} &\leq C_k |\hat{L}^{(1)} z_\xi + \xi \hat{L}^{(2)} z_\xi|_{H^{k+1}\times H^k} \\ &\leq C_k |z_\xi|_{H^{k+1}} \\ &\leq C_k. \end{aligned} \tag{4.28}$$

We next consider $z_\xi^{(2)} = R^{(2)}(\lambda, \xi) u^{(0)}$. We have $z_\xi^{(2)} = \frac{1}{\xi^2}(z_\xi - z^{(0)} - \xi z^{(1)}) \in H^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1))$ and, by (4.23), (4.26) and (4.27),

$$(\lambda + \hat{L}_0) z_\xi^{(2)} + L^{(1)} z_\xi^{(1)} + \hat{L}^{(2)} z_\xi = 0.$$

We thus obtain

$$z_\xi^{(2)} = -(\lambda + \hat{L}_0)^{-1}(\hat{L}^{(1)} z_\xi^{(1)} + \hat{L}^{(2)} z_\xi).$$

It then follows from (4.17), (4.24) and (4.28) that

$$\begin{aligned} |z_\xi^{(2)}|_{H^{k+1}\times H^{k+2}} &\leq C_k |\hat{L}^{(1)} z_\xi^{(1)} + \hat{L}^{(2)} z_\xi|_{H^{k+1}\times H^k} \\ &\leq C_k \{ |z_\xi^{(1)}|_{H^{k+1}} + |z_\xi|_{H^k} \} \\ &\leq C_k. \end{aligned} \tag{4.29}$$

We see from (4.24), (4.25) and (4.29) that

$$|\tilde{u}^{(1)}|_{H^{k+1}\times H^{k+2}} + |\tilde{u}^{(2)}(\xi)|_{H^{k+1}\times H^{k+2}} \leq C_k \tag{4.30}$$

for $k = 1, 2, \cdots$.

Similarly, one can show

$$|\tilde{u}^{(1)*}|_{H^{k+1}\times H^{k+2}} + |\tilde{u}^{(2)*}(\xi)|_{H^{k+1}\times H^{k+2}} \leq C_k |u^{(0)*}|_{H^{k+1}\times H^k} \leq C_k \tag{4.31}$$

for $k = 1, 2, \cdots$.

The desired estimates for integrands $\hat{\Pi}^{(1)}(x_2, y_2)$ and $\hat{\Pi}^{(2)}(\xi, x_2, y_2)$ are now obtained by (4.9), (4.30), (4.31) and the Sobolev embedding. This completes the proof. □

We now introduce a projection based on $\hat{\Pi}(\xi)$. For an interval $J \subset \mathbf{R}$, we denote the characteristic function of J by $\mathbf{1}_J$. We define $\hat{\chi}_1$ by

$$\hat{\chi}_1(\xi) = \mathbf{1}_{[0,r_1)}(|\xi|) = \begin{cases} 1 & (|\xi| < r_1) \\ 0 & (|\xi| \geq r_1). \end{cases}$$

Here and in what follows we take a number $r_1 > 0$ satisfying $r_1 \leq \min\{r_0, 1\}$.

We define P_1 and P_1^* by

$$P_1 u = \mathscr{F}^{-1}(\hat{\chi}_1(\xi)\hat{\Pi}(\xi)\hat{u})$$

and

$$P_1^* u = \mathscr{F}^{-1}(\hat{\chi}_1(\xi)\hat{\Pi}^*(\xi)\hat{u}).$$

It then follows that $P_1^2 = P_1$, $P_1^{*2} = P_1^*$ and

$$\langle\langle P_1 u_1, u_2 \rangle\rangle = \langle\langle u_1, P_1^* u_2 \rangle\rangle.$$

Based on Lemma 4.2 (iii), we expand P_1 as

$$P_1 = P_1^{(0)} + \partial_{x_1} P_1^{(1)} + \partial_{x_1}^2 P_1^{(2)},$$

where $P_1^{(j)} u = \mathscr{F}^{-1}(\hat{P}_1^{(j)}\hat{u})$ $(j = 0, 1, 2)$,

$$\hat{P}_1^{(0)} = \hat{\chi}_1 \hat{\Pi}^{(0)},$$

$$\hat{P}_1^{(1)} = -i\hat{\chi}_1 \hat{\Pi}^{(1)}, \quad \hat{\Pi}^{(1)} = -(\hat{\Pi}^{(0)}\hat{L}^{(1)}\hat{S} + \hat{S}\hat{L}^{(1)}\hat{\Pi}^{(0)}),$$

$$\hat{P}_1^{(2)} = -\hat{\chi}_1 \hat{\Pi}^{(2)}(\xi).$$

In what follows we will also denote $\hat{\Pi}^{(0)}$ by $\Pi^{(0)}$.

The following properties of $\Pi^{(0)}$ will be used in the subsequent analysis.

Lemma 4.4. (i) *For any* $k = 1, 2, \cdots$, $\partial_{x_1}^k \Pi^{(0)} = \Pi^{(0)} \partial_{x_1}^k$.
(ii) *If* $Q_0 u|_{x_2=0,1} = 0$, *then* $\Pi^{(0)}(\partial_{x_2} u) = 0$.
(iii) *For any* $l = 0, 1, \cdots$, *there holds* $\|\partial_{x_2}^l \Pi^{(0)} u\|_2 \leq C_l \|\Pi^{(0)} u\|_2$.
(iv) *If* $\tilde{Q} u|_{x_2=0,1} = 0$, *then* $\|(I - \Pi^{(0)})u\|_2 \leq C\|\partial_{x_2}(I - \Pi^{(0)})u\|_2$.

Proof. The assertions (i)–(iii) easily follow from the definition of $\Pi^{(0)}$:

$$\Pi^{(0)} u = \langle u, u^{(0)*} \rangle u^{(0)} = \langle Q_0 u \rangle u^{(0)}.$$

As for (iv), since $\tilde{Q}u^{(0)}|_{x_2=0,1} = {}^\top(0, w^{(0)})|_{x_2=0,1} = 0$, if $\tilde{Q}u|_{x_2=0,1} = 0$, then we have $\tilde{Q}(I - \Pi^{(0)})u|_{x_2=0,1} = 0$. By the Poincaré inequality, $\|\tilde{Q}(I - \Pi^{(0)})u\|_2 \le C\|\partial_{x_2}\tilde{Q}(I - \Pi^{(0)})u\|_2$. We also have $\langle Q_0(I - \Pi^{(0)})u\rangle = 0$. Therefore, by the Poincaré inequality, $\|Q_0(I - \Pi^{(0)})u\|_2 \le C\|\partial_{x_2}Q_0(I - \Pi^{(0)})u\|_2$. This completes the proof. □

We next state boundedness properties of P_1 and P_1^*.

Lemma 4.5. (i) *For any* $k = 1, 2, \cdots$, $\partial_{x_1}^k P_1 = P_1\partial_{x_1}^k$ *and* $\partial_{x_1}^k P_1^{(j)} = P_1^{(j)}\partial_{x_1}^k$ $(j = 0, 1, 2)$.

(ii) *There holds* $\tilde{Q}P_1u|_{x_2=0,1} = \tilde{Q}P_1^*u|_{x_2=0,1} = 0$.

(iii) *For any* $k, l = 0, 1, \cdots$, *there hold*

$$\|\partial_{x_1}^k\partial_{x_2}^l P_1u\|_2 \le C_l\|u\|_2,$$

$$\|\partial_{x_1}^k\partial_{x_2}^l P_1^*u\|_2 \le C_l\|u\|_2,$$

$$\|\partial_{x_1}^k\partial_{x_2}^l P_1^{(j)}u\|_2 \le C_l\|u\|_2 \quad (j = 0, 1, 2).$$

Proof. It is clear that (i) holds true; (ii) follows from Lemma 4.2 (iii) and Lemma 4.3 (ii); and (iii) follows from Lemma 4.3. This completes the proof. □

We next consider P_1-part of $U(t)$. We define $U_1(t)$ by

$$U_1(t) = P_1U(t) = U(t)P_1 = P_1U(t)P_1.$$

Then we have $\widehat{U_1(t)u} = \hat{\chi}_1 e^{\hat{\lambda}_0(\xi)t}\hat{\Pi}(\xi)\hat{u}$. We also define $U_1^{(j)}(t)$ $(j = 0, 1, 2)$ by

$$U_1^{(j)}(t)u = \mathscr{F}^{-1}(\hat{\chi}_1(\xi)e^{\hat{\lambda}_0(\xi)t}\hat{P}_1^{(j)}\hat{u}) \quad (j = 0, 1, 2).$$

Then

$$U_1(t) = U_1^{(0)}(t) + \partial_{x_1}U_1^{(1)}(t) + \partial_{x_1}^2 U_1^{(2)}(t).$$

Furthermore, we have the following properties.

Lemma 4.6. (i) *There hold* $\partial_{x_1}^k U_1(t) = U_1(t)\partial_{x_1}^k$, $\partial_{x_1}^k U_1^{(j)}(t) = U_1^{(j)}(t)\partial_{x_1}^k$ $(j = 0, 1, 2)$ *for* $k = 0, 1, 2, \cdots$, *and*

$$\|\partial_{x_1}^k\partial_{x_2}^l U_1(t)u\|_2 \le C_l\|u\|_2, \quad \|\partial_{x_1}^k\partial_{x_2}^l U_1^{(j)}(t)u\|_2 \le C_l\|u\|_2 \quad (j = 0, 1, 2)$$

for $t \ge 0$ *and* $k, l = 0, 1, 2, \cdots$.

(ii) *There hold*

$$\Pi^{(0)} U_1^{(0)}(t) = U_1^{(0)}(t)\Pi^{(0)} = U_1^{(0)}(t),$$
$$\Pi^{(0)} U_1^{(1)}(t) = \Pi^{(0)} U_1^{(1)}(t)(I - \Pi^{(0)}),$$

and

$$(I - \Pi^{(0)})U_1^{(1)}(t) = (I - \Pi^{(0)})U_1^{(1)}(t)\Pi^{(0)}.$$

Proof. Properties in (i) and the first relation in (ii) are easy to verify. The last two relations in (ii) follow from the fact that $\hat{S}\Pi^{(0)} = \Pi^{(0)}\hat{S} = O$, which yields relations

$$\begin{aligned} \Pi^{(0)}\hat{\Pi}^{(1)} &= -\Pi^{(0)}(\Pi^{(0)}\hat{L}^{(1)}\hat{S} + \hat{S}\hat{L}^{(1)}\Pi^{(0)}) \\ &= -(\Pi^{(0)}\hat{L}^{(1)}\hat{S} + \hat{S}\hat{L}^{(1)}\Pi^{(0)})\Pi^{(0)} = \hat{\Pi}^{(1)}(I - \Pi^{(0)}), \end{aligned}$$

and likewise, $(I - \Pi^{(0)})\hat{\Pi}^{(1)} = \hat{\Pi}^{(1)}\Pi^{(0)}$. This completes the proof. □

We next consider the decay estimates on $U_1(t)$ and $U_1^{(j)}(t)$ $(j = 0, 1, 2)$.

Lemma 4.7. *There hold the following estimates uniformly for $t \geq 0$.*

(i) $\|\partial_{x_1}^k \partial_{x_2}^l U_1(t)u\|_2 \leq C_l(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u\|_1 \quad (k, l = 0, 1, 2, \cdots),$

(ii) $\|\partial_{x_1}^k \partial_{x_2}^l \Pi^{(0)} U_1(t)u\|_2 \leq C_l(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u\|_1 \quad (k, l = 0, 1, 2, \cdots),$

(iii) $\|\partial_{x_1}^k \partial_{x_2}^l (I - \Pi^{(0)})U_1(t)u\|_2 \leq C_l(1+t)^{-\frac{1}{4}-\frac{k+1}{2}}\|u\|_1 \quad (k, l = 0, 1, 2, \cdots).$

(iv) $\|\partial_{x_1}^k \partial_{x_2}^l U_1^{(j)}(t)u\|_2 \leq C_l(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u\|_1 \quad (j = 0, 1, 2;\ k, l = 0, 1, 2, \cdots).$

Proof. Since $\hat{\lambda}_0(\xi) = -ia_0\xi - \kappa_0\xi^2 + O(|\xi|^3)$, we see from Lemma 4.3 that

$$\begin{aligned} \|\partial_{x_1}^k \partial_{x_2}^l U_1(t)u\|_2 &\leq C\Big(\int_{\mathbf{R}} \hat{\chi}_1(\xi)^2|\xi|^{2k}e^{-\kappa_0|\xi|^2 t}|\partial_{x_2}^l \hat{\Pi}(\xi)\hat{u}|_2^2\, d\xi\Big)^{\frac{1}{2}} \\ &\leq C_l\Big(\int_{\mathbf{R}} \hat{\chi}_1(\xi)^2|\xi|^{2k}e^{-\kappa_0|\xi|^2 t}|\hat{u}|_1^2\, d\xi\Big)^{\frac{1}{2}} \\ &\leq C_l\Big(\int_{\mathbf{R}} \hat{\chi}_1(\xi)^2|\xi|^{2k}e^{-\kappa_0|\xi|^2 t}\, d\xi\Big)^{\frac{1}{2}}\|u\|_1 \\ &\leq C_l \begin{cases} \|u\|_1, \\ t^{-\frac{1}{4}-\frac{k}{2}}\|u\|_1. \end{cases} \end{aligned}$$

We thus obtain (i). Estimate (ii) and (iv) can be obtained similarly. As for (iii), one can prove the desired estimate by noting

$$(I - \Pi^{(0)})\hat{\Pi}(\xi) = (I - \Pi^{(0)})(\xi\hat{\Pi}^{(1)} + \xi^2\hat{\Pi}^{(2)}(\xi)).$$

This completes the proof. □

We finally consider the asymptotic behavior of $U_1(t)$.

Let $U_0(t)$ and $\mathscr{U}(t)$ be defined by

$$U_0(t)u = \mathscr{F}^{-1}(\hat{\chi}_1 e^{\hat{\lambda}_0(\xi)t}\langle Q_0\hat{u}\rangle)$$

and

$$\mathscr{U}(t)\sigma = \mathscr{F}^{-1}(e^{-(ia_0\xi+\kappa_0\xi^2)t}\hat{\sigma}).$$

It then follows that $\Pi^{(0)}$-part of $U_1(t)$ is given by $U_0(t)$, i.e., we have

$$\Pi^{(0)}U_1(t)u = \big(U_0(t)P_1u\big)u^{(0)}. \tag{4.32}$$

One can easily see the following properties.

Lemma 4.8. *There hold the following relations.*

(i) $$U_0(t) = U_0(t)\Pi^{(0)} = U_0(t)P_1^{(0)},$$

(ii) $$\partial_{x_1}^k U_0(t) = U_0(t)\partial_{x_1}^k, \quad \partial_{x_1}^k\mathscr{U}(t) = \mathscr{U}(t)\partial_{x_1}^k \quad (k = 1, 2, \cdots),$$

(iii) $$\|\partial_{x_1}^k\mathscr{U}(t)\sigma\|_2 \le C_k t^{-\frac{1}{4}-\frac{k}{2}}\|\sigma\|_1 \quad (k = 0, 1, 2, \cdots),$$

(iv) $$\|\partial_{x_1}^k U_0(t)u\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u\|_1 \quad (k = 0, 1, 2, \cdots),$$

(v) $$\|\partial_{x_1}^k U_0(t)P_1u\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u\|_1 \quad (k = 0, 1, 2, \cdots),$$
$$\|\partial_{x_1}^k U_0(t)P_1^{(j)}u\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u\|_1 \quad (j = 1, 2; k = 0, 1, 2, \cdots).$$

We finally show that the leading part of $U_0(t)$ is given by $\mathscr{U}(t)$, which, together with Lemma 4.7 (iii) and (4.32), implies that the asymptotic behavior of $U_1(t)$ is described by $\mathscr{U}(t)$.

Lemma 4.9. *Let $u = {}^\top(\phi, w) \in (H^1 \times \tilde{H}^1) \cap L^1$ and let $\sigma = \langle Q_0u\rangle$. Then*

$$\|\partial_{x_1}^k(U_0(t)u - \mathscr{U}(t)\sigma)\|_2 \le C_k t^{-\frac{1}{4}-\frac{k+1}{2}}\|u\|_1$$

and

$$\|\partial_{x_1}^k (U_0(t)P_1 u - \mathscr{U}(t)\sigma)\|_2 \le C_k t^{-\frac{1}{4}-\frac{k+1}{2}} \|u\|_1.$$

Proof. We see that

$$\begin{aligned}\mathscr{F}(U_0(t)u) &= e^{-(ia_0\xi+\kappa_0\xi^2)t}\hat{\sigma} + (\hat{\chi}_1 - 1)e^{-(ia_0\xi+\kappa_0\xi^2)t}\hat{\sigma} \\ &\quad +\hat{\chi}_1(e^{\hat{\lambda}_0(\xi)t} - e^{-(ia_0\xi+\kappa_0\xi^2)t})\hat{\sigma}.\end{aligned}$$

Since $\hat{\lambda}_0(\xi) + (ia_0\xi + \kappa_0\xi^2) = O(\xi^3)$, we have

$$\begin{aligned}|e^{\hat{\lambda}_0(\xi)t} - e^{-(ia_0\xi+\kappa_0\xi^2)t}| &= |e^{-(ia_0\xi+\kappa_0\xi^2)t}(e^{(\hat{\lambda}(\xi)+(ia_0\xi+\kappa_0\xi^2))t} - 1)| \\ &\le C|\xi|e^{-\frac{\kappa_0}{4}\xi^2 t}.\end{aligned}$$

It then follows that

$$\|\partial_{x_1}^k (U_0(t)u - \mathscr{U}(t)\sigma)\|_2 \le C_k t^{-\frac{1}{4}-\frac{k+1}{2}} \|u\|_1.$$

This shows the first inequality. The second inequality now follows from the first one and Lemma 4.8 (i) and (v) since

$$U_0(t)P_1 = U_0(t)P_1^{(0)} + \partial_{x_1}U_0(t)P_1^{(1)} + \partial_{x_1}^2 U_0(t)P_1^{(2)}.$$

This completes the proof. □

5. Decomposition of the solution

In sections 5–7 we prove the a priori estimate in Proposition 3.3. To do so, we will decompose the solution $u(t)$ of (4.1) into several parts based on the spectral properties of L.

We first introduce some notation and projection operators. Let $\hat{\chi}_2$ and $\hat{\chi}_3$ be defined by

$$\hat{\chi}_2(\xi) = \mathbf{1}_{[r_1,1)}(|\xi|), \quad \hat{\chi}_3(\xi) = \mathbf{1}_{[1,\infty)}(|\xi|).$$

We define $P_{\infty,j}$ $(j = 1, 2, 3)$ by

$$P_{\infty,1}u = \mathscr{F}^{-1}(\hat{P}_{\infty,1}\hat{u}), \quad \hat{P}_{\infty,1}\hat{u} = \hat{\chi}_1(I - \hat{\Pi}(\xi))\hat{u},$$

$$P_{\infty,j}u = \mathscr{F}^{-1}(\hat{P}_{\infty,j}\hat{u}), \quad \hat{P}_{\infty,j}\hat{u} = \hat{\chi}_j\hat{u} \ \ (j = 2, 3).$$

Set

$$\tilde{P}_\infty = I - P_1, \quad P_\infty^{(0)} = P_{\infty,1} + P_{\infty,2}.$$

We then have

$$I = P_1 + \tilde{P}_\infty, \quad \tilde{P}_\infty = P_\infty^{(0)} + P_{\infty,3}.$$

We define $\langle f\rangle_j$ $(j = 1, 2, \infty)$ by

$$\langle f\rangle_j = \mathscr{F}^{-1}(\hat{\chi}_j \langle \hat{f}\rangle), \quad j = 1, 2,$$

$$\langle f\rangle_\infty = \langle f\rangle_1 + \langle f\rangle_2 = \mathscr{F}^{-1}((\hat{\chi}_1 + \hat{\chi}_2)\langle \hat{f}\rangle).$$

We decompose the solution $u(t)$ into

$$\begin{aligned} &u(t) = P_1 u(t) + \tilde{P}_\infty u(t), \\ &P_1 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t), \\ &\tilde{P}_\infty u(t) = (\sigma_\infty u^{(0)})(t) + u_\infty(t), \end{aligned}$$

where

$$\begin{aligned} &(\sigma_1 u^{(0)})(t) = \Pi^{(0)} P_1 u(t), \quad \sigma_1 = \langle Q_0 P_1 u(t)\rangle = \langle Q_0 P_1 u(t)\rangle_1, \\ &u_1(t) = (I - \Pi^{(0)}) P_1 u(t), \\ &(\sigma_\infty u^{(0)})(t) = \Pi^{(0)} P_\infty^{(0)} u(t), \quad \sigma_\infty = \langle Q_0 P_\infty^{(0)} u(t)\rangle = \langle Q_0 P_\infty^{(0)} u(t)\rangle_\infty, \\ &u_\infty(t) = P_\infty u, \end{aligned}$$

Here and in what follows, P_∞ denotes the operator defined by

$$P_\infty = (I - \Pi^{(0)}) P_\infty^{(0)} + P_{\infty,3}.$$

We now derive the equations for σ_1, u_1, σ_∞ and u_∞. We define $\tilde{\mathcal{M}}$ by

$$\tilde{\mathcal{M}} = L - \hat{L}_0 = \tilde{A} + \tilde{B}$$

with

$$\tilde{A} = A - \hat{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu+\tilde{\nu}}{\rho_s}\partial_{x_1}^2 & -\frac{\tilde{\nu}}{\rho_s}\partial_{x_1}\partial_{x_2} \\ 0 & -\frac{\tilde{\nu}}{\rho_s}\partial_{x_1}\partial_{x_2} & -\frac{\nu}{\rho_s}\partial_{x_1}^2 \end{pmatrix},$$

$$\tilde{B} = B - \hat{B}_0 = \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \rho_s \partial_{x_1} & 0 \\ \partial_{x_1}\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s}\,\cdot\right) & v_s^1 \partial_{x_1} & 0 \\ 0 & 0 & v_s^1 \partial_{x_1} \end{pmatrix}.$$

Proposition 5.1. *Let $T > 0$ and $u(t)$ be a solution of* (4.1) *in $Z^m(T)$. Then there hold*

$$\sigma_k \in \cap_{j=0}^{[\frac{m}{2}]} C^j([0,T]; H^l(\mathbf{R})) \quad (k = 1, \infty; \forall l = 0, 1, 2, \cdots),$$

$$u_1 \in \cap_{j=0}^{[\frac{m}{2}]} C^j([0,T]; H^l(\Omega)) \quad (\forall l = 0, 1, 2, \cdots),$$

$$u_\infty \in Z^m(T).$$

Moreover, σ_1, u_1, σ_∞ and u_∞ satisfy

$$\sigma_1(t) = U_0(t) P_1 u_0 + \int_0^t U_0(t-\tau) \Pi^{(0)} P_1 \boldsymbol{F}(\tau)\, d\tau, \tag{5.1}$$

$$u_1(t) = (I - \Pi^{(0)}) U_1(t) u_0 + \int_0^t (I - \Pi^{(0)}) U_1(t-\tau) \boldsymbol{F}(\tau)\, d\tau, \tag{5.2}$$

$$\partial_t \sigma_\infty + \langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty) \rangle_\infty = \langle Q_0 P_\infty^{(0)} \boldsymbol{F} \rangle_\infty, \tag{5.3}$$

$$\partial_t u_\infty + L u_\infty + \tilde{\mathcal{M}}(\sigma_\infty u^{(0)}) - \langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty) \rangle_\infty u^{(0)} = \boldsymbol{F}_\infty, \tag{5.4}$$

$$w_\infty|_{\partial\Omega} = 0, \tag{5.5}$$

$$\sigma_\infty|_{t=0} = \sigma_{\infty,0}, \quad u_\infty|_{t=0} = u_{\infty,0}. \tag{5.6}$$

Here $\sigma_{\infty,0} = \langle Q_0 P_\infty^{(0)} u_0 \rangle_\infty$, $u_{\infty,0} = (I - \Pi^{(0)}) P_\infty^{(0)} u_0 + P_{\infty,3} u_0$ and $\boldsymbol{F}_\infty = P_\infty \boldsymbol{F} = (I - \Pi^{(0)}) P_\infty^{(0)} \boldsymbol{F} + P_{\infty,3} \boldsymbol{F}$.

Proof. Since $u \in Z^m(T)$, the first assertion for σ_k $(k = 1, \infty)$ and u_1 follows from boundedness properties of $\Pi^{(0)}$ and P_1 given in Lemma 4.4 and Lemma 4.5. As for u_∞, it is easy to see that $P_1 Z^m(T) \subset Z^m(T)$; and so $P_{\infty,1} u(t) \in Z^m(T)$. Obviously, $P_{\infty,2} u(t)$ and $P_{\infty,3} u(t)$ are in $Z^m(T)$. Since $u^{(0)} = {}^\top(\phi^{(0)}, w^{(0)})$, $w^{(0)}|_{x_2=0,1} = 0$, we see that $\Pi^{(0)} Z^m(T) \subset Z^m(T)$, and hence, $\Pi^{(0)} P_\infty^{(0)} u(t) = \Pi^{(0)} (P_{\infty,1} + P_{\infty,2}) u(t) \in Z^m(T)$. As a consequence, $u_\infty(t) = (I - \Pi^{(0)}) P_\infty^{(0)} u(t) + P_{\infty,3} u(t) \in Z^m(T)$.

We next derive (5.1) and (5.2). Applying P_1 and $\tilde{P}_\infty$ to (4.1), we have

$$\partial_t (P_1 u) + L P_1 u = P_1 \boldsymbol{F}, \tag{5.7}$$

$$\partial_t (\tilde{P}_\infty u) + L \tilde{P}_\infty u = \tilde{P}_\infty \boldsymbol{F}. \tag{5.8}$$

It follows from (5.7) that

$$P_1u(t) = U_1(t)u_0 + \int_0^t U_1(t-\tau)\boldsymbol{F}(\tau)\,d\tau. \tag{5.9}$$

Applying $\Pi^{(0)}$ and $I - \Pi^{(0)}$ to (5.9), we obtain (5.1) and (5.2).

As for (5.3) and (5.4), we apply $P_\infty^{(0)}$ and $P_{\infty,3}$ to (5.8) to obtain

$$\partial_t(P_\infty^{(0)}u) + LP_\infty^{(0)}u = P_\infty^{(0)}\boldsymbol{F}, \tag{5.10}$$

$$\partial_t(P_{\infty,3}u) + LP_{\infty,3}u = P_{\infty,3}\boldsymbol{F}. \tag{5.11}$$

Since $\Pi^{(0)}L = \Pi^{(0)}\tilde{\mathcal{M}}$ and $L\Pi^{(0)} = \tilde{\mathcal{M}}\Pi^{(0)}$, applying $\Pi^{(0)}$ and $I - \Pi^{(0)}$ to (5.10), we have

$$\partial_t(\sigma_\infty u^{(0)}) + \Pi^{(0)}\tilde{\mathcal{M}}(P_\infty^{(0)}u) = \Pi^{(0)}P_\infty^{(0)}\boldsymbol{F}, \tag{5.12}$$

(5.13)

$$\begin{aligned}\partial_t(I - \Pi^{(0)})P_\infty^{(0)}u + L(I - \Pi^{(0)})P_\infty^{(0)}u + \tilde{\mathcal{M}}(\sigma_\infty u^{(0)}) - \Pi^{(0)}\tilde{\mathcal{M}}(P_\infty^{(0)}u) \\ = (I - \Pi^{(0)})P_\infty^{(0)}\boldsymbol{F}.\end{aligned}$$

Since

$$\begin{aligned}\Pi^{(0)}\tilde{\mathcal{M}}(P_\infty^{(0)}u) &= \langle Q_0\tilde{B}(\sigma_\infty u^{(0)} + (I - \Pi^{(0)})P_\infty^{(0)}u)\rangle_\infty u^{(0)} \\ &= \langle Q_0\tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle_\infty u^{(0)},\end{aligned}$$

(5.3) follows from (5.12). Here we used the fact that

$$\langle Q_0\tilde{B}P_{\infty,3}u\rangle_\infty = \mathscr{F}^{-1}((\hat{\chi}_1 + \hat{\chi}_2)\langle Q_0\tilde{B}_\xi\hat{P}_{\infty,3}\hat{u}\rangle) = 0,$$

where $\tilde{B}_\xi = \hat{B}_\xi - \hat{B}_0$. (5.4) now follows by adding (5.11) and (5.13). This completes the proof. □

We next state some properties of σ_∞ and u_∞ parts.

Lemma 5.2. *There hold the following inequalities.*

(i) $$\|\partial_{x_1}^k\langle Q_0P_\infty^{(0)}u\rangle_\infty\|_2 \le \|\langle Q_0P_\infty^{(0)}u\rangle_\infty\|_2 \quad (\forall k = 0, 1, 2, \cdots),$$

(ii) $$\|P_\infty u\|_2 \le C\|\partial_x P_\infty u\|_2 \quad \textit{if } \tilde{Q}u|_{x_2=0,1} = 0.$$

Proof. Inequality (i) is obvious since $\mathrm{supp}\,(\hat{\chi}_1+\hat{\chi}_2)\subset\{|\xi|\le 1\}$. As for (ii), since $\mathrm{supp}\,\hat{\chi}_3\subset\{|\xi|\ge 1\}$, we see that

$$\|P_{\infty,3}u\|_2\le\|\partial_{x_1}P_{\infty,3}u\|_2.$$

Since $\tilde{Q}u|_{x_2=0,1}=0$, we have $\tilde{Q}P_\infty^{(0)}u|_{x_2=0,1}=0$, and hence, $\tilde{Q}(I-\Pi^{(0)})P_\infty^{(0)}u|_{x_2=0,1}=0$. By the Poncaré inequality we obtain

$$\|\tilde{Q}(I-\Pi^{(0)})P_\infty^{(0)}u\|\le\|\partial_{x_2}\tilde{Q}(I-\Pi^{(0)})P_\infty^{(0)}u\|_2.$$

Furthermore, since $\langle Q_0(I-\Pi^{(0)})P_\infty^{(0)}u\rangle=0$, we see from the Poincaré inequality that

$$\|Q_0(I-\Pi^{(0)})P_\infty^{(0)}u\|\le C\|\partial_{x_2}Q_0(I-\Pi^{(0)})P_\infty^{(0)}u\|_2.$$

It then follows that

$$\|P_\infty u\|_2\le C\{\|\partial_x(I-\Pi^{(0)})P_\infty^{(0)}u\|_2+\|\partial_xP_{\infty,3}u\|_2\}\le C\|\partial_xP_\infty u\|_2.$$

Here we used $(\partial_x(I-\Pi^{(0)})P_\infty^{(0)}u,\partial_xP_{\infty,3}u)=0$, which follows from the fact $\hat{\chi}_1\hat{\chi}_3=\hat{\chi}_2\hat{\chi}_3=0$ and the Plancherel theorem. This completes the proof. □

To prove the a priori estimate in Proposition 3.3, we will estimate the following quantities.

Let $u(t)$ be a solution of (4.1) in $Z^m(T)$ and let $u(t)$ be decomposed as above:

$$u(t)=(\sigma_1u^{(0)})(t)+u_1(t)+(\sigma_\infty u^{(0)})(t)+u_\infty(t).$$

We define $M(t)\ge 0$ by

$$M(t)^2=M_1(t)^2+\sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{2}}E_\infty(\tau)\quad(t\in[0,T]).$$

Here $M_1(t)$ and $E_\infty(t)$ are defined by

$$\begin{aligned}M_1(t)&=\sup_{0\le\tau\le t}(1+\tau)^{\frac14}\|\sigma_1(\tau)\|_2\\&\quad+\sup_{0\le\tau\le t}(1+\tau)^{\frac34}\Big\{\|\partial_{x_1}\sigma_1(\tau)\|_2+\|u_1(\tau)\|_2+\sum_{j=1}^{[\frac{m}{2}]}(\|\partial_\tau^j\sigma_1(\tau)\|_2+\|\partial_\tau^ju_1(\tau)\|_2)\Big\},\\E_\infty(t)&=[\![u_\infty(t)]\!]_m^2+[\![\sigma_\infty(t)]\!]_m^2.\end{aligned}$$

We also introduce the quantity $D_\infty(t)$ for $u_\infty(t)={}^\top(\phi_\infty(t),w_\infty(t))$:

$$D_\infty(t)=|||D\phi_\infty(t)|||_{m-1}^2+|||Dw_\infty(t)|||_m^2+|||D\sigma_\infty(t)|||_{m-1}^2.$$

We will show the following estimates for $M_1(t)$ and $E_\infty(t)$.

Proposition 5.3. *There are positive constants ν_0, γ_0 and ω_0 such that if $\nu \geq \nu_0$, $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$, then the following assertion holds true.*

There exists a number $\varepsilon_2 > 0$ such that if a solution $u(t)$ of (4.1) in $Z^m(T)$ satisfies $\sup_{0\leq\tau\leq t}[\![u(\tau)]\!]_m \leq \varepsilon_2$ and $M(t) \leq 1$ for $t \in [0,T]$, then the following estimates hold uniformly for $t \in [0,T]$ with $C > 0$ independent of T.

$$M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\}, \tag{5.14}$$

$$\begin{aligned} & E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau)\, d\tau \\ \leq\ & C\big\{e^{-at}E_\infty(0) + (1+t)^{-\frac{3}{2}}M(t)^4 + \int_0^t e^{-a(t-\tau)}\tilde{R}(\tau)\, d\tau\big\}. \end{aligned} \tag{5.15}$$

Here $a = a(\nu, \tilde{\nu}, \gamma, \|\rho_s - 1\|_{C^{m+1}[0,1]})$ is a positive constant; and $\tilde{R}(t)$ is a quantity that satisfies

$$\tilde{R}(t) \leq C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\} \tag{5.16}$$

whenever $\sup_{0\leq\tau\leq t}[\![u(\tau)]\!]_m \leq \varepsilon_2$ and $M(t) \leq 1$.

The proof of Proposition 5.3 will be given in sections 6–8. We will prove (5.14), (5.15) and (5.16) in sections 6, 7 and 8, respectively.

Assuming Proposition 5.3 to hold true, we can show the following estimate.

Proposition 5.4. *If $\nu \geq \nu_0$, $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$, then the following assertion holds true.*

There exists a number $\varepsilon_3 > 0$ such that if a solution $u(t)$ of (4.1) in $Z^m(T)$ satisfies $\|u_0\|_{H^m\cap L^1} \leq \varepsilon_3$, then there holds the estimate

$$M(t) \leq C_2\|u_0\|_{H^m\cap L^1}$$

for some constant $C_2 > 0$ independent of T.

The a priori estimate in Proposition 3.3 immediately follows from Proposition 5.4. Furthermore, Proposition 5.4 also provides decay estimates

$$[\![u(t)]\!]_m \leq C(1+t)^{-\frac{1}{4}}\|u_0\|_{H^m\cap L^1}$$

and

$$\|\partial_{x_1}^k u(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_{H^m\cap L^1}$$

for $k=0,1$.

Proof of Proposition 5.4. If $\sup_{0\le\tau\le t}[\![u(\tau)]\!]_m \le \varepsilon_2$ and $M(t)\le 1$, then we see from (5.15) and (5.16) that

$$\begin{aligned}
&E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau)\,d\tau \\
&\le\ C\Big\{e^{-at}E_\infty(0) + (1+t)^{-\frac{3}{2}}M(t)^4 + M(t)^3\int_0^t e^{-a(t-\tau)}(1+\tau)^{-\frac{3}{2}}\,d\tau \\
&\qquad + M(t)\int_0^t e^{-a(t-\tau)}D_\infty(\tau)\,d\tau\Big\} \\
&\le\ C\Big\{e^{-at}E_\infty(0) + (1+t)^{-\frac{3}{2}}M(t)^3 + M(t)\int_0^t e^{-a(t-\tau)}D_\infty(\tau)\,d\tau\Big\},
\end{aligned}$$

and hence,

$$(1+t)^{\frac{3}{2}}E_\infty(t) + \mathscr{D}(t) \le C\{\|u_0\|^2_{H^m\cap L^1} + M(t)^3 + M(t)\mathscr{D}(t)\}, \tag{5.17}$$

where

$$\mathscr{D}(t) = (1+t)^{\frac{3}{2}}\int_0^t e^{-a(t-\tau)}D_\infty(\tau)\,d\tau.$$

It follows from (5.14) and (5.17) that

$$M(t)^2 + \mathscr{D}(t) \le C_3\{\|u_0\|^2_{H^m\cap L^1} + M(t)^3 + M(t)\mathscr{D}(t)\} \tag{5.18}$$

whenever $\sup_{0\le\tau\le t}[\![u(\tau)]\!]_m \le \varepsilon_2$ and $M(t)\le 1$.

We now show that there exists $\varepsilon_3>0$ such that if $\|u_0\|_{H^m\cap L^1} < \varepsilon_3$, then $M(t) < 2C_4\|u_0\|_{H^m\cap L^1}$ for all $t\in[0,T]$ with some $C_4>0$ independent of T.

We first observe that there is a constant $C_5>0$ such that $\sup_{0\le\tau\le t}[\![u(\tau)]\!]_m \le C_5 M(t)$.

Since $M(0)\le C_6\|u_0\|_{H^m}$ for some $C_6>0$ and since $M(t)$ is continuous in t, we see that there exists $t_0>0$ such that $M(t) < 2C_6\|u_0\|_{H^m}$ for all $t\in[0,t_0]$.

We set $C_4 = \max\{\sqrt{\frac{C_3}{2}}, C_6\}$ and $\varepsilon_3 = \min\{\frac{1}{2C_4}, \frac{1}{4C_3C_4}, \frac{\varepsilon_2}{2C_4C_5}\}$. Then $M(t) < 2C_4\|u_0\|_{H^m\cap L^1}$ for $t\in[0,t_0]$. Assume that there exists $t_1\in(t_0,T)$ such that $M(t) < 2C_4\|u_0\|_{H^m\cap L^1}$ for $t\in[0,t_1)$ and $M(t_1) = 2C_4\|u_0\|_{H^m\cap L^1}$. Since $M(t)\le 2C_4\|u_0\|_{H^m\cap L^1}\le 1$ for $t\in[0,t_1]$, we have $\sup_{0\le\tau\le t}[\![u(\tau)]\!]_m \le$

$C_5 M(t) \le \varepsilon_2$ for $t \in [0, t_1]$. We thus see from (5.18) that if $\|u_0\|_{H^m \cap L^1} < \varepsilon_3$, then

$$
\begin{aligned}
M(t)^2 + 2\mathscr{D}(t) &\le C_3 \big\{ \|u_0\|^2_{H^m \cap L^1} + 2C_4 \|u_0\|_{H^m \cap L^1} (M(t)^2 + \mathscr{D}(t)) \big\} \\
&< C_3 \|u_0\|^2_{H^m \cap L^1} + \tfrac{1}{2}(M(t)^2 + \mathscr{D}(t)),
\end{aligned}
$$

and hence,

$$
M(t)^2 + \mathscr{D}(t) < 2C_3 \|u_0\|^2_{H^m \cap L^1} \le 4C_4^2 \|u_0\|^2_{H^m \cap L^1}
$$

for $t \in [0, t_1]$. But this contradicts to $M(t_1) = 2C_4 \|u_0\|_{H^m \cap L^1}$. We thus conclude that $M(t) < 2C_4 \|u_0\|_{H^m \cap L^1}$ for all $t \in [0, T]$. This completes the proof. □

6. Estimates on $P_1 u(t)$

In this section we estimate the P_1-part of $u(t)$:

$$
P_1 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t),
$$

where $\sigma_1(t) = \langle Q_0 P_1 u(t) \rangle$ and $u_1(t) = (I - \Pi^{(0)}) u(t)$.

We will prove (5.14), i.e., the following estimate.

Proposition 6.1. *There exists a number $\varepsilon_4 > 0$ such that if a solution $u(t)$ of (4.1) in $Z^m(T)$ satisfies $\sup_{0 \le \tau \le t} [\![u(\tau)]\!]_m \le \varepsilon_4$ and $M(t) \le 1$ for all $t \in [0, T]$, then the estimate*

$$
M_1(t) \le C\{\|u_0\|_1 + M(t)^2\}
$$

holds uniformly for $t \in [0, T]$ with $C > 0$ independent of T.

To prove Proposition 6.1, we employ the results in section 4, together with Lemma 6.2 and Lemma 6.3 below.

Before going further we make one observation. In view of the spectral properties of the linearized operator, the most slowly decaying part of $u(t)$ is expected to be $\sigma_1(t)$; and hence, the most slowly decaying part of the nonlinearity $\boldsymbol{F}(t)$ would be given by the terms involving only $\sigma_1(t)^2$ which is, in fact, only one term $\sigma_1^2 \boldsymbol{F}_1$ with $\boldsymbol{F}_1 = \boldsymbol{F}_1(x_2)$:

$$
\boldsymbol{F}_1 = -{}^{\top}\big(0, 0, \tfrac{1}{2\gamma^4 \rho_s(x_2)} \partial_{x_2}(P''(\rho_s(x_2))\{\phi^{(0)}(x_2)\}^2)\big).
$$

We thus write $\boldsymbol{F}$ as

$$
\boldsymbol{F} = \sigma_1^2 \boldsymbol{F}_1 + \boldsymbol{F}_2,
$$

where $\boldsymbol{F}_2 = \boldsymbol{F} - \sigma_1^2 \boldsymbol{F}_1$ contains terms involving u_1, σ_∞, u_∞, their derivatives, and terms of order $O(\sigma_1 \partial_{x_1} \sigma_1)$ and $O(\sigma_1^3)$ but not just $O(\sigma_1^2)$. In particular, $\Pi^{(0)} \boldsymbol{F} = \Pi^{(0)} \boldsymbol{F}_2$.

We make two lemmas on the nonlinearities.

Lemma 6.2. *There hold the following relations.*

(i)
$$\Pi^{(0)} \boldsymbol{F} = -\partial_{x_1} \langle \phi w^1 \rangle u^{(0)},$$

(ii)
$$\Pi^{(0)} P_1 \boldsymbol{F} = -P_1^{(0)} (\partial_{x_1} \langle \phi w^1 \rangle u^{(0)}) + \Pi^{(0)} \partial_{x_1} P_1^{(1)} (I - \Pi^{(0)}) \boldsymbol{F} + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} \boldsymbol{F},$$

(iii)
$$(I - \Pi^{(0)}) P_1 \boldsymbol{F} = (I - \Pi^{(0)}) \partial_{x_1} P_1^{(1)} \Pi^{(0)} \boldsymbol{F} + (I - \Pi^{(0)}) \partial_{x_1}^2 P_1^{(2)} \boldsymbol{F}.$$

Proof. Since $w|_{x_2=0,1} = 0$, by integration by parts, we have

$$\Pi^{(0)} \boldsymbol{F} = \langle Q_0 \boldsymbol{F} \rangle u^{(0)} = -\partial_{x_1} \langle \phi w^1 \rangle u^{(0)}.$$

This shows (i).

We write $\Pi^{(0)} P_1 \boldsymbol{F}$ as

$$\begin{aligned} \Pi^{(0)} P_1 \boldsymbol{F} &= \Pi^{(0)} P_1^{(0)} \boldsymbol{F} + \Pi^{(0)} \partial_{x_1} P_1^{(1)} \boldsymbol{F} + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} \boldsymbol{F} \\ &= P_1^{(0)} \Pi^{(0)} \boldsymbol{F} + \Pi^{(0)} \partial_{x_1} P_1^{(1)} \boldsymbol{F} + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} \boldsymbol{F} \\ &= -P_1^{(0)} (\partial_{x_1} \langle \phi w^1 \rangle u^{(0)}) + \Pi^{(0)} \partial_{x_1} P_1^{(1)} \boldsymbol{F} + \Pi^{(0)} \partial_{x_1}^2 P_1^{(2)} \boldsymbol{F}. \end{aligned}$$

By Lemma 4.2 (iii), $\hat{\Pi}^{(1)}$ is given by $-(\Pi^{(0)} \hat{L}^{(1)} \hat{S} + \hat{S} \hat{L}^{(1)} \Pi^{(0)})$ with $\hat{S} = \big((I - \Pi^{(0)}) \hat{L}_0 (I - \Pi^{(0)})\big)^{-1}$. Since $(\Pi^{(0)})^2 = \Pi^{(0)}$, $\Pi^{(0)} \hat{S} = O$ and $\hat{S} = \hat{S}(I - \Pi^{(0)})$ we see that

$$\Pi^{(0)} \hat{\Pi}^{(1)} = -\Pi^{(0)} \hat{L}^{(1)} \hat{S} (I - \Pi^{(0)}) = \Pi^{(0)} \hat{\Pi}^{(1)} (I - \Pi^{(0)}),$$
$$(I - \Pi^{(0)}) \hat{\Pi}^{(1)} = -(I - \Pi^{(0)}) \hat{S} \hat{L}^{(1)} \Pi^{(0)} = (I - \Pi^{(0)}) \hat{\Pi}^{(1)} \Pi^{(0)}.$$

It follows that

$$\begin{aligned} \Pi^{(0)} \partial_{x_1} P_1^{(1)} \boldsymbol{F} &= \mathscr{F}^{-1} (\hat{\chi}_1 \xi \Pi^{(0)} \hat{\Pi}^{(1)} \hat{\boldsymbol{F}}) \\ &= \mathscr{F}^{-1} (\hat{\chi}_1 \xi \Pi^{(0)} \hat{\Pi}^{(1)} (I - \Pi^{(0)}) \hat{\boldsymbol{F}}) \\ &= \Pi^{(0)} \mathscr{F}^{-1} (\hat{\chi}_1 \xi \hat{\Pi}^{(1)} (I - \Pi^{(0)}) \hat{\boldsymbol{F}}) \\ &= \Pi^{(0)} \partial_{x_1} P_1^{(1)} (I - \Pi^{(0)}) \boldsymbol{F}. \end{aligned}$$

We thus obtain (ii).

Similarly, we can show

$$(I-\mathit{\Pi}^{(0)})\partial_{x_1}P_1^{(1)}\boldsymbol{F}=(I-\mathit{\Pi}^{(0)})\partial_{x_1}P_1^{(1)}\mathit{\Pi}^{(0)}\boldsymbol{F},$$

and obtain (iii). This completes the proof. $\square$

It is not difficult to verify the following estimates on the nonlinearities.

Lemma 6.3. *There exists a number $\varepsilon_5>0$ such that whenever a solution $u(t)$ of* (4.1) *in $Z^m(T)$ satisfies* $\sup_{0\le\tau\le t}[\![u(\tau)]\!]_m\le\varepsilon_5$ *and $M(t)\le 1$ for $t\in[0,T]$, there hold the following estimates for $t\in[0,T]$ with $C>0$ independent of T.*

(i) $$\|\partial_{x_1}\langle\phi w^1\rangle(t)\|_1\le C(1+t)^{-1}M(t)^2,$$

(ii) $$\|\langle\phi w^1\rangle(t)\|_1\le C(1+t)^{-\frac{1}{2}}M(t)^2,$$

(iii) $$\|\partial_{x_1}(\sigma_1^2(t))\|_1\le C(1+t)^{-1}M(t)^2,$$

(iv) $$\|\boldsymbol{F}(t)\|_1\le C(1+t)^{-\frac{1}{2}}M(t)^2,$$

(v) $$\|\boldsymbol{F}_2(t)\|_1\le C(1+t)^{-1}M(t)^2,$$

(vi) $$\|\boldsymbol{F}(t)\|_2\le C(1+t)^{-\frac{3}{4}}M(t)^2.$$

Proof of Proposition 6.1. We write (5.1) and (5.2) as

$$\sigma_1(t)=U_0(t)P_1u_0+I(t),$$

$$u_1(t)=(I-\mathit{\Pi}^{(0)})U_1(t)u_0+J(t),$$

where

$$I(t)=\int_0^t U_0(t-\tau)\mathit{\Pi}^{(0)}P_1\boldsymbol{F}(\tau)\,d\tau,$$

$$J(t)=\int_0^t (I-\mathit{\Pi}^{(0)})U_1(t-\tau)\boldsymbol{F}(\tau)\,d\tau.$$

Lemma 4.7 and Lemma 4.8 yield

$$\|\partial_{x_1}^k U_0(t)P_1 u_0\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_1,$$

$$\|(I-\Pi^{(0)})U_1(t)u_0\|_2 \le C(1+t)^{-\frac{3}{4}}\|u_0\|_1.$$

We next estimate $I(t)$ which we write as

$$I(t) = I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_0^{\frac{t}{2}} U_0(t-\tau)\Pi^{(0)}P_1\boldsymbol{F}(\tau)\,d\tau,$$

$$I_2(t) = \int_{\frac{t}{2}}^{t} U_0(t-\tau)\Pi^{(0)}P_1\boldsymbol{F}(\tau)\,d\tau.$$

By Lemma 4.8 (i), (ii) and Lemma 6.2 (ii), we have

$$\begin{aligned} & U_0(t-\tau)\Pi^{(0)}P_1\boldsymbol{F}(\tau) \\ = \; & \partial_{x_1}U_0(t-\tau)\big\{-P_1^{(0)}(\langle\phi w^1\rangle u^{(0)}) + P_1^{(1)}(I-\Pi^{(0)})\boldsymbol{F} + \partial_{x_1}P_1^{(2)}\boldsymbol{F}\big\}(\tau). \end{aligned}$$

It then follows from Lemma 4.8 (ii) and Lemma 6.3 that

$$\begin{aligned} \|\partial_{x_1}^k I_1(t)\|_2 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\big\{\|\langle\phi w^1\rangle(\tau)\|_1 + \|\boldsymbol{F}(\tau)\|_1\big\}\,d\tau \\ &\le CM(t)^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-\frac{1}{2}}\,d\tau \\ &\le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}M(t)^2 \end{aligned}$$

for $k=0,1$.

As for $I_2(t)$, using Lemma 4.5 (i) and Lemma 6.2 (ii), we write $\Pi^{(0)}P_1\boldsymbol{F}$ as

$$\begin{aligned} \Pi^{(0)}P_1\boldsymbol{F} &= -P_1^{(0)}((\partial_{x_1}\langle\phi w^1\rangle)u^{(0)}) \\ &\quad +\Pi^{(0)}\big(P_1^{(1)}(I-\Pi^{(0)}) + \partial_{x_1}P_1^{(2)}\big)\big(\partial_{x_1}(\sigma_1)^2\boldsymbol{F}_1\big) \qquad (6.1) \\ &\quad +\Pi^{(0)}\big(\partial_{x_1}P_1^{(1)}(I-\Pi^{(0)}) + \partial_{x_1}^2 P_1^{(2)}\big)\boldsymbol{F}_2. \end{aligned}$$

It then follows from Lemma 4.8 and Lemma 6.3 that

$$
\begin{aligned}
&\|\partial_{x_1}^k I_2(t)\|_2 \\
&\le C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac14-\frac{k}{2}}\big\{\|\partial_{x_1}\langle\phi w^1\rangle(\tau)\|_1+\|\partial_{x_1}(\sigma_1(\tau))^2\|_1+\|\boldsymbol{F}_2(\tau)\|_1\big\}\,d\tau \\
&\le CM(t)^2\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac14-\frac{k}{2}}(1+\tau)^{-1}\,d\tau \\
&\le C(1+t)^{-\frac14-\frac{k}{2}}M(t)^2
\end{aligned}
$$

for $k=0,1$.

We next consider $J(t)$. By Lemma 4.6 (ii), we have

$$
\begin{aligned}
(I-\mathit{\Pi}^{(0)})U_1(t-\tau)\boldsymbol{F}(\tau) &= (I-\mathit{\Pi}^{(0)})\partial_{x_1}U_1^{(1)}(t-\tau)\mathit{\Pi}^{(0)}\boldsymbol{F} \\
&\quad +(I-\mathit{\Pi}^{(0)})\partial_{x_1}^2U_1^{(2)}(t-\tau)\boldsymbol{F}(\tau).
\end{aligned}
$$

Furthermore, using Lemma 4.6 (i) and Lemma 6.2 (i), we rewrite this as follows:

$$
\begin{aligned}
(I-\mathit{\Pi}^{(0)})U_1(t-\tau)\boldsymbol{F}(\tau) &= -(I-\mathit{\Pi}^{(0)})\partial_{x_1}^2U_1^{(1)}(t-\tau)(\langle\phi w^1\rangle(\tau)u^{(0)}) \\
&\quad +(I-\mathit{\Pi}^{(0)})\partial_{x_1}^2U_1^{(2)}(t-\tau)\boldsymbol{F}(\tau).
\end{aligned}
$$

for $0\le\tau\le\frac{t}{2}$ and

$$
\begin{aligned}
(I-\mathit{\Pi}^{(0)})U_1(t-\tau)\boldsymbol{F}(\tau) &= -(I-\mathit{\Pi}^{(0)})\partial_{x_1}U_1^{(1)}(t-\tau)(\partial_{x_1}\langle\phi w^1\rangle(\tau)u^{(0)}) \\
&\quad +(I-\mathit{\Pi}^{(0)})\partial_{x_1}U_1^{(2)}(t-\tau)(\partial_{x_1}(\sigma_1(\tau)^2)\boldsymbol{F}_1) \\
&\quad +(I-\mathit{\Pi}^{(0)})\partial_{x_1}^2U_1^{(2)}(t-\tau)\boldsymbol{F}_2(\tau)
\end{aligned}
$$

for $\frac{t}{2}\le\tau\le t$. It then follows from Lemma 4.7 and Lemma 6.3 (iv) that

$$
\begin{aligned}
&\|J(t)\|_2 \\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac54}\big\{\|\langle\phi w^1\rangle(\tau)\|_1+\|\boldsymbol{F}(\tau)\|_1\big\}\,d\tau \\
&\quad +C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac34}\big\{\|\partial_{x_1}\langle\phi w^1\rangle(\tau)\|_1+\|\partial_{x_1}(\sigma_1(\tau)^2)\|_1+\|\boldsymbol{F}_2(\tau)\|_1\big\}\,d\tau \\
&\le CM(t)^2\left\{\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac54}(1+\tau)^{-\frac12}\,d\tau+\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac34}(1+\tau)^{-1}\,d\tau\right\} \\
&\le C(1+t)^{-\frac34}M(t)^2.
\end{aligned}
$$

We thus obtain

$$(6.2)\quad \sum_{k=0}^{1}(1+t)^{\frac{1}{4}+\frac{k}{2}}\|\partial_{x_1}^k\sigma_1(t)\|_2+(1+t)^{\frac{3}{4}}\|u_1(t)\|_2\le C\{\|u_0\|_1+M(t)^2\}.$$

It remains to estimate time derivatives. Since

$$-\widehat{LP_1u}=\hat{\chi}_1\hat{\lambda}_0(\xi)\hat{\Pi}(\xi)\hat{u}=\hat{\chi}_1\hat{\lambda}_0(\xi)\widehat{P_1u},$$

we see from (5.7) that

$$(6.3)\qquad \partial_t\widehat{P_1u}=\hat{\chi}_1\hat{\lambda}_0(\xi)\widehat{P_1u}+\widehat{P_1\boldsymbol{F}}$$

with $\hat{\lambda}_0(\xi)=-(ia_0\xi+\kappa_0\xi^2+O(\xi^3))=O(\xi)$. It then follows from (6.3) and Lemma 6.2 (vi) that

$$(6.4)\qquad \begin{aligned}\|\partial_tP_1u(t)\|_2 &\le C\{\|\partial_{x_1}P_1u(t)\|_2+\|P_1\boldsymbol{F}(t)\|_2\}\\ &\le C(1+t)^{-\frac{3}{4}}\{\|u_0\|_1+M(t)^2\}.\end{aligned}$$

Concerning $\|\partial_t^{j+1}P_1u(t)\|_2$ for $j=1,\cdots,[m/2]-1$, we obtain from (6.3)

$$\|\partial_t^{j+1}P_1u(t)\|_2\le C\{\|\partial_t^jP_1u(t)\|_2+\|\partial_t^jP_1\boldsymbol{F}\|_2\}.$$

Since

$$\|\partial_t^j\boldsymbol{F}\|_2\le C(1+t)^{-\frac{3}{4}}M(t)^2$$

for $0\le j\le[m/2]-1$ as we will see in Proposition 8.5 and Proposition 8.6 below, we find by induction on j, together with (6.4), that the estimate

$$(6.5)\qquad \|\partial_t^{j+1}P_1u(t)\|_2\le C(1+t)^{-\frac{3}{4}}\{\|u_0\|_1+M(t)^2\}$$

holds for $j=0,1,\cdots,[m/2]-1$.

The desired result now follows from (6.2) and (6.5). This completes the proof. □

7. Estimates on $\tilde{P}_\infty u(t)$

In this section we prove estimate (5.15) for σ_∞ and u_∞ by a variant of Matsumura-Nishida energy method as in [8, Section 5].

We first show the following inequality.

Proposition 7.1. *There exist ν_0, γ_0 and ω_0 such that if $\nu\ge\nu_0$, $\gamma^2/(\nu+\tilde{\nu})\ge\gamma_0^2$ and $\|\rho_s-1\|_{C^{m+1}[0,1]}\le\omega_0$, then a solution $u(t)$ of (4.1) in $Z^m(T)$ satisfies*

$$(7.1)\qquad \frac{d}{dt}\tilde{E}(t)+2D(t)\le\tilde{R}(t).$$

Here $\tilde{E}(t)$, $D(t)$ and $\tilde{R}(t)$ are quantities such that

(i) $\tilde{E}(t) + [\![w_\infty(t)]\!]_{m-2}^2$ *is equivalent to* $E_\infty(t)$*,*

(ii) $D(t)$ *is equivalent to* $D_\infty(t)$*,*

(iii) $\tilde{R}(t)$ *satisfies estimate* (5.16)*.*

The proof of Proposition 7.1 is similar to that of (5.64) in [8]. So we here give an outline only.

We introduce some quantities. Let $E^{(0)}[\tilde{P}_\infty u]$ and $D^{(0)}[w]$ be defined by

$$E^{(0)}[\tilde{P}_\infty u] = \frac{\alpha_0}{\gamma^2}\|\sigma_\infty\|_2^2 + \frac{1}{\gamma^2}\Big\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\phi_\infty\Big\|_2^2 + \|\sqrt{\rho_s}w_\infty\|_2^2$$

for $\tilde{P}_\infty u = \sigma_\infty u^{(0)} + u_\infty$ with $u_\infty = {}^\top(\phi_\infty, w_\infty)$; and

$$D^{(0)}[w] = \nu\|\nabla w\|_2^2 + \tilde{\nu}\|\mathrm{div}\, w\|_2^2.$$

Note that

$$\langle\langle Au(t), u(t)\rangle\rangle = D^{(0)}[w(t)]$$

for $u = {}^\top(\phi, w) \in Z^m(T)$.

We denote the tangential derivatives $\partial_t^j\partial_{x'}^k$ by $T_{j,k}$:

$$T_{j,k}u = \partial_t^j\partial_{x'}^k u.$$

It is easy to see that Lemma 4.4 and Lemma 4.5 of [8] hold with σ_1 and $\langle\cdot,\cdot\rangle_1$ in [8] replaced by σ_∞ and $\langle\cdot,\cdot\rangle_\infty$, respectively. Namely, we have the following lemmas.

Lemma 7.2 ([8, Lemma 4.4]). (i) $\langle\partial_{x_1} f\rangle_\infty = \partial_{x_1}\langle f\rangle_\infty$ *and* $\|\partial_{x_1}\langle f\rangle_\infty\|_2 \le \|\langle f\rangle_\infty\|_2$.

(ii) *Let* $\sigma = \sigma(x_1)$ *with* $\mathrm{supp}\,(\hat{\sigma}) \subset \{|\xi| \le 1\}$*. Then*

$$(\langle Q_0\tilde{B}u\rangle_\infty, \sigma) = -\frac{\gamma^2}{\alpha_0}\langle\langle u, \tilde{B}(\sigma u_0^{(0)})\rangle\rangle.$$

(iii) $\langle\langle\langle f\rangle_\infty u_0^{(0)}, u_\infty\rangle\rangle = 0$ *for* $u_\infty \in \mathrm{Range}\,(P_\infty)$*.*

Lemma 7.3. ([8, Lemma 4.5]). *There hold the following assertions.*

(i) $\|\langle Q_0\tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle_\infty\|_2^2 \le C(\|\partial_{x_1}\sigma_\infty\|_2^2 + \|\partial_{x_1}\phi_\infty\|_2^2 + \gamma^4\|\partial_{x_1}w_\infty\|_2^2)$.

(ii) *If* $w_\infty^2|_{x_2=0,1}=0$, *then* $\langle Q_0\tilde{B}u_\infty\rangle_\infty=\langle Q_0Bu_\infty\rangle_\infty=\langle v_s^1\partial_{x_1}\phi_\infty+\gamma^2\mathrm{div}\,(\rho_s w_\infty)\rangle_\infty$.

(iii) *If* $w_\infty^2|_{x_2=0,1}=0$, $\|\rho_s-1\|_{C^{m+1}[0,1]}\le\omega_0$ *and* $2j+k\le m$, *then*

$$\begin{aligned}&\|\partial_{x_1}^k\partial_t^j\langle Q_0\tilde{B}(\sigma_\infty u^{(0)}+u_\infty)\rangle_\infty\|_2^2\\ &\le\ C\{\|\partial_{x_1}^p\partial_t^j\sigma_\infty\|_2^2+\|\partial_{x_1}^q\partial_t^j\phi_\infty\|_2^2+\gamma^4\|\mathrm{div}\,(\partial_{x_1}^r\partial_t^j w_\infty)\|_2^2+\gamma^4\omega_0^2\|\partial_{x_1}^s\partial_t^j w_\infty\|_2^2\}\end{aligned}$$

for $0\le p,q\le k+1$, $0\le r,s\le k$.

We begin with the L^2 energy estimates for tangential derivatives. We set

$$\sigma_*=\sigma_1+\sigma_\infty,\quad \phi_*=\phi_1+\phi_\infty,\quad w_*=w_1+w_\infty,$$

$$u_*={}^\top(\phi_*,w_*)\,(=u_1+u_\infty).$$

We will write $\tilde{Q}\boldsymbol{F}={}^\top(0,\boldsymbol{f})$ in the form

$$\tilde{Q}\boldsymbol{F}=\tilde{\boldsymbol{F}}_0+\tilde{\boldsymbol{F}}_1+\tilde{\boldsymbol{F}}_2+\tilde{\boldsymbol{F}}_3,$$

where $\tilde{\boldsymbol{F}}_l={}^\top(0,\boldsymbol{f}_l)$ $(l=0,1,2,3)$ with

$$\begin{aligned}\boldsymbol{f}_0\ &=\ -w\cdot\nabla w+f_1(\rho_s,\phi)(-\partial_{x_1}^2\sigma_*w^{(0)}+\tfrac{\partial_{x_2}^2v_s}{\gamma^2\rho_s}\phi_*)\\ &\quad+f_2(\rho_s,\phi)(-\partial_{x_1}^2\sigma_*w^{(0)}-\partial_{x_1}\sigma_*\partial_{x_2}w^{(0)})\\ &\quad+\boldsymbol{f}_{01}(x_2,\phi)\phi\sigma_*+\boldsymbol{f}_{02}(x_2,\phi)\phi\partial_{x_1}\sigma_*+\boldsymbol{f}_{03}(x_2,\phi)\phi\phi_*,\end{aligned}$$

$$\begin{aligned}\boldsymbol{f}_1&=-\mathrm{div}\,(f_1(\rho_s,\phi)\nabla w_*)+{}^\top(\nabla w_*)\nabla(f_1(\rho_s,\phi)),\\ \boldsymbol{f}_2&=-\nabla(f_2(\rho_s,\phi)\mathrm{div}\,w_*)+(\mathrm{div}\,w_*)\nabla(f_2(\rho_s,\phi)),\\ \boldsymbol{f}_3&=\nabla(f_3(x_2,\phi)\phi\phi_*)-\phi_*\nabla(f_3(x_2,\phi)\phi).\end{aligned}$$

Here ∇w_* denotes the 2×2 matrix $(\partial_{x_i}w_*^j)$; $f_1=\frac{\nu\phi}{(\phi+\gamma^2\rho_s)\rho_s}$; $f_2=\frac{\tilde{\nu}\phi}{(\phi+\gamma^2\rho_s)\rho_s}$; and $\boldsymbol{f}_{0l}(x_2,\phi)$ $(l=1,2,3)$ and $f_3(x_2,\phi)$ are some smooth functions of x_2 and ϕ.

Proposition 7.4. *There is a constant* $\nu_0>0$ *such that if* $\nu\ge\nu_0$ *and* $\|\rho_s-1\|_{C^{m+1}[0,1]}\le1$, *then the following estimate holds for* $0\le2j+k\le m$:

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}E^{(0)}[T_{j,k}\tilde{P}_\infty u]+\tfrac{3}{4}D^{(0)}[T_{j,k}w_\infty]\\ &\le R_{j,k}^{(1)}+C\big\{\big(\tfrac{1}{\gamma^2}+\tfrac{\nu+\tilde{\nu}}{\gamma^4}\big)\|\partial_{x'}T_{j,k}\sigma_\infty\|_2^2+\big(\tfrac{1}{\gamma^2}+\tfrac{\nu}{\gamma^4}\big)\|\tilde{T}_{j,k}\phi_\infty\|_2^2\big\},\end{aligned}\tag{7.2}$$

where

$$\tilde{T}_{j,k}\phi_\infty = \begin{cases} \partial_x \phi_\infty & (j=k=0), \\ T_{j,k}\phi_\infty & (2j+k \geq 1); \end{cases}$$

and $R^{(1)}_{j,k}$ *is given by*

$$\begin{aligned} R^{(1)}_{j,k} &= \tfrac{\alpha_0}{\gamma^2}\{(\langle Q_0 T_{j,k}\boldsymbol{F}\rangle_\infty, T_{j,k}\sigma_\infty) - (\langle Q_0 T_{j,k} P_1 \boldsymbol{F}\rangle_\infty, T_{j,k}\sigma_\infty)\} \\ &\quad + \tilde{R}^{(1)}_{j,k} - \langle\langle \langle Q_0 T_{j,k}\boldsymbol{F}\rangle_\infty u^{(0)}, T_{j,k}u_\infty\rangle\rangle \\ &\quad + \langle\langle T_{j,k}P_1\boldsymbol{F}, T_{j,k}u_\infty\rangle\rangle - \langle\langle \langle Q_0 T_{j,k}P_1\boldsymbol{F}\rangle_\infty u^{(0)}, T_{j,k}u_\infty\rangle\rangle. \end{aligned}$$

Here

$$\tilde{R}^{(1)}_{j,k} = \langle\langle T_{j,k}\boldsymbol{F}, T_{j,k}u_\infty\rangle\rangle$$

when $2j+k \leq m-1$*; and*

$$\begin{aligned} \tilde{R}^{(1)}_{j,k} &= -(T_{j,k}(\phi \operatorname{div} w), T_{j,k}\phi_\infty \tfrac{P'(\rho_s)}{\gamma^4\rho_s}) + \tfrac{1}{2}(\operatorname{div}(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w), |T_{j,k}\phi_\infty|^2) \\ &\quad -([T_{j,k}, w_\infty]\nabla\phi_\infty, T_{j,k}\phi_\infty\tfrac{P'(\rho_s)}{\gamma^4\rho_s}) \\ &\quad + \langle\langle T_{j,k}\boldsymbol{f}_0, T_{j,k}w_\infty\rho_s\rangle\rangle + \textstyle\sum_{l=1}^3 \langle\langle T_{j,k}\boldsymbol{f}_l, T_{j,k}w_\infty\rho_s\rangle\rangle_{-1} \end{aligned}$$

when $2j+k=m$*. Here and in what follows, for* $G = g + \partial_{x_j}\tilde{g}$ *with* $g, \tilde{g} \in L^2$ *and* $v \in H^1_0$*,* $\langle\langle G, v\rangle\rangle_{-1}$ *denotes*

$$\langle\langle G, v\rangle\rangle_{-1} = (g, v) - (\tilde{g}, \partial_{x_j} v).$$

Outline of Proof. We apply $T_{j,k}$ to (5.3) and (5.4). We then take the inner products of the resulting equations with $T_{j,k}\sigma_\infty$ and $T_{j,k}u_\infty$. Using Lemma 7.2 (ii) and integration by parts with the symmetric properties of A and B:

$$\langle\langle Au, u\rangle\rangle = D^{(0)}[w],$$

$$\langle\langle Bu, u\rangle\rangle = -\langle\langle u, Bu\rangle\rangle = 0,$$

we obtain

(7.3)

$$\frac{1}{2}\frac{d}{dt}\|T_{j,k}\sigma_\infty\|_2^2 - \frac{\gamma^2}{\alpha_0}\langle\langle T_{j,k}u_\infty, \tilde{B}(T_{j,k}\sigma_\infty u_0^{(0)})\rangle\rangle = (\langle Q_0 T_{j,k}P_\infty^{(0)}\boldsymbol{F}\rangle_\infty, T_{j,k}\sigma_\infty),$$

(7.4)

$$\frac{1}{2}\frac{d}{dt}\big(\tfrac{1}{\gamma^2}\big\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\phi_\infty\big\|_2^2+\|\sqrt{\rho_s}T_{j,k}w_\infty\|_2^2\big)$$

$$+D^{(0)}[T_{j,k}w_\infty]+\langle\langle C_0T_{j,k}u_\infty,T_{j,k}u_\infty\rangle\rangle+\langle\langle\tilde{\mathcal{M}}(T_{j,k}\sigma_\infty u^{(0)}),T_{j,k}u_\infty\rangle\rangle$$

$$-\langle\langle\langle Q_0\tilde{B}(T_{j,k}\sigma_\infty u^{(0)}+T_{j,k}u_\infty)\rangle_\infty u_1^{(0)},T_{j,k}u_\infty\rangle\rangle=\langle\langle T_{j,k}\boldsymbol{F}_\infty,T_{j,k}u_\infty\rangle\rangle.$$

We add $\frac{\alpha_0}{\gamma^2}\times$ (7.3) to (7.4). Then, since

$$\begin{aligned}&\langle\langle\tilde{\mathcal{M}}(T_{j,k}\sigma_\infty u^{(0)}),T_{j,k}u_\infty\rangle\rangle\\ &\quad=\ \langle\langle\tilde{A}(T_{j,k}\sigma_\infty u^{(0)}),T_{j,k}u_\infty\rangle\rangle+\langle\langle\tilde{B}(T_{j,k}\sigma_\infty u_0^{(0)}),T_{j,k}u_\infty\rangle\rangle\\ &\qquad+\langle\langle\tilde{B}(T_{j,k}\sigma_\infty u_1^{(0)}),T_{j,k}u_\infty\rangle\rangle,\end{aligned}$$

the term $\langle\langle\tilde{B}(T_{j,k}\sigma_\infty u_0^{(0)}),T_{j,k}u_\infty\rangle\rangle$ is cancelled. We thus arrive at

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}E^{(0)}[T_{j,k}\tilde{P}_\infty u]+D^{(0)}[T_{j,k}w_\infty]\\ &\qquad+\big\{\langle\langle C_0T_{j,k}u_\infty,T_{j,k}u_\infty\rangle\rangle+\langle\langle\tilde{A}(T_{j,k}\sigma_\infty u^{(0)}),T_{j,k}u_\infty\rangle\rangle\\ &\qquad+\langle\langle\tilde{B}(T_{j,k}\sigma_\infty u_1^{(0)}),T_{j,k}u_\infty\rangle\rangle\\ &\qquad-\langle\langle\langle Q_0\tilde{B}(T_{j,k}\sigma_\infty u^{(0)}+T_{j,k}u_\infty)\rangle_\infty u_1^{(0)},T_{j,k}u_\infty\rangle\rangle\big\}\\ &\quad=\ \frac{\alpha_0}{\gamma^2}(\langle Q_0T_{j,k}P_\infty^{(0)}\boldsymbol{F}\rangle_\infty,T_{j,k}\sigma_\infty)+\langle\langle T_{j,k}\boldsymbol{F}_\infty,T_{j,k}u_\infty\rangle\rangle.\end{aligned}\tag{7.5}$$

The desired estimate for the case $2j+k\le m-1$ now follows from (7.5) by substituting

$$\boldsymbol{F}_\infty=\boldsymbol{F}-\langle Q_0\boldsymbol{F}\rangle_\infty u^{(0)}-\{P_1\boldsymbol{F}-\langle Q_0P_1\boldsymbol{F}\rangle_\infty u^{(0)}\}$$

and

$$\langle Q_0P_\infty^{(0)}\boldsymbol{F}\rangle_\infty=\langle Q_0\boldsymbol{F}\rangle_\infty-\langle Q_0P_1\boldsymbol{F}\rangle_\infty$$

into the right-hand side of (7.5) with the aid of Lemma 5.2 (ii) and Lemma 7.3. In case $2j+k=m$, the computation above is formal; and the desired result can formally be obtained by further integration by parts on $\tilde{R}_{j,k}$. This can be justified by an argument of commutator estimate for the transport equation (5.3) and a theory of weak solutions of the parabolic equation (5.4) as in, e.g., [9]. □

We next derive the H^1-parabolic estimates for w_∞. We define $J[\tilde{P}_\infty u]$ by

$$J[\tilde{P}_\infty u]=-2\langle\langle\sigma_\infty u^{(0)}+u_\infty,B\tilde{Q}u_\infty\rangle\rangle\ \text{ for }\tilde{P}_\infty u=\sigma_\infty u^{(0)}+u_\infty.$$

A direct computation shows that if $\gamma^2 \geq 1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq 1$, then

$$|J[\tilde{P}_\infty u]| \leq \frac{b_0\gamma^2}{\nu} E^{(0)}[\tilde{P}_\infty u] + \frac{1}{2} D^{(0)}[w_\infty]$$

for some constant $b_0 > 0$.

Let b_1 be a positive constant (to be determined later) and define $E^{(1)}[\tilde{P}_\infty u]$ by

$$E^{(1)}[\tilde{P}_\infty u] = \tfrac{2b_1\gamma^2}{\nu} E^{(0)}[\tilde{P}_\infty u] + D^{(0)}[w_\infty] + J[\tilde{P}_\infty u].$$

Note that if $b_1 \geq b_0$, then $E^{(1)}[\tilde{P}_\infty u]$ is equivalent to $E^{(0)}[\tilde{P}_\infty u] + D^{(0)}[w_\infty]$.

Proposition 7.5. *There exists $b_1 \geq b_0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq 1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq 1$, then the following estimate holds for $0 \leq 2k + j \leq m-1$:*

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt} E^{(1)}[T_{j,k}\tilde{P}_\infty u] + \tfrac{b_1\gamma^2}{\nu} D^{(0)}[T_{j,k}w_\infty] + \tfrac{1}{2}\|\sqrt{\rho_s}\partial_t T_{j,k} w_\infty\|_2^2 \\ &\leq R^{(2)}_{j,k} + C\big\{\big(\tfrac{1}{\nu} + \tfrac{\nu+\tilde{\nu}}{\gamma^2}\big)\|\partial_{x'} T_{j,k}\sigma_\infty\|_2^2 \\ &\quad + \big(\tfrac{1}{\nu} + \tfrac{1}{\gamma^2} + \tfrac{\nu^2}{\gamma^4}\big)\|\tilde{T}_{j,k}\phi_\infty\|_2^2 + \tfrac{1}{\gamma^2}\|\partial_{x_1} T_{j,k}\phi_\infty\|_2^2\big\}, \end{aligned} \tag{7.6}$$

where

$$R^{(2)}_{j,k} = \frac{2b_1\gamma^2}{\nu} R^{(1)}_{j,k} + C\|T_{j,k}\boldsymbol{F}\|_2^2.$$

Outline of Proof. We consider the case $j = k = 0$. We take the inner product of (5.4) with $\partial_t \tilde{Q} u_\infty$ to obtain

$$\begin{aligned} &\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + \langle\langle L u_\infty, \partial_t \tilde{Q} u_\infty\rangle\rangle + \langle\langle \tilde{\mathcal{M}}(\sigma_\infty u^{(0)}), \partial_t \tilde{Q} u_\infty\rangle\rangle \\ &-\langle\langle\langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle\rangle_\infty u^{(0)}, \partial_t \tilde{Q} u_\infty\rangle\rangle = \langle\langle \boldsymbol{F}_\infty, \partial_t \tilde{Q} u_\infty\rangle\rangle. \end{aligned} \tag{7.7}$$

By using the symmetric property of A we have

$$\begin{aligned} \langle\langle L u_\infty, \partial_t \tilde{Q} u_\infty\rangle\rangle &= \tfrac{1}{2}\tfrac{d}{dt} D^{(0)}[w_\infty] + \langle\langle B u_\infty, \partial_t \tilde{Q} u_\infty\rangle\rangle \\ &\quad + \langle\langle C_0 u_\infty, \partial_t \tilde{Q} u_\infty\rangle\rangle. \end{aligned} \tag{7.8}$$

As for the second term on the right of (7.8), we rewrite it as

$$\begin{aligned} \langle\langle B u_\infty, \partial_t \tilde{Q} u_\infty\rangle\rangle &= -\tfrac{d}{dt}\langle\langle u_\infty, B\tilde{Q} u_\infty\rangle\rangle + \langle\langle \partial_t u_\infty, B\tilde{Q} u_\infty\rangle\rangle \\ &= -\tfrac{d}{dt}\langle\langle u_\infty, B\tilde{Q} u_\infty\rangle\rangle + \langle\langle \partial_t Q_0 u_\infty, B\tilde{Q} u_\infty\rangle\rangle \\ &\quad + \langle\langle \partial_t \tilde{Q} u_\infty, B\tilde{Q} u_\infty\rangle\rangle. \end{aligned} \tag{7.9}$$

By (5.4), we have

$$\partial_t \phi_\infty = -\{v_s^1 \partial_{x_1}\phi_\infty + \gamma^2 \mathrm{div}\,(\rho_s w_\infty) + (v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1})\partial_{x_1}\sigma_\infty$$
$$-\langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle_\infty \phi^{(0)}\} + f_\infty^0.$$

We thus obtain, if $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq 1$, then

$$\begin{aligned}
&|\langle\langle \partial_t Q_0 u_\infty, B\tilde{Q}u_\infty\rangle\rangle| \\
&\leq C\big\{\|\partial_{x_1}\phi_\infty\|_2 + \gamma^2\|\partial_x w_\infty\|_2 + \|\partial_{x_1}\sigma_\infty\|_2\big\}\|\mathrm{div}\,(\rho_s w_\infty)\|_2 \\
&\quad + C\|Q_0\boldsymbol{F}_\infty\|_2\|\mathrm{div}\,(\rho_s w_\infty)\|_2 \\
&\leq C\big\{\tfrac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \tfrac{1}{\gamma^2}(\|\partial_{x_1}\phi_\infty\|_2^2 + \|\partial_{x_1}\sigma_\infty\|_2^2 + \|Q_0\boldsymbol{F}_\infty\|_2^2)\big\}.
\end{aligned}$$

It then follows from (7.9) that

$$\begin{aligned}
&\langle\langle Bu_\infty, \partial_t\tilde{Q}u_\infty\rangle\rangle \\
&\geq -\tfrac{d}{dt}\langle\langle u_\infty, B\tilde{Q}u_\infty\rangle\rangle - |\langle\langle \partial_t\tilde{Q}u_\infty, B\tilde{Q}u_\infty\rangle\rangle| \\
&\quad -C\big\{\tfrac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \tfrac{1}{\gamma^2}(\|\partial_{x_1}\phi_\infty\|_2^2 + \|\partial_{x_1}\sigma_1\|_2^2 + \|Q_0\boldsymbol{F}_\infty\|_2^2)\big\}.
\end{aligned} \tag{7.10}$$

We deduce from (7.8) and (7.10) that

$$\begin{aligned}
\langle\langle Lu_\infty, \partial_t\tilde{Q}u_\infty\rangle\rangle &\geq \tfrac{1}{2}\tfrac{d}{dt}\big(D^{(0)}[w_\infty] - 2\langle\langle u_\infty, B\tilde{Q}u_\infty\rangle\rangle\big) \\
&\quad -|\langle\langle C_0 u_\infty, \partial_t\tilde{Q}u_\infty\rangle\rangle| - |\langle\langle \partial_t\tilde{Q}u_\infty, B\tilde{Q}u_\infty\rangle\rangle| \\
&\quad -C\big\{\tfrac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \tfrac{1}{\gamma^2}(\|\partial_{x_1}\phi_\infty\|_2^2 + \|\partial_{x_1}\sigma_\infty\|_2^2 + \|Q_0\boldsymbol{F}_\infty\|_2^2)\big\},
\end{aligned} \tag{7.11}$$

provided that $\gamma^2 \geq 1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq 1$.

We next consider $\langle\langle \tilde{\mathcal{M}}(\sigma_\infty u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\rangle$ on the left-hand side of (7.7):

$$\langle\langle \tilde{\mathcal{M}}(\sigma_\infty u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\rangle = \langle\langle \tilde{A}(\sigma_\infty u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\rangle + \langle\langle \tilde{B}(\sigma_\infty u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\rangle. \tag{7.12}$$

By Lemma 5.2 (ii) we have $\|\partial_{x_1}^k\sigma_\infty\|_2 \leq \|\partial_{x_1}\sigma_\infty\|_2$ for $k \geq 1$. This, together with the fact $w^{(0),1} = O(\gamma^{-2})$, yields

$$\begin{aligned}
|\langle\langle \tilde{A}(\sigma_\infty u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\rangle| &\leq C\tfrac{\nu+\tilde{\nu}}{\gamma^2}(\|\partial_{x_1}\sigma_1\|_2 + \|\partial_{x_1}^2\sigma_1\|_2)\|\sqrt{\rho_s}\partial_t w_\infty\|_2 \\
&\leq \tfrac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + C\tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}\|\partial_{x_1}\sigma_\infty\|_2^2.
\end{aligned} \tag{7.13}$$

The second term on the right of (7.12) is treated in a similar manner to the estimate (7.10). Since $Lu^{(0)}=0$, we have $\partial_{x_n}\left(\frac{P'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)}\right)=0$. It then follows that $\tilde{B}(\sigma_\infty u^{(0)})=B(\sigma_\infty u^{(0)})$, and hence,

$$\begin{aligned}\langle\langle \tilde{B}(\sigma_\infty u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\rangle &= \langle\langle B(\sigma_\infty u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\rangle \\ &= -\tfrac{d}{dt}\langle\langle \sigma_\infty u^{(0)}, B\tilde{Q}u_\infty\rangle\rangle + \langle\langle \partial_t\sigma_\infty u^{(0)}, B\tilde{Q}u_\infty\rangle\rangle.\end{aligned}$$

By (5.3) we have

$$\partial_t\sigma_\infty = -\langle Q_0\tilde{B}(\sigma_\infty u^{(0)}+u_\infty)\rangle_\infty + \langle Q_0 P_\infty^{(0)}\boldsymbol{F}\rangle_\infty.$$

Substituting this into $\langle\langle \partial_t\sigma_\infty u^{(0)}, B\tilde{Q}u_\infty\rangle\rangle$ we obtain

(7.15)
$$\begin{aligned}&|\langle\langle \partial_t\sigma_\infty u^{(0)}, B\tilde{Q}u_\infty\rangle\rangle| \\ &\quad\leq\ C\big\{\tfrac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \tfrac{1}{\gamma^2}(\|\partial_{x_1}\sigma_\infty\|_2^2 + \|\partial_{x_1}\phi_\infty\|_2^2 + \|\langle Q_0P_\infty^{(0)}\boldsymbol{F}\rangle_\infty\|_2^2)\big\},\end{aligned}$$

provided that $\|\rho_s-1\|_{C^{m+1}[0,1]}\leq 1$.

We thus find from (7.12)–(7.15) that

(7.16)
$$\begin{aligned}&\langle\langle \tilde{\mathcal{M}}(\sigma_\infty u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\rangle \\ &\quad\geq\ -\tfrac{d}{dt}\langle\langle \sigma_\infty u^{(0)}, B\tilde{Q}u_\infty\rangle\rangle - \tfrac{1}{4}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 \\ &\qquad -C\big\{\tfrac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \tfrac{1}{\gamma^2}(\|\partial_{x_1}\sigma_\infty\|_2^2 + \|\partial_{x_1}\phi_\infty\|_2^2 + \|\langle Q_0P_\infty^{(0)}\boldsymbol{F}\rangle_\infty\|_2^2) \\ &\qquad +\tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}\|\partial_{x_1}\sigma_\infty\|_2^2\big\}\end{aligned}$$

if $\|\rho_s-1\|_{C^{m+1}[0,1]}\leq 1$.

It follows from (7.7), (7.11) and (7.16) that

(7.17)
$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}(D^{(0)}[w_\infty] + J[\tilde{P}_\infty u]) + \tfrac{3}{4}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 \\ &\quad\leq\ |\langle\langle C_0u_\infty, \partial_t\tilde{Q}u_\infty\rangle\rangle| + \langle\langle \partial_t\tilde{Q}u_\infty, B\tilde{Q}u_\infty\rangle\rangle| \\ &\qquad +|\langle\langle\langle Q_0\tilde{B}(\sigma_\infty u^{(0)}+u_\infty)\rangle_\infty u^{(0)}, \partial_t\tilde{Q}u_\infty\rangle\rangle| + |\langle\langle \boldsymbol{F}_\infty, \partial_t\tilde{Q}u_\infty\rangle\rangle| \\ &\qquad +C\big\{\tfrac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \tfrac{1}{\gamma^2}\big(\|\partial_{x_1}\phi_\infty\|_2^2 + \|\partial_{x_1}\sigma_\infty\|_2^2 + \|Q_0\boldsymbol{F}_\infty\|_2^2 + \|\langle Q_0P_\infty^{(0)}\boldsymbol{F}\rangle_\infty\|_2^2\big) \\ &\qquad +\tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}\|\partial_{x_1}\sigma_\infty\|_2^2\big\}.\end{aligned}$$

We estimate the right-hand side of (7.17) by using the Schwartz inequality and Lemma 7.3 to obtain

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(D^{(0)}[w_\infty] + J[u]) + \tfrac{1}{2}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 \\
&\quad\le\ C\|\boldsymbol{F}\|_2^2 + C\tfrac{\gamma^2}{\nu}D^{(0)}[w_\infty] \\
&\qquad + C\big\{\big(\tfrac{1}{\gamma^2} + \tfrac{(\nu+\tilde\nu)^2}{\gamma^4}\big)\|\partial_{x_1}\sigma_\infty\|_2^2 + \tfrac{\nu^2}{\gamma^4}\|\phi_\infty\|_2^2 + \tfrac{1}{\gamma^2}\|\partial_{x_1}\phi_\infty\|_2^2\big\}.
\end{aligned}
\tag{7.18}
$$

Taking $b_1 > 0$ in such a way that $b_1 \ge \max\{b_0, 2C\}$ and adding $\frac{2b_1\gamma^2}{\nu} \times (7.2)$ to (7.18), we obtain the desired estimate. The case $1 \le 2j + k \le m - 1$ can be treated similarly. $\square$

As for the dissipative estimates for x_2-derivatives of ϕ_∞, we have the following inequality.

Proposition 7.6. *If $\|\rho_s - 1\|_{C^{m+1}[0,1]} \le \min\{1, \frac{(\nu+\tilde\nu)^2}{\gamma^4}, \omega_0\}$, then the following estimate holds for $0 \le 2j + k + l \le m - 1$:*

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\tfrac{1}{\gamma^2}\Big\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\partial_{x_2}^{l+1}\phi_\infty\Big\|_2^2 + \tfrac{1}{2(\nu+\tilde\nu)}\Big\|\tfrac{\tilde P'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_2}^{l+1}\phi_\infty\Big\|_2^2 \\
&\quad \le R_{j,k,l}^{(3)} + C\tfrac{\nu+\tilde\nu}{\gamma^4}\|K_{j,k,l}\|_2^2,
\end{aligned}
\tag{7.19}
$$

where

$$
R_{j,k,l}^{(3)} = \left|\frac{1}{2\gamma^2}(\mathrm{div}\,(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}w), |T_{j,k}\partial_{x_2}^{l+1}\phi_\infty|^2)\right| + \frac{\nu+\tilde\nu}{\gamma^4}\|H_{j,k,l}\|_2^2
$$

with

$$
\|H_{j,k,l}\|_2^2 \le C\big\{\|[T_{j,k}\partial_{x_2}^{l+1}, w]\cdot\nabla\phi_\infty\|_2^2 + \|T_{j,k}\partial_{x_2}^{l+1}\tilde f_\infty^0\|_2^2 + \|\frac{\gamma^2\rho_s^2}{\nu+\tilde\nu}T_{j,k}\partial_{x_2}^{l}f_\infty^2\|_2^2\big\},
$$

and

$$
\tilde f_\infty^0 = -\phi\mathrm{div}\,w - w\cdot\nabla(\sigma_*\phi^{(0)} + \phi_1) - \{f_1^0 - \langle Q_0P_1\boldsymbol{F}\rangle_\infty\phi^{(0)}\}
$$

with $P_1\boldsymbol{F} = {}^\top(f_1^0, f_1^1, f_1^2)$; and $K_{j,k,l}$ is estimated as

$$
\begin{aligned}
&\tfrac{\nu+\tilde\nu}{\gamma^4}\|K_{j,k,l}\|_2^2 \\
&\quad\le\ C\big\{\tfrac{\nu^2}{\nu+\tilde\nu}\|T_{j,k+1}\partial_{x_2}^{l}\partial_x w_\infty\|_2^2 + \tfrac{1}{\nu+\tilde\nu}\|\sqrt{\rho_s}\partial_t T_{j,k}\partial_{x_2}^{l}w_\infty\|_2^2 \\
&\qquad + (\nu+\tilde\nu)\omega_0^2\big(\textstyle\sum_{p=0}^{l-1}\|T_{j,k+1}\partial_{x_2}^{p}\partial_x w_\infty\|_2^2 + \sum_{p=0}^{l}\|T_{j,k}\partial_{x_2}^{p}\partial_x w_\infty\|_2^2\big) \\
&\qquad + \tfrac{1}{\nu+\tilde\nu}\textstyle\sum_{p=0}^{l}\|T_{j,k+1}\partial_{x_2}^{p}w_\infty\|_2^2 \\
&\qquad + \tfrac{\nu+\tilde\nu}{\gamma^4}\big(\textstyle\sum_{p=0}^{l}\|T_{j,k}\partial_{x_2}^{p}\partial_x\phi_\infty\|_2^2 + \|\partial_{x_1}T_{j,k}\sigma_\infty\|_2^2\big)\big\}.
\end{aligned}
$$

Outline of Proof. The first equation of (5.4) is written as

$$\partial_t\phi_\infty + (v_s + w)\cdot\nabla\phi_\infty + \gamma^2\rho_s\partial_{x_1}w^1_\infty + \gamma^2\partial_{x_2}(\rho_s w^2_\infty)$$
$$+(v^1_s\phi^{(0)} + \gamma^2\rho_s w^{(0),1})\partial_{x_1}\sigma_\infty - \langle Q_0\tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle_\infty\phi^{(0)}$$
$$= \tilde{f}^0_\infty.$$

Applying $T_{j,k}\partial^{l+1}_{x_2}$ to this equation, we have

$$\begin{aligned}(7.20)\qquad &\partial_t(T_{j,k}\partial^{l+1}_{x_2}\phi_\infty) + (v_s + w)\cdot\nabla(T_{j,k}\partial^{l+1}_{x_2}\phi_\infty) + \gamma^2\rho_s T_{j,k}\partial^{l+2}_{x_2}w^n_\infty\\ &= -[T_{j,k}\partial^{l+1}_{x_2}, v_s + w]\cdot\nabla\phi_\infty - \gamma^2\partial^{l+1}_{x_2}(\rho_s\partial_{x_1}T_{j,k}w^1_\infty)\\ &\quad -\gamma^2[\partial^{l+2}_{x_2}, \rho_s]T_{j,k}w^2_\infty - \partial^{l+1}_{x_2}(v^1_s\phi^{(0)} + \gamma^2\rho_s w^{(0),1})\partial_{x_1}T_{j,k}\sigma_\infty\\ &\quad +\langle Q_0\tilde{B}(T_{j,k}\sigma_\infty u^{(0)} + T_{j,k}u_\infty)\rangle_\infty\partial^{l+1}_{x_2}\phi^{(0)} + T_{j,k}\partial^{l+1}_{x_2}\tilde{f}^0_\infty.\end{aligned}$$

The third equation of (5.4) is written as

$$-\tfrac{\nu+\tilde{\nu}}{\rho_s}\partial^2_{x_2}w^2_\infty + \partial_{x_2}(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi_\infty)$$
$$= -\big\{\partial_t w^2_\infty - \tfrac{\nu}{\rho_s}\partial^2_{x_1}w^2_\infty - \tfrac{\tilde{\nu}}{\rho_s}\partial_{x_2}\partial_{x_1}w^1_\infty + v^1_s\partial_{x_1}w^2_\infty - \tfrac{\tilde{\nu}}{\rho_s}(\partial_{x_2}w^{(0),1})\partial_{x_1}\sigma_\infty\big\} + f^2_\infty.$$

Applying $T_{j,k}\partial^l_{x_2}$ to this equation, we have

$$\begin{aligned}(7.21)\qquad &-\tfrac{\nu+\tilde{\nu}}{\rho_s}T_{j,k}\partial^{l+2}_{x_2}w^2_\infty + \tfrac{P'(\rho_s)}{\gamma^2\rho_s}T_{j,k}\partial^{l+1}_{x_2}\phi_\infty\\ &= (\nu+\tilde{\nu})[\partial^l_{x_2}, \tfrac{1}{\rho_s}]\partial^2_{x_2}T_{j,k}w^2_\infty - [\partial^{l+1}_{x_2}, \tfrac{P'(\rho_s)}{\gamma^2\rho_s}]T_{j,k}\phi_\infty\\ &\quad -\big\{\partial_t T_{j,k}\partial^l_{x_2}w^2_\infty - T_{j,k}\partial^l_{x_2}\big(\tfrac{\nu}{\rho_s}\partial^2_{x_1}w^2_\infty + \tfrac{\tilde{\nu}}{\rho_s}\partial_{x_2}\partial_{x_1}w^1_\infty\big)\\ &\quad +T_{j,k}\partial^l_{x_2}(v^1_s\partial_{x_1}w^2_\infty) - \partial^l_{x_2}(\tfrac{\tilde{\nu}}{\rho_s}\partial_{x_2}w^{(0),1})\partial_{x_1}T_{j,k}\sigma_\infty\big\} + T_{j,k}\partial^l_{x_2}f^2_\infty.\end{aligned}$$

Adding $\frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\times$ (7.21) to (7.20), we have

$$\begin{aligned}(7.22)\qquad &\partial_t(T_{j,k}\partial^{l+1}_{x_2}\phi_\infty) + (v_s + w)\cdot\nabla(T_{j,k}\partial^{l+1}_{x_2}\phi_\infty) + \tfrac{\rho_s P'(\rho_s)}{\nu+\tilde{\nu}}T_{j,k}\partial^{l+1}_{x_2}\phi_\infty\\ &= H_{j,k,l} + K_{j,k,l},\end{aligned}$$

where

$$H_{j,k,l} = -[T_{j,k}\partial^{l+1}_{x_2}, w]\cdot\nabla\phi_\infty + T_{j,k}\partial^{l+1}_{x_2}\tilde{f}^0_\infty + \frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}T_{j,k}\partial^l_{x_2}f^2_\infty,$$

$$
\begin{aligned}
K_{j,k,l} &= -[\partial_{x_2}^{l+1}, v_s^1]\cdot\nabla T_{j,k}\phi_\infty - \gamma^2[\partial_{x_2}^{l+1},\rho_s]\partial_{x_1}T_{j,k}w'_\infty \\
&\quad -\gamma^2[\partial_{x_2}^{l+2},\rho_s]T_{j,k}w_\infty^2 - \partial_{x_2}^{l+1}(v_s^1\phi^{(0)}+\gamma^2\rho_s w^{(0),1})\partial_{x_1}T_{j,k}\sigma_\infty \\
&\quad +\langle Q_0\tilde{B}(T_{j,k}\sigma_\infty u^{(0)}+T_{j,k}u_\infty)\rangle_\infty \partial_{x_2}^{l+1}\phi^{(0)} \\
&\quad +\gamma^2\rho_s^2[\partial_{x_2}^l,\tfrac{1}{\rho_s}]\partial_{x_2}^2 T_{j,k}w_\infty^2 - \tfrac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}[\partial_{x_2}^{l+1},\tfrac{P'(\rho_s)}{\gamma^2\rho_s}]T_{j,k}\phi_\infty \\
&\quad -\tfrac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\big\{\partial_t T_{j,k}\partial_{x_2}^l w_\infty^2 - \tfrac{\nu}{\rho_s}\partial_{x_1}^2 T_{j,k}\partial_{x_2}^l w_\infty^2 + \tfrac{\nu}{\rho_s}\partial_{x_2}^{l+1}\partial_{x_1}T_{j,k}w_\infty^1 \\
&\quad -\nu[\partial_{x_2}^l,\tfrac{1}{\rho_s}]\partial_{x_1}^2 T_{j,k}w_\infty^2 - \tilde{\nu}[\partial_{x_2}^l,\tfrac{1}{\rho_s}]\partial_{x_2}\partial_{x_1}T_{j,k}w_\infty^1 \\
&\quad +T_{j,k}\partial_{x_n}^l(v_s^1\partial_{x_1}w_\infty^2) - \partial_{x_2}^l(\tfrac{\tilde{\nu}}{\rho_s}\partial_{x_2}w^{(0),1})\partial_{x_1}T_{j,k}\sigma_\infty\big\}.
\end{aligned}
$$

Multiplying (7.22) by $\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}T_{j,k}\partial_{x_2}^{l+1}\phi_\infty$ and integrating over Ω, we have the desired inequality for $2j+k+l\le m-2$. In case $2j+k+l=m-1$ the computation is formal, but it can be done rigorously by using an argument of commutator estimate for the transport equation (7.22) as in, e.g., [9].

□

The following estimate for the material derivative of ϕ_∞ plays an important role to obtain the dissipative estimate for higher order x_2-derivatives of w_∞. We denote the material derivative of ϕ_∞ by $\dot{\phi}_\infty$:

$$
\dot{\phi}_\infty = \partial_t\phi_\infty + (v_s+w)\cdot\nabla\phi_\infty.
$$

Proposition 7.7. (i) *If* $\|\rho_s-1\|_{C^{m+1}[0,1]}\le\min\{1,\frac{(\nu+\tilde{\nu})^2}{\gamma^4},\omega_0\}$, *then the following estimate holds for* $0\le 2j+k+l\le m-1$*:*

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\tfrac{1}{\gamma^2}\Big\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\partial_{x_2}^{l+1}\phi_\infty\Big\|_2^2 + \tfrac{1}{4(\nu+\tilde{\nu})}\Big\|\tfrac{P'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_2}^{l+1}\phi_\infty\Big\|_2^2 \\
&\qquad +c_0\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\partial_{x_2}^{l+1}\dot{\phi}_\infty\|_2^2 \\
&\qquad \le R_{j,k,l}^{(3)} + C\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|K_{j,k,l}\|_2^2,
\end{aligned}
\tag{7.23}
$$

where c_0 *is a positive constant; and* $R_{j,k,l}^{(3)}$ *and* $K_{j,k,l}$ *satisfy the same estimates as in Proposition* 7.6.

(ii) *Let* $0\le q\le k$*. Then*

$$
\begin{aligned}
\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty\|_2^2 &\le C\big\{R_{j,k}^{(4)} + D^{(0)}[T_{j,k}w_\infty] + (\nu+\tilde{\nu})\omega_0^2\|T_{j,k}w_\infty\|_2^2 \\
&\qquad +\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x_1}T_{j,k}\sigma_\infty\|_2^2 + \tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x_1}T_{j,q}\phi_\infty\|_2^2\big\},
\end{aligned}
\tag{7.24}
$$

where $R^{(4)}_{j,k} = \frac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\tilde{f}^0_\infty\|_2^2$.

Outline of Proof. By (7.22), we have

$$T_{j,k}\partial_{x_2}^{l+1}\dot{\phi}_\infty = -\frac{\rho_s\tilde{P}'(\rho_s)}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_2}^{l+1}\phi_\infty + \tilde{H}_{j,k,l} + \tilde{K}_{j,k,l},$$

where

$$\tilde{H}_{j,k,l} = [T_{j,k}\partial_{x_2}^{l+1}, w]\cdot\nabla\phi_\infty + H_{j,k,l},$$

$$\tilde{K}_{j,k,l} = [\partial_{x_2}^{l+1}, v_s]\cdot\nabla T_{j,k}\phi_\infty + K_{j,k,l}.$$

This, together with (7.19), gives (7.23).

The estimate (7.24) follows from the first equation of (5.4) which is written as

$$\begin{aligned} T_{j,k}\dot{\phi}_\infty &= -\rho_s\gamma^2\mathrm{div}\,(T_{j,k}w_\infty) - \gamma^2(\partial_{x_2}\rho_s)T_{j,k}w^2_\infty \\ &\quad -(v^1_s\phi^{(0)} + \gamma^2\rho_s w^{(0),1})\partial_{x_1}T_{j,k}\sigma_\infty \\ &\quad +\langle Q_0\tilde{B}(T_{j,k}\sigma_\infty u^{(0)} + T_{j,k}u_\infty)\rangle_\infty\phi^{(0)} + T_{j,k}\tilde{f}^0_\infty. \end{aligned}$$

Applying $T_{j,k}$ to this equation, we can obtain (7.24). □

Let us derive the dissipative estimates for σ_∞.

Proposition 7.8. *There are positive constants* ν_0 *and* γ_0 *such that if* $\nu \geq \nu_0$ *and* $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$*, then the following estimate holds for* $0 \leq 2j+k \leq m-1$*:*

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\tfrac{\nu}{\gamma^2(\nu+\tilde{\nu})}\|T_{j,k}\sigma_\infty\|_2^2 + \tfrac{\alpha_1}{2(\nu+\tilde{\nu})}\|\partial_{x_1}T_{j,k}\sigma_\infty\|_2^2 \\ &\leq R^{(5)}_{j,k} + C\big\{\tfrac{1}{\nu+\tilde{\nu}}\|\partial_t T_{j,k}w_\infty\|_2^2 + D^{(0)}[T_{j,k}w_\infty] \\ &\quad +\big(\tfrac{1}{\nu+\tilde{\nu}} + \tfrac{\nu^2}{\gamma^4(\nu+\tilde{\nu})}\big)\|\partial_{x_2}T_{j,p}\phi_\infty\|_2^2\big\}, \end{aligned} \tag{7.25}$$

where $\alpha_1 > 0$ *is a constant;* p *is any integer satisfying* $0 \leq 2j+p+1 \leq m$ *and* $0 \leq p \leq k$*; and*

$$R^{(5)}_{j,k} = \tfrac{\nu}{\gamma^2(\nu+\tilde{\nu})}(Q_0T_{j,k}P^{(0)}_\infty\boldsymbol{F}, T_{j,k}\sigma_\infty) - \tfrac{1}{\nu+\tilde{\nu}}(\rho_s(-\Delta)^{-1}(\rho_s\partial_{x_1}T_{j,k}\boldsymbol{f}^1_\infty), T_{j,k}\sigma_\infty).$$

Here $(-\Delta)^{-1}$ *is the inverse of* $-\Delta$ *on* $L^2(\Omega)$ *with domain* $D(-\Delta) = H^2(\Omega)\cap H^1_0(\Omega)$*.*

Outline of Proof. We rewrite (5.3) as

$$\begin{aligned} &\partial_t\sigma_\infty + \gamma^2\partial_{x_1}\langle\rho_s w^1_\infty\rangle_\infty + \langle Q_0\tilde{B}(\sigma_\infty u^{(0)})\rangle_\infty \\ &= \langle Q_0P^{(0)}_\infty\boldsymbol{F}\rangle_\infty - \partial_{x_1}\langle v^1_s\phi_\infty\rangle_\infty. \end{aligned} \tag{7.26}$$

Since $Lu^{(0)} = 0$, we have $\partial_{x_2}\left(\frac{P'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)}\right) = 0$. Furthermore, since $1 = \langle u^{(0)}, u^{(0)*}\rangle = \langle \phi^{(0)}\rangle$, we have $\frac{P'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)} = \alpha_0$. Therefore, the second equation of (5.4) is written as

$$-\Delta w_\infty^1 = -\frac{\alpha_0}{\nu}\rho_s\partial_{x_1}\sigma_\infty + \tilde{f}_\infty^1 + \frac{\rho_s}{\nu}f_\infty^1, \tag{7.27}$$

where

$$\begin{aligned}\tilde{f}_\infty^1 \;=\; & -\tfrac{\rho_s}{\nu}\big\{\partial_t w_\infty^1 - \tfrac{\tilde{\nu}}{\rho_s}\partial_{x_1}\mathrm{div}\, w_\infty + \tfrac{P'(\rho_s)}{\gamma^2\rho_s}\partial_{x_1}\phi_\infty + v_s^1\partial_{x_1}w_\infty^1 \\ & +\tfrac{\nu\partial_{x_2}^2 v_s^1}{\gamma^2\rho_s^2}\phi_\infty + (\partial_{x_2}v_s^1)w_\infty^1 - \tfrac{\nu+\tilde{\nu}}{\rho_s}(\partial_{x_1}^2\sigma_\infty)w^{(0),1} + v_s^1 w^{(0),1}\partial_{x_1}\sigma_\infty \\ & -\langle Q_0\tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle_\infty w^{(0),1}\big\}.\end{aligned}$$

It follows from (7.27) that

$$w_\infty^1 = -\frac{\alpha_0}{\nu}(-\Delta)^{-1}(\rho_s\partial_{x_1}\sigma_\infty) + (-\Delta)^{-1}(\tilde{f}_\infty^1 + \frac{\rho_s}{\nu}f_\infty^1). \tag{7.28}$$

Substituting (7.28) into the second term on the left-hand side of (7.26), we have

(7.29)

$$\begin{aligned}&\partial_t\sigma_\infty - \tfrac{\alpha_0\gamma^2}{\nu}\langle\rho_s(-\Delta)^{-1}(\rho_s\partial_{x_1}^2\sigma_\infty)\rangle_\infty + \langle Q_0\tilde{B}(\sigma_\infty u^{(0)})\rangle_\infty \\ &\quad = \langle Q_0P_\infty^{(0)}\boldsymbol{F}\rangle_\infty - \partial_{x_1}\langle v_s^1\phi_\infty\rangle_\infty - \gamma^2\partial_{x_1}\langle\rho_s(-\Delta)^{-1}(\tilde{f}_\infty^1 + \tfrac{1}{\nu}\rho_s f_\infty^1)\rangle_\infty.\end{aligned}$$

By the Plancherel theorem,

$$-(\langle\rho_s(-\Delta)^{-1}(\rho_s\partial_{x_1}^2\sigma_\infty)\rangle_\infty, \sigma_\infty) = (2\pi)^{-1}((\hat{\chi}_1+\hat{\chi}_2)|\xi|^2\langle\rho_s(|\xi|^2-\partial_{x_2}^2)^{-1}\rho_s\rangle\hat{\sigma}_\infty, \hat{\sigma}_\infty).$$

Since there exists a constant $c > 0$ such that

$$\langle\rho_s(|\xi|^2 - \partial_{x_2}^2)^{-1}\rho_s\rangle = |(|\xi|^2 - \partial_{x_2}^2)^{-\frac{1}{2}}\rho_s|_2^2 \geq c > 0$$

for $|\xi| \leq 1$, we have

$$-(\langle\rho_s(-\Delta)^{-1}(\rho_s\partial_{x_1}^2\sigma_\infty)\rangle_\infty, \sigma_\infty) \geq (2\pi)^{-1}c\||\xi|\hat{\sigma}_\infty\|_2^2 = c\|\partial_{x_1}\sigma_\infty\|_2^2.$$

Furthermore,

$$(\langle Q_0\tilde{B}(\sigma_\infty u^{(0)})\rangle_\infty, \sigma_\infty) = \langle v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1}\rangle(\partial_{x_1}\sigma_\infty, \sigma_\infty) = 0.$$

Therefore, taking the inner product of (7.29) with σ_∞, we can obtain the desired estimate. $\square$

We next estimate the higher order derivatives.

Proposition 7.9. *If $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$, then there holds the following estimate for $0 \leq 2j + k + l \leq m - 1$:*

$$
\begin{aligned}
&\tfrac{\nu^2}{\nu+\tilde{\nu}}\|\partial_x^{l+2} T_{j,k} w_\infty\|_2^2 + \tfrac{1}{\nu+\tilde{\nu}}\|\partial_x^{l+1} T_{j,k}\phi_\infty\|_2^2 \\
&\leq \; C R^{(6)}_{j,k,l} + C\big\{\big(\tfrac{\nu+\tilde{\nu}}{\gamma^4} + \tfrac{1}{\nu+\tilde{\nu}}\big)\|\partial_{x_1} T_{j,k}\sigma_\infty\|_2^2 \\
&\quad + \big(\tfrac{\nu+\tilde{\nu}}{\gamma^4} + \tfrac{\omega_0^2}{\nu+\tilde{\nu}}\big)\|T_{j,k}\phi_\infty\|_{H^l}^2 + \tfrac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty\|_{H^{l+1}}^2 \\
&\quad + \tfrac{1}{\nu+\tilde{\nu}}\|\partial_t T_{j,k} w_\infty\|_{H^l}^2 \\
&\quad + \left(\tfrac{1}{\nu+\tilde{\nu}} + (\nu+\tilde{\nu})\omega_0^2\right)\|T_{j,k} w_\infty\|_{H^{l+1}}^2 + D^{(0)}[T_{j,k} w_\infty]\big\},
\end{aligned}
\tag{7.30}
$$

where

$$R^{(6)}_{j,k,l} = \tfrac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\tilde{f}^0_\infty\|_{H^{l+1}}^2 + \tfrac{1}{\nu+\tilde{\nu}}\|T_{j,k}\boldsymbol{f}_\infty\|_{H^l}^2.$$

Outline of Proof. We use the estimates for the Stokes system. Let ${}^\top(\tilde{\phi}, \tilde{w})$ be the solution of the Stokes system

$$
\begin{aligned}
\operatorname{div}\tilde{w} &= F \quad \text{in } \Omega, \\
-\Delta\tilde{w} + \nabla\tilde{\phi} &= G \quad \text{in } \Omega, \\
\tilde{w}|_{\partial\Omega} &= 0.
\end{aligned}
$$

Then for any $l \in \mathbf{Z}$, $l \geq 0$, there exists a constant $C > 0$ such that

$$\|\partial_x^{l+2}\tilde{w}\|_2^2 + \|\partial_x^{l+1}\tilde{\phi}\|_2^2 \leq C\{\|F\|_{H^{l+1}}^2 + \|G\|_{H^l}^2 + \|\partial_x\tilde{w}\|_2^2\}. \tag{7.31}$$

(See, e.g., [1], [6, Appendix].)

By (5.4), we have

$$
\begin{aligned}
\operatorname{div}(T_{j,k} w_\infty) &= F_{j,k} \quad \text{in } \Omega, \\
-\Delta(T_{j,k} w_\infty) + \nabla\left(\tfrac{P'(\rho_s)}{\nu\gamma^2} T_{j,k}\phi_\infty\right) &= G_{j,k} \quad \text{in } \Omega, \\
T_{j,k} w_\infty|_{\partial\Omega} &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
F_{j,k} &= \tfrac{1}{\gamma^2\rho_s} T_{j,k}\tilde{f}^0_\infty - \tfrac{\partial_{x_2}\rho_s}{\rho_s} T_{j,k} w^2_\infty \\
&\quad - \tfrac{1}{\gamma^2\rho_s}\big\{T_{j,k}\dot{\phi}_\infty + (v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1})\partial_{x_1} T_{j,k}\sigma_\infty \\
&\quad - \langle Q_0\tilde{B}(T_{j,k}\sigma_\infty u^{(0)} + T_{j,k} u_\infty)\rangle_\infty \phi^{(0)}\big\},
\end{aligned}
$$

$$
\begin{aligned}
G_{j,k} &= \tfrac{\rho_s}{\nu} T_{j,k}\boldsymbol{f}_\infty + \tfrac{P'(\rho_s)\nabla\rho_s}{\nu\gamma^2\rho_s} T_{j,k}\phi_\infty \\
&\quad - \tfrac{\rho_s}{\nu}\big\{\partial_t T_{j,k} w_\infty - \tfrac{\tilde{\nu}}{\rho_s}\nabla F_{j,k} + v_s^1 \partial_{x_1} T_{j,k} w_\infty + \tfrac{\nu\partial_{x_1}^2 v_s^1}{\gamma^2\rho_s^2} T_{j,k}\phi_\infty \boldsymbol{e}_1 \\
&\quad + (\partial_{x_2} v_s^1) T_{j,k} w_\infty^2 \boldsymbol{e}_1 - \tfrac{\nu}{\rho_s} w^{(0)} \partial_{x_1}^2 T_{j,k}\sigma_\infty - \tfrac{\tilde{\nu}}{\rho_s}\nabla(w^{(0),1}\partial_{x_1} T_{j,k}\sigma_\infty) \\
&\quad + \alpha_0 \nabla T_{j,k}\sigma_\infty + v_s^1 w^{(0)}\partial_{x_1} T_{j,k}\sigma_\infty - \langle Q_0 \tilde{B}(T_{j,k}\sigma_\infty u^{(0)} + T_{j,k} u_\infty)\rangle_\infty w^{(0)}\big\}.
\end{aligned}
$$

Applying (7.31) and noting that

$$
\begin{aligned}
\|\partial_x^{l+1}\big(\tfrac{P'(\rho_s)}{\gamma^2} T_{j,k}\phi_\infty\big)\|_2^2 &\geq \|\tfrac{P'(\rho_s)}{\gamma^2}\partial_{x_2}^{l+1} T_{j,k}\phi_\infty\|_2^2 - \|[\partial_{x_2}^{l+1}, \tfrac{P'(\rho_s)}{\gamma^2}] T_{j,k}\phi_\infty\|_2^2 \\
&\geq \|\partial_{x_2}^{l+1} T_{j,k}\phi_\infty\|_2^2 - C\omega_0^2 \|T_{j,k}\phi_\infty\|_{H^{l+1}}^2 \\
&\geq \tfrac{1}{2}\|\partial_{x_2}^{l+1} T_{j,k}\phi_\infty\|_2^2 - C\omega_0^2 \|T_{j,k}\phi_\infty\|_{H^l}^2,
\end{aligned}
$$

we can obtain the desired estimate. □

We finally estimate the time derivatives of σ_∞ and ϕ_∞.

Proposition 7.10. (i) *If $0 \leq 2j \leq m-1$, then there holds the following estimate:*

$$
\begin{aligned}
&\|\partial_t^{j+1}\sigma_\infty\|_2^2 \\
&\quad \leq C\big\{R_{j,k}^{(7)} + \|\partial_{x_1}\partial_t^j \sigma_\infty\|_2^2 + \|\partial_{x_1}\partial_t^j \phi_\infty\|_2^2 + \gamma^4 \|\partial_{x_1}\partial_t^j w_\infty\|_2^2\big\}.
\end{aligned} \tag{7.32}
$$

Here $R_j^{(7)} = \|\langle Q_0 \partial_t^j P_\infty^{(0)} \boldsymbol{F}\rangle_\infty\|_2^2$.

(ii) *If $0 \leq 2j \leq m-1$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq 1$, then there holds the following estimate:*

$$
\begin{aligned}
&\|\partial_t^{j+1}\phi_\infty\|_{H^{m-1-2j}}^2 \\
&\quad \leq C\big\{R_j^{(8)} + \|\partial_{x_1}\partial_t^j \phi_\infty\|_{H^{m-1-2j}}^2 + \gamma^4 \|\partial_x \partial_t^j w_\infty\|_{H^{m-1-2j}}^2 + \|\partial_{x_1}\partial_t^j \sigma_\infty\|_2^2\big\}.
\end{aligned} \tag{7.33}
$$

Here $R_j^{(8)} = \|\partial_t^j Q_0 \boldsymbol{F}_\infty\|_{H^{m-1-2j}}^2$.

Outline of Proof. The estimates (7.32) and (7.33) follows from (5.3) and the first equation of (5.4)

$$
\partial_t \sigma_\infty = \langle Q_0 P_\infty^{(0)} \boldsymbol{F}\rangle_\infty - \langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle_\infty
$$

and

$$
\begin{aligned}
\partial_t \phi_\infty &= f_\infty^0 - \{v_s^1 \partial_{x_1}\phi_\infty + \gamma^2 \mathrm{div}\,(\rho_s w_\infty) + (v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1})\partial_{x_1}\sigma_\infty \\
&\quad - \langle Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle_\infty \phi^{(0)}\}.
\end{aligned}
$$

□

Proposition 7.1 now follows from combining Propositions 7.4–7.10.

Outline of Proof of Proposition 7.1. We proceed as in [8, Section 5.2]. We define

$$\tilde{E}^{(0)}(t) = \sum_{\substack{2j+k\le m\\ 2j\neq m}} E^{(0)}[T_{j,k}\tilde{P}_\infty u(t)], \quad E^{(1)}(t) = \sum_{2j+k\le m-1} E^{(1)}[T_{j,k}u(t)]$$

$$E^{(2)}(t) = \sum_{2j+k\le m-1} \tfrac{1}{\gamma^2}\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\partial_{x_2}T_{j,k}\phi_\infty(t)\|_2^2,$$

$$E^{(3)}(t) = \sum_{2j+k\le m-1} \frac{\nu}{\gamma^2(\nu+\tilde{\nu})}\|T_{j,k}\sigma_\infty\|_2^2$$

and

$$\tilde{D}^{(0)}(t) = \sum_{\substack{2j+k\le m\\ 2j\neq m}} D^{(0)}[T_{j,k}w_\infty(t)],$$

$$D^{(1)}(t) = \sum_{2j+k\le m-1}\left(\tfrac{2b_1\gamma^2}{\nu(\nu+\tilde{\nu})}D^{(0)}[T_{j,k}w_\infty(t)] + \tfrac{1}{\nu+\tilde{\nu}}\|\sqrt{\rho_s}\partial_t T_{j,k}w_\infty(t)\|_2^2\right),$$

$$D^{(2)}(t) = \sum_{2j+k\le m-1}\left(\tfrac{1}{\nu+\tilde{\nu}}\|\tfrac{P'(\rho_s)}{\gamma^2}\partial_{x_2}T_{j,k}\phi_\infty(t)\|_2^2 + \tfrac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty\|_{H^1}^2\right),$$

$$D^{(3)}(t) = \sum_{2j+k\le m-1}\tfrac{\alpha_1}{\nu+\tilde{\nu}}\|\partial_{x'}T_{j,k}\sigma_\infty u\|_2^2.$$

Let b_j $(j=2,\cdots,6)$ be positive numbers and consider

$$\begin{aligned}&\sum_{\substack{2j+k\le m\\ 2j\neq m}}\big\{(7.2) + b_2\times(7.24)\big\}\\ &\quad+\sum_{2j+k\le m-1}\big\{\tfrac{1}{\nu+\tilde{\nu}}\times(7.6) + b_2\times(7.23)_{l=0} + b_3\times(7.25) + b_4\times(7.30)_{l=0}\big\}\\ &\quad+\sum_{2j\le m-2}\tfrac{\nu}{\gamma^4}b_5\times\big\{(7.32)+(7.33)\big\} + \tfrac{b_6\nu}{\gamma^2}\times(7.2)_{2j=m}.\end{aligned}$$

Taking $\frac{1}{\nu}$, $\frac{\nu+\tilde{\nu}}{\gamma^2}$, ω_0 and b_j $(j=2,\cdots,6)$ are suitably small, we can obtain

$$\begin{aligned}&\tfrac{d}{dt}E^{(4)}(t) + \tfrac{1}{2}D^{(4)}(t) + \tfrac{\nu b_5}{\gamma^4}\sum_{2j\le m-2}(\|\partial_t^{j+1}\sigma_\infty(t)\|_2^2 + \|\partial_t^{j+1}\phi_\infty(t)\|_2^2)\\ &\quad\le C\sum_{j=1}^{8}R^{(j)}(t).\end{aligned}\tag{7.34}$$

Here

$$E^{(4)}(t) = \tilde{E}^{(0)}(t) + \frac{1}{\nu+\tilde{\nu}}E^{(1)}(t) + b_2E^{(2)}(t) + b_3E^{(3)}(t) + \frac{b_6\nu}{\gamma^2}E^{(0)}[\partial_t^{[\frac{m}{2}]}\tilde{P}_\infty u(t)],$$

$$\begin{aligned}D^{(4)}(t) &= \tfrac{1}{2}(\tilde{D}^{(0)}(t) + D^{(1)}(t) + b_2D^{(2)}(t) + b_3D^{(3)}(t))\\ &\quad+b_4\sum_{2j+k\le m-1}\big(\tfrac{\nu^2}{\nu+\tilde{\nu}}\|\partial_x^2T_{j,k}w_\infty\|_2^2 + \tfrac{1}{\nu+\tilde{\nu}}\|\partial_xT_{j,k}\phi_\infty\|_2^2\big)\\ &\quad+\tfrac{b_6\nu}{\gamma^2}D^{(0)}[\partial_t^{[\frac{m}{2}]}w_\infty(t)]\end{aligned}$$

and

$$R^{(1)}=\sum_{2j+k\le m}R^{(1)}_{j,k},\ \ R^{(p)}=\sum_{2j+k\le m-1}R^{(p)}_{j,k}\ \ (p=2,5),\ \ R^{(4)}=\sum_{\substack{2j+k\le m\\ 2j\ne m}}R^{(4)}_{j,k},$$

$$R^{(p)}=\sum_{2j+k+l\le m-1}R^{(p)}_{j,k,l}\ \ (p=3,6),\quad R^{(p)}=\sum_{2j\le m-1}R^{(8)}_{j}\ \ (p=7,8).$$

To establish the desired estimate (7.1) we need to estimate higher order derivatives in x_2. For $1\le l\le m-1$, we set

$$E^{(4)}_l(t)=\sum_{2j+k\le m-1-l}\frac{1}{\gamma^2}\Big\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\partial^{l+1}_{x_2}\phi_\infty(t)\Big\|_2^2$$

and

$$\begin{aligned}D^{(4)}_l(t)\ \ =\ \ &\tfrac12\sum_{2j+k\le m-1-l}\Big(\tfrac{1}{\nu+\tilde\nu}\Big\|\tfrac{P'(\rho_s)}{\gamma^2}T_{j,k}\partial^{l+1}_{x_2}\phi_\infty(t)\Big\|_2^2+\tfrac{c_0(\nu+\tilde\nu)}{\gamma^4}\|T_{j,k}\partial^{l+1}_{x_2}\dot\phi_\infty(t)\|_2^2\Big)\\ &+\tfrac{b_7}{2}\sum_{2j+k\le m-1-l}\Big(\tfrac{\nu^2}{\nu+\tilde\nu}\Big\|\partial^{l+2}_{x}T_{j,k}w_\infty(t)\Big\|_2^2+\tfrac{1}{\nu+\tilde\nu}\|\partial^{l+1}_{x}T_{j,k}\phi_\infty(t)\|_2^2\Big).\end{aligned}$$

Here b_7 is a positive number. Using (7.23), (7.30) with $1\le l\le m-1$ and (7.34), one can find small numbers b_7 and b_8 by induction on l in such a way that the following estimate holds for $1\le l\le m-1$:

$$\begin{aligned}&\tfrac{d}{dt}\Big(2^lE^{(4)}(t)+\textstyle\sum_{p=1}^{l}2^{l+1-p}b_8^pE^{(4)}_p(t)\Big)+\sum_{p=0}^{l}b_8^pD^{(4)}_p(t)\\ &\quad\le\Big(\textstyle\sum_{p=0}^{l}2^p\Big)C_1\sum_{j=1}^{8}R^{(j)}(t),\end{aligned}\tag{7.36}$$

where $D^{(4)}_0(t)=D^{(4)}(t)$. See [8, Section 5.2] for the details. The desired estimate (7.1) now follows from (7.36) with $l=m-1$; and Proposition 7.1 is proved. □

To prove (5.15) we employ the following lemma.

Lemma 7.11. *There exists a positive number $\tilde r_0=\tilde r_0(\nu,\tilde\nu,\gamma)$ such that if $r_1\le\tilde r_0$, then there holds the estimate*

$$\|\varPi^{(0)}P_{\infty,1}u\|_2\le C\|\partial_{x_1}(I-\varPi^{(0)})P_{\infty,1}u\|_2.$$

Proof. We set $\tilde\varPi^{(1)}(\xi)=\xi\hat\varPi^{(1)}+\xi^2\hat\varPi^{(2)}(\xi)$. Since

$$\varPi^{(0)}(I-\hat\varPi(\xi))=\varPi^{(0)}(I-\varPi^{(0)}-\tilde\varPi^{(1)}(\xi))=-\varPi^{(0)}\tilde\varPi^{(1)}(\xi),$$

we see that

$$\widehat{\varPi^{(0)}P_{\infty,1}u}=\varPi^{(0)}\hat\chi_1(\xi)(I-\hat\varPi(\xi))\hat u=-\hat\chi_1(\xi)\varPi^{(0)}\tilde\varPi^{(1)}(\xi)\hat u.$$

It then follows that

$$\begin{aligned} |\widehat{\Pi^{(0)}P_{\infty,1}u}|_2 &\le C|\xi||\hat{\chi}_1\hat{u}|_2 \\ &\le C|\xi|\big(\hat{\chi}_1|(I-\Pi^{(0)})\hat{u}|_2 + \hat{\chi}_1|\Pi^{(0)}\hat{u}|_2\big). \end{aligned}$$

Since $(P_{\infty,1})^2 = P_{\infty,1}$, we find that

$$|\widehat{\Pi^{(0)}P_{\infty,1}u}|_2 \le C|\xi|\big(|(I-\Pi^{(0)})\widehat{P_{\infty,1}u}|_2 + |\Pi^{(0)}\widehat{P_{\infty,1}u}|_2\big)$$

for $|\xi| \le r_1$. Therefore, there exists a positive number $\tilde{r}_0$ such that if $r_1 \le \tilde{r}_0$, then

$$|\widehat{\Pi^{(0)}P_{\infty,1}u}|_2 \le C|\xi||(I-\Pi^{(0)})\widehat{P_{\infty,1}u}|_2$$

for $|\xi| \le r_1$, from which we obtain

$$\|\Pi^{(0)}P_{\infty,1}u\|_2 \le C\|\partial_{x_1}(I-\Pi^{(0)})P_{\infty,1}u\|_2.$$

This completes the proof. □

We now prove (5.15).

Proof of (5.15). We fix ν, $\tilde{\nu}$, γ and ω_0 so that Proposition 7.1 holds true; and set $r_1 = \min\{r_0, \tilde{r}_0, 1\}$. Then, obviously,

$$\|\Pi^{(0)}P_{\infty,2}u\|_2 \le \frac{1}{r_1}\|\partial_{x_1}\Pi^{(0)}P_{\infty,2}u\|_2.$$

By Lemma 7.11 we also have

$$\|\Pi^{(0)}P_{\infty,1}u\|_2 \le C\|\partial_{x_1}(I-\Pi^{(0)})P_{\infty,1}u\|_2.$$

We thus obtain

$$\begin{aligned} \|\sigma_\infty\|_2 &\le C\{\|\partial_{x_1}(I-\Pi^{(0)})P_{\infty,1}u\|_2 + \tfrac{1}{r_1}\|\partial_{x_1}\Pi^{(0)}P_{\infty,2}u\|_2\} \\ &\le C(\|\partial_{x_1}\sigma_\infty\|_2 + \|\partial_{x_1}u_\infty\|_2). \end{aligned} \tag{7.37}$$

By Lemma 5.2 (ii) we have

$$\|u_\infty\|_2 \le C\|\partial_x u_\infty\|_2. \tag{7.38}$$

It follows from (7.37) and (7.38) that

$$D(t) \ge \tilde{a}\tilde{E}(t) \tag{7.39}$$

for a constant $\tilde{a} > 0$. We thus deduce from (7.1) and (7.39) that

$$\frac{d}{dt}\tilde{E}(t) + \tilde{a}\tilde{E}(t) + D(t) \leq \tilde{R}(t). \tag{7.40}$$

This gives

$$\tilde{E}(t) + \int_0^t e^{-\tilde{a}(t-\tau)} D(\tau)\, d\tau \leq e^{-\tilde{a}t}\tilde{E}(0) + \int_0^t e^{-\tilde{a}(t-\tau)}\tilde{R}(\tau)\, d\tau. \tag{7.41}$$

Writing the second and third equation of (5.4) as

$$\begin{aligned}
-\nu\partial_{x_2}^2 w_\infty^1 \;&=\; \rho_s f_\infty^1 - \rho_s\big\{\partial_t w_\infty^1 - \tfrac{\nu}{\rho_s}\partial_{x_1}^2 w_\infty^1 - \tfrac{\tilde{\nu}}{\rho_s}\partial_{x_1}\mathrm{div}\, w_\infty + \partial_{x_1}\big(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi_\infty\big) \\
&\quad + v_s^1\partial_{x_1} w_\infty^1 + \tfrac{\nu\partial_{x_2}^2 v_s^1}{\gamma^2\rho_s^2}\phi_\infty + (\partial_{x_2} v_s^1)\partial_{x_1} w_\infty^2 \\
&\quad - \tfrac{\nu+\tilde{\nu}}{\rho_s} w^{(0),1}\partial_{x_1}^2\sigma_\infty + \alpha_0\partial_{x_1}\sigma_\infty + v_s^1 w^{(0),1}\partial_{x_1}\sigma_\infty \\
&\quad - \langle Q_0\tilde{B}(\sigma_\infty u^{(0)} + u_\infty)\rangle_\infty w^{(0),1}\big\}, \\
-(\nu+\tilde{\nu})\partial_{x_2}^2 w_\infty^2 \;&=\; \rho_s f_\infty^2 - \rho_s\big\{\partial_t w_\infty^2 - \tfrac{\nu}{\rho_s}\partial_{x_1}^2 w_\infty^2 - \tfrac{\tilde{\nu}}{\rho_s}\partial_{x_2}\partial_{x_1} w_\infty^1 \\
&\quad + \partial_{x_2}\big(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi_\infty\big) + v_s^1\partial_{x_1} w_\infty^2 - \tfrac{\tilde{\nu}}{\rho_s}(\partial_{x_2} w^{(0),1})\partial_{x_1}\sigma_\infty\big\},
\end{aligned}$$

we can obtain

$$\frac{\nu^2}{\nu+\tilde{\nu}}[\![\partial_{x_2}^2 w_\infty(t)]\!]_{m-2}^2 \leq C\tilde{E}(t) + CR^{(9)}(t), \tag{7.42}$$

where $R^{(9)}(t) = [\![\tilde{Q}\boldsymbol{F}_\infty(t)]\!]_{m-2}^2$. It then follows from (7.41) and (7.42) that

$$\begin{aligned}
&\tilde{E}(t) + \frac{\nu^2}{\nu+\tilde{\nu}}[\![\partial_{x_2}^2 w_\infty(t)]\!]_{m-2}^2 + \int_0^t e^{-\tilde{a}(t-\tau)} D(\tau)\, d\tau \\
&\quad\leq\; C\big\{e^{-\tilde{a}t}\tilde{E}(0) + R^{(9)}(t) + \int_0^t e^{-\tilde{a}(t-\tau)}\tilde{R}(\tau)\, d\tau\big\}.
\end{aligned} \tag{7.43}$$

Since

$$R^{(9)}(t) \leq C(1+t)^{-\frac{3}{2}} M(t)^4 \tag{7.44}$$

as we will see in Proposition 8.6 below, we deduce (5.15) from (7.43) and (7.44). This completes the proof. □

8. Estimates on the nonlinearities

In this section we establish the estimates on the nonlinearities, namely, (5.16) and (7.44)

Proposition 8.1. *There exists a number $\varepsilon_6 > 0$ such that the following assertion holds true.*

Let $u(t)$ be a solution of (4.1) *in $Z^m(T)$. If $\sup_{0\le\tau\le t}[\![u(\tau)]\!]_m \le \varepsilon_6$ and $M(t) \le 1$ for all $t \in [0,T]$, then the following estimates hold for all $t \in [0,T]$ with $C > 0$ independent of T.*

(i)
$$[\![\tilde{Q}\boldsymbol{F}_\infty(t)]\!]_{m-2} \le C(1+t)^{-\frac{3}{4}}M(t)^2,$$

(ii)
$$\tilde{R}(t) \le C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\}.$$

To prove Proposition 8.1 we employ the following inequalities.

Lemma 8.2. (i) *Let $2 \le p \le \infty$ and let j and k be integers satisfying*

$$0 \le j < k, \quad k > j + n\Big(\frac{1}{2} - \frac{1}{p}\Big).$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^j f\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{L^2(\mathbf{R}^n)}^{1-a}\|\partial_x^k f\|_{L^2(\mathbf{R}^n)}^a,$$

where $a = \frac{1}{k}(j + \frac{n}{2} - \frac{n}{p})$.

(ii) *Let $2 \le p \le \infty$ and let j and k be integers satisfying*

$$0 \le j < k, \quad k > j + 2\Big(\frac{1}{2} - \frac{1}{p}\Big).$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^j f\|_p \le C\|f\|_{H^k}.$$

(iii) *If $f \in H^1$ and $f = f(x_1)$ is independent of x_2. Then*

$$\|f\|_\infty \le C\|f\|_2^{\frac{1}{2}}\|\partial_{x_1} f\|_2^{\frac{1}{2}}.$$

Proof. The inequality in (i) is a special case of the Gagliardo-Nirenberg-Sobolev inequality which can be proved by using Fourier transform. Inequality in (ii) can be obtained by (i) with $n = 2$ and the standard extension argument. As for (iii), we note that if $f = f(x_1)$, then

$$\|f\|_p = \|f\|_{L^p(\mathbf{R})} \ (1 \le p \le \infty), \quad \|\partial_{x_1} f\|_2 = \|\partial_{x_1} f\|_{L^2(\mathbf{R})}.$$

Therefore, the inequality in (iii) is a simple consequence of (i) with $n=1$, $p=\infty$, $j=0$ and $k=1$. $\square$

Lemma 8.3. (i) *Let m and m_k $(k=1,\cdots,\ell)$ be nonnegative integers and let α_k $(k=1,\cdots,\ell)$ be multi-indices. Suppose that*

$$m \ge [\tfrac{n}{2}]+1, \quad 0 \le |\alpha_k| \le m_k \le m+|\alpha_k| \quad (k=1,\cdots,\ell)$$

and

$$m_1+\cdots+m_\ell \ge (\ell-1)m+|\alpha_1|+\cdots+|\alpha_\ell|.$$

Then there exists a constant $C>0$ such that

$$\|\partial_x^{\alpha_1} f_1 \cdots \partial_x^{\alpha_\ell} f_\ell\|_2 \le C \prod_{1\le k\le \ell} \|f_k\|_{H^{m_k}}.$$

(ii) *Let $1 \le k \le m$. Suppose that $F(x,t,y)$ is a smooth function on $\Omega \times [0,\infty) \times I$, where I is a compact interval in $\mathbf{R}$. Then for $|\alpha|+2j=k$ there hold*

$$\begin{aligned} &\left\| \left[\partial_x^\alpha \partial_t^j, F(x,t,f_1)\right] f_2 \right\|_2 \\ &\le \left\{ \begin{array}{l} C_0(t,f_1(t))[\![f_2]\!]_{k-1} + C_1(t,f_1(t))\{1+|||Df_1|||_{m-1}^{|\alpha|+j-1}\}|||Df_1|||_{m-1}[\![f_2]\!]_k, \\ C_0(t,f_1(t))[\![f_2]\!]_{k-1} + C_1(t,f_1(t))\{1+|||Df_1|||_{m-1}^{|\alpha|+j-1}\}|||Df_1|||_{m}[\![f_2]\!]_{k-1}. \end{array} \right. \end{aligned}$$

Here

$$C_0(t,f_1(t)) = \sum\nolimits_{\substack{(\beta,l)\le(\alpha,j) \\ (\beta,l)\ne(0,0)}} \sup_x |(\partial_x^\beta \partial_t^l F)(x,t,f_1(x,t))|$$

and

$$C_1(t,f_1(t)) = \sum\nolimits_{\substack{(\beta,l)\le(\alpha,j) \\ 1\le p\le j+|\alpha|}} \sup_x |(\partial_x^\beta \partial_t^l \partial_y^p F)(x,t,f_1(x,t))|.$$

The proof of Lemma 8.3 can be found in [10, 11].

We recall that $u(t)$ is decomposed into

$$u(t) = \sigma_1 u^{(0)} + u_1 + \sigma_\infty u^{(0)} + u_\infty;$$

and we write

$$\sigma_* = \sigma_1 + \sigma_\infty, \quad \phi_* = \phi_1 + \phi_\infty, \quad w_* = w_1 + w_\infty,$$

$$u_* = {}^\top(\phi_*, w_*) \,(= u_1 + u_\infty)$$

as in section 7.

Before estimating the nonlinearities we make a simple observation.

Lemma 8.4. *Let* $u(t) = {}^\top(\phi(t), w(t)) = (\sigma_* u^{(0)})(t) + u_*(t)$ *be a solution of* (4.1) *in* $Z^m(T)$. *Then there hold the following estimates for all* $t \in [0, T]$ *with* $C > 0$ *independent of* T.

(i) $$\|\sigma_*(t)\|_2 \le C(1+t)^{-\frac{1}{4}} M(t),$$

(ii) $$|||D\sigma_*(t)|||_{m-1} + [\![u_*(t)]\!]_m \le C(1+t)^{-\frac{3}{4}} M(t),$$

(iii) $$[\![\phi(t)]\!]_m + [\![w(t)]\!]_m \le C(1+t)^{-\frac{1}{4}} M(t),$$

(iv) $$\|\sigma_*(t)\|_\infty \le C(1+t)^{-\frac{1}{2}} M(t),$$

(v) $$\|\phi(t)\|_\infty + \|w(t)\|_\infty \le C(1+t)^{-\frac{1}{2}} M(t).$$

Proof. Estimates (i), (ii) and (iii) immediately follow from the definition of $M(t)$. As for (iv), we see from Lemma 8.2 (iii) that

$$\|\sigma_*(t)\|_\infty \le C\|\sigma_*(t)\|_2^{\frac{1}{2}} \|\partial_{x_1}\sigma_*(t)\|_2^{\frac{1}{2}} \le C(1+t)^{-\frac{1}{2}} M(t).$$

This shows (iv). Estimate (v) now follows from (iv) and (ii). This completes the proof. □

Let us consider estimates on $Q_0\boldsymbol{F}$.

Proposition 8.5. *Let* $u(t)$ *be a solution of* (4.1) *in* $Z^m(T)$ *and assume that* $M(t) \le 1$ *for all* $t \in [0, T]$. *Then the following estimates hold with* $C > 0$ *independent of* T.

(i)

$$[\![\phi \operatorname{div} w]\!]_l \le C \begin{cases} (1+t)^{-1} M(t)^2 + (1+t)^{-\frac{1}{4}} M(t) |||Dw_\infty(t)|||_m & (l = m), \\ (1+t)^{-1} M(t)^2 & (l = m-1), \end{cases}$$

(ii) $$[\![w \cdot \nabla(\sigma_* \phi^{(0)} + \phi_1)]\!]_m \le C(1+t)^{-1} M(t)^2,$$

(iii) $$[\![w\cdot\nabla\phi_\infty]\!]_{m-1}\le C(1+t)^{-1}M(t)^2,$$

(iv) $$\left|\left(\mathrm{div}\,\left(\frac{P'(\rho_s)}{\gamma^4\rho_s}w\right),|\partial_t^j\partial_x^k\phi_\infty|^2\right)\right|\le C(1+t)^{-\frac{1}{2}}M(t)D_\infty(t)$$

for $2j+k=m$,

(v) $$\|[\partial_t^j\partial_x^k,w]\cdot\nabla\phi_\infty\|_2\le C(1+t)^{-\frac{1}{4}}M(t)\sqrt{D_\infty(t)}$$

for $2j+k=m$,

(vi) $$\|\partial_t^j(\phi w)\|_2\le(1+t)^{-1}M(t)^2$$

for $1\le j\le[\frac{m}{2}]$.

Proof. We write $\phi\mathrm{div}\,w$ as

$$\begin{aligned}\phi\mathrm{div}\,w &= \sigma_*\phi^{(0)}w^{(0),1}\partial_{x_1}\sigma_*+\sigma_*\phi^{(0)}\mathrm{div}\,w_*+\phi_*w^{(0),1}\partial_{x_1}\sigma_*\\ &\quad+\phi_*\mathrm{div}\,w_*.\end{aligned}\tag{8.1}$$

By Lemma 4.5 and Lemma 5.2, we have $[\![\partial_{x_1}\sigma_*(t)]\!]_m\le C|||D\sigma_*(t)|||_{m-1}$. So applying Lemma 8.3 to each term on the right-hand side of (8.1) and using Lemma 8.4, we obtain estimate (i). For example, let $2j+k\le m$. Then

$$\begin{aligned}&\|\partial_t^j\partial_x^k(\sigma_*\phi^{(0)}w^{(0),1}\partial_{x_1}\sigma_*)\|_2\\ &\quad\le\|\sigma_*\phi^{(0)}w^{(0),1}\partial_t^j\partial_x^k\partial_{x_1}\sigma_*\|_2+\|[\partial_t^j\partial_x^k,\sigma_*\phi^{(0)}w^{(0),1}]\partial_{x_1}\sigma_*\|_2\\ &\quad\le C\{\|\sigma_*\|_\infty[\![\partial_{x_1}\sigma_*]\!]_m+\|\sigma_*\|_\infty[\![\partial_{x_1}\sigma_*]\!]_{m-1}+|||D\sigma_*|||_{m-1}[\![\partial_{x_1}\sigma_*]\!]_m\}\\ &\quad\le C(\|\sigma_*\|_\infty+|||D\sigma_*|||_{m-1})|||D\sigma_*|||_{m-1}\\ &\quad\le C(1+t)^{-\frac{5}{4}}M(t)^2.\end{aligned}$$

As for (ii), we write

$$\begin{aligned}w\cdot\nabla(\sigma_*\phi^{(0)}+\phi_1) &= \sigma_*\phi^{(0)}w^{(0),1}\partial_{x_1}\sigma_*+w_*^1\partial_{x_1}\sigma_*\phi^{(0)}+w_*^n\sigma_1\partial_{x_2}\phi^{(0)}\\ &\quad+\sigma_*w^{(0),1}\partial_{x_1}\phi_1+w_*\cdot\nabla\phi_1.\end{aligned}\tag{8.2}$$

The first three terms on the right of (8.2) can be estimated similarly to the right of (8.1). As for the last two terms on the right of (8.2), we employ the estimate $[\![\nabla\phi_1]\!]_m \le C[\![u_1]\!]_m$, which follows from Lemma 4.5. Using this inequality and Lemma 8.3 and Lemma 8.4, we have, for $2j+k\le m$,

$$\begin{aligned}\|\partial_t^j\partial_x^k(w_*\cdot\nabla\phi_1)\|_2 &\le \|w_*\partial_t^j\partial_x^k\nabla\phi_1\|_2+\|[\partial_t^j\partial_x^k,w_*]\cdot\nabla\phi_1\|_2\\ &\le C\{\|w_*\|_\infty[\![\nabla\phi_1]\!]_m+|||Dw_*|||_{m-1}[\![\nabla\phi_1]\!]_m\}\\ &\le C[\![w_*]\!]_m[\![\nabla\phi_1]\!]_m\\ &\le C(1+t)^{-\frac{3}{2}}M(t)^2.\end{aligned}$$

Similarly we can obtain

$$\|\partial_t^j\partial_x^k(\sigma_* w^{(0),1}\partial_{x_1}\phi_1)\|_2\le C(1+t)^{-\frac{5}{4}}M(t)^2.$$

Estimate (iii) can be obtained by applying Lemma 8.3 and Lemma 8.4 to

$$w\cdot\nabla\phi_\infty=\sigma_* w^{(0),1}\partial_{x_1}\phi_\infty+w_*\cdot\nabla\phi_\infty.$$

As for (iv), since $j\ge 1$ or $k\ge 1$, by the Hölder and Sobolev inequalities, we have

$$\begin{aligned}&\left|\left(\mathrm{div}\left(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\right),|\partial_t^j\partial_x^k\phi_\infty|^2\right)\right|\\ &\qquad\le C\{\|w\|_\infty+\|\partial_x w\|_\infty\}|||D\phi_\infty|||_{m-1}^2\\ &\le C\{\|w\|_\infty+|||Dw_\infty|||_m\}|||D\phi_\infty|||_{m-1}^2\\ &\le C(1+t)^{-\frac{1}{2}}M(t)D_\infty(t).\end{aligned}$$

Estimate (v) is obtained by Lemma 8.3 and Lemma 8.4:

$$\|[\partial_t^j\partial_x^k,w]\cdot\nabla\phi_\infty\|_2\le C|||Dw|||_m[\![\partial_x\phi_\infty]\!]_{m-1}\le C(1+t)^{-\frac{1}{2}}M(t)\sqrt{D_\infty(t)}.$$

This completes the proof. □

We next consider $\tilde{Q}\boldsymbol{F}={}^\top(0,\boldsymbol{f})$. We first observe that since $m\ge 2(=[n/2]+1)$ we have the Sobolev inequality

$$\|f\|_\infty\le C_S\|f\|_{H^m}.$$

Let $\varepsilon_6>0$ be a number such that $C_S\varepsilon_6\le\frac{\gamma^2\rho_1}{2}$. Then whenever $[\![u(t)]\!]_m\le\varepsilon_6$, we have

$$\|\phi(t)\|_\infty\le C_S[\![u(t)]\!]_m\le C_S\varepsilon_6\le\frac{\gamma^2\rho_1}{4},$$

and hence,

$$\rho(x,t)=\rho_s(x_n)+\gamma^{-2}\phi(x,t)\ge\rho_1-\gamma^{-2}\|\phi(t)\|_\infty\ge\frac{3\rho_1}{4}(>0).$$

We thus see that $\tilde{Q}\boldsymbol{F}(t)$ is smooth whenever $[\![u(t)]\!]_m \le \varepsilon_6$. So, we assume that $[\![u(t)]\!]_m \le \varepsilon_6$ for $t \in [0,T]$.

Recall that $\tilde{Q}\boldsymbol{F}$ is written in the form

$$\tilde{Q}\boldsymbol{F} = \tilde{\boldsymbol{F}}_0 + \tilde{\boldsymbol{F}}_1 + \tilde{\boldsymbol{F}}_2 + \tilde{\boldsymbol{F}}_3.$$

Here $\tilde{\boldsymbol{F}}_l = {}^\top(0, \boldsymbol{f}_l)$ $(l = 0,1,2,3)$, where

$$\begin{aligned}
\boldsymbol{f}_0 &= -w\cdot\nabla w + f_1(\rho_s,\phi)(-\partial_{x_1}^2\sigma_* w^{(0)} + \tfrac{\partial_{x_2}^2 v_s}{\gamma^2\rho_s}\phi_*) \\
&\quad + f_2(\rho_s,\phi)(-\partial_{x_1}^2\sigma_* w^{(0)} - \partial_{x_1}\sigma_*\partial_{x_2}w^{(0)}) \\
&\quad + \boldsymbol{f}_{01}(x_2,\phi)\phi\sigma_* + \boldsymbol{f}_{02}(x_2,\phi)\phi\partial_{x_1}\sigma_* + \boldsymbol{f}_{03}(x_2,\phi)\phi\phi_*,
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{f}_1 &= -\mathrm{div}\,(f_1(\rho_s,\phi)\nabla w_*) + {}^\top(\nabla w_*)\nabla(f_1(\rho_s,\phi)), \\
\boldsymbol{f}_2 &= -\nabla(f_2(\rho_s,\phi)\mathrm{div}\,w_*) + (\mathrm{div}\,w_*)\nabla(f_2(\rho_s,\phi)), \\
\boldsymbol{f}_3 &= \nabla(f_3(x_2,\phi)\phi\phi_*) - \phi_*\nabla(f_3(x_2,\phi)\phi).
\end{aligned}$$

Here $f_1 = \frac{\nu\phi}{(\phi+\gamma^2\rho_s)\rho_s}$; $f_2 = \frac{\tilde{\nu}\phi}{(\phi+\gamma^2\rho_s)\rho_s}$; and $\boldsymbol{f}_{0l}(x_2,\phi)$ $(l = 1,2,3)$ and $f_3(x_2,\phi)$ are smooth functions of x_2 and ϕ.

Proposition 8.6. *Let $u(t)$ be a solution of (4.1) in $Z^m(T)$ and assume that $\sup_{0\le\tau\le t}[\![u(\tau)]\!]_m \le \varepsilon_6$ and $M(t) \le 1$ for all $t \in [0,T]$. Then the following estimates hold with $C > 0$ independent of T.*

(i) $$[\![\tilde{Q}\boldsymbol{F}]\!]_{m-2} \le C(1+t)^{-\frac{3}{4}}M(t)^2,$$

(ii) $$[\![\boldsymbol{f}_0]\!]_m \le C\{(1+t)^{-\frac{3}{4}}M(t)^2 + (1+t)^{-\frac{3}{4}}M(t)|||Dw_\infty(t)|||_m\},$$

(iii) $$\sum_{l=1}^{3}[\![\boldsymbol{f}_l]\!]_{m-1} \le C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\},$$

(iv) $$\sum_{l=1}^{3}\|T_{j,k}\boldsymbol{f}_l\|_{H^{-1}} \le C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}$$

for $2j+k = m$. Here we regard $T_{j,k}\boldsymbol{f}_l$ with $2j+k = m$ as an element in H^{-1} by $(T_{j,k}\boldsymbol{f}_l)[v] = \langle\langle T_{j,k}\boldsymbol{f}_l, v\rangle\rangle_{-1}$ for $v \in H_0^1$.

Proof. Estimates (i), (ii), (iii) can be obtained similarly to the proof of Proposition 8.5. So we omit the proof.

Let us prove estimate (iv). Let $2j+k=m$ and let $v \in H_0^1$. If $k \geq 1$, then we see from (iii) that

$$
\begin{aligned}
|\langle\langle T_{j,k}\boldsymbol{f}_l, v\rangle\rangle_{-1}| &= |-(T_{j,k-1}\boldsymbol{f}_l, T_{0,1}v)| \\
&\leq \|T_{j,k-1}\boldsymbol{f}_l\|_2 \|T_{0,1}v\|_2 \\
&\leq C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H_0^1}.
\end{aligned}
$$

We thus obtain

$$\|T_{j,k}\boldsymbol{f}_l\|_{H^{-1}} \leq C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H_0^1}.$$

If $k=0$, i.e., $m=2j$, then, by definition,

$$
\begin{aligned}
&\langle\langle \partial_t^j \boldsymbol{f}_1, v\rangle\rangle_{-1} \\
&= (\partial_t^j(f_1(\rho_s,\phi)\nabla w_*), \nabla v) + (\partial_t^j({}^\top(\nabla w_*)\partial_{\rho_s} f_1(\rho_s,\phi)\nabla\rho_s), v) \\
&+ (\partial_t^j({}^\top(\nabla w_*)\partial_\phi f_1(\rho_s,\phi)\nabla\phi), v) \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}
$$

As for I_1, we have

$$|I_1| \leq \|\partial_t^j(f_1(\rho_s,\phi)\nabla w_*)\|_2 \|\nabla v\|_2.$$

As in the proof of Proposition 8.5 one can estimate $\|\partial_t^j(f_1(\rho_s,\phi)\nabla w_*)\|_2$ to obtain

$$|I_1| \leq C\{(1+t)^{-\frac{5}{4}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H_0^1}.$$

Similarly we have

$$|I_2| \leq C\{(1+t)^{-\frac{5}{4}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H_0^1}.$$

We next consider I_3 which we write as follows:

$$
\begin{aligned}
I_3 &= (\partial_\phi f_1(\rho_s,\phi)\partial_t^j({}^\top(\nabla w_*)\nabla\phi), v) + ([\partial_t^j, \partial_\phi f_1(\rho_s,\phi)]({}^\top(\nabla w_*)\nabla\phi), v) \\
&\equiv J_1 + J_2.
\end{aligned}
$$

As for J_1, we have

$$
\begin{aligned}
|J_1| &\leq |(\partial_\phi f_1(\rho_s,\phi)^\top(\nabla\partial_t^j w_*)\nabla\phi, v)| + |(\partial_\phi f_1(\rho_s,\phi)^\top([\partial_t^j, {}^\top\nabla\phi]^\top(\nabla w_*)), v)| \\
&\equiv J_{11} + J_{12}.
\end{aligned}
$$

Using Lemma 8.2 and Lemma 8.4 we have

$$\begin{aligned} J_{11} &\le C\|\nabla\phi\|_4\|\partial_t^j\partial_x w_*\|_2\|v\|_4 \\ &\le C\|\phi\|_{H^2}\{[\![\partial_x w_1]\!]_m + |||Dw_\infty|||_m\}\|v\|_{H_0^1} \\ &\le C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H_0^1}. \end{aligned}$$

As for J_{12}, we have

$$|[\partial_t^j, {}^\top\nabla\phi]^\top(\nabla w_*)| \le C\sum_{l=1}^{j}|\partial_t^l\nabla\phi||\partial_t^{j-l}\partial_x w_*|.$$

Since

$$\frac{1}{2} - \frac{m-1-2l}{2} + \frac{1}{2} - \frac{m-2(j-l)}{2} = 1 - \frac{m-1}{2} < 1,$$

we can find $p_{1,l}$, $p_{2,l} \ge 2$ satisfying

$$\frac{1}{p_{1,l}} > \frac{1}{2} - \frac{m-1-2l}{2}, \quad \frac{1}{p_{2,l}} > \frac{1}{2} - \frac{m-2(j-l)}{2}, \quad \frac{1}{2} \le \frac{1}{p_{1,l}} + \frac{1}{p_{2,l}} < 1.$$

We now take a number $p_{3,l} \ge 2$ satisfying $\frac{1}{p_{3,l}} = 1 - (\frac{1}{p_{1,l}} + \frac{1}{p_{2,l}}) > 0$. It then follows from Lemma 8.2 that

$$\begin{aligned} |J_{12}| &\le C\textstyle\sum_{l=1}^{j}\|\partial_t^j\partial_x\phi\|_{p_{1,l}}\|\partial_t^{j-l}\partial_x w_*\|_{p_{2,l}}\|v\|_{p_{3,l}} \\ &\le C\textstyle\sum_{l=1}^{j}\|\partial_t^j\partial_x\phi\|_{H^{m-1-2l}}\|\partial_t^{j-l}\partial_x w_*\|_{H^{m-2(j-l)}}\|v\|_{H_0^1} \\ &\le C[\![\phi]\!]_m\{[\![\partial_x w_1]\!]_m + |||Dw_\infty|||_m\}\|v\|_{H_0^1} \\ &\le C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H_0^1}. \end{aligned}$$

It remains to estimate J_2. By Lemma 8.3 we have

$$|J_2| \le C[\![\partial_t\phi]\!]_{m-1}[\![{}^\top(\nabla w_*)\nabla\phi]\!]_{m-1}\|v\|_2,$$

and, also,

$$\begin{aligned} [\![{}^\top(\nabla w_*)\nabla\phi]\!]_{m-1} &\le C\{\|\nabla w_*\|_\infty[\![\partial_x\phi]\!]_{m-1} + |||D\nabla w_*|||_{m-1}[\![\nabla\phi]\!]_{m-1}\} \\ &\le C\{[\![\partial_x w_1]\!]_m + |||Dw_\infty|||_m\}[\![\partial_x\phi]\!]_{m-1} \\ &\le C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}. \end{aligned}$$

As for $[\![\partial_t\phi]\!]_{m-1}$, we see from (6.3) and Proposition 7.10 that

$$\begin{aligned}[\![\partial_t\phi]\!]_{m-1} &\le C\{[\![\partial_t\sigma_*]\!]_{m-1}+[\![\partial_t\phi_*]\!]_{m-1}\}\\ &\le C\{(1+t)^{-\frac{3}{4}}M(t)+[\![\partial_{x_1}\phi_\infty]\!]_{m-1}+[\![\partial_x w_\infty]\!]_{m-1}\\ &\quad +[\![\partial_{x_1}\sigma_\infty]\!]_{m-1}+[\![P_1\boldsymbol{F}]\!]_{m-1}+[\![\langle Q_0P_\infty^{(0)}\boldsymbol{F}\rangle_\infty]\!]_{m-1}+[\![Q_0\boldsymbol{F}_\infty]\!]_{m-1}\}\\ &\le C\{(1+t)^{-\frac{3}{4}}M(t)+[\![P_1\boldsymbol{F}]\!]_{m-1}+[\![Q_0\boldsymbol{F}]\!]_{m-1}\}.\end{aligned}$$

Since $2j=m$, we have $[\frac{m-1}{2}]=\frac{m-2}{2}$, and hence, by Lemma 4.5,

$$[\![P_1\boldsymbol{F}]\!]_{m-1}\le C[\![\boldsymbol{F}]\!]_{m-2}.$$

It then follows from Proposition 8.5 and Proposition 8.6 (i) that

$$[\![P_1\boldsymbol{F}]\!]_{m-1}+[\![Q_0\boldsymbol{F}]\!]_{m-1}\le C\{[\![Q_0\boldsymbol{F}]\!]_{m-1}+[\![\tilde{Q}\boldsymbol{F}]\!]_{m-2}\}\le C(1+t)^{-\frac{3}{4}}M(t)^2,$$

which implies that

$$[\![\partial_t\phi]\!]_{m-1}\le C(1+t)^{-\frac{3}{4}}M(t).$$

We thus obtain

$$|J_2|\le C\{(1+t)^{-\frac{7}{4}}M(t)^2+(1+t)^{-1}M(t)|||Dw_\infty(t)|||_m\}\|v\|_2.$$

Consequently,

$$|I_3|\le C\{(1+t)^{-1}M(t)^2+(1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H_0^1}.$$

Therefore, we arrive at

$$|\langle\langle\partial_t^j\boldsymbol{f}_1,v\rangle\rangle_{-1}|\le C\{(1+t)^{-1}M(t)^2+(1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H_0^1}.$$

This gives

$$\|\partial_t^j\boldsymbol{f}_1\|_{H^{-1}}\le C\{(1+t)^{-1}M(t)^2+(1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m\}.$$

Clearly, one can obtain the same estimate for $\|\partial_t^j\boldsymbol{f}_2\|_{H^{-1}}$. Concerning $\|\partial_t^j\boldsymbol{f}_3\|_{H^{-1}}$, one can obtain the desired estimate by replacing ∇w_* with ϕ_* in the argument above for $\|\partial_t^j\boldsymbol{f}_1\|_{H^{-1}}$. This completes the proof. $\square$

We are now in a position to prove Proposition 8.1.

Proof of Proposition 8.1. Since $[\![\tilde{Q}\boldsymbol{F}_\infty]\!]_{m-2}\le C[\![\boldsymbol{F}]\!]_{m-2}$, assertion (i) follows from Proposition 8.5 and Proposition 8.6 (i).

Let us consider $\tilde{R}(t)$. We write $R^{(1)}_{j,k}$ as

$$\begin{aligned} R^{(1)}_{j,k} &= \tfrac{\alpha_0}{\gamma^2}(\langle Q_0 T_{j,k}\boldsymbol{F}\rangle_\infty, T_{j,k}\sigma_\infty) - \tfrac{\alpha_0}{\gamma^2}(\langle Q_0 T_{j,k}P_1\boldsymbol{F}\rangle_\infty, T_{j,k}\sigma_\infty) \\ &\quad +\tilde{R}^{(1)}_{j,k} - \langle\langle \langle Q_0 T_{j,k}\boldsymbol{F}\rangle_\infty u^{(0)}, T_{j,k}u_\infty\rangle\rangle \\ &\quad +\langle\langle T_{j,k}P_1\boldsymbol{F}, T_{j,k}u_\infty\rangle\rangle - \langle\langle \langle Q_0 T_{j,k}P_1\boldsymbol{F}\rangle_\infty u^{(0)}, T_{j,k}u_\infty\rangle\rangle. \\ &\equiv \textstyle\sum_{l=1}^6 I_l. \end{aligned}$$

We first consider I_3. If $2j+k \le m-1$, then applying Proposition 8.5 (i)–(iii) and Proposition 8.6 (ii), (iii), we have

$$|I_3| \le C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||^2_m\}.$$

Here we have used the Poincaré inequality $\|w_\infty(t)\|_2 \le \|\partial_x w_\infty(t)\|_2$ for $j = k = 0$. In case $2j+k = m$, applying Proposition 8.5 (i), (ii), (iv), (v), and Proposition 8.6 (ii), (iv), we obtain

$$\begin{aligned} |I_3| &\le C\{(1+t)^{-1}M(t)^3\sqrt{D_\infty(t)} + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\} \\ &\le C\{(1+t)^{-\frac{7}{4}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\} \end{aligned}$$

if $M(t) \le 1$.

We next consider I_5. We write I_5 as

$$I_5 = \langle\langle T_{j,k}\boldsymbol{F}, P_1^* T_{j,k}u_\infty\rangle\rangle.$$

If $2j+k \le m-1$, one can estimate I_5 as I_3. If $2j+k = m$ and $k \ge 1$, then, by Lemma 4.5 (iii),

$$\begin{aligned} |I_5| &\le |-\langle\langle T_{j,k-1}\boldsymbol{F}, \partial_{x_1}P_1^* T_{j,k}u_\infty\rangle\rangle| \\ &\le C\|T_{j,k-1}\boldsymbol{F}\|_2\|T_{j,k}u_\infty\|_2 \\ &\le C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||^2_m\}. \end{aligned}$$

In case $2j = m$, let Φ be defined by $Q_0 P_1^* \partial_t^j u_\infty = {}^\top(\Phi, 0)$. Then, by integration by parts, we have

$$\langle\langle \partial_t^j Q_0\boldsymbol{F}, P_1^*\partial_t^j u_\infty\rangle\rangle = -(\partial_t^j(\phi w), \nabla(\Phi\tfrac{P'(\rho_s)}{\gamma^4\rho_s})).$$

Since $\|\nabla(\Phi\frac{P'(\rho_s)}{\gamma^4\rho_s})\|_2 \le C\|\partial_t^j u_\infty\|_2$ by Lemma 4.5 (iii), we see from Proposition 8.5 (vi) that

$$|\langle\langle \partial_t^j Q_0\boldsymbol{F}, P_1^*\partial_t^j u_\infty\rangle\rangle| \le C(1+t)^{-\frac{7}{4}}M(t)^3.$$

Since $\tilde{Q}P_1^*\partial_t^j u_\infty \in H_0^1$ by Lemma 4.5 (ii), one can estimate $\langle\langle \partial_t^j \tilde{Q}\boldsymbol{F}, P_1^*\partial_t^j u_\infty\rangle\rangle$ as in the estimate of I_3 and we obtain

$$|I_5| \le C\{(1+t)^{-\frac{7}{4}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\}.$$

As for I_4, we have

$$\begin{aligned} I_4 &= -(2\pi)^{-1}\langle\langle(\hat{\chi}_1+\hat{\chi}_2)\varPi^{(0)}\widehat{T_{j,k}\boldsymbol{F}}, \widehat{T_{j,k}u_\infty}\rangle\rangle \\ &= -(2\pi)^{-1}\langle\langle\widehat{T_{j,k}\boldsymbol{F}}, (\hat{\chi}_1+\hat{\chi}_2)\varPi^{(0)*}\widehat{T_{j,k}u_\infty}\rangle\rangle \\ &= \langle\langle T_{j,k}\boldsymbol{F}, \langle T_{j,k}u_\infty, u^{(0)}\rangle_\infty u^{(0)*}\rangle\rangle. \end{aligned}$$

Here and in what follows we denote $\langle f, u^{(0)}\rangle_\infty = \mathscr{F}^{-1}((\hat{\chi}_1+\hat{\chi}_2)\langle\hat{f}, u^{(0)}\rangle)$. Since $\tilde{Q}(\langle T_{j,k}u_\infty, u^{(0)}\rangle_\infty u^{(0)*}) = 0\,(\in H_0^1)$, as in the estimate for I_5, we obtain

$$|I_4| \le C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)|||Dw_\infty(t)|||_m^2\}.$$

The same estimate also holds for I_1, since

$$\begin{aligned} I_1 = \tfrac{\alpha_0}{\gamma^2}(\langle Q_0T_{j,k}\boldsymbol{F}\rangle_\infty, T_{j,k}\sigma_\infty) &= \langle\langle\langle Q_0T_{j,k}\boldsymbol{F}\rangle_\infty, T_{j,k}\sigma_\infty\rangle\rangle \\ &= \langle\langle T_{j,k}\boldsymbol{F}, \langle T_{j,k}\sigma_\infty, u^{(0)}\rangle_\infty u^{(0)*}\rangle\rangle. \end{aligned}$$

Similarly, I_2 and I_6 can be estimated as I_5, since

$$I_2 = \langle\langle T_{j,k}P_1\boldsymbol{F}, \langle T_{j,k}\sigma_\infty, u^{(0)}\rangle_\infty u^{(0)*}\rangle\rangle,$$

$$I_6 = \langle\langle T_{j,k}P_1\boldsymbol{F}, \langle T_{j,k}u_\infty, u^{(0)}\rangle_\infty u^{(0)*}\rangle\rangle.$$

We thus obtain

$$R^{(1)}(t) \le C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\}.$$

The remaining terms can be estimated in a similar manner. This completes the proof. □

9. Asymptotic behavior

In this section we prove that the asymptotic behavior of the solution of (4.1) is described by a solution of a 1-dimensional viscous Burgers equation.

Let $\sigma = \sigma(x_1, t)$ be the solution of the following problem:

$$\partial_t\sigma - \kappa_0\partial_{x_1}^2\sigma + a_0\partial_{x_1}\sigma + a_1\partial_{x_1}(\sigma^2) = 0, \tag{9.1}$$

(9.2) $$\sigma|_{t=0} = \sigma_0,$$

where κ_0 and a_0 are the numbers given in Lemma 4.2; $a_1 = \langle \phi^{(0)} w^{(0),1} \rangle - i\langle \hat{L}^{(1)} \hat{S} \boldsymbol{F}_1 \rangle$; and $\sigma_0 = \langle Q_0 u_0 \rangle = \langle \phi_0 \rangle$.

We will show the following estimate.

Proposition 9.1. *If $\nu \geq \nu_0$, $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ and $\|\rho_s - 1\|_{C^{m+1}[0,1]} \leq \omega_0$, then the following assertion holds. For any $\delta > 0$ there exists $\varepsilon_7 > 0$ such that if $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_7$, then*

$$\|\sigma_1(t) - \sigma(t)\|_2 \leq C(1+t)^{-\frac{3}{4}-\delta} \|u_0\|_{H^m \cap L^1}.$$

To prove Proposition 9.1, we will employ the following well-known decay properties of $\sigma(t)$.

Lemma 9.2. *Let $\sigma(t)$ be a solution of* (9.1)–(9.2) *with $\sigma_0 \in H^1 \cap L^1$. Then*

$$\|\partial_{x_1}^k \sigma(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|\sigma_0\|_{H^1 \cap L^1} \quad (k = 0, 1),$$

$$\|\sigma(t)\|_\infty \leq C(1+t)^{-\frac{1}{2}} \|\sigma_0\|_{H^1 \cap L^1}.$$

We introduce a quantity. Let $\sigma_1(t)$ and $\sigma(t)$ be solutions of (5.1) and (9.1)–(9.2), respectively. We define $N(t)$ by

$$N(t) = \sup_{0 \leq \tau \leq t} (1+\tau)^{\frac{3}{4}-\delta} \|\sigma_1(t) - \sigma(t)\|_{H^1}.$$

Proposition 9.1 would then follow if we could show that $N(t) \leq C\|u_0\|_{H^m \cap L^1}$.

Proof of Proposition 9.1. In terms of $\mathscr{U}(t)$ the solution $\sigma(t)$ of (9.1)–(9.2) is written as

(9.3) $$\sigma(t) = \mathscr{U}(t)\sigma_0 - a_1 \int_0^t \mathscr{U}(t-\tau) \partial_{x_1}(\sigma^2)(\tau)\, d\tau.$$

We next rewrite $U_0(t-\tau)\mathit{\Pi}^{(0)} P_1 \boldsymbol{F}(\tau)$ in (5.1). By Lemma 6.2 (ii), we have

$$\begin{aligned}
\mathit{\Pi}^{(0)} P_1 \boldsymbol{F} &= -P_1^{(0)}(\partial_{x_1} \langle \phi w^1 \rangle u^{(0)}) + \mathit{\Pi}^{(0)} \partial_{x_1} P_1^{(1)} (I - \mathit{\Pi}^{(0)}) \boldsymbol{F} + \mathit{\Pi}^{(0)} \partial_{x_1}^2 P_1^{(2)} \boldsymbol{F} \\
&= -P_1^{(0)}(a_{11} \partial_{x_1}(\sigma_1^2) u^{(0)}) - P_1^{(0)}(\partial_{x_1}(\langle \phi w^1 \rangle - \langle \phi^{(0)} w^{(0),1} \sigma_1^2 \rangle) u^{(0)}) \\
&\quad + \mathit{\Pi}^{(0)} \partial_{x_1} P_1^{(1)} (I - \mathit{\Pi}^{(0)}) \boldsymbol{F} + \mathit{\Pi}^{(0)} \partial_{x_1}^2 P_1^{(2)} \boldsymbol{F}.
\end{aligned}$$

Here $a_{11} = \langle \phi^{(0)} w^{(0),1} \rangle$. Since

$$\begin{aligned} \mathscr{F}(P_1^{(1)}(I - \Pi^{(0)})(\sigma_1^2 \boldsymbol{F}_1)) &= i\hat{\chi}_1 \Pi^{(0)} \hat{L}^{(1)} \hat{S}(\widehat{(\sigma_1^2)} \boldsymbol{F}_1) \\ &= i\hat{\chi}_1 \langle \hat{L}^{(1)} \hat{S} \boldsymbol{F}_1 \rangle \widehat{(\sigma_1^2)} u^{(0)} = -a_{12} \mathscr{F}(P_1^{(0)}(\sigma_1^2 u^{(0)})), \end{aligned}$$

where $a_{12} = -i\langle \hat{L}^{(1)} \hat{S} \boldsymbol{F}_1 \rangle$, substituting $\boldsymbol{F} = \sigma_1^2 \boldsymbol{F}_1 + \boldsymbol{F}_2$ into $\Pi^{(0)} \partial_{x_1} P_1^{(1)}(I - \Pi^{(0)}) \boldsymbol{F}$, we have

$$\Pi^{(0)} \partial_{x_1} P_1^{(1)}(I - \Pi^{(0)}) \boldsymbol{F} = -a_{12} P_1^{(0)}(\partial_{x_1}(\sigma_1^2) u^{(0)}) + \Pi^{(0)} \partial_{x_1} P_1^{(1)}(I - \Pi^{(0)}) \boldsymbol{F}_2.$$

Using Lemma 4.8 (i), we thus arrive at

$$\begin{aligned} & U_0(t-\tau) \Pi^{(0)} P_1 \boldsymbol{F}(\tau) \\ = \; & -a_1 U_0(t-\tau)(\partial_{x_1}(\sigma_1^2(\tau)) u^{(0)}) \\ & -U_0(t-\tau)(\partial_{x_1}(\langle \phi w^1 \rangle(\tau) - \langle \phi^{(0)} w^{(0),1} \sigma_1^2(\tau) \rangle) u^{(0)})) \\ & +U_0(t-\tau) h_4(\tau) + U_0(t-\tau) h_5(\tau), \end{aligned}$$

where $a_1 = a_{11} + a_{12}$,

$$h_4 = \partial_{x_1} P_1^{(1)}(I - \Pi^{(0)}) \boldsymbol{F}_2 + \partial_{x_1}^2 P_1^{(2)} \boldsymbol{F}_2,$$

$$h_5 = \partial_{x_1}^2 P_1^{(2)}(\sigma_1^2 \boldsymbol{F}_1).$$

It then follows from (5.1) and (9.3) that

$$\sigma_1(t) - \sigma(t) = \sum_{j=0}^{5} I_j(t). \tag{9.4}$$

Here

$$I_0(t) = U_0(t) u_0 - \mathscr{U}(t) \sigma_0,$$

$$I_1(t) = -a_1 \int_0^t \mathscr{U}(t-\tau)(\partial_{x_1}(\sigma_1^2) - \partial_{x_1}(\sigma^2))(\tau) \, d\tau,$$

$$I_2(t) = -a_1 \int_0^t (U_0(t-\tau)(\partial_{x_1}(\sigma_1^2(\tau)) u^{(0)}) - \mathscr{U}(t-\tau) \partial_{x_1} \sigma_1^2(\tau)) \, d\tau,$$

$$I_3(t) = -\int_0^t U_0(t-\tau))(\partial_{x_1}(\langle \phi w^1 \rangle(\tau) - \langle \phi^{(0)} w^{(0),1} \sigma_1^2(\tau) \rangle) u^{(0)}) \, d\tau,$$

$$I_j(t) = \int_0^t U_0(t-\tau)) h_j(\tau) \, d\tau, \quad j = 4, 5.$$

Lemma 4.9 implies that

$$\|I_0(t)\|_{H^1} \le C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^m\cap L^1}.$$

Let us consider $I_1(t)$. By Lemma 9.1 and the definition of $M(t)$ and $N(t)$, we have

$$\begin{aligned}\|(\sigma_1^2-\sigma^2)(\tau)\|_1 &\le \|(\sigma_1+\sigma)(\tau)\|_2\|(\sigma_1-\sigma_2)(\tau)\|_2\\ &\le C(1+\tau)^{-1+\delta}\|u_0\|_{H^m\cap L^1}N(t).\end{aligned}$$

Furthermore, by Lemma 8.2, we have $\|(\sigma_1-\sigma)(\tau)\|_\infty \le C(1+\tau)^{-\frac{3}{4}+\delta}N(t)$, and hence,

$$\begin{aligned}\|\partial_{x_1}(\sigma_1^2-\sigma^2)(\tau)\|_2 &\le C\{\|(\sigma_1+\sigma)(\tau)\|_\infty\|\partial_{x_1}(\sigma_1-\sigma_2)(\tau)\|_2\\ &\qquad +\|(\sigma_1-\sigma)(\tau)\|_\infty\|\partial_{x_1}(\sigma_1+\sigma_2)(\tau)\|_2\}\\ &\le C(1+\tau)^{-\frac{5}{4}+\delta}\|u_0\|_{H^m\cap L^1}N(t).\end{aligned}$$

It then follows from Lemma 4.8 that for $k=0,1$,

$$\begin{aligned}\|\partial_{x_1}^k I_1(t)\|_2 &\le C\Big\{\int_0^{t-1}(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1+\delta}\,d\tau\\ &\qquad +\int_{t-1}^{t}(t-\tau)^{-\frac{k}{2}}(1+\tau)^{-\frac{5}{4}+\delta}\,d\tau\Big\}\|u_0\|_{H^m\cap L^1}N(t)\\ &\le C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^m\cap L^1}N(t).\end{aligned}$$

As for $I_2(t)$, since $\langle\partial_{x_1}(\sigma_1^2)u^{(0)}\rangle = \partial_{x_1}(\sigma_1^2)$, we see from Lemma 4.9 that for $k=0,1$,

$$\begin{aligned}\|\partial_{x_1}^k I_2(t)\|_2 &\le C\Big\{\int_0^{\frac{t-1}{2}}(1+t-\tau)^{-\frac{5}{4}}\|\sigma_1^2(\tau)\|_1\,d\tau\\ &\qquad +\int_{\frac{t-1}{2}}^{t-1}(1+t-\tau)^{-\frac{3}{4}}\|(\sigma_1\partial_{x_1}\sigma_1)(\tau)\|_1\,d\tau\\ &\qquad +\int_{t-1}^{t}(t-\tau)^{-\frac{k}{2}}\|\sigma_1^2(\tau)\|_2\,d\tau\Big\}\\ &\le C\Big\{\int_0^{\frac{t-1}{2}}(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{1}{2}}\,d\tau\\ &\qquad +\int_{\frac{t-1}{2}}^{t-1}(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1}\,d\tau\\ &\qquad +\int_{t-1}^{t}(t-\tau)^{-\frac{k}{2}}(1+\tau)^{-\frac{3}{4}}\,d\tau\Big\}M(t)^2\\ &\le C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^m\cap L^1}^2.\end{aligned}$$

As for I_3, we have

$$\|\langle\phi w^1\rangle(\tau)-\langle\phi^{(0)}w^{(0)}\sigma_1^2\rangle(\tau)\|_1$$
$$\le C\{\|\sigma_1(\tau)\|_2\|u(\tau)-(\sigma_1 u^{(0)})(\tau)\|_2+\|u(\tau)-(\sigma_1 u^{(0)})(\tau)\|_2^2\}$$
$$\le C(1+\tau)^{-1}M(t)^2.$$

Lemma 4.8 then gives

$$\begin{aligned}\|\partial_{x_1}^k I_3(t)\|_2 &\le C\int_0^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1}\,d\tau\\ &\le C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m\cap L^1}^2.\end{aligned}$$

To estimate $I_4(t)$, we write $h_4(\tau)$ as

$$h_4(\tau)=\partial_{x_1}(P_1^{(1)}(I-\Pi^{(0)})\boldsymbol{F}_2+\partial_{x_1}P_1^{(2)}\boldsymbol{F}_2)(\tau).$$

Using Lemma 4.5, Lemma 4.8 and Lemma 6.3, we have

$$\begin{aligned}\|\partial_{x_1}^k I_4(t)\|_2 &\le C\int_0^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1}\,d\tau\\ &\le C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m\cap L^1}^2.\end{aligned}$$

As for $I_5(t)$, we write $h_5(\tau)$ as

$$h_5(\tau)=\begin{cases}\partial_{x_1}^2P_1^{(2)}(\sigma_1^2\boldsymbol{F}_1)(\tau) & \text{for } \tau\in[0,\frac{t}{2}],\\ 2\partial_{x_1}P_1^{(2)}(\sigma_1\partial_{x_1}\sigma_1\boldsymbol{F}_1)(\tau) & \text{for } \tau\in[\frac{t}{2},t].\end{cases}$$

As in the estimate on $I_2(t)$, we see from Lemma 4.5 and Lemma 4.8 that

$$\begin{aligned}\|\partial_{x_1}^k I_5(t)\|_2 &\le C\{\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{1}{2}}\,d\tau\\ &\qquad +\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1}\,d\tau\\ &\le C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^m\cap L^1}^2.\end{aligned}$$

We thus obtain

$$\|\partial_{x_1}(\sigma_1-\sigma)(t)\|_2\le C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^m\cap L^1}\{\|u_0\|_{H^m\cap L^1}+N(t)\},$$

which yields

$$N(t)\le C\|u_0\|_{H^m\cap L^1}\{\|u_0\|_{H^m\cap L^1}+N(t)\}.$$

The desired result now follows by taking $\|u_0\|_{H^m\cap L^1}$ suitably small. This completes the proof. □

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