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Dynamic mathematical modeling of cell-fractone interactions

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Abstract. Within the last 20 years, new biological structures called *fractones*, named in honor of the late Dr. Benoit Mandelbrot due to their fractal-like appearance, have been discovered by cell biologists. Their primary purposes are theorized to pertain to the major processes of the life cycle of cells, namely cell division, migration, and differentiation. Building on the back of the discretized diffusion equations, we built a mathematical model of how these fractones interact with the cells and the associated growth factors produced in order to gain insight into the growth process as a whole. As it is shown in this paper, the complexity of this biological process opens the door to entirely new questions in the field of control theory.

Keywords. Morphogenesis, Fractones, Control Theory, Computational Modeling.

1. Introduction

The primary goal of this paper is, motivated by the hypothesis that new biological structures named fractones pertain to the major process of the life cycle, to formulate questions in the field of control theory for a new category of systems.

Neurulation and subsequent events of brain's formation involve multiple neurogenic growth factors that in turn induce neuroepithelial cell proliferation, differentiation and migration. From the distribution and activation of these growth factors in space and time will depend the morphogenic events of the developing mamalian brain. The distribution and activation of those growth factors in space and time will depend the morphogenic events of the developing mamalian brain. However, the process organizing the distribution/availability of growth factors within the neuroepithelium is not understood. It is known that extracellular matrix (ECM) molecules such as heparan sulfate proteoglycans (HSPG) interact and strongly influence growth factors in the extracellular space prior to the recognition of growth factors by cell surface receptors and subsequent biological activation. The HSPG capture, concentrate and protect growth factors in specialized locations [1, 7, 17, 26]. For numerous heparin-binding growth factors such as fibroblast growth factors (FGF) and bone morphogenetic protein (BMP), very active families of growth factors for developmental neurogenesis, HSPG are even thought to be obligatory for presenting the growth factors to the cell surface receptors of the target cells [1, 2, 15, 26]. Therefore, the ECM may play a crucial role as a topological organizer of growth factor capture and activation, and as a potential driver of morphogenic events. We have characterized novel ECM structures, termed fractones, that directly contact neural stem and progenitor cells in the adult brain neurogenic zone [10, 13, 14]. Adult fractones are highly immunoreactive for HPSG, capture and concentrate FGF-2 [10] and BMP-7 [6] and are associated with cell proliferation in the neurogenic zone [10]. In addition, we have recently shown that FGF-2 and BMP-7 must bind fractones in the neural stem cell niche to regulate neural stem cell proliferation and the production of new neural cells in adulthood [6]. Our preliminary results indicate the presence of fractones (laminin and HSPG immunoreactive punctae) associated with neuroepithelial cell proliferation during development, capable of capturing FGF-2, and similar to laminin immunoreactive punctae characterized by others during development [11, 12, 20]. Therefore, "embryonic fractones" are good candidates to collect and promote growth factors at the surface of the neuroepitelial cells to spatially control their proliferation. Our hypothesis is that fractones are the captors that spatially control the activation of growth factors in a precise location to generate a morphogenic event.

To validate this hypothesis, we propose to develop and analyze a mathematical model predicting cell proliferation from the spatial distribution of fractones in a developing mouse. Dynamic mathematical modeling, i.e. models that represents change in rates over time, serves several purposes [8]. Using computer simulations, by mimicking the assumed forces resulting in a system behavior, the dynamic model helps us to understand the nonlinear dynamic of the system under study. Such approach is especially well suited for biological systems whose complexity renders a purely analytical approach unrealistic. Moreover, it allows us to overcome the excessively demanding purely experimental

approach to understand a biological system. Our primary goal in this paper is to develop a model that contains the crucial features of our hypothesis and at the same time that is sufficiently simple to allow an understanding of the underlying principles of the observed system.

We propose to model this biological process as a control system, the control depicting the spatial distribution of the active fractones. This is a novel approach with respect to the most commonly reaction-diffusion models seen in the literature on morphogenesis, however it is not that surprising. Indeed, control theory is instrumental to overcome many challenges faced by scientists to design systems with a very high degree of complexity and interaction with the environment [4, 5, 16]. Examples of its applicability in physical and biological systems are numerous [18, 19]. In this paper, we will highlight the fact that due to the specific nature of morphogenesis and in particular of the cell's proliferation, our innovative model opens an entirely new area in control theory. The reason comes from the fact that in this proposal the state space of our control system is dynamic; this is an intrinsic property of biological systems. In physics, for instance, the state space is static and the equations of motion are derived from minimizing a Lagrangian. In engineering, the configuration manifold is fixed and, one either attempts to determine the evolution of the system while minimizing a prescribed cost or one tries to design controls to take into account uncertainties of the system. As a result, new methods have to be proposed to analyze biological systems from the control theory point of view. This will advance the field of control theory by considering new problems and by providing insight toward the development of innovative ideas and methods to solve these types of problems. In this paper, we take a first step in that direction.

2. Background

2.1. Neurulation

A fundamental problem is to understand how growth factors control the topology of cell proliferation and direct the construction of the forming neural tissue. It has been demonstrated that ECM molecules, particularly proteoglycans, strongly influence growth factor-mediated cell proliferation. ECM proteoglycans can capture and present growth factors to the cell surface receptors to ultimately trigger the biological response of growth factors.

F. Mercier and his collaborators have discovered ECM structures that are associated with proliferating cells in the stem cell niche of the adult mammalian brain [13, 14, 10]. These structures, termed fractones, display a complex branched morphology by transmission electron microscopy [13, 14] but appear as small punctae by immunofluorescence microscopy after immunolabeling for laminin, collagen IV or heparan sulfate proteoglycans (HSPG) [13, 10]. Fractones hold a high potential as captors of mitotic and neurogenic growth factors via their HSPG [10]. The most recent results indicate that fibroblast growth factor-2 (FGF-2) and bone morphogenetic factor-7 (BMP-7) must bind fractones to respectively stimulate and inhibit cell proliferation in the adult neurogenic niche [6]. Do fractones exist during development? Several authors demonstrated the occurence of laminin immunoreactive punctae during development [12, 23, 24]. However, the function of these punctae during development has not been reported. F. Mercier investigated the presence of these punctae at several stages of mouse development, and their potential as captors of FGF-2 (personal communication).

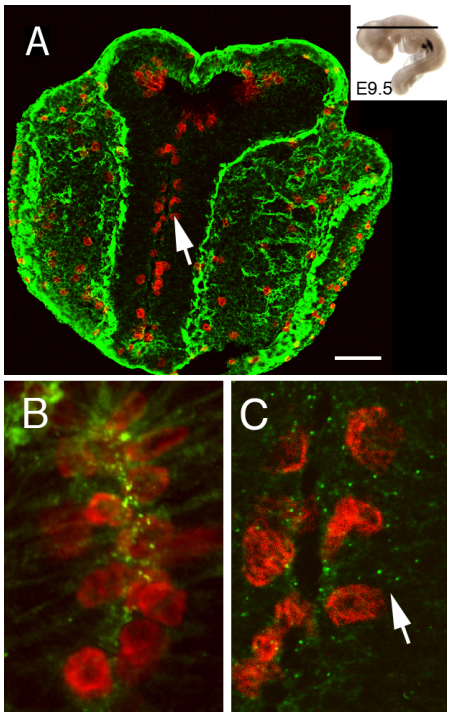

Figure 1: **Characterization of fractones in the mouse neuroepithelium during brain morphogenesis.** *A. Laser scanning confocal microscopy image showing the section of the whole head of an E9.5 embryo (9.5 days post-coitum). Proliferating neuroepithelial cells were visualized by phosphorylated histone-3 (PH3, a marker of mitosis) immunofluorescence cytochemistry (red). The extracellular matrix material was revealed by immunoreactivity for laminin, a ubiquitous glycoprotein found in basement membranes and fractones. However, fractones are too small to be visualized at this level of magnification. Note that cells proliferate near the lumen of the forming cavity (arrow, neural groove). The plan of section is indicated in the inset. B. High magnification confocal microscopy field showing proliferating neuroepithelial cells (PH3 immunoreactivity, red) associated with fractone (green punctae) at E8.5. C. Magnification of the area indicated by an arrow in A showing that neuroepithelial cells also proliferate (red) next to fractones (green punctae, arrow) at E9.5. Scale bars. A: 50 m; B and C: 10 m.*

2.2. Control Theory

The history of mathematics used to solve problems arising from biology dates back several hundred years to the times of Bernoulli and Euler. Prior to the mid 1900s, though, biology served primarily as the inspiration to understanding larger problems rather than as a practical field to be studied under the rigors of applied mathematics. Many problems in the field, even simplified with strong assumptions and in their least-complex forms, were unable to be solved using traditional techniques of mathematicians due to their complexity. Within the last thirty years, and mostly due to the advent of the computer, researchers are now analyzing complex systems without unnecessary simplifications. With the capability to undertake more and more complex systems came a new era for mathematical and computational biology.

The appearance and usage of control theory in the field of biology is a relatively new idea, dating back only a few decades. The first real evidence of the usage of control theory to understand a biological process originates with Norbert Wiener [25], who developed many of the ideas of feedback and filtering in the early 1940s in collaboration with the Harvard physiologist Arturo Rosenblueth, who was, in turn, heavily influenced by the work of his colleague Walter Cannon [3], who coined the term homeostasis in 1932 to refer to feedback mechanisms for set-point regulation in living organisms. Rudolf Kalman [9] often used biological analogies in his discussion of control systems theory, and so did many other early researchers. Modern biological control, enveloped in the more general field of systems biology, eminates from the work of Ludwig von Bertalanffy[22] with his general systems theory. Born even more recently, the field of systems biology (the merging of biology, physics, engineering, and/or mathematics) is large and encompassing, so much so that it, at times, is hard to define what is and is not part of the field. Among the problems addressed by the field are: complex molecular systems; quantitative modeling of enzyme kinetics; mathematical modeling of population growth; simulations developed to study neurophysiology; control theory and cybernetics. Some recent problems approached by those studying control theory in the field of biology have been to model, among others: internal workings of the cells; molecular signaling or energy transfer (among RNA, DNA, proteins, etc.); cell signal transduction processes; neural pathways; regulation versus homeostasis; RNA/DNA transcription with an emphasis on mutation; and gene function and interactions. The breadth and variety of problems that can be modeled using control theory runs the gamut, from the molecular through the microscopic up to the macroscopic.

Many areas of biology have been affected by many areas of mathematical science, and the challenges of biology have also prompted advances of importance to the mathematical sciences themselves. The rapidly developing field of systems biology is tremendously exciting, and full of unique research opportunities and challenges, especially for the application of control theory.

Mathematically, the classical models attempting to describe morphogenesis are based on reaction-diffusion equations with the pioneering work of Turing [21]. Growth factors—then called chemicals—were allowed to freely and continually diffuse in the intracellular space. Reaction would occur in this soup of chemicals and those events would induce the cell to divide. Although Turing made a great attempt to mathematically portray morphogenesis, it is not an adequate model to describe the system given new discoveries and developments since the 1950s. Indeed, with his model Turing was describing how reactive chemicals produced in a static, living structure interact in a continuous medium (and, surprisngly, form wave-like patterns). Clearly, reaction-diffusion equations cannot be used to study the mechanisms of morphogenesis during development. Morphogenesis involves the capture and activation of growth factors by fractones at specific locations according to a precise timing. Moreover, the distribution of fractones is constantly changing during development, reflecting the dynamic of the morphogenic events. Therefore, the organizing role of fractones in morphogenesis must be analyzed by alternative mathematical models.

3. Mathematical Model

For the preliminary stages of our study we will provide only an approximate model of brain development. Additional parameters reflecting the neurulation process such as production of growth factors in precise locations will be incorporated later on. Let us present here a two-dimensional model of the morphogenesis process based on the existence of fractones.

3.1. Configuration Space

The morphogenic event will start from an initial configuration of cells immerged in what we call the ambiant space. Growth factors diffuse within the ambiant space. The ambiant space is discretised with a precision to be chosen by the user (eventually it will be determined by the experimental biological maps).

Definition 1. We call a square of our discretisation a unit. In the sequel, each unit will be identified to an integer pair (i, j). The origin of the discretisation is choosen arbitrarily and will be identified to $(0, 0)$.

Assumption 1. In the following, we assume that the space between the cells account for 20% of the total space occupied by the cells. This is reflected in our discretisation by representing a cell as a square composed of 81 units (i.e. a 9 by 9 square), while the "in-between cells" space is represented by single unit-rows and unit-columns. Notice that at this stage of the work it is an arbitrary choice and it will be straighforward to adjust it to reflect the observations from the experimental maps. Finally, in our discretisation, a fractone is represented as one unit, and will always be represented in green. In Figure 2, we represent two such discretisations for two different configurations of cells.

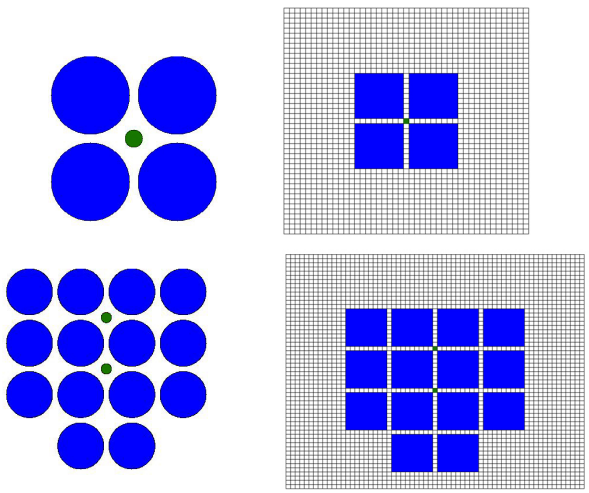

Figure 2: **Discretisation of given configurations of cells.** *The fractones are in green.*

The diffusion of growth factors occurs in the extra-cellular space around the cells as well as in the reminder of the ambiant space not filled with cells. Where a cell exists, the diffusion of growth factor is prevented since the cell acts like a wall. In our representation each new forming cell can be seen as the formation of a new obstacle in the ambiant space.

Assumption 2. In this paper, for simplicity we assume that the boundary of the ambiant space in which the biological process takes place is fixed but our definitions allow for boundaries that vary with time as well. Moreover, as a first step we assume the cells to be vertically and horizontally aligned.

Definition 2. At each time t of the biological process, we define the current configuration of cells associated with the remainder of the ambiant space as the Configuration Space of our system. By construction, in our case the configuration space is a topological space with holes (the cells), and will be denoted by $Conf(t)$.

Remark 1. Due to the morphogenic nature of the biological process under study, the configuration space is constantly evolving. This distinguishes in a very non trivial way our problem for the traditional problems in engineering or physics whose systems are usually defined on a static configuration space.

Prior to introducing the model, let us describe more precisely $Conf(t)$. To the discretisation of the initial configuration of cells and the ambiant space, i.e. $Conf(0)$, we associate a collection of indices $(i,j) \in I_0 \times J_0$ where each index is represented by an integer. Each pair of indexes represent a unit of our discretisation. For instance in Figure 2, the upper left configuration of cells assumed to be in squared ambiant space is associated to $I_0 = J_0 = \{0,1,\cdots,44\}\backslash(\{13,\cdots,21\}\cup\{23,\cdots,31\})$. As mentioned before, since cells are constantly forming, the configuration space evolves constantly as well, however it will always be formed by the product of unions of subsets of $\mathbb{Z}$. We introduce $Conf(t) = I_t \times J_t \subset \mathbb{Z}\times\mathbb{Z}$, where I_t, J_t are both unions of finite subsets of $\mathbb{Z}$. Note that with this equality we identify the configuration to its discretisation and that will be the case in all that follows. Indeed there is a one-to-one correspondance between both.

In our proposed model, the morphogenic events will be governed by a control system defined on a state space. The state space differs from the configuration space as follows. The state space represents the concentration of growth factor in each unit of our discretisation of the configuration space.

Assumption 3. For simplicity, in this paper we assume the diffusion of a unique type of growth factor and equal sensitivity of the fractones with respect to that growth factor. However, our model will be developed such that expanding to several type of growth factors and fractone sensitivity can be added in a straightforward way.

Definition 3. To each unit (i,j), and at each time t we associate a concentration of growth factor that we denote by $X_{ij}(t)$. The rate of change in the concentration of growth factor is described using classical diffusion equations.

3.2. Diffusion of Growth Factors in $Conf(t)$

Assume at first that there is no cells, therefore the growth factors diffuse freely in the ambiant space. The pure dissipation is then described by:

$$\dot{X}(t) = F^0(X(t)) \tag{1}$$

where the component of $X(t)$ are given by $X_{ij}(t)$ which represents the quantity of growth factor in unit (i,j) at time t as described in definition 3, and assuming diffusion occurs between a unit (i,j) and its heigth-neighbors we have:

$$\dot{X}_{ij}(t) = \nu_X[\sum_{k,l=-1}^{1}(X_{i+k,j+l}(t) - X_{ij}(t))], \quad (i,j)\in I_0\times J_0. \tag{2}$$

Here the parameter ν_X represents the diffusion parameter associate to the considered growth factor. Assume now that a cell forms in the ambiant space. The cell therefore becomes an obstacle to the diffusion process. Mathematically, rather than looking at a cell as an obstacle, we identify the cell to a hole in a topological space. The hole depicting the location of the cell insures that the diffusion of the growth factor takes place in the remainder of the ambiant space only. By doing so we do not have to perturbe the diffusion process, instead we continuously modify the topological space within which the diffusion process takes place.

Let us describe the new state space on which the diffusion process takes place. Assume the cell is centered at unit (a,b). This means that at the time t at which the cell formed, the configuration space $I_t \times J_t$ transforms into a new configuration space $\tilde{I}_t \times \tilde{J}_t = I_t \times J_t\backslash(\{a-4,\cdots,a+$

$4\}\times\{b-4,\cdots,b+4\}$). Notice that since several cells might be forming at the same time, the topological changes in the state space will reflect all the created holes. We then have:

$$\dot{X}_{ij}(t) = \nu_X[\sum_{\substack{k,l=-1 \\ (i+k,j+l)\in\tilde{I}_t\times\tilde{J}_t}}^{1} (X_{i+k,j+l}(t) - X_{ij}(t))], \tag{3}$$

for $(i,j) \in \tilde{I}_t \times \tilde{J}_t$.

3.3. Fractones as Controllers

The next step is to introduce the fractones into our model. As mentioned before, a fractone is represented as a one unit (i,j) of our discretisation.

Assumption 4. The assumption is that the fractones store the quantity of growth factors that they capture, and that this quantity becomes unavailable to the diffusion process.

In other terms, if a fractone is associate to a unit (i,j) there is perturbation to the diffusion process as follows. We introduce a control function $u(t) = (u_{ij}(t)) \in \{0,1\}^{I_t\times J_t}$ defined on a time interval $[0,T]$, with T representing the duration of the cascade of morphogenic events under study. When a fractone becomes active in unit (i,j) at time t, the component $u_{ij}(t)$ of the control is turned on to 1 while it is set to zero otherwise (i.e. no fractone is associated to unit (i,j) or is not yet into an active mode). Once active the fractone stores the current quantity of growth factors available in unit (i,j) and acts as a captor for the diffusion process. In other words, diffusion from the unit associated to an active fractone to its neighbords is prevented. To represent this perturbed-diffusion process we define a control system:

$$\dot{X}(t) = F^0(X(t)) + \sum_{(i,j)\in I_t\times J_t} F^{(i,j)}(X(t))u_{(i,j)}(t), \tag{4}$$

where $X(t)$ is the state variable and denotes the concentration of growth factor X in the discretized configuration space $I_t \times J_t$ at time t, the drift vector field F^0 is given by the right-hand side of (3) and represents the regular diffusion of growth factors taking place in the configuration space, and finally the control vector fields perturbes the regular diffusion to account for the possible prescence of an active fractone. More precisely, we have:

$$F_{ij}^{(i,j)}(X(t)) = \nu_X(\sum_{\substack{k,l=-1,(k,l)\neq(0,0) \\ (i+k,j+l)\in I_t\times J_t}}^{1} X_{i,j}(t)), \tag{5}$$

and

$$F_{i+k,j+l}^{(i,j)}(X(t)) = -\nu_X X_{ij}(t), \tag{6}$$

for $k,l \in \{-1,0,1\}$, $(i+k,j+l) \in I_t \times J_t$. In our notation, $F_{mn}^{(i,j)}$ represents the component mn of the vector field $F^{(i,j)}$. What we have done is to artificially hide for the diffusion process the value of the quantity of growth factor available in unit (i,j) in which a fractone is active. This implies that continuously, the neighboring units will diffuse to unit (i,j) and that the quantity of growth factor collected there is then stored over time and hidden for the diffusion process. Once the stored quantity reach a given threshold, mitosis can happen.

Definition 4. An admissible control is a mesureable function $u : [0,T] \to \{0,1\}^{n(t)}$ where T represents the duration of the morphogenic event under study, and $n(t)$ is the number of pairs included in $I_t \times J_t$.

3.4. Mitosis

The motivation behind the introduction of fractones as controllers comes from the hypothesis that it is the fractones that proceed with the order to the cell to undergo mitosis. Indeed, an active fractone stores quantity of growth factors through the diffusion process, and once this quantity reaches a prescribed threshold all the cells associated to this active fractone duplicate. In other words, the spatial distribution of fractones determines the morphogenic events.

To translate this mathematically, we can equivalently state that the spatial distribution of fractones and the diffusion process of growth factors regulate the appearance and the location of holes in our topological space, namely the configuration space. A natural question arises: when a cell undergoes mitosis, how does the existing mass of cells deform? At this stage, we will limit ourselves to simple assumptions to avoid to unnecessarily complexify the problem.

Based on assumption 2, in the sequel we identify a cell C to a unit of our discretisation. Indeed, since we assume our cells to be squares of 9×9 units of our discretisation and to be vertically and horizontally aligned, a cell C is completely determined by its middle unit (a,b). We write $C = (a,b)$. The following assumptions that regulate the deformation of the existing mass of cells once mitosis occurs is arbitrary and can be modified easily.

Assumption 5. Let unit (i,j) represent an active fractone. Associated to this fractone, there are at most four cells. The existing cells are represented by their central units $(i+\alpha 5, j+\alpha 5)$, $\alpha = \pm 1$. When the fractone (i,j) reaches the threshold value for the quantity of growth factor, all cells associated to the fractones are given the order to duplicate. Mathematically, it can be expressed as follows. If $(i+\alpha 5, j+\alpha 5) \in I_t \times J_t$, where $\alpha = \pm 1$, then the cell $(i+\alpha 5, j+\alpha 5)$ duplicates.

Assume cell (a,b) receives the order to duplicate. The next step is to model the deformation of the current configuration of cells or equivalently to define the location of a new hole in our topological space, namely the state space. First we introduce the following notion of distance.

Definition 5. Let $a = (a_1,a_2)$ and $b = (b_1,b_2)$ be two units such that $a_1 = b_1 \bmod 10$ and $a_2 = b_2 \bmod 10$. The linear distance between a and b is defined by $d_L(a,b) =$

$(|a_1 - b_1| + |a_2 - b_2|)$ and the geometric distance is defined by $d_G(a,b) = \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}$.

The geometric distance helps to determine a hierachy between units that are at the same linear distance from a given unit. This notion of geometric distance is based on the assumption that the mass of cell is optimizing its shape by prioritizing compactness. Clearly, we have that $d_G(a,b) = 10\sqrt{n^2+m^2}$ with $n, m \in \mathbb{Z}$. Notice that given unit (a,b), the closest units mod 10 from (a,b) are at a distance 1, and there are 4 of them. The next closest units are at a distance $\sqrt{2}$ and there are also 4 of them. The table below describes the possible distances, only one quandrant is displayed since it is symmetrical with respect to the other quadrants, and the table is symmetrical about its diagonal. The pattern is very clear. For the distances represented either by a number in the diagonal in Table 1 or by by an integer that is not issued from a Pythagorean triple, globally there are exactly 4 units at such distance mod 10 from unit (a,b). If the integer comes from a Pythagorean triple (such as 5) there are 12 such units and for any other number there are exactly 8 choices.

0	1	2	3	4	5	6
1	$\sqrt{2}$					
2	$\sqrt{5}$	$2\sqrt{2}$				
3	$\sqrt{10}$	$\sqrt{13}$	$3\sqrt{2}$			
4	$\sqrt{17}$	$2\sqrt{5}$	5	$4\sqrt{2}$		
5	$\sqrt{26}$	$\sqrt{29}$	$\sqrt{34}$	$\sqrt{41}$	$5\sqrt{2}$	
6	$\sqrt{37}$	$2\sqrt{10}$	$3\sqrt{5}$	$2\sqrt{13}$	$\sqrt{61}$	$6\sqrt{2}$

Table 1: Distance distribution for the deformation of the mass of cell as measured from the active fractone.

Let us describe our strategy in details. We identify the active fractone to unit (i,j). To this fractone there are at most 4 cells that are connected, those are described by $C_1 = (i+10, j-10), C_2 = (i-10, j-10), C_3 = (i-10, j+10)$ and $C_4 = (i+10, j+10)$. At time t, the active fractone (i,j) reaches the threshold for the configuration of growth factor. The connected cells then get the order to duplicate. The algorithm works as follows: first, it checks if $(i+5, j+5) \in I_t \times J_t$, if not there is a cell $C = (i+5, j+5)$, duplication takes place and the mass of cells deforms in the direction of the closest free space in terms of the geometric distance to contain a cell. Since there might be more than one such spot, we must make an arbitrary choice. Then, the algorithm checks for units in the first quadrant $(i+5)^+, (j+5)^-$ where $(i+5)^+$(resp. $(i+5)^-$)=set of indices k such that $k = (i+5)+\mathbb{N}$ (resp. $k = (i+5)-\mathbb{N}$) and $(j+5)^-$(resp. $(j+5)^+$)=set of indices l such that $l = (j+5) - \mathbb{N}$ (resp. $l = (j+5) + \mathbb{N}$). For a distance represented by mutliple units in this first quadrant, it will choose the one with the second index that is the closest to $j+5$. In case no unit are at the prescribed distance from $(i+5, j+5)$ mod 10, the algorithm does a similar search into the second quadrant $(i+5)^-, (j+5)^-$ and the units are considered clockwise starting from the first quandrant. If still no such units are found in the second quadrant, it then moves the search to the third quadrant $(i+5)^-, (j+5)^+$ and finally to the fourth quadrant $(i+5)^+, (j+5)^+$. Once this cycle is complete, the algorithm does the same calculations for a potential cell at $(i+5, j-5)$ associated to the active fractone, to be followed by the same two cycles for $(i-5, j-5)$ and $(i-5, j+5)$.

4. Statement of the Problem

This section is the main objective of the paper now that we have described mathematically the important structures of our study we can state the problem in a formal way.

To summarize what was done in the previous sections, the morphogenic events have been modeled as an affine control system of the form:

$$\dot{x}(t) = F^0(x(t)) + \sum_{i=1}^{n} u_i(t) F^i(x(t)), x(t) \in M(t) \tag{7}$$

where the state space $M(t)$ varies with time, $\dim M(t) = n(t)$, and such that $u(t)$ is an admissible control. In other words, it is a fully actuated system defined on a dynamical state space. Notice that the dimension of $M(t)$ is arbitrary since it depends on our discretisation, but the important feature of our system is that this dimension varies with respect to time. The problem is now the following:

Problem. *Given an initial and final configuration of cells in a prescribed ambiant space, determine an initial concentration of growth factors and a dynamic spatial distribution of fractones such that the mass of cells transforms from its initial configuration to its final configuration.*

An initial answer will be determined through experimental work. Indeed, the experimental maps will provide information about the control function used by nature to produce morphogenic events. On Figure 3 we represents a discretisation of a fractone map (obtained experimentally) with a prescribed precision (which is equivalent to choose the size of the units in our model). This discretised map will be used to extract information about the control function.

Restated in mathematical terms, we have:

> *Given $I_{t_0} \times J_{t_0}$ and $I_{t_f} \times J_{t_f}$, determine $X(t_0)$ and an admissible control $u(\cdot)$ such that $I_{t_0} \times J_{t_0}$ transforms into $I_{t_f} \times J_{t_f}$ under the evolution of system (4) and the rules for mitosis described in section 3.4.*

Due to the morphogenic nature of the system under study that implies a dynamic state space, this problem opens a completely new area in the field of control theory. New methods will have to be developed to answer such questions, and these type of problems are highly nontrivial. To understand the nature of the problem, we take a look at a simple case in Section 5.

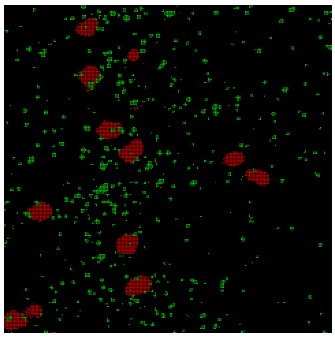

Figure 3: **Discretisation of a fractone map**.

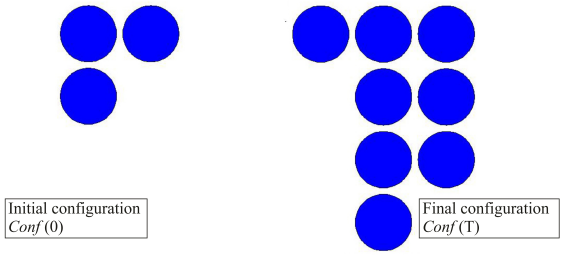

Figure 4: **Initial and final configuration of cells.** *For those configurations, there exist multiple sequences of morphogenic events to reach the desired configuration from the initial one.*

After stating the problem as above, we can go into more sophisticated versions. Indeed, as it can be seen in Section 5, there might not exist a solution to this problem for a given set of initial and final configurations. In this case, how do we modify the question? One solution is to introduce a notion of distance between configurations of cells and to ask how to reach a final configuration that is at the shortest distance from the desired one. There is also clearly the possibility that several controls lead to the same final configuration of cells. In that case, how do we select one? What is the criterion to be used to determine the most efficient control function? One strategy to be explored in a forthcoming work is based on the experimental observations collected in the lab through the fractone's maps. Based on those observations as well as assumptions such as minimizing the number of times mitosis can take place during the entire duration of the morphogenic event or minimizing the number of switching in the control function (which is equivalent to minimize the changes in the spatial distribution of the fractones), we can ascribe a cost function to be minimized. Our problem then becomes an optimal control problem. The bottom line is that, due to the complexity of the biological system that we study, there is an extremly large number of questions associated to this problem, and as said previously new methods need to be developed.

The current model is based on what we believe are the most critical features of our hypothesis. However, some of our assumptions are very restrictive and we also need to add some complexity to produce a more realistic model. For instance, to be added in the model in a forthcoming work is the following: first, we want to relax the assumption that the cells are horizontally and vertically aligned to allow broader configuration of cells; second, it is important to introduce a penality function for the diffusion of growth factors in the intracellular space found between the cells with respect to the diffusion in the free ambiant space; and third, we would like to add the possibility of having multiple growth factors diffusing in the ambiant space at different rates as well as having active fractones with varying sensitivities to each respective growth factor. However, despite the new features to be added, the statement of the problem will generally remain the same.

5. Example

On Fig. 4 we represent the desired initial and final configuration of cells, respectively $Conf(0)$ and $Conf(T)$. Let us at this stage not discuss the diffusion of growth factor but focus on the spatial distribution of fractones, i.e. on the control function that creates the morphogenic events to transform $Conf(0)$ into $Conf(T)$. Also, to limit complexity, we assume that there is only one fractone active at a time. We have that:
$I_0 \times J_0 = \{-20, \cdots, 22\} \times \{-30, \cdots, 12\} \backslash (A \cup B)$ where $A = \{-9, \cdots, 1\} \times (\{-9, \cdots, 1\} \cup \{1, \cdots, 9\})$ and $B = \{1, \cdots, 9\} \times \{1, \cdots, 9\}$ and $I_T \times J_T = I_0 \times J_0 \backslash (C \cup D \cup E)$ where we have $C = \{-19, \cdots, -11\} \times \{1, \cdots, 9\}$, $D = \{-9, \cdots, -1\} \times (\{-29, \cdots, -21\} \cup \{-19, \cdots, 11\})$ and $E = \{1, \cdots, 9\} \times (\{-19, \cdots, -11\} \cup \{-9, \cdots, -1\})$.
Notice that we arbitrarily choose the origin of our discretisation as the unit between the three initial cells and that the ambiant space is assumed to be a square, each side being composed of 42 units. Assume, for instance, that an active fractone is located in unit $(-10, -10)$ and that it reaches the threshold value for duplication, then the mass of cells transforms into four cells in a square configuration. In other words, we have $I_{t_1} \times J_{t_1} =$ where t_1 is the time needed for the stored concentration of growth factor in the unit associated to the active fractone to reach the threshold. The following control function $u(\cdot)$, where the time duration for each piecewise constant component depends on the diffusion process of the growth factor, generates morphogenic events starting with the prescribed configuration and reaching the desired one:
(8)

$$u(t) = \begin{cases} u_{(-10,-10)}(t) = 1, \text{ and 0 otherwise for } 0 \leq t < t_1 \\ u_{(-10,0)}(t) = 1, \text{ and 0 otherwise for } t_1 \leq t < t_2 \\ u_{(-10,-20)}(t) = 1, \text{ and 0 otherwise for } t_2 \leq t < t_3 \\ u_{(-10,-20)}(t) = 1, \text{ and 0 otherwise for } t_3 \leq t \leq T \end{cases}$$

A graphic representation can be seen on Fig. 5. It is easy to see that the control is not unique, i.e. there is another sequence of morphogenic events leading to the same desired cell configuration.

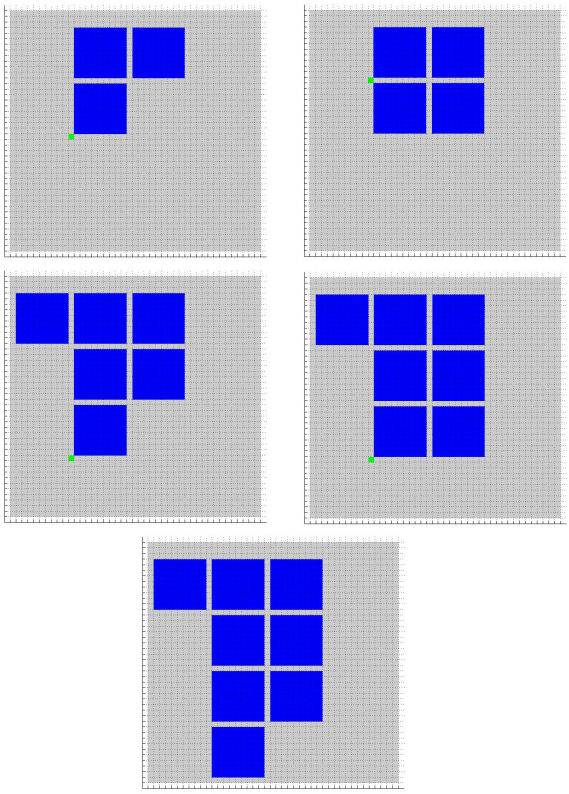

Figure 5: **Morphogenic Events Starting as** $Conf(0)$ **and reaching** $Conf(T)$**.** *Notice that each step between pictures incorporates duplication of cells as well as a possible change in the distribution of the active fractone.*

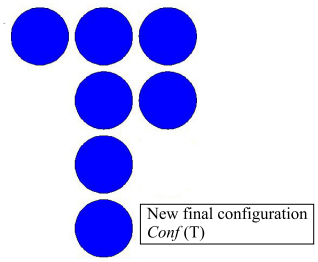

Figure 6: **Representation of the perturbed final configuration.** *Notice that it differs only by one cell from the previous one. In this case, there exists no morphogenic event based on our assumptions for mitosis that can lead exactly to this configuration.*

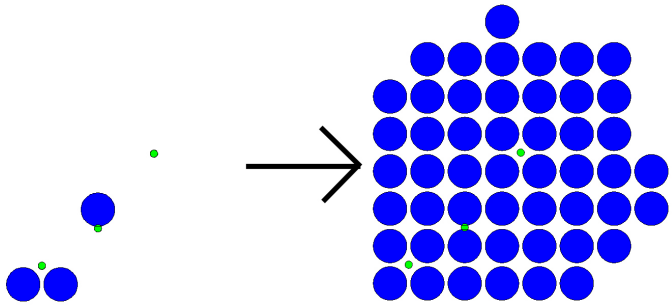

Figure 7: **Initial and Final Configurations for a series of Morphogenic Events.**

Let us now perturbe very slightly the desired configuration of cells, see Fig 6, i.e. we take $\tilde{I}_T \times \tilde{J}_T = I_T \times J_T \cup \{1, \cdots, 9\} \times \{-19, \cdots, -11\}$. In that case, given the rules for mitosis from section 3.4, it can be shown that there is no spatial distribution of fractones to solve the problem (in other words there is no admissible control function $u(.)$). However, as seen previously, there is an admissible control to reach the final configuration displayed in Fig. 4. This configuration differs from the desired one by only one cell. As suggested above, it will therefore makes sense to introduce in the future a notion of distance between configurations of cells and to ask how to reach a final configuration that is at the shortest distance from the desired one.

On Figure 7, we represent an initial configuration with 3 cells and 3 fractones and its final configuration after a series of morphogenic events. Notice that the square shape of the final mass of cells on thebottom left is due to the fact that the boundary of the ambiant space is fixed. This can easily be relaxed if necessary.

6. Future Work

There are mainly two directions of work that we are planning to undertake at this stage. First, from a purely mathematical perspective an open question is the development of new techniques to answer controllability and optimality questions for control systems such as the one introduced in this paper. Second, the interplay between the biological motivation and the mathematics must be refined to predict neurulation and post-neurulation growth by the mathematical model using fractone maps. More precisely, we need to first incorporate into the model important features that have not yet been taken into account. During the same time, we will use dual and triple immunocytochemistry for laminin (fractone marker) and for phosphorylated histone-3/bromodeoxyuridine (both markers of proliferating cells) and epifluorescence microscopy recording to map fractones and cell proliferation throughout the developing brain from embryonic stage E7.5 to E14.5. Then, after discretizing the fractone maps, we will determine whether the prediction of the mathematical model reflects the growth of the neural tissue observed in the maps. The observation of spatial dis-

tribution of fractones provided by the maps will determine the control function to be used in the mathematical model to produce our simulations.

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