Anderson-Darling test and the Malliavin calculus

Hirotsu, Takehiro
Nisshin Fire & Marine Insurance Co., Ltd

Kondo, Hiroki
Nisshin Fire & Marine Insurance Co., Ltd

Saito, Shingo
Faculty of Mathematics, Kyushu University

Sato, Takuya
Nisshin Fire & Marine Insurance Co., Ltd

他

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Takehiro Hirotsu, Hiroki Kondo, Shingo Saito,
Takuya Sato, Tatsushi Tanaka, and Setsuo Taniguchi

Abstract. The quadratic Wiener functional coming from the Anderson-Darling test statistic is investigated in the framework of the Malliavin calculus. The functional gives a completely new and important example of a quadratic Wiener functional.

Keywords. Anderson-Darling test, the Malliavin calculus, quadratic Wiener functional, Legendre polynomial

1. INTRODUCTION

In 1952, T. Anderson and D. Darling ([1]) introduced a statistical test, which is now called the Anderson-Darling test, to assert if a sample of data arises from a paricular probability distribution. Under the null hypothesis that a sample of data $\{x_1 < x_2 < \cdots < x_n\}$ comes from a distribution with cumulative distribution function F, the Anderson-Darling test statistic is defined by

$$A_n^2 = -n - \frac{1}{n}\sum_{j=1}^{n}(2j-1)\{\log u_j + \log(1 - u_{n-j+1})\},$$

$$\text{where } u_i = F(x_i), \ i = 1, \ldots, n.$$

If A_n^2 is large, the null hypothesis is rejected ([1, 2, 9]).

The test statistic A_n^2 admits the integral expression that

$$A_n^2 = n\int_0^1\left(\frac{\nu_n(t)}{n} - t\right)\psi(t)dt \quad \left(\nu_n(t) \equiv \#\{k \,|\, u_k \le t\}\right).$$

Then, due to Donsker's invariance principle ([3, 4]), as $n \to \infty$, A_n^2 converges in law to the random variable

$$\int_0^1\frac{b_t^2}{t(1-t)}dt,$$

$\{b_t\}_{t\in[0,1]}$ being the pinned Brownian motion ([1]). In [1], Anderson-Darling studied the above random variable and its extension

$$\mathfrak{a}^2(\psi) = \int_0^1 b_t^2\psi(t)dt,$$

where $\psi : (0,1) \to [0,\infty)$ is Borel measurable, by using the Fourier series decomposition of the Gaussian process $\{b_t\sqrt{\psi(t)}\}_{t\in[0,1]}$.

The aim of of this paper is to investigate $\mathfrak{a}^2(\psi)$ from the viewpoint of the Malliavin calculus, which enables one to deal with such Wiener functional as $\mathfrak{a}^2(\psi)$ in a systematic manner ([6, 7, 10]). A Wiener functional is said to be quadratic if its third derivative in the sense of the Malliavin calculus vanishes. In the study of stochastic oscillatory integrals, which are a probabilistic counterpart of Feynman path integrals, quadratic Wiener functionals play an essential role. Typical examples of quadratic Wiener functionals are the square norm and the variance on the time interval of the standard Brownian motion, and Lévy's stochastic area. See for example [7]. The Wiener functional $\mathfrak{a}^2(\psi)$ gives a new example of a quadratic Wiener functional.

The Wiener functional $\mathfrak{a}^2(\psi)$ is a not only new but also very important example. In our knowledge, the concrete examples of quadratic Wiener functionals all relate to the standard Brownian motion. As we shall see later, for ψ which explodes at the order of $(1 - t)^{-\alpha}$ at $t = 1$ for some $0 < \alpha < 2$, $\mathfrak{a}^2(\psi)$ is well-defined for the pinned Brownian motion, but not for the standard Brownian motion.

In Section 2, a brief review on the Malliavin calculus, in particular, stochastic oscillatory integrals will be given. Section 3 is devoted to the studies of explicit expressions of the stochastic oscillatory integral associated with $\mathfrak{a}^2(\psi)$.

2. THE MALLIAVIN CALCULUS

2.1. WIENER SPACE OF PINNED BROWNIAN MOTION

In this subsection, we shall give a brief review on the one dimensional classical Wiener space of pinned Brownian motion on $[0,1]$. Let $\mathcal{W}_0$ be the space of real-valued continuous functions w on $[0,1]$ with $w(0) = w(1) = 0$. Being equipped with the uniform convergence norm, $\mathcal{W}_0$ is a real separable Banach space. Denote by $\mathcal{H}_0$ the subspace of $\mathcal{W}_0$ consisting of h which admits square integrable derivative h'; $h(t) = \int_0^t h'(s)ds$, $t \in [0,1]$, and $\int_0^1 (h'(s))^2 ds < \infty$.

$\mathcal{H}_0$ is a separable Hilbert space with inner product

$$\langle h, g \rangle = \int_0^1 h'(t)g'(t)dt, \quad h, g \in \mathcal{H}_0.$$

The associated norm of $\mathcal{H}_0$ is denoted by $\|\cdot\|$. It is easily seen that $\mathcal{H}_0$ is embedded continuously and densely in $\mathcal{W}_0$. The dual space $\mathcal{W}_0'$ of $\mathcal{W}_0$ is included in $\mathcal{H}_0$ continuously and densely under the standard identification of $\mathcal{H}_0$ and and its dual space $\mathcal{H}_0'$. Let μ be a Gaussian measure on $(\mathcal{W}_0, \mathcal{B}(\mathcal{W}_0))$, $\mathcal{B}(\mathcal{W}_0)$ being the topological σ-field of $\mathcal{W}_0$, such that

$$(1) \quad \int_{\mathcal{W}_0} \exp(\sqrt{-1}\,\ell)d\mu = \exp(-\|\ell\|^2/2) \quad \text{for any } \ell \in \mathcal{W}_0',$$

where an element of $\mathcal{W}_0'$ is regarded as a random variable on $\mathcal{W}_0$ in the natural manner. The triplet $(\mathcal{W}_0, \mathcal{H}_0, \mu)$ is the one dimensional classical Wiener space of pinned Brownian motion on $[0, 1]$.

Let b_t, $t \in [0, 1]$, be the coordinate mapping on $\mathcal{W}_0$; $b_t(w) = w(t)$, $w \in \mathcal{W}_0$. Due to (1), $\{b_t\}_{t \in [0,1]}$ is a continuous Gaussian process with mean zero and covariance function

$$\int_{\mathcal{W}_0} b_t b_s d\mu = t \wedge s - ts, \quad t, s \in [0, 1],$$

where $t \wedge s = \min\{t, s\}$.

2.2. QUADRATIC WIENER FUNCTIONALS

For a separable Hilbert space K, we say that a K-valued Wiener functional F belongs to $\boldsymbol{D}^\infty(K)$ if F is infinitely differentiable in the sense of the Malliavin calculus, and its derivatives of all orders including itself are p-th integrable with respect to μ for any $p > 1$ (cf. [5, 8]). The k-th derivative of F is denoted by $\nabla^k F$. Denoting by $\mathcal{H}_0 \otimes K$ the Hilbert space of Hilbert-Schmidt operators of $\mathcal{H}_0$ to K equipped with the Hilbert-Schmidt norm, we define the adjoint operator $\nabla^* : \boldsymbol{D}^\infty(\mathcal{H}_0 \otimes K) \to \boldsymbol{D}^\infty(K)$ of ∇ by

$$\int_{\mathcal{W}_0} \langle \nabla^* F, G \rangle d\mu = \int_{\mathcal{W}_0} \langle F, \nabla G \rangle d\mu,$$
$$F \in \boldsymbol{D}^\infty(\mathcal{H}_0 \otimes K), G \in \boldsymbol{D}^\infty(K),$$

where, for the sake of simplicity, we have also used $\langle \cdot, \cdot \rangle$ to denote the inner products on $\mathcal{H}_0 \otimes K$ and K.

Thinking of a symmetric $U \in \mathcal{H}_0^{\otimes 2} \equiv \mathcal{H}_0 \otimes \mathcal{H}_0$ as a constant function on $\mathcal{W}_0$ with values in $\mathcal{H}_0^{\otimes 2}$, we set

$$Q_U = (\nabla^*)^2 U.$$

If U is of trace class, we define

$$q_U = Q_U + \operatorname{tr} U.$$

It is easily checked ([5, 8]) that the third derivative of functional $F \in \boldsymbol{D}^\infty(\boldsymbol{R})$ vanishes, i.e., $\nabla^3 F = 0$, if and only if F admits an expression as

$$F = \frac{1}{2} Q_U + \nabla^* h + c,$$

where

$$U = \nabla^2 F, \quad h = \int_X \nabla F d\nu, \quad \text{and} \quad c = \int_X F d\nu.$$

It is known (cf. [6]) that

Proposition 1. *Let $U \in \mathcal{H}_0^{\otimes 2}$ be symmetric, and $h \in \mathcal{H}_0$. For $\zeta \in \boldsymbol{C}$ with $|\zeta| < 1/\|U\|_{\mathrm{op}}$, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm, it holds that*

$$\int_{\mathcal{W}_0} e^{(\zeta/2)Q_U + \eta \nabla^* h} d\mu$$
$$= \left\{ \det{}_2(I - \zeta U) \right\}^{-1/2} e^{\eta^2 \langle (I - \zeta U)^{-1} h, h \rangle / 2}$$

for every $\eta \in \boldsymbol{C}$, where $\det_2$ is the Carleman-Fredholm determinant, and $\langle \cdot, \cdot \rangle$ was extended complex bi-linearly to the complexified Hilbert space $\mathcal{H}_0 \otimes \boldsymbol{C}$ of $\mathcal{H}_0$. If, in addition, U is of trace class, then

$$\int_{\mathcal{W}_0} e^{(\zeta/2)q_U + \eta \nabla^* h} d\mu$$
$$= \left\{ \det(I - \zeta U) \right\}^{-1/2} e^{\eta^2 \langle (I - \zeta U)^{-1} h, h \rangle / 2},$$

where $\det$ is the Fredholm determinant.

It is a routine to extend the above identity to ζ's in a much larger domain in $\boldsymbol{C}$ by holomorphic continuation.

Such a Wiener integral as in the above proposition is called a stochastic oscillatory integral, and is a probabilistic counterpart to the Feynman path integrals.

2.3. SQUARE NORM ON MEASURE SPACE

We review briefly the result in [10].

Let E be a topological space, $\mathcal{E}$ its Borel σ-field, and σ a σ-finite measure on $(E, \mathcal{E})$. Consider a continuous mapping $E \ni e \mapsto f_e \in \mathcal{H}_0$, where the topology of $\mathcal{H}_0$ is the strong one, i.e., comes from the norm. Assume that

$$(A0) \qquad \int_E \|f_e\|^2 \sigma(de) < \infty.$$

Due to Assumption (A0), one can define $F \in L^2(\mu)$ by the Bochner integral

$$F = \frac{1}{2} \int_E (\nabla^* f_e)^2 \sigma(de),$$

and the non-negative definite, symmetric Hilbert-Schmidt operator $A : \mathcal{H}_0 \to \mathcal{H}_0$ of trace class by

$$\langle Ah, g \rangle = \int_E \langle f_e, h \rangle \langle f_e, g \rangle \sigma(de), \quad h, g \in \mathcal{H}_0.$$

Then

$$\operatorname{tr} A = \int_E \|f_e\|^2 \sigma(de) \quad \text{and} \quad F = q_A/2.$$

Denote by $a_1, a_2, \ldots$ the eigenvalues of A, counted with multiplicity, and $\{h_n\}_{n=1}^\infty$ the corresponding orthonormal

basis (ONB in short) of $\mathcal{H}_0$. In conjunction with Proposition 1, we have that

$$(2) \quad \int_{\mathcal{W}_0} e^{\zeta F + \eta \nabla^* h} d\mu$$

$$= \left\{ \prod_{n=1}^{\infty} (1 - \zeta a_n) \right\}^{-1/2} \exp\left(\frac{\eta^2}{2} \sum_{n=1}^{\infty} \frac{\langle h_n, h \rangle_H^2}{1 - \zeta a_n} \right)$$

for $\zeta, \eta \in \boldsymbol{C}$ with $|\zeta| < 1/\max\{a_n : n = 1, 2, \dots\}$.

3. STOCHASTIC OSCILLATORY INTEGRAL ASSOCIATED WITH $\mathfrak{a}^2(\psi)$

In this section, we investigate the stochastic oscillatory integral associated with general $\mathfrak{a}^2(\psi)$.

3.1. GENERAL $\mathfrak{a}^2(\psi)$

In this subsection, we apply the results in the previous section to general $\mathfrak{a}^2(\psi)$.

Let $\psi : (0, 1) \to [0, \infty)$ be Borel measurable and assume that

$$(A1) \qquad \int_0^1 t(1 - t)\psi(t)dt < \infty.$$

For each $t \in [0, 1]$, we consider the coordinate mapping b_t as an element of $\mathcal{W}_0' \subset \mathcal{H}_0$, and, to emphasize this, write f_t for b_t;

$$f_t(w) = w(t), \quad w \in \mathcal{W}_0.$$

For square integrable $v : [0, 1] \to \boldsymbol{R}$, defining $h_v \in \mathcal{H}_0$ by

$$h_v(t) = \int_0^t v(s)ds - \left(\int_0^1 v(s)ds \right)t, \quad t \in [0, 1],$$

we have that

$$(3) \qquad \langle h_v, g \rangle = \int_0^1 v(s)g'(s)ds, \quad g \in \mathcal{H}_0.$$

Hence $f_t \in \mathcal{H}_0$ is represented as

$$f_t(s) = t \wedge s - ts, \quad s \in [0, 1].$$

In particular,

$$\|f_t\|^2 = t(1 - t).$$

By Assumption (A1),

$$\int_0^1 \|f_t\|^2 \psi(t)dt < \infty,$$

and hence Assumption (A0) in Subsection 2.3 is satisfied with $E = [0, 1]$ and $\sigma(dt) = \psi(t)dt$. Thus, we can now define the quadratic Wiener functional

$$F_\psi = \frac{1}{2} \int_0^1 (\nabla^* f_t)^2 \psi(t)dt.$$

It is easily seen (cf. [5, 8]) that

$$\nabla^* \ell = \ell \quad \text{for every } \ell \in \mathcal{W}_0'.$$

This yields that $\nabla^* f_t(w) = w(t)$, $w \in \mathcal{W}_0$, and hence

$$F_\psi = \frac{1}{2} \mathfrak{a}^2(\psi)$$

As was seen in Subsection 2.3,

$$(4) \qquad \mathfrak{a}^2(\psi) = 2F_\psi = q_{A_\psi},$$

where the Hilbert-Schmidt operator $A_\psi : \mathcal{H}_0 \to \mathcal{H}_0$ is given by

$$\langle A_\psi h, g \rangle = \int_0^1 h(t)g(t)\psi(t)dt, \quad h, g \in \mathcal{H}_0.$$

In what follows, we investigate the eigenvalues of A_ψ. Since A_ψ is non-negative definite and symmetric, the eigenvalues of A_ψ are all non-negative real numbers. Consider the following conditions;

$$(A2) \qquad \int_0^1 \{t(1 - t)\}^{1/2} \psi(t)dt < \infty.$$

$$(A3) \qquad \psi > 0 \text{ a.e. on } (0, 1).$$

$$(A4) \qquad \psi \text{ is continuous on } (0, 1).$$

Since $|h(t)| \le \sqrt{2t(1 - t)}\, \|h\|$ for $t \in [0, 1]$, under (A2), $h\psi$ is integrable on $(0, 1)$.

Proposition 2. (i) *Assume* (A2). *Then it holds that*

$$(5) \quad (A_\psi h)(t) = -\int_0^t \int_0^s h(u)\psi(u)duds$$

$$+ \left(\int_0^1 \int_0^s h(u)\psi(u)duds \right)t, \quad t \in [0, 1].$$

(ii) *Assume* (A2) *and* (A3). *Then* $\ker A_\psi = \{0\}$.
(iii) *Assume* (A2) *and* (A4). *Let* $h \in \mathcal{H}_0$. *Then* h *is an eigenfunction corresponding to an eigenvalue* $\lambda \ge 0$ *of* A_ψ *if and only if* $h \in C^1([0, 1]) \cap C^2((0, 1))$ *and satisfies that*

$$(6) \qquad \lambda h'' = -\psi h \quad on (0, 1).$$

Before proceeding to the proof, using the assertion (iii), we give the theorem on the decomposition of A_ψ and the infinite product expression of the oscillatory integral associated with $\mathfrak{a}^2(\psi)$. To do so, we introduce some notation. For $\varphi \in C([0, 1])$ and $a \in \boldsymbol{R}$, we denote by $\sigma(\varphi; a)$ the set of all $g \in C([0, 1]) \cap C^2((0, 1))$ such that $g \not\equiv 0$, $g(0) = g(1) = 0$ and $\varphi g'' = ag$ on $(0, 1)$, and set

$$\sigma(\varphi) = \{a \in \boldsymbol{R} \mid \sigma(\varphi; a) \ne \emptyset\}.$$

Theorem 1. *Assume* (A2), (A4), *and that*

$$(A5) \qquad \psi > 0 \text{ on } (0, 1) \text{ and } 1/\psi \in C([0, 1]),$$

where "$1/\psi \in C([0,1])$" means that $1/\psi : (0,1) \to (0,\infty)$ admits a continuous extension to $[0,1]$.

(i) If $-\lambda < 0, \in \sigma(1/\psi)$ and there is a $g \in \sigma(1/\psi; -\lambda)$ such that $g\psi$ is integrable on $[0,1]$, then $1/\lambda$ is an eigenvalue of A_ψ, and g is an eigenfunction associated with $1/\lambda$.

(ii) Suppose that there exist $-\lambda_n < 0, \in \sigma(1/\psi)$ and $g_n \in \sigma(1/\psi; -\lambda_n)$, $n = 1, 2, \ldots$, such that each $g_n\psi$ is integrable on $[0,1]$ and $\{g_n\}_{n=1}^\infty$ is an ONB of $\mathcal{H}_0$. Then

$$A_\psi = \sum_{n=1}^\infty \frac{1}{\lambda_n} g_n \otimes g_n,$$

where $h \otimes h \in \mathcal{H}_0 \otimes \mathcal{H}_0$ is defined by $(h \otimes h)(g) = \langle h, g \rangle h$, $g \in \mathcal{H}_0$. Moreover, it holds that

$$(7) \quad \int_{\mathcal{W}_0} e^{\zeta \mathfrak{a}^2(\psi) + \eta \nabla^* h} d\mu$$
$$= \left\{ \prod_{n=1}^\infty \left(1 - \frac{2\zeta}{\lambda_n} \right) \right\}^{-1/2} \exp\left(\frac{\eta^2}{2} \sum_{n=1}^\infty \frac{\langle h, g_n \rangle^2}{1 - (2\zeta/\lambda_n)} \right)$$

for $\zeta, \eta \in \mathbf{C}$ with $|\zeta| < 1/(2 \max\{1/\lambda_n : n = 1, 2, \ldots\})$ and $h \in \mathcal{H}_0$.

Proof. (i) By the assumption, it holds that

$$g'' = -\frac{1}{\lambda} g\psi \quad \text{on } (0,1).$$

Since $g\psi$ is integrable on $[0,1]$, this implies that g' extends to a continuous function on $[0,1]$. Hence $g \in C^1([0,1])$, and hence $g \in \mathcal{H}_0$. By Proposition 2 (iii), λ is an eigenvalue of A_ψ, and g is a corresponding eigenfunction.

(ii) In conjunction with (2) and (4), the assertion (i) yields the second assertion. $\square$

Proof of Proposition 2. (i) By an elementary change of variables, we see that

$$\int_0^1 h(t)g(t)\psi(t)dt = \int_0^1 g'(t)\left(-\int_0^t h(u)\psi(u)du \right)dt.$$

By (3), this implies (5).

(ii) Suppose that $A_\psi h = 0$. By Assumption (A2) and (5), $h\psi = 0$ a.e. Then Assumption (A3) and the continuity of h implies that $h = 0$.

(iii) The necessity is an immediate consequence of (i) and the continuity of $h\psi$ on $(0,1)$.

Conversely, suppose that $h \in \mathcal{H}_0$ belongs to $C^1([0,1]) \cap C^2((0,1))$ and satisfies (6). Due to the continuity of h' on $[0,1]$ and the integrability of $h\psi$ on $[0,1]$, integrating (6), we obtain that

$$\lambda h'(s) = \lambda h'(0) - \int_0^s h(u)\psi(u)du, \quad 0 \le s \le 1.$$

Since $h(0) = 0$, integrating again, we see that

$$\lambda h(t) = \lambda h'(0)t - \int_0^t \int_0^s h(u)\psi(u)duds, \quad 0 \le t \le 1.$$

This yields that

$$\lambda h(t) = -\int_0^t \int_0^s h(u)\psi(u)duds$$
$$+ \left(\int_0^1 \int_0^s h(u)\psi(u)duds \right)t, \quad 0 \le t \le 1,$$

which, by the assertion (i), means that $A_\psi h = \lambda h$. $\square$

It is an easy exercise of Calculus to see that the eigenvalues of A_ψ is also computable via the following integral kernel which was used in [1] to obtain the Fourier decomposition of the Gaussian process $\{b_t \sqrt{\psi(t)}\}_{t \in [0,1]}$.

Proposition 3. *Assume* (A2). *Then it holds that*

$$(8) \quad A_\psi h(t) = \int_0^1 (t \wedge s - ts)\psi(s)h(s)ds, \quad t \in [0,1].$$

In particular, if $\lambda > 0$ is an eigenvalue of A_ψ and $h \in \mathcal{H}_0$ is a corresponding eigenfunction, then $g(t) = \sqrt{\psi(t)}\, h(t)$ satisfies the integral equation with symmetric kernel $(t \wedge s - ts)\sqrt{\psi(t)}\,\sqrt{\psi(s)}$;

$$\lambda g(t) = \int_0^1 (t \wedge s - ts)\sqrt{\psi(t)}\,\sqrt{\psi(s)}\, g(s)ds, \quad t \in [0,1].$$

3.2. Anderson-Darling test statistic

We now restrict our attention to the case where

$$\psi(t) = \psi_{\mathrm{AD}}(t) = \frac{1}{t(1-t)},$$

i.e. the case corresponding to the Anderson-Darling test statistic. ψ_{AD} satisfies Assumptions (A1)–(A5). Our aim of this subsection is to calculate eigenvalues and eigenfunctions of $A_{\psi_{\mathrm{AD}}}$. To state the main result of this subsection, we need some notation. We denote by $P_n(x)$ $n = 0, 1, \ldots$, the Legendre polynomials defined as the coefficients of the Taylor expansion of $(1 - 2x\eta + \eta^2)^{-1/2}$ in η ([11]);

$$(1 - 2x\eta + \eta^2)^{-1/2} = \sum_{n=0}^\infty P_n(x)\eta^n.$$

It admits the explicit expression as

$$P_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{(2n-2j)!}{2^n j!(n-j)!(n-2j)!} x^{n-2j},$$

where $[a]$ stands for the largest integer less than or equal to a. Our first goal of this subsection is that

Theorem 2. *Let*

$$h_n(t) = 4t(1-t)P_n'(2t-1), \quad t \in [0,1]$$

and $\widetilde{h}_n = h_n/\|h_n\|$, $n = 1, 2, \ldots$ It then holds that

$$(9) \quad A_{\psi_{\mathrm{AD}}} = \sum_{n=1}^\infty \frac{1}{n(n+1)} \widetilde{h}_n \otimes \widetilde{h}_n,$$

Moreover, it holds that

$$(10) \quad \int_{\mathcal{W}_0} e^{\zeta \mathfrak{a}^2(\psi_{\mathrm{AD}}) + \eta \nabla^* h} d\mu$$

$$= \left\{ \prod_{n=1}^{\infty} \left(1 - \frac{2\zeta}{n(n+1)} \right) \right\}^{-1/2}$$

$$\times \exp \left(\frac{\eta}{2} \sum_{n=1}^{\infty} \frac{\langle h, h_n \rangle^2}{1 - (2\zeta/\{n(n+1)\})} \right)$$

for $\zeta, \eta \in \mathbf{C}$ with $|\zeta| < 1$ and $h \in \mathcal{H}_0$.

Proof. The proof is carried out by adapting the observation in [1] to our framework. By Theorem 1, it suffices to study $\sigma(1/\psi_{\mathrm{AD}}; a)$. Thus we need to investigate the ordinary differential equation (ODE in short);

$$t(1-t)h''(t) = \lambda h(t), \quad t \in (0,1), \quad h(0) = h(1) = 0.$$

Through the change of variables $t = (x+1)/2$, we shall investigate the ODE

$$(11) \quad \begin{cases} (1-x^2)g''(x) = \lambda g(x), & x \in (-1,1), \\ g(-1) = g(1) = 0. \end{cases}$$

The Ferrers' associated Legendre function of degree n and order 1 given by

$$P_{n,1}(x) = (1-x^2)^{1/2} P_n'(x)$$

is an eigenfunction associated with the eigenvalue $-n(n+1)$ of the differential operator

$$\mathcal{L}_1 = (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{1}{1-x^2} \quad \text{on } (-1,1).$$

For details, see [11]. Moreover, notice that, for any $\phi \in C^2((-1,1))$, $g \equiv (1-x^2)^{1/2}\phi$ satisfies that

$$(1-x^2)g'' = (1-x^2)^{1/2}\mathcal{L}_1\phi \quad \text{on } (-1,1).$$

Thus

$$g_n = (1-x^2)^{1/2} P_{n,1} = (1-x^2) P_n'$$

solves the ODE (11) with $\lambda = -n(n+1)$.

Define

$$h_n(t) = 4t(1-t) P_n'(2t-1) = g_n(2t-1), \quad t \in [0,1].$$

Then

$$h_n \in \sigma(1/\psi_{AD}; -n(n+1)).$$

Since $h_n \in C^{\infty}([0,1])$ and $h_n(0) = h_n(1) = 0$, $h_n\psi_{\mathrm{AD}}$ is integrable. By Theorem 1, $h_n \in \mathcal{H}_0$ and

$$(12) \quad A_{\psi_{\mathrm{AD}}} h_n = \frac{1}{n(n+1)} h_n.$$

We shall now show that $\{h_n\}_{n=1}^{\infty}$ is dense in $\mathcal{H}_0$. To do so, let

$$k_n(t) = t^{n+1} - t, \quad t \in [0,1], n = 1, 2, \ldots$$

Suppose that $h \in \mathcal{H}_0$ is perpendicular to all k_n, $n = 1, 2, \ldots$ Then

$$0 = \langle h, k_n \rangle = (n+1) \int_0^1 h'(t) t^n dt, \quad n = 1, 2, \ldots$$

This implies that h' is constant, and hence $h = 0$. Thus $\{k_n\}_{n=1}^{\infty}$ spans $\mathcal{H}_0$. Since P_n' and $k_n/\{t(1-t)\}$ are both polynomials of order $n-1$, $\{h_n\}_{n=1}^{\infty}$ also spans $\mathcal{H}_0$.

Thus the expression (9) holds. The identity (10) is an immediate consequence of (9) and Theorem 1. $\square$

The infinite product in Theorem 2 can be replaced by an elementary function;

Corollary 1. *It holds that*

$$(13) \quad \int_{\mathcal{W}_0} e^{\zeta \mathfrak{a}^2(\psi_{\mathrm{AD}}) + \eta \nabla^* h} d\mu = \left(\frac{-2\pi\zeta}{\cos(\frac{\pi}{2}\sqrt{1+8\zeta})} \right)^{1/2}$$

$$\times \exp \left(\frac{\eta}{2} \sum_{n=1}^{\infty} \frac{\langle h, h_n \rangle^2}{1 - (2\zeta/\{n(n+1)\})} \right)$$

for $\zeta, \eta \in \mathbf{C}$ with $|\zeta| < 1$ and $\operatorname{Re} \zeta > -1/8$, and $h \in \mathcal{H}_0$.

Proof. Recall the well known identity that

$$\cos\left(\frac{\pi}{2} z \right) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(2n-1)^2} \right).$$

Substitute $z = \sqrt{1+4x}$ to have

$$\cos\left(\frac{\pi}{2}\sqrt{1+4x} \right) = (-4x) \prod_{n=2}^{\infty} \left(1 - \frac{4x+1}{(2n-1)^2} \right).$$

Since

$$1 - \frac{4x+1}{(2n-1)^2} = \left(1 - \frac{x}{n(n-1)} \right)\left(1 - \frac{1}{(2n-1)^2} \right),$$

we obtain that

$$\cos\left(\frac{\pi}{2}\sqrt{1+4x} \right)$$

$$= (-4x) \prod_{n=1}^{\infty} \left(1 - \frac{x}{n(n+1)} \right) \lim_{x \to 1} \frac{\cos((\pi/2)x)}{1-x^2}$$

$$= -\pi x \prod_{n=1}^{\infty} \left(1 - \frac{x}{n(n+1)} \right).$$

Thus, by Theorem 2, we obtain the desired identity. $\square$

Acknowledgments

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Takehiro Hirotsu, Hiroki Kondo and Takuya Sato
Nisshin Fire & Marine Insurance Co., Ltd,
2-3 Kanda Surugadai, Chiyoda-ku, Tokyo, 101-8329, Japan
E-mail: takehiro.hirotsu(at)nisshinfire.co.jp
        hiroki.kondou(at)nisshinfire.co.jp
        takuya.satou(at)nisshinfire.co.jp

Shingo Saito, Tatsushi Tanaka and Setsuo Taniguchi
Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan
E-mail: ssaito(at)math.kyushu-u.ac.jp
        t.tanaka(at)math.kyushu-u.ac.jp
        taniguch(at)math.kyushu-u.ac.jp