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An Explicit Formula for the Discrete Power Function Associated with Circle Patterns of Schramm Type

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Abstract

We present an explicit formula for the discrete power function introduced by Bobenko, which is expressed in terms of the hypergeometric τ functions for the sixth Painlevé equation. The original definition of the discrete power function imposes strict conditions on the domain and the value of the exponent. However, we show that one can extend the value of the exponent to arbitrary complex numbers except even integers and the domain to a discrete analogue of the Riemann surface.

1 Introduction

The theory of discrete analytic functions has been developed in recent years based on the theory of circle packings or circle patterns, which was initiated by Thurston's idea of using circle packings as an approximation of the Riemann mapping [17]. So far many important properties have been established for discrete analytic functions, such as the discrete maximum principle and Schwarz's lemma [5], the discrete uniformization theorem [14], and so forth. For a comprehensive introduction to the theory of discrete analytic functions, we refer to [16].

It is known that certain circle patterns with fixed regular combinatorics admit rich structure. For example, it has been pointed out that the circle patterns with square grid combinatorics introduced by Schramm [15] and the hexagonal circle patterns [4, 7, 8] are related to integrable systems. Some explicit examples of discrete analogues of analytic functions have been presented which are associated with Schramm's patterns: $\exp(z)$, $\mathrm{erf}(z)$, Airy function [15], z^γ, $\log(z)$ [3]. Also, discrete analogues of z^γ and $\log(z)$ associated with hexagonal circle patterns are discussed in [4, 7, 8].

Among those examples, it is remarkable that the discrete analogue of the power function z^γ associated with the circle patterns of Schramm type has a close relationship with the sixth Painlevé equation ($\mathrm{P_{VI}}$) [6], and this fact has been used to establish the immersion property [3] and embeddedness [1] of the discrete power function. It is desirable to construct a representation formula for the discrete power function in terms of the Painlevé transcendents as was mentioned in [6]. The discrete power function can be formulated as a solution to a system of difference equations on the square lattice $(n, m) \in \mathbb{Z}^2$ with a certain initial condition. A correspondence between the dependent variable of this system and the Painlevé transcendents can be found in [13], but the formula seems somewhat indirect. Agafonov has constructed an explicit representation formula in terms of the Gauss hypergeometric function [2], however, this formula is valid only on some special points on $\mathbb{Z}^2$. In this paper, generalizing Agafonov's result, we aim to establish an explicit representation formula of the discrete power function in terms of the hypergeometric τ function of $\mathrm{P_{VI}}$ which is valid on $\mathbb{Z}^2_+ = \{(n, m) \in \mathbb{Z}^2 \,|\, n, m \geq 0\}$ and for $\gamma \in \mathbb{C}\backslash 2\mathbb{Z}$. Based on this formula, we generalize the domain of the discrete power function to a discrete analogue of the Riemann surface.

This paper is organized as follows. In section 2, we give a brief review of the definition of the discrete power function and its relation to $\mathrm{P_{VI}}$. The main result and its proof are given in section 3. We discuss the extension of the domain of the discrete power function in section 4. Section 5 is devoted to concluding remarks.

2 Discrete power function

2.1 Definition of the discrete power function

For maps, a discrete analogue of conformality has been proposed by Bobenko and Pinkall in the framework of discrete differential geometry [9].

Definition 2.1 *A map $f : \mathbb{Z}^2 \to \mathbb{C}\,;\ (n, m) \mapsto f_{n,m}$ is called discrete conformal if the cross-ratio with respect to every elementary quadrilateral is equal to -1:*

$$\frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1. \tag{2.1}$$

The condition (2.1) is a discrete analogue of the Cauchy-Riemann relation. Actually, a smooth map $f : D \subset \mathbb{C} \to \mathbb{C}$ is conformal if and only if it satisfies

$$\lim_{\epsilon \to 0} \frac{(f(x, y) - f(x + \epsilon, y))(f(x + \epsilon, y + \epsilon) - f(x, y + \epsilon))}{(f(x + \epsilon, y) - f(x + \epsilon, y + \epsilon))(f(x, y + \epsilon) - f(x, y))} = -1 \tag{2.2}$$

for all $(x, y) \in D$. However, using Definition 2.1 alone, one cannot exclude maps whose behavior is far from that of usual holomorphic maps. Because of this, an additional condition for a discrete conformal map has been considered [1, 3, 6, 10].

Definition 2.2 *A discrete conformal map $f_{n,m}$ is called embedded if inner parts of different elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ do not intersect.*

An example of an embedded map is presented in Figure 1. This condition seems to require that $f = f_{n,m}$ is a univalent function in the continuous limit, and is too strict to capture a wide class of discrete holomorphic functions. In fact, a relaxed requirement has been considered as follows [1,3].

Definition 2.3 *A discrete conformal map $f_{n,m}$ is called immersed, or an immersion, if inner parts of adjacent elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are disjoint.*

See Figure 2 for an example of an immersed map.

Let us give the definition of the discrete power function proposed by Bobenko [3,6,10].

Definition 2.4 *Let $f : \mathbb{Z}_+^2 \to \mathbb{C}\,;\,(n,m) \mapsto f_{n,m}$ be a discrete conformal map. If $f_{n,m}$ is the solution to the difference equation*

$$\gamma f_{n,m} = 2n\frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{f_{n+1,m} - f_{n-1,m}} + 2m\frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{f_{n,m+1} - f_{n,m-1}} \tag{2.3}$$

with the initial conditions

$$f_{0,0} = 0, \quad f_{1,0} = 1, \quad f_{0,1} = e^{\gamma\pi i/2} \tag{2.4}$$

for $0 < \gamma < 2$, then we call f a discrete power function.

The difference equation (2.3) is a discrete analogue of the differential equation $\gamma f = z\dfrac{\partial f}{\partial z}$ for the power function $f(z) = z^\gamma$, which means that the parameter γ corresponds to the exponent of the discrete power function.

It is easy to get the explicit formula of the discrete power function for $m = 0$ (or $n = 0$). When $m = 0$, (2.3) is reduced to a three-term recurrence relation. Solving it with the initial condition $f_{0,0} = 0$, $f_{1,0} = 1$, we have

$$f_{n,0} = \begin{cases} \dfrac{2l}{2l+\gamma}\displaystyle\prod_{k=1}^{l}\frac{2k+\gamma}{2k-\gamma} & (n = 2l), \\ \displaystyle\prod_{k=1}^{l}\frac{2k+\gamma}{2k-\gamma} & (n = 2l+1), \end{cases} \tag{2.5}$$

for $n \in \mathbb{Z}_+$. When $m = 1$ (or $n = 1$), Agafonov has shown that the discrete power function can be expressed in terms of the hypergeometric function [2]. One of the aims of this paper is to give an explicit formula for the discrete power function $f_{n,m}$ for arbitrary $(n,m) \in \mathbb{Z}_+^2$.

In Definition 2.4, the domain of the discrete power function is restricted to the “first quadrant” $\mathbb{Z}_+^2$, and the exponent γ to the interval $0 < \gamma < 2$. Under this condition, it has been shown that the discrete power function is embedded [1]. For our purpose, we do not have to persist with such a restriction. In fact, the explicit formula we will give is applicable to the case $\gamma \in \mathbb{C}\backslash 2\mathbb{Z}$. Regarding the domain, one can extend it to a discrete analogue of the Riemann surface.

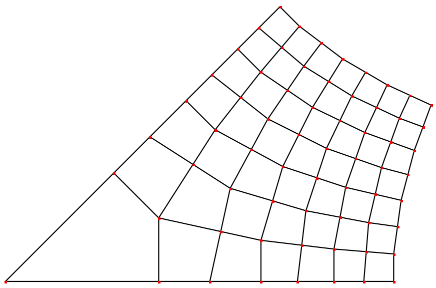

Figure 1: An example of the embedded discrete conformal map.

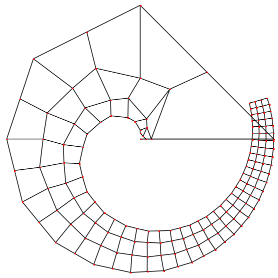

Figure 2: An example of the discrete conformal map that is not embedded but immersed.

2.2 Relationship to P_{VI}

In order to construct an explicit formula for the discrete power function $f_{n,m}$, we will move to a more general setting. The cross-ratio condition (2.1) can be regarded as a special case of the discrete Schwarzian KdV equation

$$\frac{(f_{n,m}-f_{n+1,m})(f_{n+1,m+1}-f_{n,m+1})}{(f_{n+1,m}-f_{n+1,m+1})(f_{n,m+1}-f_{n,m})}=\frac{p_n}{q_m}, \tag{2.6}$$

where p_n and q_m are arbitrary functions in the indicated variables. Some of the authors have constructed various special solutions to the above equation [11]. In particular, they have shown that an autonomous case

$$\frac{(f_{n,m}-f_{n+1,m})(f_{n+1,m+1}-f_{n,m+1})}{(f_{n+1,m}-f_{n+1,m+1})(f_{n,m+1}-f_{n,m})}=\frac{1}{t}, \tag{2.7}$$

where t is independent of n and m, can be regarded as a part of the Bäcklund transformations of P_{VI}, and given special solutions to (2.7) in terms of the τ functions of P_{VI}.

We here give a brief account of the derivation of P_{VI} according to [13]. The derivation is achieved by imposing a certain similarity condition on the discrete Schwarzian KdV equation (2.7) and the difference equation (2.3) simultaneously. The discrete Schwarzian KdV equation (2.7) is automatically satisfied if there exists a function $v_{n,m}$ satisfying

$$f_{n,m}-f_{n+1,m}=t^{-1/2}v_{n,m}v_{n+1,m},\quad f_{n,m}-f_{n,m+1}=v_{n,m}v_{n,m+1}. \tag{2.8}$$

By eliminating the variable $f_{n,m}$, we get for $v_{n,m}$ the following equation

$$t^{1/2}v_{n,m}v_{n,m+1}+v_{n,m+1}v_{n+1,m+1}=v_{n,m}v_{n+1,m}+t^{1/2}v_{n+1,m}v_{n+1,m+1}, \tag{2.9}$$

which is equivalent to the lattice modified KdV equation. It can be shown that the difference equation (2.3) is reduced to

$$n\frac{v_{n+1,m}-v_{n-1,m}}{v_{n+1,m}+v_{n-1,m}}+m\frac{v_{n,m+1}-v_{n,m-1}}{v_{n,m+1}+v_{n,m-1}}=\mu-(-1)^{m+n}\lambda \tag{2.10}$$

with $\gamma=1+2\mu$, where $\lambda\in\mathbb{C}$ is an integration constant. In the following we take $\lambda=\mu$ so that (2.10) is consistent when $n=m=0$ and $v_{1,0}+v_{-1,0}\neq 0\neq v_{0,1}+v_{0,-1}$.

Assume that the dependence of the variable $v_{n,m} = v_{n,m}(t)$ on the deformation parameter t is given by

$$-2t\frac{d}{dt}\log v_{n,m} = n\frac{v_{n+1,m} - v_{n-1,m}}{v_{n+1,m} + v_{n-1,m}} + \chi_{n+m}, \tag{2.11}$$

where $\chi_{n+m} = \chi_{n+m}(t)$ is an arbitrary function satisfying $\chi_{n+m+2} = \chi_{n+m}$. Then we have the following Proposition.

Proposition 2.5 *Let $q = q_{n,m} = q_{n,m}(t)$ be the function defined by $q_{n,m} = t^{1/2}\frac{v_{n+1,m}}{v_{n,m+1}}$. Then q satisfies P_{VI}*

$$\begin{aligned} \frac{d^2q}{dt^2} = &\frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt} \\ &+ \frac{q(q-1)(q-t)}{2t^2(t-1)^2}\left[\kappa_\infty^2 - \kappa_0^2\frac{t}{q^2} + \kappa_1^2\frac{t-1}{(q-1)^2} + (1-\theta^2)\frac{t(t-1)}{(q-t)^2}\right], \end{aligned} \tag{2.12}$$

with

$$\begin{aligned} \kappa_\infty^2 &= \frac{1}{4}(\mu - \nu + m - n)^2, & \kappa_0^2 &= \frac{1}{4}(\mu - \nu - m + n)^2, \\ \kappa_1^2 &= \frac{1}{4}(\mu + \nu - m - n - 1)^2, & \theta^2 &= \frac{1}{4}(\mu + \nu + m + n + 1)^2, \end{aligned} \tag{2.13}$$

where we denote $\nu = (-1)^{m+n}\mu$.

In general, $\mathrm{P_{VI}}$ contains four complex parameters denoted by $\kappa_\infty, \kappa_0, \kappa_1$ and θ. Since $n, m \in \mathbb{Z}_+$, a special case of $\mathrm{P_{VI}}$ appears in the above proposition, which corresponds to the case where $\mathrm{P_{VI}}$ admits special solutions expressible in terms of the hypergeometric function. In fact, the special solutions to $\mathrm{P_{VI}}$ of hypergeometric type are given as follows:

Proposition 2.6 *[12] Define the function $\tau_{n'}(a,b,c;t)$ $(c \notin \mathbb{Z},\ n' \in \mathbb{Z}_+)$ by*

$$\tau_{n'}(a,b,c;t) = \begin{cases} \det\left(\varphi(a+i-1, b+j-1, c; t)\right)_{1\le i,j\le n'} & (n' > 0), \\ 1 & (n' = 0), \end{cases} \tag{2.14}$$

with

$$\begin{aligned} \varphi(a,b,c;t) = &c_0\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b,c;t) \\ &+ c_1\frac{\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(2-c)}t^{1-c}F(a-c+1, b-c+1, 2-c; t). \end{aligned} \tag{2.15}$$

Here, $F(a,b,c;t)$ is the Gauss hypergeometric function, $\Gamma(x)$ is the Gamma function, and c_0 and c_1 are arbitrary constants. Then

$$q = \frac{\tau_{n'}^{0,-1,0}\tau_{n'+1}^{-1,-1,-1}}{\tau_{n'}^{-1,-1,-1}\tau_{n'+1}^{0,-1,0}} \tag{2.16}$$

with $\tau_{n'}^{k,l,m} = \tau_{n'}(a+k+1, b+l+2, c+m+1; t)$ gives a family of hypergeometric solutions to $\mathrm{P_{VI}}$ with the parameters

$$\kappa_\infty = a + n', \quad \kappa_0 = b - c + 1 + n', \quad \kappa_1 = c - a, \quad \theta = -b. \tag{2.17}$$

We call $\tau_{n'}(a,b,c;t)$ or $\tau_{n'}^{k,l,m}$ the hypergeometric τ function of $\mathrm{P_{VI}}$.

3 Main Results

3.1 Explicit formulae for $f_{n,m}$ and $v_{n,m}$

We present the solution to the simultaneous system of the discrete Schwarzian KdV equation (2.7) and the difference equation (2.3) under the initial conditions

$$f_{0,0} = 0, \quad f_{1,0} = c_0, \quad f_{0,1} = c_1 t^r, \tag{3.1}$$

where $\gamma = 2r$, and c_0 and c_1 are arbitrary constants. We set $c_0 = c_1 = 1$ and $t = e^{\pi i}(= -1)$ to obtain the explicit formula for the original discrete power function. Note that $\tau_{n'}(b, a, c; t) = \tau_{n'}(a, b, c; t)$ by the definition. Moreover, we interpret $F(k, b, c; t)$ for $k \in \mathbb{Z}_{>0}$ as $F(k, b, c; t) = 0$ and $\Gamma(-k)$ for $k \in \mathbb{Z}_{\geq 0}$ as $\Gamma(-k) = \dfrac{(-1)^k}{k!}$.

Theorem 3.1 *For $(n, m) \in \mathbb{Z}_+^2$, the function $f_{n,m} = f_{n,m}(t)$ can be expressed as follows.*

(1) *Case where $n \leq m$ (or $n' = n$). When $n + m$ is even, we have*

$$f_{n,m} = c_1 t^{r-n} N \frac{(r+1)_{N-1}}{(-r+1)_N} \frac{\tau_n(-N, -r-N+1, -r; t)}{\tau_n(-N+1, -r-N+2, -r+2; t)}, \tag{3.2}$$

where $N = \dfrac{n+m}{2}$ and $(u)_j = u(u+1)\cdots(u+j-1)$ is the Pochhammer symbol. When $n + m$ is odd, we have

$$f_{n,m} = c_1 t^{r-n} \frac{(r+1)_{N-1}}{(-r+1)_{N-1}} \frac{\tau_n(-N+1, -r-N+1, -r; t)}{\tau_n(-N+2, -r-N+2, -r+2; t)}, \tag{3.3}$$

where $N = \dfrac{n+m+1}{2}$.

(2) *Case where $n \geq m$ (or $n' = m$). When $n + m$ is even, we have*

$$f_{n,m} = c_0 N \frac{(r+1)_{N-1}}{(-r+1)_N} \frac{\tau_m(-N+2, -r-N+1, -r+2; t)}{\tau_m(-N+1, -r-N+2, -r+2; t)}, \tag{3.4}$$

where $N = \dfrac{n+m}{2}$. When $n + m$ is odd, we have

$$f_{n,m} = c_0 \frac{(r+1)_{N-1}}{(-r+1)_{N-1}} \frac{\tau_m(-N+2, -r-N+1, -r+1; t)}{\tau_m(-N+1, -r-N+2, -r+1; t)}, \tag{3.5}$$

where $N = \dfrac{n+m+1}{2}$.

Proposition 3.2 *For $(n, m) \in \mathbb{Z}_+^2$, the function $v_{n,m} = v_{n,m}(t)$ can be expressed as follows.*

(1) *Case where $n \leq m$ (or $n' = n$). When $n + m$ is even, we have*

$$v_{n,m} = t^{-\frac{n}{2}} \frac{(r)_N}{(-r+1)_N} \frac{\tau_n(-N+1, -r-N+1, -r+1; t)}{\tau_n(-N+1, -r-N+2, -r+2; t)}, \tag{3.6}$$

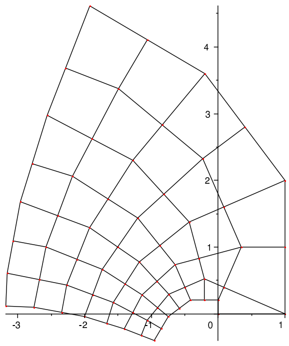

Figure 3: The discrete power function with $\gamma = 1 + i$.

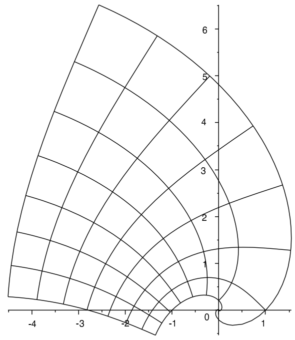

Figure 4: The ordinary power function z^{1+i}.

where $N = \dfrac{n+m}{2}$. *When* $n+m$ *is odd, we have*

$$v_{n,m} = -c_1 t^{r-\frac{n}{2}} \frac{\tau_n(-N+1, -r-N+2, -r+1; t)}{\tau_n(-N+2, -r-N+2, -r+2; t)}, \tag{3.7}$$

where $N = \dfrac{n+m+1}{2}$.

(2) *Case where* $n \geq m$ *(or* $n' = m$*). When* $n+m$ *is even, we have*

$$v_{n,m} = t^{-\frac{m}{2}} \frac{(r)_N}{(-r+1)_N} \frac{\tau_m(-N+1, -r-N+1, -r+1; t)}{\tau_m(-N+1, -r-N+2, -r+2; t)}, \tag{3.8}$$

where $N = \dfrac{n+m}{2}$. *When* $n+m$ *is odd, we have*

$$v_{n,m} = -c_0 t^{\frac{m+1}{2}} \frac{\tau_m(-N+2, -r-N+2, -r+2; t)}{\tau_m(-N+1, -r-N+2, -r+1; t)}, \tag{3.9}$$

where $N = \dfrac{n+m+1}{2}$.

Note that these expressions are applicable to the case where $r \in \mathbb{C}\backslash\mathbb{Z}$. A typical example of the discrete power function and its continuous counterpart are illustrated in Figure 3 and Figure 4, respectively. Figure 5 shows an example of the case suggesting multivalency of the map. The proof of the above theorem and proposition is given in the next subsection.

Remark 3.3

(1) *When* $m = 1$ *(or* $n = 1$*), the above results correspond to the case where* $P_{\rm VI}$ *is reduced to a Riccati equation and solved by the hypergeometric function. This case recovers the result obtained by Agafonov [2].*

(2) *Agafonov also has shown that the generalized discrete power function* $f_{n,m}$*, under the setting of* $c_0 = c_1 = 1$*,* $t = e^{2i\alpha}$ $(0 < \alpha < \pi)$ *and* $0 < r < 1$*, is embedded [2].*

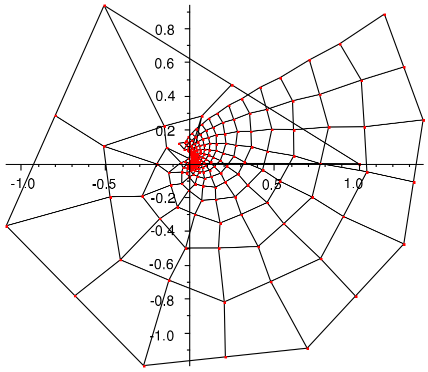

Figure 5: The discrete power function with $\gamma = 0.25 + 3.35i$.

Remark 3.4 *As we mention above, some special solutions to (2.7) in terms of the τ functions of $P_{\rm VI}$ have been presented [11]. It is easy to show that these solutions also satisfy a difference equation which is a deformation of (2.3) in the sense that the coefficients n and m of (2.3) are replaced by arbitrary complex numbers. For instance, a class of solutions presented in Theorem 6 of [11] satisfies*

$$\begin{aligned}&(\alpha_0+\alpha_2+\alpha_4)f_{n,m}\\&=(n-\alpha_2)\frac{(f_{n+1,m}-f_{n,m})(f_{n,m}-f_{n-1,m})}{f_{n+1,m}-f_{n-1,m}}-(\alpha_1+\alpha_2+\alpha_4-m)\frac{(f_{n,m+1}-f_{n,m})(f_{n,m}-f_{n,m-1})}{f_{n,m+1}-f_{n,m-1}},\end{aligned} \tag{3.10}$$

where α_i are parameters of $P_{\rm VI}$ introduced in Appendix A. Setting the parameters as $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (r,0,0,-r+1,0)$, we see that the above equation is reduced to (2.3) and that the solutions are given by the hypergeometric τ functions under the initial conditions (3.1).

3.2 Proof of main results

In this subsection, we give the proof of Theorem 3.1 and Proposition 3.2. One can easily verify that $f_{n,m}$ satisfies the initial condition (3.1) by noticing $\tau_0(a,b,c;t) = 1$. We then show that $f_{n,m}$ and $v_{n,m}$ given in Theorem 3.1 and Proposition 3.2 satisfy the relation (2.8), the difference equation (2.3), the compatibility condition (2.9) and the similarity condition (2.11) by means of the various bilinear relations for the hypergeometric τ function. Note in advance that we use the bilinear relations by specializing the parameters a, b and c as

$$a=-N,\quad b=-r-N,\quad c=-r+1,\quad N=\frac{n+m}{2}, \tag{3.11}$$

when $n+m$ is even, or

$$a=-r-N+1,\quad b=-N,\quad c=-r+1,\quad N=\frac{n+m+1}{2}, \tag{3.12}$$

when $n+m$ is odd.

We first verify the relation (2.8). Note that we have the following bilinear relations

$$\begin{aligned}(c-1)\tau_n^{0,-1,-1}\tau_{n+1}^{-1,-1,-1} &= (c-b-1)t\tau_{n+1}^{0,-1,0}\tau_n^{-1,-1,-2} + b\tau_n^{0,0,0}\tau_{n+1}^{-1,-2,-2},\\ (c-1)\tau_n^{-1,-1,-1}\tau_n^{0,-1,-1} &= (c-b-1)\tau_n^{0,-1,0}\tau_n^{-1,-1,-2} + b\tau_n^{0,0,0}\tau_n^{-1,-2,-2},\end{aligned} \tag{3.13}$$

$$\begin{aligned}(a-b)\tau_m^{0,-1,-1}\tau_m^{0,-1,0} &= a\tau_m^{-1,-1,-1}\tau_m^{1,-1,0} - b\tau_m^{0,0,0}\tau_m^{0,-2,-1},\\ (a-b)t\tau_{m+1}^{0,-1,0}\tau_m^{0,-1,-1} &= a\tau_{m+1}^{-1,-1,-1}\tau_m^{1,-1,0} - b\tau_m^{0,0,0}\tau_{m+1}^{0,-2,-1},\end{aligned} \tag{3.14}$$

$$\begin{aligned}(b-a+1)\tau_m^{0,0,0}\tau_m^{-1,-1,-1} &= (b-c+1)\tau_m^{0,-1,0}\tau_m^{-1,0,-1} + (c-a)\tau_m^{0,-1,-1}\tau_m^{-1,0,0},\\ (b-a+1)\tau_{m+1}^{-1,-1,-1}\tau_m^{0,0,0} &= (b-c+1)\tau_{m+1}^{0,-1,0}\tau_m^{-1,0,-1} + (c-a)\tau_m^{0,-1,-1}\tau_{m+1}^{-1,0,0},\end{aligned} \tag{3.15}$$

for the hypergeometric τ functions, the derivation of which is discussed in Appendix A. Let us consider the case where $n' = n$. When $n+m$ is even, the relation (2.8) is reduced to

$$\begin{aligned}-r\tau_n^{[1,1,1]}\tau_{n+1}^{[0,1,1]} &= Nt\tau_{n+1}^{[1,1,2]}\tau_n^{[0,1,0]} - (r+N)\tau_n^{[1,2,2]}\tau_{n+1}^{[0,0,0]},\\ -r\tau_n^{[0,1,1]}\tau_n^{[1,1,1]} &= N\tau_n^{[1,1,2]}\tau_n^{[0,1,0]} - (r+N)\tau_n^{[1,2,2]}\tau_n^{[0,0,0]},\end{aligned} \tag{3.16}$$

where we denote

$$\tau_{n'}^{[i_1,i_2,i_3]} = \tau_{n'}(-N+i_1, -r-N+i_2, -r+i_3; t), \tag{3.17}$$

for simplicity. We see that the relations (3.16) can be obtained from (3.13) with the parameters specialized as (3.11). In fact, the hypergeometric τ functions can be rewritten as

$$\tau_n^{0,-1,-1} = \tau_n(a+1, b+1, c) = \tau_n(-N+1, -r-N+1, -r+1) = \tau_n^{[1,1,1]}, \tag{3.18}$$

for instance. When $n+m$ is odd, (2.8) yields

$$\begin{aligned}-r\tau_n^{[1,2,1]}\tau_{n+1}^{[1,1,1]} &= (-r+N)t\tau_{n+1}^{[1,2,2]}\tau_n^{[1,1,0]} - N\tau_n^{[2,2,2]}\tau_{n+1}^{[0,1,0]},\\ -r\tau_n^{[1,1,1]}\tau_n^{[1,2,1]} &= (-r+N)\tau_n^{[1,2,2]}\tau_n^{[1,1,0]} - N\tau_n^{[2,2,2]}\tau_n^{[0,1,0]},\end{aligned} \tag{3.19}$$

which is also obtained from (3.13) by specializing the parameters as (3.12). Note that the hypergeometric τ functions can be rewritten as

$$\begin{aligned}\tau_n^{0,-1,-1} = \tau_n(a+1, b+1, c) &= \tau_n(-r-N+2, -N+1, -r+1)\\ &= \tau_n(-N+1, -r-N+2, -r+1) = \tau_n^{[1,2,1]},\end{aligned} \tag{3.20}$$

this time. In the case where $n' = m$, one can similarly verify the relation (2.8) by using the bilinear relations (3.14) and (3.15).

Next, we prove that (2.3) is satisfied, which is rewritten by using (2.8) as

$$-r\frac{f_{n,m}}{v_{n,m}} = \frac{nt^{-\frac{1}{2}}}{v_{n+1,m}^{-1} + v_{n-1,m}^{-1}} + \frac{m}{v_{n,m+1}^{-1} + v_{n,m-1}^{-1}}. \tag{3.21}$$

We use the bilinear relations

$$\begin{aligned}n'\tau_{n'}^{0,0,0}\tau_{n'}^{0,-1,-1} &= (b-c+1)\tau_{n'+1}^{0,-1,0}\tau_{n'-1}^{0,0,-1} + at^{-1}\tau_{n'+1}^{-1,-1,-1}\tau_{n'-1}^{1,0,0},\\ (a+b-c+n'+1)\tau_{n'}^{0,0,0}\tau_{n'}^{0,-1,-1} &= a\tau_{n'}^{-1,-1,-1}\tau_{n'}^{1,0,0} + (b-c+1)\tau_{n'}^{0,-1,0}\tau_{n'}^{0,0,-1},\end{aligned} \tag{3.22}$$

and

$$
\begin{aligned}
&\tau_n^{0,0,0}\tau_n^{-1,-1,-2} = -t^{-1}\tau_{n+1}^{-1,-1,-1}\tau_{n-1}^{0,0,-1} + \tau_n^{-1,-1,-1}\tau_n^{0,0,-1},\\
&\tau_m^{0,0,0}\tau_m^{1,-1,0} = \tau_m^{0,-1,0}\tau_m^{1,0,0} - \tau_{m+1}^{0,-1,0}\tau_{m-1}^{1,0,0},\\
&\tau_m^{0,-1,-1}\tau_m^{-1,0,-1} = -\tau_{m+1}^{-1,-1,-1}\tau_{m-1}^{0,0,-1} + \tau_m^{-1,-1,-1}\tau_m^{0,0,-1},
\end{aligned} \tag{3.23}
$$

for the proof. Their derivation is also shown in Appendix A. Let us consider the case where $n' = n$. When $n + m$ is even, we have

$$
\begin{aligned}
-n\tau_n^{[1,2,2]}\tau_n^{[1,1,1]} &= N\tau_{n+1}^{[1,1,2]}\tau_{n-1}^{[1,2,1]} + Nt^{-1}\tau_{n+1}^{[0,1,1]}\tau_{n-1}^{[2,2,2]},\\
m\tau_n^{[1,2,2]}\tau_n^{[1,1,1]} &= N\tau_n^{[0,1,1]}\tau_n^{[2,2,2]} + N\tau_n^{[1,1,2]}\tau_n^{[1,2,1]},
\end{aligned} \tag{3.24}
$$

from the bilinear relations (3.22) by specializing the parameters a, b and c as given in (3.11). These lead us to

$$
\begin{aligned}
v_{n+1,m}^{-1} + v_{n-1,m}^{-1} &= c_1^{-1}t^{-r+\frac{n+1}{2}}\frac{n}{N}\frac{\tau_n^{[1,2,2]}\tau_n^{[1,1,1]}}{\tau_{n+1}^{[0,1,1]}\tau_{n-1}^{[1,2,1]}},\\
v_{n,m+1}^{-1} + v_{n,m-1}^{-1} &= -c_1^{-1}t^{-r+\frac{n}{2}}\frac{m}{N}\frac{\tau_n^{[1,2,2]}\tau_n^{[1,1,1]}}{\tau_n^{[0,1,1]}\tau_n^{[1,2,1]}}.
\end{aligned} \tag{3.25}
$$

By using

$$
\tau_n^{[1,2,2]}\tau_n^{[0,1,0]} = -t^{-1}\tau_{n+1}^{[0,1,1]}\tau_{n-1}^{[1,2,1]} + \tau_n^{[0,1,1]}\tau_n^{[1,2,1]}, \tag{3.26}
$$

which is obtained from the first relation in (3.23), one can verify (3.21). When $n + m$ is odd, we have the bilinear relations

$$
\begin{aligned}
-n\tau_n^{[2,2,2]}\tau_n^{[1,2,1]} &= (-r+N)\tau_{n+1}^{[1,2,2]}\tau_{n-1}^{[2,2,1]} + (r+N-1)t^{-1}\tau_{n+1}^{[1,1,1]}\tau_{n-1}^{[2,3,2]},\\
m\tau_n^{[2,2,2]}\tau_n^{[1,2,1]} &= (r+N-1)\tau_n^{[1,1,1]}\tau_n^{[2,3,2]} + (-r+N)\tau_n^{[1,2,2]}\tau_n^{[2,2,1]},
\end{aligned} \tag{3.27}
$$

from (3.22) with (3.12), and

$$
\tau_n^{[2,2,2]}\tau_n^{[1,1,0]} = -t^{-1}\tau_{n+1}^{[1,1,1]}\tau_{n-1}^{[2,2,1]} + \tau_n^{[1,1,1]}\tau_n^{[2,2,1]}, \tag{3.28}
$$

from the first relation in (3.23). These lead us to (3.21). We next consider the case where $n' = m$. When $n + m$ is even, we get the bilinear relations

$$
\begin{aligned}
-m\tau_m^{[1,2,2]}\tau_m^{[1,1,1]} &= N\tau_{m+1}^{[1,1,2]}\tau_{m-1}^{[1,2,1]} + Nt^{-1}\tau_{m+1}^{[0,1,1]}\tau_{m-1}^{[2,2,2]},\\
n\tau_m^{[1,2,2]}\tau_m^{[1,1,1]} &= N\tau_m^{[0,1,1]}\tau_m^{[2,2,2]} + N\tau_m^{[1,1,2]}\tau_m^{[1,2,1]},
\end{aligned} \tag{3.29}
$$

and

$$
\tau_m^{[1,2,2]}\tau_m^{[2,1,2]} = \tau_m^{[1,1,2]}\tau_m^{[2,2,2]} - \tau_{m+1}^{[1,1,2]}\tau_{m-1}^{[2,2,2]}, \tag{3.30}
$$

from (3.22) and the second relation in (3.23), respectively. By using these relations, one can show (3.21) in a similar way to the case where $n' = n$. When $n + m$ is odd, we use the bilinear relations

$$
\begin{aligned}
-m\tau_m^{[2,2,2]}\tau_m^{[1,2,1]} &= (-r+N)\tau_{m+1}^{[1,2,2]}\tau_{m-1}^{[2,2,1]} + (r+N-1)t^{-1}\tau_{m+1}^{[1,1,1]}\tau_{m-1}^{[2,3,2]},\\
n\tau_m^{[2,2,2]}\tau_m^{[1,2,1]} &= (r+N-1)\tau_m^{[1,1,1]}\tau_m^{[2,3,2]} + (-r+N)\tau_m^{[1,2,2]}\tau_m^{[2,2,1]},
\end{aligned} \tag{3.31}
$$

and

$$
\tau_m^{[1,2,1]}\tau_m^{[2,1,1]} = -\tau_{m+1}^{[1,1,1]}\tau_{m-1}^{[2,2,1]} + \tau_m^{[1,1,1]}\tau_m^{[2,2,1]}, \tag{3.32}
$$

which are obtained from (3.22) and the third relation in (3.23), respectively, to show (3.21).

We next give the verification of the compatibility condition (2.9) by using the bilinear relations

$$\begin{aligned}(c-a)\tau_{n'}^{0,-1,-1}\tau_{n'+1}^{-1,-1,0} - b\tau_{n'}^{0,0,0}\tau_{n'+1}^{-1,-2,-1} &= (t-1)\tau_{n'}^{-1,-1,-1}\tau_{n'+1}^{0,-1,0},\\ (c-a)t\tau_{n'}^{0,-1,-1}\tau_{n'+1}^{-1,-1,0} - b\tau_{n'}^{0,0,0}\tau_{n'+1}^{-1,-2,-1} &= (t-1)\tau_{n'}^{0,-1,0}\tau_{n'+1}^{-1,-1,-1}.\end{aligned} \tag{3.33}$$

The derivation of these is discussed in Appendix A. We first consider the case where $n' = n$. When $n+m$ is even, we get

$$\begin{aligned}(-r+N+1)\tau_n^{[1,1,1]}\tau_{n+1}^{[0,1,2]} + (r+N)\tau_n^{[1,2,2]}\tau_{n+1}^{[0,0,1]} &= (t-1)\tau_n^{[0,1,1]}\tau_{n+1}^{[1,1,2]},\\ (-r+N+1)t\tau_n^{[1,1,1]}\tau_{n+1}^{[0,1,2]} + (r+N)\tau_n^{[1,2,2]}\tau_{n+1}^{[0,0,1]} &= (t-1)\tau_n^{[1,1,2]}\tau_{n+1}^{[0,1,1]},\end{aligned} \tag{3.34}$$

from the bilinear relations (3.33). Then we have

$$\begin{aligned}t^{\frac{1}{2}}v_{n,m} + v_{n+1,m+1} &= t^{-\frac{n+1}{2}}(t-1)\frac{(r)_N}{(-r+1)_{N+1}}\frac{\tau_n^{[1,1,2]}\tau_{n+1}^{[0,1,1]}}{\tau_n^{[1,2,2]}\tau_{n+1}^{[0,1,2]}},\\ v_{n,m} + t^{\frac{1}{2}}v_{n+1,m+1} &= t^{-\frac{n}{2}}(t-1)\frac{(r)_N}{(-r+1)_{N+1}}\frac{\tau_n^{[0,1,1]}\tau_{n+1}^{[1,1,2]}}{\tau_n^{[1,2,2]}\tau_{n+1}^{[0,1,2]}},\end{aligned} \tag{3.35}$$

from which we arrive at the compatibility condition (2.9). When $n+m$ is odd, we have

$$\begin{aligned}N\tau_n^{[1,2,1]}\tau_{n+1}^{[1,1,2]} + N\tau_n^{[2,2,2]}\tau_{n+1}^{[0,1,1]} &= (t-1)\tau_n^{[1,1,1]}\tau_{n+1}^{[1,2,2]},\\ Nt\tau_n^{[1,2,1]}\tau_{n+1}^{[1,1,2]} + N\tau_n^{[2,2,2]}\tau_{n+1}^{[0,1,1]} &= (t-1)\tau_n^{[1,2,2]}\tau_{n+1}^{[1,1,1]},\end{aligned} \tag{3.36}$$

from (3.33). Calculating $t^{\frac{1}{2}}v_{n,m} + v_{n+1,m+1}$ and $v_{n,m} + t^{\frac{1}{2}}v_{n+1,m+1}$ by means of these relations, we see that we have (2.9). In the case where $n' = m$, one can verify the compatibility condition (2.9) in a similar manner.

Let us finally verify the similarity condition (2.11), which can be written as

$$\frac{n}{2} - \frac{1}{2}\chi_{n+m} - t\frac{d}{dt}\log v_{n,m} = \frac{nv_{n+1,m}}{v_{n+1,m}+v_{n-1,m}}. \tag{3.37}$$

Here, we take the factor χ_{n+m} as $\chi_{n+m} = r\left[(-1)^{n+m}-1\right]$. The relevant bilinear relations for the hypergeometric τ function are

$$\begin{aligned}(D+n)\tau_n^{0,0,0}\cdot\tau_n^{0,-1,-1} &= at^{-1}\tau_{n+1}^{-1,-1,-1}\tau_{n-1}^{1,0,0},\\ (D+b-c+1)\tau_m^{0,-1,-1}\cdot\tau_m^{0,0,0} &= (b-c+1)\tau_m^{0,-1,0}\tau_m^{0,0,-1},\\ (D+a+m)\tau_m^{0,0,0}\cdot\tau_m^{0,-1,-1} &= a\tau_m^{-1,-1,-1}\tau_m^{1,0,0}.\end{aligned} \tag{3.38}$$

The derivation of these is obtained in Appendix A. We first consider the case where $n' = n$. When $n+m$ is even, it is easy to see that we have

$$n\frac{v_{n+1,m}}{v_{n+1,m}+v_{n-1,m}} = -Nt^{-1}\frac{\tau_{n+1}^{[0,1,1]}\tau_{n-1}^{[2,2,2]}}{\tau_n^{[1,2,2]}\tau_n^{[1,1,1]}}, \tag{3.39}$$

from the bilinear relation (3.24). We get

$$(D+n)\tau_n^{[1,2,2]}\cdot\tau_n^{[1,1,1]} = -Nt^{-1}\tau_{n+1}^{[0,1,1]}\tau_{n-1}^{[2,2,2]}, \tag{3.40}$$

from the first relation in (3.38) with (3.11). From this we can obtain the similarity condition (3.37) as follows. When $n+m$ is odd, we have

$$(D+n)\tau_n^{[2,2,2]}\cdot\tau_n^{[1,2,1]} = -t^{-1}(r+N-1)\tau_{n+1}^{[1,1,1]}\tau_{n-1}^{[2,3,2]}, \tag{3.41}$$

from the first relation in (3.38). This relation together with the first relation in (3.27) leads us to (3.37). Next, we discuss the case where $n'=m$. When $n+m$ is even, we have

$$(D+N)\tau_m^{[1,2,2]}\cdot\tau_m^{[1,1,1]} = N\tau_m^{[1,1,2]}\tau_m^{[1,2,1]}, \tag{3.42}$$

from the second relation in (3.38). Then we arrive at (3.37) by virtue of the second relation in (3.29). When $n+m$ is odd, we get

$$(D+r+\tfrac{n-m-1}{2})\tau_m^{[1,2,1]}\cdot\tau_m^{[2,2,2]} = (r+N-1)\tau_m^{[1,1,1]}\tau_m^{[2,3,2]}, \tag{3.43}$$

from the third relation in (3.38). Then we derive the similarity condition (3.37) by using the second relation in (3.31). This completes the proof of Theorem 3.1 and Proposition 3.2.

4 Extension of the domain

First, we extend the domain of the discrete power function to $\mathbb{Z}^2$. To determine the values of $f_{n,m}$ in the second, third and fourth quadrants, we have to give the values of $f_{-1,0}$ and $f_{0,-1}$ as the initial conditions. Set the initial conditions as

$$f_{-1,0} = c_2 t^{2r}, \quad f_{0,-1} = c_3 t^{3r}, \tag{4.1}$$

where c_2 and c_3 are arbitrary constants. This is natural because these conditions reduce to

$$f_{1,0} = 1, \quad f_{0,1} = e^{\pi i r}, \quad f_{-1,0} = e^{2\pi i r}, \quad f_{0,-1} = e^{3\pi i r} \tag{4.2}$$

at the original setting. Due to the symmetry of equations (2.7) and (2.3), we immediately obtain the explicit formula of $f_{n,m}$ in the second and third quadrant.

Corollary 4.1 *Under the initial conditions $f_{0,1} = c_1 t^r$ and (4.1), we have*

$$f_{-n,m} = f_{n,m}|_{c_0\mapsto c_2 t^{2r}}, \quad f_{-n,-m} = f_{n,m}|_{c_0\mapsto c_2 t^{2r}, c_1\mapsto c_3 t^{2r}}, \tag{4.3}$$

for $n,m\in\mathbb{Z}_+$.

Next, let us discuss the explicit formula in the fourth quadrant. Naively, we use the initial conditions $f_{0,-1} = c_3 t^{3r}$ and $f_{1,0} = c_0$ to get the formula $f_{n,-m} = f_{n,m}|_{c_1\mapsto c_3 t^{2r}}$. However, this setting makes the discrete power function $f_{n,m}$ become a single-valued function on $\mathbb{Z}^2$. In order to allow $f_{n,m}$ to be multi-valued on $\mathbb{Z}^2$, we introduce a discrete analogue of the Riemann surface by the following procedure. Prepare an infinite number of $\mathbb{Z}^2$-planes, cut the positive part of the "real axis" of each $\mathbb{Z}^2$-plane and glue them in a similar way to the continuous case. The next step is to write the initial conditions (3.1) and (4.1) in polar form as

$$f(1,\pi k/2) = c_k t^{kr} \quad (k=0,1,2,3), \tag{4.4}$$

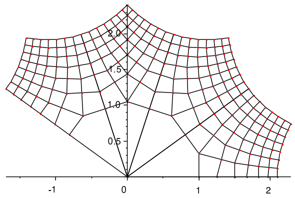

Figure 6: The discrete power function with $\gamma = 5/2$ whose domain is $\mathbb{Z}^2$.

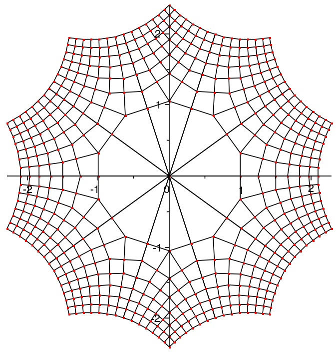

Figure 7: The discrete power function with $\gamma = 5/2$ whose domain is the discrete Riemann surface.

where the first component, 1, denotes the absolute value of $n+im$ and the second component, $\pi k/2$, is the argument. We must generalize the above initial conditions to those for arbitrary $k \in \mathbb{Z}$ so that we obtain the explicit expression of $f_{n,m}$ for each quadrant of each $\mathbb{Z}^2$-plane. Let us illustrate a typical case. When $\frac{3}{2}\pi \le \arg(n+im) \le 2\pi$, we solve the equations (2.7) and (2.3) under the initial conditions

$$f(1, 3\pi/2) = c_3 t^{3r}, \quad f(1, 2\pi) = c_4 t^{4r}, \tag{4.5}$$

to obtain the formula

$$f_{-n,-m} = f_{n,m}|_{c_0 \mapsto c_4 t^{4r}, c_1 \mapsto c_3 t^{2r}} \quad (n, m \in \mathbb{Z}_+). \tag{4.6}$$

We present the discrete power function with $\gamma = 5/2$ whose domain is $\mathbb{Z}^2$ and the discrete Riemann surface in Figure 6 and 7, respectively. Note that the necessary and sufficient condition for the discrete power function to reduce to a single-valued function on $\mathbb{Z}^2$ is ($c_k = c_{k+4}$ and) $e^{4\pi i r} = 1$, which means that the exponent γ is an integer.

5 Concluding remarks

The discrete logarithmic function and cases where $\gamma \in 2\mathbb{Z}$ were excluded from the considerations in the previous sections. From the viewpoint of the theory of hypergeometric functions, these cases lead to integer differences in the characteristic exponents. Thus we need a different treatment for precise description of these cases. However, they may be obtained by some limiting procedures in principle. In fact, Agafonov has examined the case where $\gamma = 2$ and $\gamma = 0$ by using a limiting procedure [1, 2], the former is the discrete power function Z^2 and latter is the discrete logarithmic function. In general, one may obtain a description of these cases by introducing the functions $\widetilde{f}_{n,m}$ and $\widehat{f}_{n,m}$ as

$$\widetilde{f}_{n,m} := \begin{cases} \lim\limits_{r\to j} \dfrac{1}{j} \dfrac{(-r+1)_j}{(r+1)_{j-1}} f_{n,m}, & \text{for } \gamma = 2j \in 2\mathbb{Z}_{>0} \\ \lim\limits_{r\to -j} \dfrac{(-r+1)_j}{(r+1)_j} f_{n,m}, & \text{for } \gamma = -2j \in 2\mathbb{Z}_{<0} \end{cases} \tag{5.1}$$

and

$$\widehat{f}_{n,m} = \lim_{r\to 0} \frac{f_{n,m}-1}{r}, \tag{5.2}$$

respectively. The function $\widetilde{f}_{n,m}$ might coincide with the counterpart defined in section 6 of [3].

Moreover, it has been shown that the discrete power function and logarithmic function associated with hexagonal patterns are also described by some discrete Painlevé equations [4]. It may be an interesting problem to construct the explicit formula for them.

Acknowledgements

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A Bäcklund transformations of the sixth Painlevé equation

As a preparation, we give a brief review of the Bäcklund transformations and some of the bilinear equations for the τ functions [12]. It is well-known that P_{VI} (2.12) is equivalent to the Hamilton system

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad ' = t(t-1)\frac{d}{dt}, \tag{A.1}$$

whose Hamiltonian is given by

$$H = f_0 f_3 f_4 f_2^2 - [\alpha_4 f_0 f_3 + \alpha_3 f_0 f_4 + (\alpha_0 - 1) f_3 f_4] f_2 + \alpha_2(\alpha_1+\alpha_2) f_0. \tag{A.2}$$

Here f_i and α_i are defined by

$$f_0 = q - t, \quad f_3 = q - 1, \quad f_4 = q, \quad f_2 = p, \tag{A.3}$$

and

$$\alpha_0 = \theta, \quad \alpha_1 = \kappa_\infty, \quad \alpha_3 = \kappa_1, \quad \alpha_4 = \kappa_0 \tag{A.4}$$

with $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$. The Bäcklund transformations of P_{VI} are described by

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \quad (i,j = 0,1,2,3,4), \tag{A.5}$$

$$s_2(f_i) = f_i + \frac{\alpha_2}{f_2}, \quad s_i(f_2) = f_2 - \frac{\alpha_i}{f_i} \ (i = 0,3,4), \tag{A.6}$$

$$\begin{aligned} s_5 &: \ \alpha_0 \leftrightarrow \alpha_1, \quad \alpha_3 \leftrightarrow \alpha_4, \quad f_2 \mapsto -\frac{f_0(f_2 f_0 + \alpha_2)}{t(t-1)}, \quad f_4 \mapsto t\frac{f_3}{f_0}, \\ s_6 &: \ \alpha_0 \leftrightarrow \alpha_3, \quad \alpha_1 \leftrightarrow \alpha_4, \quad f_2 \mapsto -\frac{f_4(f_4 f_2 + \alpha_2)}{t}, \quad f_4 \mapsto \frac{t}{f_4}, \\ s_7 &: \ \alpha_0 \leftrightarrow \alpha_4, \quad \alpha_1 \leftrightarrow \alpha_3, \quad f_2 \mapsto \frac{f_3(f_3 f_2 + \alpha_2)}{t-1}, \quad f_4 \mapsto \frac{f_0}{f_3}, \end{aligned} \tag{A.7}$$

where $A=(a_{ij})_{i,j=0}^{4}$ is the Cartan matrix of type $D_4^{(1)}$. Then the group of birational transformations $\langle s_0,\dots,s_7\rangle$ generate the extended affine Weyl group $\widetilde{W}(D_4^{(1)})$. In fact, these generators satisfy the fundamental relations

$$s_i^2=1 \quad (i=0,\dots,7), \qquad s_is_2s_i=s_2s_is_2 \quad (i=0,1,3,4), \tag{A.8}$$

and

$$\begin{gathered} s_5s_{\{0,1,2,3,4\}}=s_{\{1,0,2,4,3\}}s_5, \quad s_6s_{\{0,1,2,3,4\}}=s_{\{3,4,2,0,1\}}s_6, \quad s_7s_{\{0,1,2,3,4\}}=s_{\{4,3,2,1,0\}}s_7, \\ s_5s_6=s_6s_5, \quad s_5s_7=s_7s_5, \quad s_6s_7=s_7s_6. \end{gathered} \tag{A.9}$$

We add a correction term to the Hamiltonian H as follows,

$$H_0=H+\frac{t}{4}\left[1+4\alpha_1\alpha_2+4\alpha_2^2-(\alpha_3+\alpha_4)^2\right]+\frac{1}{4}\left[(\alpha_1+\alpha_4)^2+(\alpha_3+\alpha_4)^2+4\alpha_2\alpha_4\right]. \tag{A.10}$$

This modification gives a simpler behavior of the Hamiltonian with respect to the Bäcklund transformations. From the corrected Hamiltonian, we introduce a family of Hamiltonians h_i ($i=0,1,2,3,4$) as

$$h_0=H_0+\frac{t}{4}, \quad h_1=s_5(H_0)-\frac{t-1}{4}, \quad h_3=s_6(H_0)+\frac{1}{4}, \quad h_4=s_7(H_0), \quad h_2=h_1+s_1(h_1). \tag{A.11}$$

Next, we also introduce τ functions τ_i ($i=0,1,2,3,4$) by $h_i=(\log\tau_i)'$. Imposing the condition that the action of the s_i's on the τ functions also commute with the derivation $'$, one can lift the Bäcklund transformations to the τ functions. The action of $\widetilde{W}(D_4^{(1)})$ is given by

$$s_0(\tau_0)=f_0\frac{\tau_2}{\tau_0}, \quad s_1(\tau_1)=\frac{\tau_2}{\tau_1}, \quad s_2(\tau_2)=\frac{f_2}{\sqrt{t}}\frac{\tau_0\tau_1\tau_3\tau_4}{\tau_2}, \quad s_3(\tau_3)=f_3\frac{\tau_2}{\tau_3}, \quad s_4(\tau_4)=f_4\frac{\tau_2}{\tau_4}, \tag{A.12}$$

and

$$\begin{aligned} s_5:\ & \tau_0\mapsto[t(t-1)]^{\frac14}\tau_1, \quad \tau_1\mapsto[t(t-1)]^{-\frac14}\tau_0, \\ & \tau_3\mapsto t^{-\frac14}(t-1)^{\frac14}\tau_4, \quad \tau_4\mapsto t^{\frac14}(t-1)^{-\frac14}\tau_3, \quad \tau_2\mapsto[t(t-1)]^{-\frac12}f_0\tau_2, \end{aligned} \tag{A.13}$$

$$s_6:\ \tau_0\mapsto it^{\frac14}\tau_3, \quad \tau_3\mapsto -it^{-\frac14}\tau_0, \quad \tau_1\mapsto t^{-\frac14}\tau_4, \quad \tau_4\mapsto t^{\frac14}\tau_1, \quad \tau_2\mapsto t^{-\frac12}f_4\tau_2, \tag{A.14}$$

$$\begin{aligned} s_7:\ & \tau_0\mapsto(-1)^{-\frac34}(t-1)^{\frac14}\tau_4, \quad \tau_4\mapsto(-1)^{\frac34}(t-1)^{-\frac14}\tau_0, \\ & \tau_1\mapsto(-1)^{\frac34}(t-1)^{-\frac14}\tau_3, \quad \tau_3\mapsto(-1)^{-\frac34}(t-1)^{\frac14}\tau_1, \\ & \tau_2\mapsto -i(t-1)^{-\frac12}f_3\tau_2. \end{aligned} \tag{A.15}$$

We note that some of the fundamental relations are modified

$$s_is_2(\tau_2)=-s_2s_i(\tau_2) \quad (i=5,6,7), \tag{A.16}$$

and

$$\begin{aligned} s_5s_6\tau_{\{0,1,2,3,4\}}&=\{i,-i,-1,-i,i\}s_6s_5\tau_{\{0,1,2,3,4\}}, \\ s_5s_7\tau_{\{0,1,2,3,4\}}&=\{i,-i,-1,i,-i\}s_7s_5\tau_{\{0,1,2,3,4\}}, \\ s_6s_7\tau_{\{0,1,2,3,4\}}&=\{-i,-i,-1,i,i\}s_7s_6\tau_{\{0,1,2,3,4\}}. \end{aligned} \tag{A.17}$$

Let us introduce the translation operators

$$\begin{aligned} \widehat{T}_{13} &= s_1 s_2 s_0 s_4 s_2 s_1 s_7, & \widehat{T}_{40} &= s_4 s_2 s_1 s_3 s_2 s_4 s_7, \\ \widehat{T}_{34} &= s_3 s_2 s_0 s_1 s_2 s_3 s_5, & T_{14} &= s_1 s_4 s_2 s_0 s_3 s_2 s_6, \end{aligned} \tag{A.18}$$

whose action on the parameters $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is given by

$$\begin{aligned} \widehat{T}_{13}(\vec{\alpha}) &= \vec{\alpha} + (0, 1, 0, -1, 0), \\ \widehat{T}_{40}(\vec{\alpha}) &= \vec{\alpha} + (-1, 0, 0, 0, 1), \\ \widehat{T}_{34}(\vec{\alpha}) &= \vec{\alpha} + (0, 0, 0, 1, -1), \\ T_{14}(\vec{\alpha}) &= \vec{\alpha} + (0, 1, -1, 0, 1). \end{aligned} \tag{A.19}$$

We denote $\tau_{k,l,m,n'} = T_{14}^{n'} \widehat{T}_{34}^{m} \widehat{T}_{40}^{l} \widehat{T}_{13}^{k}(\tau_0)$ $(k, l, m, n' \in \mathbb{Z})$. By using this notation, we have

$$\begin{aligned} \tau_{0,0,0,0} &= \tau_0, & \tau_{-1,-1,-1,0} &= [t(t-1)]^{\frac{1}{4}} \tau_1, \\ \tau_{0,-1,-1,0} &= (-1)^{-\frac{3}{4}} t^{\frac{1}{4}} \tau_3, & \tau_{0,-1,0,0} &= (-1)^{-\frac{3}{4}} (t-1)^{\frac{1}{4}} \tau_4, \\ \tau_{-1,-2,-1,1} &= (-1)^{-\frac{1}{4}} s_0(\tau_0), & \tau_{0,-1,0,1} &= (-1)^{-\frac{3}{4}} [t(t-1)]^{\frac{1}{4}} s_1(\tau_1), \\ \tau_{-1,-1,0,1} &= -i t^{\frac{1}{4}} s_3(\tau_3), & \tau_{-1,-1,-1,1} &= (t-1)^{\frac{1}{4}} s_4(\tau_4), \end{aligned} \tag{A.20}$$

for instance. When the parameters $\vec{\alpha}$ take the values

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-b, a + n', -n', c - a, b - c + 1 + n'), \tag{A.21}$$

the function $\tau_{k,l,m,n'}$ relates to the hypergeometric τ function $\tau_{n'}^{k,l,m}$ introduced in Proposition 2.6 by [12]

$$\tau_{k,l,m,n'} = \omega_{k,l,m,n'} \tau_{n'}^{k,l,m} t^{-(\hat{a}+\hat{b}-\hat{c}+2n')^2/4-(\hat{a}-\hat{b}-n')^2/4+n'(\hat{b}+n')-n'(n'-1)/2} (t-1)^{(\hat{a}+\hat{b}-\hat{c}+2n')^2/4+1/2}, \tag{A.22}$$

where we denote $\hat{a} = a + k$, $\hat{b} = b + l + 1$ and $\hat{c} = c + m$, and the constants $\omega_{k,l,m,n'} = \omega_{k,l,m,n'}(a, b, c)$ are determined by the recurrence relations

$$\begin{aligned} \omega_{k+1,l,m,i}\omega_{k-1,l,m,i} &= i\hat{a}(\hat{c} - \hat{a})\omega_{k,l,m,i}^2, \\ \omega_{k,l+1,m,i}\omega_{k,l-1,m,i} &= -i\hat{b}(\hat{c} - \hat{b})\omega_{k,l,m,i}^2, \qquad (i = 0, 1) \\ \omega_{k,l,m+1,i}\omega_{k,l,m-1,i} &= (\hat{c} - \hat{a})(\hat{c} - \hat{b})\omega_{k,l,m,i}^2 \end{aligned} \tag{A.23}$$

and

$$\omega_{k,l,m,n'+1}\omega_{k,l,m,n'-1} = -\omega_{k,l,m,n'}^2 \tag{A.24}$$

with initial conditions

$$\begin{aligned} \omega_{-1,-2,-1,1} &= (-1)^{-1/4} b, & \omega_{0,-2,-1,1} &= b, \\ \omega_{-1,-1,-1,1} &= 1, & \omega_{0,-1,-1,1} &= (-1)^{-1/4}, \\ \omega_{-1,0,0,1} &= -(-1)^{-3/4}(c - a), & \omega_{0,0,0,1} &= -i, \\ \omega_{-1,-1,0,1} &= -i(c - a), & \omega_{0,-1,0,1} &= (-1)^{-3/4}, \end{aligned} \tag{A.25}$$

and

$$\begin{aligned} \omega_{-1,-2,-1,0} &= (-1)^{-3/4} b, & \omega_{0,-2,-1,0} &= -b, \\ \omega_{-1,-1,-1,0} &= 1, & \omega_{0,-1,-1,0} &= (-1)^{-3/4}, \\ \omega_{-1,0,0,0} &= (-1)^{-3/4}(c - a), & \omega_{0,0,0,0} &= 1, \\ \omega_{-1,-1,0,0} &= c - a, & \omega_{0,-1,0,0} &= (-1)^{-3/4}. \end{aligned} \tag{A.26}$$

From the above formulation, one can obtain the bilinear equations for the τ functions. For examle, let us express the Bäcklund transformations $s_2(f_i) = f_i + \dfrac{\alpha_2}{f_2}$ $(i = 0, 3, 4)$ in terms of the τ functions τ_j $(j = 0, 1, 3, 4)$. We have by using (A.12)

$$\begin{aligned}
&\alpha_2 t^{-\frac{1}{2}}\tau_3\tau_4 - s_1(\tau_1)s_2 s_0(\tau_0) + s_0(\tau_0)s_2 s_1(\tau_1) = 0,\\
&\alpha_2 t^{-\frac{1}{2}}\tau_0\tau_4 - s_1(\tau_1)s_2 s_3(\tau_3) + s_3(\tau_3)s_2 s_1(\tau_1) = 0,\\
&\alpha_2 t^{-\frac{1}{2}}\tau_0\tau_3 - s_1(\tau_1)s_2 s_4(\tau_4) + s_4(\tau_4)s_2 s_1(\tau_1) = 0.
\end{aligned} \tag{A.27}$$

Applying the affine Weyl group $\widetilde{W}(D_4^{(1)})$ on these equations, we obtain

$$\begin{aligned}
&(\alpha_0 + \alpha_2 + \alpha_4)\, t^{-\frac{1}{2}}\tau_3 s_4(\tau_4) - s_1(\tau_1)s_4 s_2 s_0(\tau_0) + \tau_0 s_0 s_4 s_2 s_1(\tau_1) = 0,\\
&(\alpha_0 + \alpha_2 + \alpha_4)\, t^{\frac{1}{2}}\tau_1\tau_3 - \tau_4 s_4 s_2 s_0(\tau_0) + \tau_0 s_0 s_2 s_4(\tau_4) = 0,
\end{aligned} \tag{A.28}$$

$$\begin{aligned}
&(\alpha_0 + \alpha_1 + \alpha_2)\, t^{-\frac{1}{2}}\tau_3\tau_4 - \tau_1 s_1 s_2 s_0(\tau_0) + \tau_0 s_0 s_2 s_1(\tau_1) = 0,\\
&(\alpha_0 + \alpha_1 + \alpha_2)\, t^{\frac{1}{2}} s_1(\tau_1)\tau_3 - s_4(\tau_4)s_1 s_2 s_0(\tau_0) + \tau_0 s_0 s_1 s_2 s_4(\tau_4) = 0,
\end{aligned} \tag{A.29}$$

$$\begin{aligned}
&(\alpha_2 + \alpha_3 + \alpha_4)\, t^{-\frac{1}{2}}\tau_0\tau_1 - \tau_4 s_4 s_2 s_3(\tau_3) + \tau_3 s_3 s_2 s_4(\tau_4) = 0,\\
&(\alpha_2 + \alpha_3 + \alpha_4)\, t^{-\frac{1}{2}} s_4(\tau_4)\tau_0 - s_1(\tau_1)s_4 s_2 s_3(\tau_3) + \tau_3 s_3 s_4 s_2 s_1(\tau_1) = 0,
\end{aligned} \tag{A.30}$$

and

$$\begin{aligned}
&\alpha_2 t^{-\frac{1}{2}}\tau_0\tau_3 - s_1(\tau_1)s_2 s_4(\tau_4) + s_4(\tau_4)s_2 s_1(\tau_1) = 0,\\
&(\alpha_1 + \alpha_4 + \alpha_2)t^{-\frac{1}{2}}\tau_0\tau_3 - \tau_1 s_1 s_2 s_4(\tau_4) + \tau_4 s_4 s_2 s_1(\tau_1) = 0.
\end{aligned} \tag{A.31}$$

For instance, the first equation in (A.28) can be obtained by applying $s_0 s_4$ on the first one in (A.27). We also get the second equation in (A.28) by applying $s_0 s_4 s_6$ on the second one in (A.27). Other equations can be derived in a similar manner. By applying the translation $T_{14}^{n'}\widehat{T}_{34}^{m}\widehat{T}_{40}^{l}\widehat{T}_{13}^{k}$ to the bilinear relations (A.28) and noticing (A.20), we get

$$\begin{aligned}
&(\alpha_0 + \alpha_2 + \alpha_4 - m)\, t^{-\frac{1}{2}}\tau_{k,l-1,m-1,n'}\tau_{k-1,l-1,m-1,n'+1}\\
&\qquad\qquad +\tau_{k,l-1,m,n'+1}\tau_{k-1,l-1,m-2,n'} + \tau_{k,l,m,n'}\tau_{k-1,l-2,m-2,n'+1} = 0,\\
&(\alpha_0 + \alpha_2 + \alpha_4 - m)\tau_{k-1,l-1,m-1,n'}\tau_{k,l-1,m-1,n'}\\
&\qquad\qquad +\tau_{k,l-1,m,n'}\tau_{k-1,l-1,m-2,n'} - \tau_{k,l,m,n'}\tau_{k-1,l-2,m-2,n'} = 0,
\end{aligned} \tag{A.32}$$

and then (3.13) for the hypergeometric τ functions. Similarly, we obtain for the hypergeometric τ functions (3.14), (3.15) and (3.22) from (A.29), (A.30) and (A.31), respectively. The constraints

$$f_0 = f_4 - t, \quad f_3 = f_4 - 1, \tag{A.33}$$

yield

$$\begin{aligned}
\tau_0 s_4 s_2 s_0(\tau_0) &= s_4(\tau_4)s_2 s_4(\tau_4) - t\tau_1 s_4 s_2 s_1(\tau_1),\\
\tau_0 s_1 s_2 s_0(\tau_0) &= \tau_4 s_1 s_2 s_4(\tau_4) - t s_1(\tau_1)s_2 s_1(\tau_1),\\
\tau_3 s_4 s_2 s_3(\tau_3) &= s_4(\tau_4)s_2 s_4(\tau_4) - \tau_1 s_4 s_2 s_1(\tau_1),
\end{aligned} \tag{A.34}$$

and

$$\begin{aligned}
\tau_3 s_3(\tau_3) - \tau_0 s_0(\tau_0) &= (t-1)\tau_1 s_1(\tau_1),\\
t\tau_3 s_3(\tau_3) - \tau_0 s_0(\tau_0) &= (t-1)\tau_4 s_4(\tau_4),
\end{aligned} \tag{A.35}$$

from which we obtain (3.23) and (3.33), respectively. Due to (A.11) we have the relation

$$h_0 - h_3 = (t-1)\left[f_2 f_4 + \frac{1}{2}(1-\alpha_3-\alpha_4)\right]. \tag{A.36}$$

Then we get the bilinear relations

$$\begin{aligned} D\tau_0\cdot\tau_3 &= t^{\frac{1}{2}} s_4(\tau_4) s_2 s_1(\tau_1) + \frac{1}{2}(1-\alpha_3-\alpha_4)\tau_0\tau_3,\\ D\tau_0\cdot\tau_3 &= t^{\frac{1}{2}} \tau_4 s_4 s_2 s_1(\tau_1) + \frac{1}{2}(1-\alpha_3+\alpha_4)\tau_0\tau_3,\\ D\tau_0\cdot\tau_3 &= t^{\frac{1}{2}} \tau_1 s_1 s_2 s_4(\tau_4) + \frac{1}{2}(\alpha_0-\alpha_1)\tau_0\tau_3, \end{aligned} \tag{A.37}$$

where D denotes Hirota's differential operator defined by $D\,g\cdot f = t\left(\frac{dg}{dt}f - g\frac{df}{dt}\right)$. By applying the translation $T_{14}^{n'}\widehat{T}_{34}^{m}\widehat{T}_{40}^{l}\widehat{T}_{13}^{k}$ to the first bilinear relation of (A.37), one gets

$$\left[D+\frac{1}{2}\left(\alpha_3+\alpha_4-k+l+n'-\frac{1}{2}\right)\right]\tau_{k,l,m,n'}\cdot\tau_{k,l-1,m-1,n'} = -t^{\frac{1}{2}}(t-1)^{-\frac{1}{2}}\tau_{k-1,l-1,m-1,n'+1}\tau_{k+1,l,m,n'-1}, \tag{A.38}$$

which is reduced to the first relation of (3.38). The second and third relations of (A.37) also yield their counterparts in (3.38).

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