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ON PROJECTIVE SPACE BUNDLE WITH NEF NORMALIZED TAUTOLOGICAL LINE BUNDLE

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Abstract. In this paper, we study the structure of projective space bundles whose relative anti-canonical line bundle is nef. As an application, we get a characterization of abelian varieties up to finite étale covering.

Introduction

For a morphism between smooth projective varieties $\pi : Y \to X$ the relative anti-canonical divisor $-K_\pi$ on Y is defined by the difference of anticanonical divisors $-K_\pi := -K_Y - \pi^*(-K_X)$. J. Kollár, Y. Miyaoka and S. Mori proved that the relative anti-canonical divisor of a non-constant generically smooth morphism cannot be ample in arbitrary characteristic [7], [11]. In the case where $\pi : Y = \mathbb{P}_X(\mathcal{E}) \to X$ is a projectivization of vector bundle on X, we know that the relative anti-canonical divisor is positive proportion of the normalized tautological divisor. Miyaoka studied the case where Y is a curve and showed that the nefness of the normalized tautological divisor is equal to the semistability of vector bundle [10]. Nakayama generalized this to the arbitrary dimension in [13]. In this paper we study the more explicit structure of vector bundles with nef normalized tautological divisor. In Section 1, we review the definition and some known results. In Section 2, we treat semiample cases and show that a pullback of such a bundle by some finite unramified covering is trivial up to twist by some line bundle. In Section 3, we treat the case where X is a blow-up of a smooth variety Z along smooth subvariety or a projective bundle over a smooth variety Z. In these cases we show that the vector bundle with nef normalized tautological divisor on X is isomorphic to the pullback of vector bundle on K having the same property up to twist by the exceptional divisor. In Section 4 we study manifolds whose tangent bundle have a nef normalized tautological divisor. We prove such surfaces are isomorphic to a quotient of abelian surface by some finite étale morphism. Moreover under the assumption that such a divisor is semiample, we can show that finite étale covering of abelian varieties are all varieties satisfying this property.

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notation

We will work, throughout this paper, over the complex number field $\mathbb{C}$ unless otherwise mentioned. We freely use the customary terminology in algebraic geometry. Vector bundles are often identified with the locally free sheaves of their sections, and these words are used interchangeably. Line bundles are identified with linear equivalence classes of Cartier divisors , and their tensor products are denoted additively.

1. Preliminary

At first we recall the notion of normalized tautological divisor defined in [13] which is originally introduced by Y. Miyaoka as normalized hyperplane class in [10].

Definition 1.1. Let X be a smooth projective variety over algebraic closed field of arbitrary characteristic and $\mathcal{E}$ a rank r vector bundle on X. A normalized tautological divisor $\Lambda_{\mathcal{E}}$ of $\mathcal{E}$ is a $\mathbb{Q}$-divisor on $X = \mathbb{P}_M(\mathcal{E})$ such that $r\Lambda_{\mathcal{E}} := r\xi_{\mathcal{E}} - \pi^*(det(\mathcal{E}))$ where $\xi_{\mathcal{E}}$ is the tautological divisor on $X = \mathbb{P}_M(\mathcal{E})$ and π is the natural projection $\pi : \mathbb{P}_M(\mathcal{E}) \to M$. In particular, $r\Lambda_{\mathcal{E}}$ is linearly equivalent to the relative anti-canonical divisor $-K_\pi$.

By virtue of the following theorem, we know that the normalized tautological divisor $\Lambda_{\mathcal{E}}$ of projective space bundle $\pi : \mathbb{P}_M(\mathcal{E}) \to M$ cannot be ample.

Theorem 1.2. ([7],[11],[15],[4], $char \geqslant 0$) *Let X and Y be smooth projective varieties over an algebraically closed field of arbitrary charactristic and let $\pi : X \to Y$ be a non-constant generically smooth morphism. Let H be an ample divisor on Y. For any positive ϵ, the divisor $-K_\pi - \epsilon\pi^*H$ is not nef. In particular, $-K_\pi$ is not ample.*

Moreover N.Nakayama show that the relative anti-canonical divisor $-K_\pi$ of projective space bundle $\pi : \mathbb{P}_M(\mathcal{E}) \to M$ cannot be nef and big in characteristic 0. To state his theorem, we recall the definition of numerical D-dimension of nef divisors on smooth projective varieties.

Definition 1.3. Let X be an n-dimensional smooth projective variety, A an ample divisor on X and D a nef divisor on X. We define the numerical D-dimension $\nu(D, X)$ of D by

$$\nu(D, X) = \max\{k \in \mathbb{N} \mid D^k \cdot A^{n-k} \neq 0\}.$$

We call D big if $\nu(D, X) = n$.

Theorem 1.4 ([13]). *Let X be a smooth projective variety and $\mathcal{E}$ a rank r vector bundle on X. Assume that the relative anti-canonical line bundle $-K_\pi$ of projective space bundle $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ is nef. Then $\nu(-K_\pi) = r-1$. In particular $-K_\pi$ cannot be nef and big.*

2. Vector bundles with semiample normalized tautological divisor

First we consider the case where $-K_\pi$ of projective bundle $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ is semiample. If $\mathcal{E} \cong \mathcal{O}_X^r$ is a trivial vector bundle on smooth projective variety X, the relative anti-canonical divisor $-K_\pi = p^*(-K_{\mathbb{P}^{r-1}})$ is basepoint-free where $p : \mathbb{P}_X(\mathcal{E}) \cong X \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the second projection. In the case where dimX= 1 N. Nakayama shows the following result.

Theorem 2.1 ([12]). *Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to C$ be a $\mathbb{P}^1$-bundle over a smooth curve C. Then the following two conditions are equivalent:*

1. *there is a finite étale morphism $f : \tilde{C} \to C$ such that $X \times_C \tilde{C} \cong \mathbb{P}^1 \times \tilde{C}$ over $\tilde{C}$;*
2. *the normalized tautological line bundle $\Lambda_{\mathcal{E}}$ is semiample.*

Remark 2.2. Forthermore N. Nakayama shows very interesting fact; for $\mathbb{P}^1$-bundle $\mathbb{P}_C(\mathcal{E})$ over a smooth curve C of genus $g(C) > 1$ the following two conditions are equivalent:

1. the normalized tautological line bundle $\Lambda_{\mathcal{E}}$ is semiample.
2. X has a surjective endomorphism $g : X \to X$ that is not isomorphism.

We generalize the result of N. Nakayama to arbitrary rank vector bundles and arbitrary dimensional base manifolds.

Theorem 2.3. *Let X be a smooth projective variety and $\mathcal{E}$ a rank r vector bundle on X. Assume that the relative anti-canonical divisor $-K_\pi$ of projective space bundle $\pi : Y = \mathbb{P}_X(\mathcal{E}) \to X$ is semiample. Then there exist a finite étale morphism $f : X' \to X$ such that $f^*\mathcal{E}$ is trivial up to twist by a line bundle.*

Proof. From the semiampleness of $-K_\pi$ we have a fibration $\varphi : Y \to Z \subseteq \mathbb{P}^N$ defined by the basepoint-free divisor $-mK_\pi$ for $m \gg 0$ and the Stein factorization.

$$\begin{array}{ccc} Y = \mathbb{P}_X(\mathcal{E}) & \xrightarrow[\varphi]{} & Z \subseteq \mathbb{P}^N \\ \downarrow{\scriptstyle \pi} & & \\ X & & \end{array}$$

By Theorem 1.4 we have dim Im$\varphi(Y) = r-1$. We may asuume that Z is smooth by the replacement Z with $\mathbb{P}^N$. Let $S = \varphi^{-1}(z)$ be a general fiber of φ then S is smooth and $\pi|_S : S \to X$ is finite surjective. We will show that $\pi|_S$ is unramified i.e. $\Omega_S \cong \pi^*\Omega_X|_S$. In this situation we have the morphism $\Phi : \mathcal{T}_\pi \to \mathcal{T}_Y \to \varphi^*\mathcal{T}_Z$. By the restriction of this morphism to S we have the generically injective morphism $\mathcal{T}_\pi|_S \to \varphi^*\mathcal{T}_Z|_S \cong \mathcal{O}_S^N$ since

φ is generically smooth. Taking some direct summand we have a generically injective morphism between vector bundles of same rank $\mathcal{T}_\pi|_S \to \mathcal{O}_S^{r-1}$. From the semiampleness of $-K_\pi$ we know that the determinant morphism $-K_\pi|_S \to \mathcal{O}_S$ is isomorphism. Therefore we get an isomorphism $\mathcal{T}_\pi|_S \cong \mathcal{O}_S^{r-1}$. Taking the dual we also have an isomorphism $\Omega_\pi|_S \cong \mathcal{O}_S^{r-1}$. Next we consider the exact sequence

$$0 \to (\pi|_S)^*\Omega_X \to \Omega_S \to \Omega_{\pi|_S} \to 0.$$

It is sufficient that $\Omega_{\pi|_S} = 0$. From the argument mentioned above we know the morphism $\Phi^\vee : \varphi^*\Omega_Z|_S \cong \mathcal{O}_S^N \to \Omega_Y|_S \to \Omega_\pi|_S \cong \mathcal{O}_S^{r-1}$ is surjective. Since S is smooth subscheme of Y we have an exact sequence

$$0 \to \mathcal{N}_{S/Y}^\vee \to \Omega_Y|_S \to \Omega_S \to 0.$$

The image of $\varphi^*\Omega_Z|_S$ in $\Omega_Y|_S$ is contained in $\mathcal{N}_{S/Y}^\vee$ since the image of $\varphi^*\Omega_Z|_S$ in Ω_S is 0. Therefore we have a surjection $\mathcal{N}_{S/Y}^\vee \to \Omega_\pi|_S \to 0$. From the following commutative diagram we have $\Omega_{\pi|_S} = 0$.

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathcal{N}_{S/Y}^\vee & \longrightarrow & \Omega_S & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Omega_\pi|_S & \longrightarrow & \Omega_{\pi|_S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Taking a base-change by the étale morphism $\pi|_S : S \to X$, we have a finite étale morphism $g : \mathbb{P}_S(\pi|_S^*(\mathcal{E})) \to \mathbb{P}_X(\mathcal{E})$ and the natural projection $\tilde{\pi} : \mathbb{P}_S(\pi|_S^*(\mathcal{E})) \to S$ has a smooth section. By using this argument repeatedly, we have a finite étale morphism $f : X' \to X$ such that the natural projection $\pi' : \mathbb{P}_{X'}(f^*(\mathcal{E})) \to X'$ has sufficiently many disjoint sections. Therefore we can show that $\mathbb{P}_{X'}(f^*(\mathcal{E})) \cong X' \times \mathbb{P}^{r-1}$ and we have $f^*\mathcal{E}$ is trivial up to twist by a line bundle. □

3. VECTOR BUNDLES WITH NEF NORMALIZED TAUTOLOGICAL DIVISOR ON SOME SPECIAL VARIETIES

Next we consider the case where the normalized tautological divisor $\Lambda_\mathcal{E}$ is nef. To state Theorems proved by Y.Miyaoka and N.Nakayama we review the definition of stability of vector bundles.

DEFINITION 3.1. Let $\mathcal{E}$ be a vector bundle on smooth projective variety X of dimension n and A an ample line bundle on X. $\mathcal{E}$ is said to be A-semistable in the sense of Takemoto-Mumford if

$$\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$$

for every non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$ where $\mu(\mathcal{F}) := c_1(\mathcal{F}).A^{n-1}/rank(\mathcal{F})$.

THEOREM 3.2. ([10], *char* $\geqslant 0$) *Let* $\mathcal{E}$ *be a rank* r *vector bundle on smooth projective curve* C *over a field* k *of characteristic* $p \geqslant 0$.

(1) *If* $p = 0$, $\Lambda_{\mathcal{E}}$ *is nef if and only if* $\mathcal{E}$ *is* μ*-semistable.*

(2) *If* $p > 0$, $\Lambda_{\mathcal{E}}$ *is nef if and only if* $\mathcal{E}$ *is strongly* μ*-semistable i.e. all the Frobenius pull backs of* $\mathcal{E}$ *are* μ*-semistable.*

THEOREM 3.3 ([13]). *Let* $\mathcal{E}$ *be a rank* r *vector bundle on smooth complex projective variety* X *of dimension* d. *Then the following conditions are equivalent:*

(1) $\Lambda_{\mathcal{E}}$ *is nef;*

(2) $\mathcal{E}$ *is* μ*-semistable and*

$$(c_2(\mathcal{E}) - \frac{2r}{r-1}c_1^2(\mathcal{E})).A^{d-2} = 0$$

for an ample divisor A*;*

(3) *There is a filtration of vector subbundles*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ *are projectively flat and the averaged first Chern classes* $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ *are numerically equivalent to* $\mu(\mathcal{E}) := c_1(\mathcal{E})/rank\mathcal{E}$ *for any* i.

Here, a vector bundle $\mathcal{E}$ is called projectively flat if it admits a projectively flat Hermitian metric. Nakayama shows that projectively flat vector bundles are induced from some representations of the fundamental group of the base space.

PROPOSITION 3.4 ([13]). *Let* $\mathcal{E}$ *be a vector bundle of rank* r *on a smooth complex projective variety* X. *Then* $\mathcal{E}$ *is projectively flat if and only if the associated* $\mathbb{P}^{r-1}$*-bundle* $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ *is induced from a projective unitary representation* $\pi_1(X) \to PU(r)$.

By using this fact, we immediately get the following result.

THEOREM 3.5. *Let* X *be a smooth projective variety and* $\mathcal{E}$ *a rank* r *vector bundle on* X. *Assume that* X *is simply connected and the normalized tautological line bundle* $\Lambda = \Lambda_{\mathcal{E}}$ *of* $\mathcal{E}$ *is nef. Then* $\mathcal{E}$ *is trivial up to twist by a line bundle.*

Proof of Theorem 3.5. On a simply connected manifold we can prove that there are only trivial projectively flat vector bundles up to twist by Proposition 3.4. From Theorem 3.3 (3), we know that $\mathcal{E}$ is constructed by an extension of such vector bundles. We can denote $\mathcal{E}_1 = \mathcal{L}^{\oplus r_1}$ for some line bundle $\mathcal{L}$. For the bundle $\mathcal{E}_2$ we have an exact sequence

$$0 \to \mathcal{E}_1 = \mathcal{L}^{\oplus r_1} \to \mathcal{E}_2 \to \mathcal{E}_2/\mathcal{E}_1 = \mathcal{M}^{\oplus r_2} \to 0.$$

By virtue of Theorem 3.3 (3) we know that the averaged first Chern class $\mu(\mathcal{E}_2)$ is numerical equivalence to $\mu(\mathcal{E}_2/\mathcal{E}_1)$. From this we easily get $\mathcal{L}$ is isomorphic to $\mathcal{M}$. Since $h^1(\mathcal{O}_X) = 0$ the extension above split and we have $\mathcal{E}_2 \cong \mathcal{L}^{\oplus r_1+r_2}$. Using this argument repeatedly we can show that $\mathcal{E}$ is trivial up to twist by a line bundle. □

Corollary 3.6. *Let X is a rationally connected variety and $\mathcal{E}$ a vector bundle on X. If the normalized tautological line bundle $\Lambda = \Lambda_{\mathcal{E}}$ of $\mathcal{E}$ is nef then $\mathcal{E}$ is trivial up to twist by a line bundle.*

In positive characteristic case, If the base manifold is surface we can prove the same statement as in Corollary 3.6.

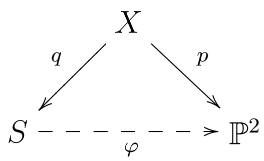
Theorem 3.7. ($char > 0$) *let S be a smooth projective rational surface over an algebraically closed field k of positive characteristic $p > 0$ and $\mathcal{E}$ a vector bundle of rank r on S. Assume that the anticanonical bundle of the projection $\pi : \mathbb{P}_S(\mathcal{E}) \to S$ is nef, then $\mathcal{E}$ is isomorphic to a trivial bundle up to twist.*

To show this we prepare the following lemma.

Lemma 3.8. ($char \geqslant 0$) *Let S be a smooth projective surface $\mathcal{E}$ a rank r vector bundle on S and $f : S' \to S$ a blow-up of S at a point p. If $f^*(\mathcal{E})$ is isomorphic to a vector bundle $\mathcal{L}^{\oplus r}$ for some line bundle $\mathcal{L}$ on S', then $\mathcal{E}$ is also isomorphic to a vector bundle $\mathcal{M}^{\oplus r}$ for some line bundle $\mathcal{M}$ on S.*

Proof. Let C be a exceptional divisor of f. Then $f^*(\mathcal{E})$ is trivial on C i.e. $f^*(\mathcal{E})|_C \cong \mathcal{L}|_C{}^{\oplus r} \cong \mathcal{O}_C{}^{\oplus r}$. By the Krull-Schmidt theorem of vector bundle [2], we have $\mathcal{L}|_C \cong \mathcal{O}_C$. Therefore there exists a line bundle $\mathcal{M}$ such that $f^*(\mathcal{M}) \cong \mathcal{L}$. Hence we have $f^*(\mathcal{E} \otimes \mathcal{M}^{-1}) \cong \mathcal{O}_{S'}{}^{\oplus r}$. Therefore we have $\mathcal{E} \otimes \mathcal{M}^{-1} \cong \mathcal{O}_S{}^{\oplus r}$. □

Proof of Theorem 3.7. Since S is rational, we have a birational map $f : S \dashrightarrow \mathbb{P}^2$. Let



be a resolution of indeterminacy of φ. By assumption we can show that $\Lambda_{q^*\mathcal{E}}$ is nef. Since p is a composition of blow-ups of a point, we can use Theorem 3.11 and we get a vector bundle $\mathcal{E}'$ on $\mathbb{P}^2$ such that $q^*\mathcal{E} \otimes \mathcal{L} = p^*\mathcal{E}'$ for some line bundle $\mathcal{L}$ on X. Moreover $\Lambda_{\mathcal{E}'}$ is nef. For any line l in $\mathbb{P}^2$ we have a decomposition $\mathcal{E}'|_l \cong \bigoplus \mathcal{O}_l^r$ by the argument as in proof of Theorem 3.6. In particular $\mathcal{E}'$ is a uniform vector bundle on $\mathbb{P}^2$. Hence $\mathcal{E}'$ is isomorphic to a trivial vector bundle. Therefore $q^*\mathcal{E} \otimes \mathcal{L} = p^*\mathcal{E}'$ is a trivial vector bundle on X. From Lemma 3.8 we know that $\mathcal{E}$ is trivial up to twist by a line bundle on S. □

If X is not rationally connected, even if X is uniruled, there is a non-trivial vector bundle $\mathcal{E}$ such that the normalized tautological divisor is nef.

Example 3.9. (char $\geqslant 0$) Let C be a smooth elliptic curve and $Y = C \times \mathbb{P}^1$. Then Y is uniruled and $h^1(Y, \mathcal{O}_Y) = h^1(C, \mathcal{O}_C) = 1$. Let $\mathcal{E}$ be a nontrivial extension of trivial line bundles $0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0$. Then $\Lambda_{\mathcal{E}} = \xi_{\mathcal{E}}$ is nef.

However we get following results.

THEOREM 3.10. (*char* $\geqslant 0$) *Let* $\varphi : X = \mathbb{P}_Y(\mathcal{F}) \to Y$ *be a* $\mathbb{P}^d$*-bundle on a smooth projective variety* Y*,* $\mathcal{E}$ *a vector bundle of rank* r *on* X *and* $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ *the natural projection. Assume the normalized tautological divisor* $\Lambda_{\mathcal{E}}$ *is nef. Then there exists a vector bundle* $\mathcal{E}'$ *on* Y *such that* $\mathcal{E} = \varphi^*\mathcal{E}'$ *such that* $\Lambda_{\mathcal{E}'}$ *is nef.*

PROOF. Let F be a fiber of π. Then we have $\mathcal{E}|_F \cong \bigoplus^r \mathcal{O}_{\mathbb{P}^d}(a)$ where a is an integer. Since a is independent from the choice of the fiber F. Therefore $\mathcal{E} \otimes \mathcal{O}(-a\xi_{\mathcal{F}}) = \pi^*\mathcal{E}'$ where $\mathcal{E}'$ is a rank r vector bundle on Y. Because $\pi' : \mathbb{P}_X(\mathcal{E}) \to \mathbb{P}_Y(\mathcal{E}')$ is surjective and $\Lambda_{\mathcal{E}} = \pi'^*\Lambda_{\mathcal{E}'}$, we have $\Lambda_{\mathcal{E}'}$ is nef. □

THEOREM 3.11. (*char* $\geqslant 0$) *Let* $\varphi : X \to X'$ *be the blow-up of smooth variety along a smooth subvariety* $Y \subset X'$ *and* $\mathcal{E}$ *a vector bundle of rank* r *on* X*. Assume that* $\Lambda_{\mathcal{E}}$ *is nef. Then there exists a vector bundle* $\mathcal{E}'$ *such that* $\mathcal{E} \otimes \mathcal{L} = \varphi^*\mathcal{E}'$ *where* $\mathcal{L}$ *is a line bundle on* X*. Moreover* $\Lambda_{\mathcal{E}'}$ *is nef.*

To prove this we use the following results.

PROPOSITION 3.12. (*char* $\geqslant 0$) *Let* X *be a smooth projective variety and* $\mathcal{E}$ *a vector bundle on* X*. Assume that the normalized tautological line bundle* $\Lambda = \Lambda_{\mathcal{E}}$ *of* $\mathcal{E}$ *is nef. Then for any rational curve* $\gamma : \mathbb{P}^1 \to X$ *on* X*,* $\gamma^*(\mathcal{E})$ *is trivial up to twist by a line bundle.*

PROOF. We have a splitting :

$$\gamma^*\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \mathcal{O}_{\mathbb{P}^1}(a_r),$$

where $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$. We have the morphism $\gamma' : \mathbb{P}_{\mathbb{P}^1}(\gamma^*\mathcal{E}) \to \mathbb{P}_X(\mathcal{E})$ indeced by γ. We denote natural projections by $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ and $\pi' : \mathbb{P}_{\mathbb{P}^1}(\gamma^*\mathcal{E}) \to \mathbb{P}^1$. Then we have

$$r\Lambda_{\gamma^*\mathcal{E}} = r\xi_{\gamma^*\mathcal{E}} - \pi'^*det(\gamma^*\mathcal{E}) = r\gamma^*\xi_{\mathcal{E}} - \gamma^*\pi^*det(\mathcal{E}) = r\gamma^*\Lambda_{\mathcal{E}}.$$

Since $\Lambda_{\mathcal{E}}$ is nef by assumption, $\Lambda_{\gamma^*\mathcal{E}}$ is also nef. Let C be the section of $\gamma^*\mathcal{E}$ associated with the quotient line bundle $\gamma^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(a_1) \to 0$. We have

$$r\Lambda_{\gamma^*\mathcal{E}}.C = r\xi_{\gamma^*\mathcal{E}}.C - \pi'^*det(\gamma^*\mathcal{E}).C = ra_1 - \sum_{i=1}^{r} a_i \geqslant 0.$$

Therefore we get $a_1 = a_2 = \cdots = a_r$. Hence $\gamma^*\mathcal{E}$ is trivial up to twist by a line bundle. □

THEOREM 3.13. ([1], *char* $\geqslant 0$) *Let* $\varphi : X \to X'$ *be the blow-up of smooth variety along a smooth subvariety* $Y \subset X'$ *and* $\mathcal{E}$ *a vector bundle on* X*. Assume that* $\mathcal{E}$ *is trivial for any fiber of* φ*. Then there exists a vector bundle* X' *such that* $\mathcal{E} = \varphi^*\mathcal{E}'$*.*

Proof of Theorem 3.11. By virtue of theorem 3.13, we only have to show that $\mathcal{E}$ is trivial on the fiber F of φ. In this case F is isomorphic to $\mathbb{P}^{s-1}$ where s is the codimension of Y in X'. From Proposisio 3.12 we know that $\mathcal{E}|_F$ is uniform bundle on F. Therefore we have $\mathcal{E}|_F \cong \oplus^r \mathcal{O}_{\mathbb{P}^{s-1}}(a)$. For two distinct fibers F_1 and F_2 of φ, we have $c_1(\mathcal{E}|_{F_1}) = c_1(\mathcal{E})|_{F_1} = c_1(\mathcal{E})|_{F_2} = c_1(\mathcal{E}|_{F_2})$. Therefore a is independent from the choice of F. Hence there exist a vector bundle $\mathcal{E}'$ on X' and a integer k such that $\mathcal{E} \otimes \mathcal{O}_X(aE) = \varphi^*\mathcal{E}'$ where E is the exceptional divisor of φ. In this case we have $\Lambda_{\mathcal{E}} = \varphi^*\Lambda_{\mathcal{E}'}$. Since $\Lambda_{\mathcal{E}}$ is nef and φ is surjective, $\Lambda_{\mathcal{E}'}$ is also nef. □

4. NEFNESS OF NORMALIZED TAUTOLOGICAL BUNDLE OF TANGENT BUNDLE

In this section we consider the nefness of the normalized tautological line bundle of tangent bundle of manifolds.

PROPOSITION 4.1. ($char \geqslant 0$) *Let X be a manifold contains a rational curve $f : \mathbb{P}^1 \to X$. Then the normalized tautological line bundle of tangent bundle $\Lambda_{\mathcal{T}_X}$ of X is not nef.*

PROOF. By Lemma 3.12 we know that $f^*(\mathcal{T}_X) \cong \mathcal{O}(a)^{\oplus n}$ for some integer $a \geqslant 2$ for any rational curve $f : \mathbb{P}^1 \to X$. Therefore we have $deg(f^*(-K_X)) = na > 0$. In particular K_X is not nef. If K_X is not nef, then we can find a rational curve $g : \mathbb{P}^1 \to X$ on X such that $deg(g^*(-K_X)) \leqslant n+1$ by Theorem 1.13 in [9]. This is a contradiction. □

From this Proposition, we immediately prove the following result.

COROLLARY 4.2. ($char \geqslant 0$) *Let X be a smooth projective variety over algebraically closed field $k = \bar{k}$ of arbitrary characteristic. If Kodaira dimension $\kappa(X) = -\infty$ or X is not minimal then the normalized tautological line bundle of tangent bundle $\Lambda_{\mathcal{T}_X}$ of X is not nef.*

In the case where the Kodaira dimension $\kappa(X) = 0$ and X is minimal, the canonical divisor is numerical trivial. Therefore the nefness of the normalized tautological line bundle of tangent bundle is equivalent to the nefness of tangent bundle. F.Campana and T.Peternell proved that in this case there is an *étale* covering $T \to X$ from abelian variety T.

THEOREM 4.3 ([3] Theorem 2.3). *Let X be a smooth projectie manifold. Assume that the tangent bundle $\mathcal{T}_X$ is nef and K_X is nef, then the canonical bundle is numerical trivial $K_X \equiv 0$ and there is an étale covering $T \to X$ from abelian variety T.*

The case where Kodaira dimension $\kappa(X) \geqslant 1$ is very difficult in general. But the case where $dimX = 2$ we can obtain the following result.

PROPOSITION 4.4. *Let S be a smooth projective minimal surface with the Kodaira dimension $\kappa(S) \geqslant 1$. Then the normalized tautological line bundle of tangent bundle $\Lambda_{\mathcal{T}_S}$ is not nef.*

PROOF. If the Kodaira dimension $\kappa(S) = 1$, then S is a relatively minimal elliptic surface $\pi : S \to C$. By virtue of Theorem 3.3 if $\Lambda_{\mathcal{T}_S}$ is nef we have $c_1(S)^2 = c_2(S) = 0$. Therefore we have the Euler character $\chi(\mathcal{O}_S) = 0$ and a singular fiber of S is multiple fiber from corollary 16 and 17 in [6]. By canonical bundle formula for elliptic surfaces (c.f Theorem 15 in [6]) we obtain

$$K_S = \pi^*(K_C \otimes \mathcal{L}) \otimes \mathcal{O}_S(\sum_i (m_i - 1)F_i)$$

where $\mathcal{L}$ is a line bundle on C of $deg(\mathcal{L}) = 0$ and F_i is the multiple fiber of π with the multiplicity m_i. We consider the exact sequence of sheaves $0 \to \pi^*(\Omega_C) \to \Omega_S$. This sheaf morphism drops rank on multible fibers. Therefore we can get the torsion-free subsheaf $0 \to \pi^*(\Omega_C) \otimes \mathcal{O}_S(\sum_i (m_i - 1)F_i) \to \Omega_S$. In this case we have $\pi^*(\Omega_C) \otimes \mathcal{O}_S(\sum_i (m_i - 1)F_i) \equiv_{num} K_S$. Since K_S is not numerical trivial, for an ample divisor A we have $\mu_A(\pi^*(\Omega_C) \otimes \mathcal{O}_S(\sum_i (m_i - 1)F_i)) = A.K_S > A.K_S/2 = \mu_A(\Omega_S)$. This is a contradiction to the semistability of Ω_S.

If the Kodaira dimension $\kappa(S) = 2$ i.e. S is of general type, we have $0 < c_1^2(S) \leqslant 3c_2(S) < 4c_2(S) = c_1^2(S)$ by Theorem 3.3 and Miyaoka-Yau inequality. It is a contradiction. □

Combining with these result we have a characterization of abelian surface up to finite étale covering.

THEOREM 4.5. *Let S be a smooth projective surface. If the normalized tautological line bundle of tangent bundle is nef, then there is an étale covering $T \to S$ from abelian surface T.*

In higher dimensional case we have the following characterization of abelian variety up to finite étale covering .

THEOREM 4.6. (*E. Sato*) *Let X be a smooth projective variety of n dimensional. If the normalized tautological line bundle of tangent bundle is semiample, then there is an étale covering $T \to X$ from abelian variety T.*

PROOF. By virtue of Theorem 4.1 we may assume that X is minimal. From Theorem 2.3 we have a finite étale covering $f : \tilde{X} \to X$ such that $\mathcal{T}_{\tilde{X}} \cong f^*\mathcal{T}_X \cong \mathcal{L}^{\oplus n}$ for some line bundle $\mathcal{L}$ on $\tilde{X}$. If $\kappa(X) \geqslant 1$ we have $K_{\tilde{X}}.A^{n-1} = -n\mathcal{L}.A^{n-1} > 0$ for an ample divisor on $\tilde{X}$. By Proposition 2 in [5] we know that the universal covering space of X is isomorphic to the direct product D^n where D is an open disc $D = \{z \in \mathbb{C} | |z| < 1\}$. It contradicts to the decomposition into same line bundles $\mathcal{T}_{\tilde{X}} \cong \mathcal{L}^{\oplus n}$. Therefore we may consider only the case where $\kappa(X) \leqslant 0$. In this case there is an *étale* covering $T \to X$ from abelian variety T by Corollary 4.2 and Theorem 4.3. □

References

[1] M. Andreatta and G. Occhetta, Extending extremal contractions from an ample section, Adv. Geom. 2 (2002), 133–146.

[2] M. Atiyah, On the Krull-Schmidt theorem with application to sheaves, Bull. Soc. Math. France 84 (1956), 307–317.

[3] F. Campana and T. Peternell, Projective manifolds whose tangent bundles are numerically effective, Math. Ann. 289 (1991), 169–187.

[4] O. Debarre, Higher-Dimensional Algebraic Geometry, Springer-Verlag, New York (2001)

[5] S. Druel, Variétés algébriques dont le fibtré tangent est totalement décomposé. J. reine angew. Math. 522 (2000), 161–171.

[6] R. Friedman, Algebraic surfaces and holomorphic vector bundles. Universitext. Springer, New York (1998)

[7] J. Kollár, Y. Miyaoka and S. Mori, Rational connectedness and boundedness of Fano manifolds, J. Diff. Geom. 36 (1992), 765–769, .

[8] J. Kollár, Rational Curves on Algebraic Varieties. Springer-Verlag, Berlin (1996)

[9] J. Kollár and S. Mori, Birational Geometry of Algebraic Varieties, Cambridge Tracts in Math. 134, Cambridge Univ. Press, Cambridge, 1998, with the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[10] Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety, Advanced Studies in Pure Math. 10 (1987) 449–476.

[11] Y. Miyaoka, Relative deformations of morphisms and applications to fibre spaces, Comment. Math. Univ. St. Pauli 42 (1993), no.1, 1–7.

[12] N. Nakayama, Ruled surfaces with non-trivial surjective endomorphisms. Kyushu J. Math. 56 (2002), 433–446.

[13] N. Nakayama, Zariski-decomposition and abundance, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[14] C. Okonek, M. Schneider and H. Spindler, Vector Bundles on Complex Projective Spaces, Birkhäuser, 1980.

[15] Q. Zhang, On projective varieties with nef anticanonical divisors, Math. Ann. 332 (2005), 697–703.

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