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Brezina, Jan
Graduate School of Mathematics, Kyushu University

Kagei, Yoshiyuki
Faculty of Mathematics, Kyushu University

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Jan Brezina & Yoshiyuki Kagei

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Decay properties of solutions to the linearized compressible Navier-Stokes equation around time-periodic parallel flow

Jan Březina *   Yoshiyuki Kagei †

Abstract

Decay estimates on solutions to the linearized compressible Navier-Stokes equation around time-periodic parallel flow are established. It is shown that if the Reynolds and Mach numbers are sufficiently small, solutions of the linearized problem decay in L^2 norm as an $n-1$ dimensional heat kernel. Furthermore, it is proved that the asymptotic leading part of solutions is given by solutions of an $n-1$ dimensional linear heat equation with a convective term multiplied by time-periodic function.

Mathematics Subject Classification

Keywords. Compressible Navier-Stokes equation, decay estimates, asymptotic behavior, time-periodic.

1 Introduction

This paper is concerned with the asymptotic behavior of solutions to the compressible Navier-Stokes equation with time-periodic external force and (or) time-periodic boundary conditions.

We consider the system of equations

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div}\, v + \nabla P(\rho) = \rho \boldsymbol{g}, \tag{1.2}$$

in an n dimensional infinite layer $\Omega_\ell = \mathbb{R}^{n-1} \times (0, \ell)$:

$$\begin{aligned}\Omega_\ell \;\; = \;\; & \{x = (x', x_n)\,; \\ & \qquad x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\, 0 < x_n < \ell\}.\end{aligned}$$

Here $n \geq 2$; $\rho = \rho(x,t)$ and $v = (v^1(x,t), \ldots, v^n(x,t))$ denote the unknown density and velocity at time $t \geq 0$ and position $x \in \Omega_\ell$, respectively; P is the pressure, smooth function of ρ, where for given ρ_* positive number we assume $P'(\rho_*) > 0$; μ and μ' are the viscosity

*Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, JAPAN
†Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, JAPAN

coefficients that are assumed to be constants satisfying $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$; $\boldsymbol{g}$ is a time-periodic external force of the form

$$\boldsymbol{g} = (g^1(x_n, t), 0, \ldots, 0, g^n(x_n)),$$

with g^1 being τ-periodic function in time, where $\tau > 0$.

The system (1.1)–(1.2) is considered under the boundary condition

$$v|_{x_n=0} = v|_{x_n=\ell} = 0, \tag{1.3}$$

and the initial condition

$$(\rho, v)|_{t=0} = (\rho_0, v_0). \tag{1.4}$$

Under suitable smallness conditions on g^n, problem (1.1)–(1.3) has a time-periodic solution $\overline{u}_p = (\overline{\rho}_p, \overline{v}_p)$

$$\begin{aligned} \overline{\rho}_p &= \overline{\rho}_p(x_n), \\ \overline{v}_p &= (\overline{v}_p^1(x_n, t), 0, \ldots, 0), \end{aligned}$$

where $\overline{\rho}_p$ and $\overline{v}_p$ satisfy

$$|\overline{\rho}_p - \rho^*|_{C^0[0,\ell]} \leq C \frac{\mu |g^n|_{C^0[0,\ell]}}{P'(\rho_*)\rho_* V}, \quad V = \frac{\rho_* \ell^2}{\mu} |g^1|_{C^0(\mathbb{R}; L^2(0,\ell))},$$

and

$$|\overline{v}_p^1|_{C^0(\mathbb{R}; L^2[0,\ell])} \leq C \frac{\rho_* \ell^2 |g^1|_{C^0(\mathbb{R}; L^2(0,\ell))}}{\mu}.$$

We are interested in large time behavior of solutions to the problem (1.1)–(1.4) when the initial value (ρ_0, v_0) is sufficiently close to the value of time-periodic solution $\overline{u}_p = (\overline{\rho}_p, \overline{v}_p)$ at some fixed time. In this paper we deal with the linearized problem around the time-periodic flow.

Lately stability of stationary parallel flows for compressible Navier-Stokes equation has been investigated in [2, 3, 4, 5, 6]. It was proved in [4, 5], that stationary parallel flow is asymptotically stable for sufficiently small initial disturbances if the Reynolds number and Mach number are sufficiently small. Furthermore, when $n \geq 3$ the disturbances behave in large time as solutions of the linearized problem, whose asymptotic leading parts are given by solutions of an $n-1$ dimensional linear heat equation with convective term. On the other hand, when $n = 2$ the asymptotic behavior is no longer described by the linearized problem; and it is described by a nonlinear diffusion equation, namely, by a 1-dimensional viscous Burgers equation.

In this paper we will extend previously obtained results for stationary parallel flows to the case of general time-periodic parallel flows. Combining techniques used in [2, 3, 6] and [1] we will treat the linearized problem around time-periodic parallel flow and establish decay estimates on solutions similar to those in the case of the stationary parallel flows. Whereas [2, 3, 6] are concerned with the stability of the stationary parallel flows, in [1] the diffusive stability of oscillations in reaction-diffusion systems is treated. Our main result in this paper reads as follows. We set $V = \frac{\rho_* \ell^2}{\mu} |g^1|_{\infty,2} > 0$ and introduce parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \gamma = \frac{\sqrt{P'(\rho_*)}}{V},$$

with $|\cdot|_{\infty,2}$ being the norm in space $C^0(\mathbb{R}; L^2(0,1))$. We note that the Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively. After a suitable non-dimensionalisation, the linearized problem is written as follows:

$$\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\,(\rho_p w) = 0, \tag{1.5}$$

$$\begin{aligned} &\partial_t w - \tfrac{\nu}{\rho_p} \Delta w - \tfrac{\nu+\nu'}{\rho_p} \nabla \mathrm{div}\, w + v_p^1 \partial_{x_1} w^n + (\partial_{x_n} v_p^1) w^n \boldsymbol{e}_1 \\ &\qquad\qquad + \tfrac{\nu}{\gamma^2 \rho_p^2} (\partial_{x_n}^2 v_p^1) \phi \boldsymbol{e}_1 + \nabla \left(\tfrac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p} \phi \right) = 0, \end{aligned} \tag{1.6}$$

$$w|_{x_n=0} = w|_{x_n=1} = 0, \tag{1.7}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \tag{1.8}$$

in Ω. Here $\phi = \phi(x,t)$ and $w = (w^1(x,t), \ldots, w^n(x,t))$ denote the unknowns. The domain Ω_l is transformed into $\Omega = \mathbb{R}^{n-1} \times (0,1)$; and $(\overline{\rho}_p, \overline{v}_p)$ is transformed into (ρ_p, v_p), where $v_p = (v_p^1(x_n,t), 0, \ldots, 0)$ is T-periodic in t with $T = \frac{V}{\ell}\tau$. We write (1.5)–(1.8) in the form

$$\partial_t u + L(t) u = \mathbf{0}, \quad w|_{x_n=0,1} = 0, \quad u|_{t=s} = u_0, \tag{1.9}$$

where $u = {}^T(\phi, w)$; $L(t)$ is the linearized operator and $u_0 = {}^T(\phi_0, w_0)$. We will prove that if Re and Ma are sufficiently small, then the solution $u_s(t) = (\phi(t), w(t))$ of the linearized problem (1.9) satisfies

$$\begin{aligned} &\|\partial_{x'}^k \partial_{x_n}^l u_s(t)\|_2 \\ &\le C\{(t-s)^{-\frac{n-1}{4}-\frac{k}{2}} \|u_0\|_{L^1(\mathbb{R}^{n-1}; H^1(0,1)\times L^2(0,1))} + e^{-d(t-s)}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'} w_0\|_2)\}, \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &\|\partial_{x'}^k \partial_{x_n}^l (u_s(t) - \sigma_{t,s} u^{(0)}(t))\|_2 \\ &\le C\{(t-s)^{-\frac{n-1}{4}-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{L^1(\mathbb{R}^{n-1}; H^1(0,1)\times L^2(0,1))} + e^{-d(t-s)}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'} w_0\|_2)\}, \end{aligned} \tag{1.11}$$

for $t - s \ge 4T$, $s \ge 0$, $k, l = 0, 1$, where $u^{(0)}(t) = u^{(0)}(x_n, t)$ is a function T-periodic in t and $\sigma_{t,s} = \sigma_{t,s}(x')$ is a function whose Fourier transform in x' is given by

$$\mathscr{F}(\sigma_{t,s}) = e^{-(i\kappa_0 \xi_1 + \kappa_1 |\xi'|^2)(t-s)} [\widehat{\phi}_0(\xi')].$$

Here $[\widehat{\phi}_0(\xi')]$ is a quantity given by

$$[\widehat{\phi}_0(\xi')] = \int_0^1 \widehat{\phi}_0(\xi', x_n)\, dx_n,$$

with $\widehat{\phi}_0$ being the Fourier transform of ϕ_0 in x' and κ_0, κ_1 are positive constants depending on ρ_*, l, V, μ, μ' and $P'(\rho_*)$. Precise statement of the results will be given in Section 3.

As in the case of the stationary parallel flows [6] these decay estimates as well as a decomposition argument in the proof will be useful for the nonlinear problem, which will be treated elsewhere.

To obtain decay estimates as in [6], we consider the Fourier transform of (1.9) in $x' \in \mathbb{R}^{n-1}$. That can be written as

$$\frac{d}{dt}u + \widehat{L}_{\xi'}(t)u = 0, \quad u|_{t=s} = u_0, \tag{1.12}$$

on $H^1(0,1) \times L^2(0,1)$. Here $\widehat{L}_{\xi'}(t)$ is an operator on $H^1(0,1) \times L^2(0,1)$ with domain $D(\widehat{L}_{\xi'}(t)) = H^1(0,1) \times (H^2(0,1) \cap H^1_0(0,1))$ with a dual variable $\xi' \in \mathbb{R}^{n-1}$. We denote $\widehat{U}_{\xi'}(t,s)$ the solution operator for (1.12). The operator $\widehat{U}_{\xi'}(t,s)$ has different characters between cases $|\xi'| \ll 1$ and $|\xi'| \gg 1$. We thus decompose the solution operator $\mathscr{U}(t,s)$ associated with (1.9) into three parts: $\mathscr{U}(t,s) = \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{|\xi'|\le r}\right) + \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{r\le|\xi'|\le R}\right) + \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{|\xi'|\ge R}\right)$ for some $0 < r \ll 1 \ll R$, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Since $\widehat{L}_{\xi'}(t)$ is periodic in t, we investigate the monodromy operator $\widehat{U}_{\xi'}(T) = \widehat{U}_{\xi'}(T,0)$ for $|\xi'| \le r \ll 1$ as in [1] and $\widehat{U}_{\xi'}(T)$ can be regarded as a perturbation from $\widehat{U}_0(T) = \widehat{U}_{\xi'}(T)|_{\xi'=0}$. We will find that the spectrum of $\mathscr{U}(t,s)$ near 1 is given by that of monodromy operator $\widehat{U}_{\xi'}(T)$ with $|\xi'| \le r \ll 1$, which is parameterized as $1 - i\kappa_0\xi_1 T - \kappa_1|\xi'|^2 T + O(|\xi'|^3)T$ with some $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$, provided Re and Ma are sufficiently small. On the other hand, if $|\xi'| \ge R \gg 1$, we can derive the exponential decay property of the corresponding part of the solution operator $\mathscr{U}(t,s)$ by the Fourier transformed version of Matsumura-Nishida's energy method (see [3, 8]), provided that Re and Ma are sufficiently small. As for the bounded frequency part $r \le |\xi'| \le R$, we employ a certain time-dependent decomposition argument and apply a variant of Matsumura-Nishida's energy method as in [6] to show the exponential decay. As a result, one can see that the solution of the linearized problem behaves as $\sigma_{t,s}(x')u^{(0)}(x_n,t) = \mathscr{F}^{-1}\left(e^{-(i\kappa_0\xi_1+\kappa_1|\xi'|^2)(t-s)}[\widehat{\phi}_0(\xi')]\right)u^{(0)}(x_n,t)$, provided that Re and Ma are sufficiently small.

Problem (1.1)–(1.4) with $\boldsymbol{g} = (g^1(x_n,t), 0, \ldots, 0, g^n(x_n))$ also covers another particularly interesting problem. Let us for a moment consider problem (1.1)–(1.2) together with $\boldsymbol{g} = (0, \ldots, 0, g^n(x_n))$ and boundary condition

$$v|_{x_n=0} = V^1(t)\boldsymbol{e}_1, \quad v|_{x_n=\ell} = 0,$$

where V^1 is τ-periodic function of time and $\boldsymbol{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. This problem is a natural extension of Stokes' second problem from half space to infinite strip for compressible fluid. The motion of a fluid is caused by the periodic oscillation of a boundary plate. The study of the flow of a viscous fluid over an oscillating plate is not only of theoretical interest, but it also occurs in many applied problems and since Stokes (1851) it has received much attention under various settings. It is straightforward to see, that if we seek solution to this problem in the form $v = (1 - x_n)V^1(t)\boldsymbol{e}_1 + \tilde{v}$, then behavior of $\tilde{v}$ is governed by the same linearized problem as we get for (1.1)–(1.3) with $\boldsymbol{g} = (-(1-x_n)\partial_t V^1(t)/\rho, 0, \ldots, 0, g^n(x_n))$ and therefore our result also holds for this particular problem.

This paper is organized as follows. In Section 2 we rewrite the problem into a non-dimensional one and state the existence of time-periodic parallel flows. Our main results are stated in Section 3. In Section 4 we prove the exponential decay estimates. Finally, in Section 5 we prove the asymptotic behavior (1.10) and (1.11).

2 Periodic Solution and the Linearized Problem

In this section we state the existence of time-periodic solution and then we rewrite the problem into the one for the disturbance in a non-dimensional form. At the end of this section we introduce notation that will be used throughout this paper. Let ρ_* be a given positive number. Throughout the paper we assume that

$$P'(\rho_*) > 0.$$

We introduce the following dimensionless variables:

$$x = \ell\widetilde{x}, \quad t = \frac{\ell}{V}\widetilde{t}, \quad v = V\widetilde{v}, \quad \rho = \rho_*\widetilde{\rho}, \quad P = \rho_* V^2 \widetilde{P}, \quad \boldsymbol{g}(x_n, t) = \frac{\mu V}{\rho_* \ell^2}\widetilde{\boldsymbol{g}}(\widetilde{x}_n, \widetilde{t}),$$

where

$$V = \frac{\rho_* \ell^2}{\mu}|g_1|_{\infty,2}.$$

Then the problem (1.1)–(1.4) is transformed into the following dimensionless problem on the layer $\Omega = \mathbb{R}^{n-1} \times (0,1)$:

$$\partial_t\widetilde{\rho} + \mathrm{div}\,(\widetilde{\rho}\widetilde{v}) = 0, \tag{2.1}$$

$$\widetilde{\rho}(\partial_t\widetilde{v} + (\widetilde{v}\cdot\nabla)\widetilde{v}) - \nu\Delta\widetilde{v} - (\nu+\nu')\nabla\mathrm{div}\,\widetilde{v} + \widetilde{P}'(\widetilde{\rho})\nabla\widetilde{\rho} = \nu\widetilde{\rho}\widetilde{\boldsymbol{g}} \tag{2.2}$$

$$\widetilde{v}|_{x_n=0} = \widetilde{v}|_{x_n=1} = 0, \tag{2.3}$$

$$(\widetilde{\rho}, \widetilde{v})|_{t=0} = (\widetilde{\rho}_0, \widetilde{v}_0). \tag{2.4}$$

Here ν and ν' are the non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_*\ell V}, \quad \nu' = \frac{\mu'}{\rho_*\ell V}.$$

We also introduce parameters γ and T:

$$\gamma = \sqrt{\widetilde{P}'(1)}, \quad T = \frac{V}{\ell}\tau,$$

where T is period of $\widetilde{g}_1$ in time. We note that the Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively. In what follows we omit "tilde" of $\widetilde{x}_n$ and $\widetilde{t}$.

One can see that if $|\widetilde{g}^n|_\infty$ is small enough then a time-periodic solution $(\rho_p, v_p) = (\rho_p(x_n), v_p^1(x_n,t)\boldsymbol{e}_1)$ exists. More precisely, substituting $(\widetilde{\rho}, \widetilde{v}) = (\rho_p(x_n), v_p^1(x_n,t)\boldsymbol{e}_1)$ into (2.1)–(2.3), we have

$$\partial_t v_p^1 - \frac{\nu}{\rho_p}\partial_{x_n}^2 v_p^1 = \nu \widetilde{g}^1 \tag{2.5}$$

$$\partial_{x_n}(\widetilde{P}(\rho_p)) = \nu\rho_p\widetilde{g}^n \tag{2.6}$$

$$v_p^1|_{x_n=0} = v_p^1|_{x_n=1} = 0. \tag{2.7}$$

We state the existence of a time-periodic solution to (2.5)–(2.7) with

$$1 = \int_0^1 \widetilde{\rho}(x_n)\, dx_n. \tag{2.8}$$

Proposition 2.1. *Assume that $\widetilde{P}'(\rho) > 0$ for $\rho_1 \le \rho \le \rho_2$ with some $0 < \rho_1 < 1 < \rho_2 < 2$. Let $\Phi(\rho) = \int_1^\rho \frac{\widetilde{P}'(\eta)}{\eta}\, d\eta$ for $\rho_1 \le \rho \le \rho_2$ and let $\Psi(r) = \Phi^{-1}(r)$ for $r_1 \le r \le r_2$. Here Φ^{-1} denotes the inverse function of Φ and $r_j = \Phi(\rho_j)$ $(j = 1, 2)$. If*

$$\nu|\widetilde{g}^n|_\infty \le C\min\left\{|r_1|, r_2, \frac{1}{4\gamma^2|\Psi''|_{C^0[r_1,r_2]}}\right\} \le C, \tag{2.9}$$

then there exists a smooth time-periodic solution $(\rho_p, v_p) = (\rho_p(x_n), v_p^1(x_n,t)\boldsymbol{e}_1)$ of (2.5)–(2.8) satisfying

$$\rho_1 \le \rho_p(x_n) \le \rho_2, \quad |\rho_p - 1|_\infty \le C\frac{\nu}{\gamma^2}|\widetilde{g}^n|_\infty,$$

$$v_p^1(x_n, t) = \nu\int_{-\infty}^t e^{-\nu A(t-s)}\widetilde{g}^1(x_n, s)\, ds,$$

where A denotes the uniformly elliptic operator on $L^2(0,1)$ with domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$ and $Av = -\frac{1}{\rho_p(x_n)}\partial_{x_n}^2 v$ for $v \in D(A)$. Furthermore we have estimates

$$|\partial_t^k v_p^1|_{\infty,2} \le C|\partial_t^k \widetilde{g}^1|_{\infty,2}, \quad k \ge 0,$$

$$|\partial_{x_n} v_p^1|_{\infty,2} \le C\left(\frac{1}{\nu}|\partial_t \widetilde{g}^1|_{\infty,2} + |\widetilde{g}^1|_{\infty,2}\right),$$

$$|\partial_t^k\partial_{x_n}^2 v_p^1|_{\infty,2} \le C\left(\frac{1}{\nu}|\partial_t^{k+1}\widetilde{g}^1|_{\infty,2} + |\partial_t^k\widetilde{g}^1|_{\infty,2}\right), \quad k \ge 0.$$

Additionally, if $\nu|\widetilde{g}^n|_{C^{k-1}[0,1]} \le \eta$, then

$$|\partial_{x_n}^k \rho_p|_\infty \le C_k\nu|\widetilde{g}^n|_{C^{k-1}[0,1]} \text{ for } k = 1, 2, \ldots$$

Here C_k are positive constants depending on $k, \eta, |\Psi|_{C^k[r_1,r_2]}$ and ρ_2. In particular,

$$|\partial_{x_n}\rho_p|_\infty \le C\frac{\nu}{\gamma^2}|\widetilde{g}^n|_\infty,$$

$$|\widetilde{P}'(\rho_p)-\gamma^2|_\infty \le C|\widetilde{P}''|_{C^0[\rho_1,\rho_2]}\frac{\nu}{\gamma^2}|\widetilde{g}^n|_\infty.$$

Remark. Operator A satisfies estimates

$$|e^{-\nu At}v|_2 \le C\, e^{-\frac{\nu}{2}t}\,|v|_2,$$

$$|\partial_{x_n}e^{-\nu At}v|_2 \le C\frac{1}{t^{\frac{1}{2}}}\,e^{-\frac{\nu}{2}t}\,|v|_2,$$

for some $C>0$ and all $t>0$.

Proof. Proof of existence and properties of ρ_p is the same as in [4], if we substitute $\nu\widetilde{g}^n$ for $\widetilde{g}^n$, so we omit it. Once we obtained ρ_p we easily get v_p^1 solution of (2.5) and (2.7). To obtain estimates on v_p^1 we combine iteratively following relations:

$$\partial_t\int_{-\infty}^t e^{-\nu A(t-s)}f(s)\,ds = f(t)+\int_{-\infty}^t \partial_t[e^{-\nu A(t-s)}f(s)]\,ds$$

$$= f(t)-\int_{-\infty}^t \partial_s[e^{-\nu A(t-s)}f(s)]\,ds+\int_{-\infty}^t e^{-\nu A(t-s)}\partial_s f(s)\,ds = \int_{-\infty}^t e^{-\nu A(t-s)}\partial_s f(s)\,ds,$$

$$\partial_{x_n}^2 v_p^1 = \frac{\rho_p}{\nu}(\partial_t v_p^1-\nu\widetilde{g}^1),$$

$$|\partial_{x_n}v_p^1(t)|_{L^2} \le |v_p^1(t)|_{L^2}^{\frac{1}{2}}\;|\partial_{x_n}^2 v_p^1(t)|_{L^2}^{\frac{1}{2}}.$$

□

Setting $\widetilde{\rho}=\rho_p+\gamma^{-2}\phi$ and $\widetilde{v}=v_p+w$ in (2.1)–(2.4) and neglecting nonlinear terms of $u={}^T(\phi,w)$, we arrive at the linearized problem:

$$\partial_t\phi+v_p^1\partial_{x_1}\phi+\gamma^2\mathrm{div}\,(\rho_p w)=0, \tag{2.10}$$

$$\begin{aligned}\partial_t w-\tfrac{\nu}{\rho_p}\Delta w-\tfrac{\nu+\nu'}{\rho_p}\nabla\mathrm{div}\,w+v_p^1\partial_{x_1}w+(\partial_{x_n}v_p^1)w^n\boldsymbol{e}_1&\\ +\tfrac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1)\phi\boldsymbol{e}_1+\nabla\left(\tfrac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\phi\right)&=\boldsymbol{0},\end{aligned} \tag{2.11}$$

$$w|_{x_n=0}=w|_{x_n=1}=0, \tag{2.12}$$

$$(\phi,w)|_{t=0}=(\phi_0,w_0). \tag{2.13}$$

Our main concern in this paper is the estimates of solutions to the problem (2.10)–(2.13).

In the remaining part of this section we introduce some notation which will be used throughout the paper. For a domain N we denote by $L^2(N)$ the usual Lebesgue space on N and its norm is denoted by $\|\cdot\|_{L^2(N)}$. Let m be a nonnegative integer. $H^m(N)$ denotes the m-th order L^2 Sobolev space on N with norm $\|\cdot\|_{H^m(N)}$. $C_0^m(N)$ stands for the set of all C^m functions which have compact support in N. We denote by $H_0^1(N)$ the completion of $C_0^1(N)$ in $H^1(N)$.

We simply denote by $L^2(N)$ (resp., $H^m(N)$) the set of all vector fields $w = {}^T(w^1, \dots, w^n)$ on N with $w^j \in L^2(N)$ (resp., $H^m(N)$), $j = 1, \dots, n$, and its norm is also denoted by $\|\cdot\|_{L^2(N)}$ (resp., $\|\cdot\|_{H^m(N)}$). For $u = {}^T(\phi, w)$ with $\phi \in H^k(N)$ and $w = {}^T(w^1, \dots, w^n) \in H^m(N)$, we define $\|u\|_{H^k(N)\times H^m(N)}$ by $\|u\|_{H^k(N)\times H^m(N)} = \|\phi\|_{H^k(N)} + \|w\|_{H^m(N)}$. When $k = m$, we simply write $\|u\|_{H^k(N)\times H^k(N)} = \|u\|_{H^k(N)}$.

In case $N = \Omega$ we abbreviate $L^2(\Omega)$ (resp., $H^m(\Omega)$) as L^2 (resp., H^m). In particular, the norm $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|_{L^2}$ is denoted by $\|\cdot\|_p$.

In case $N = (0,1)$ we denote the norm of $L^2(0,1)$ by $|\cdot|_2$. The inner product of $L^2(0,1)$ is denoted by

$$(f, g) = \int_0^1 f(x_n)\overline{g(x_n)}\, dx_n, \quad f, g \in L^2(0,1).$$

Here $\overline{g}$ denotes the complex conjugate of g. For $u_j = {}^T(\phi_j, w_j) \in L^2(0,1)$ with $w_j = {}^T(w_j^1, \dots, w_j^n)$ $(j = 1, 2)$, we also define a weighted inner product $\langle u_1, u_2\rangle$ by

$$\langle u_1, u_2\rangle = \int_0^1 \phi_1\overline{\phi}_2 \tfrac{\widetilde{P}'(\rho_p)}{\gamma^4\rho_p}\, dx_n + \int_0^1 w_1\overline{w}_2\rho_p\, dx_n.$$

Furthermore, for $f \in L^1(0,1)$ we denote the mean value of f in $(0,1)$ by $[f]$:

$$[f] = (f, 1) = \int_0^1 f(x_n)\, dx_n.$$

For $u = {}^T(\phi, w) \in L^1(0,1)$ with $w = {}^T(w^1, \dots, w^n)$ we define $[u]$ by

$$[u] = [\phi] + [w^1] + \cdots + [w^n].$$

The norm of $H^m(0,1)$ is denoted by $|\cdot|_{H^m}$.

We often write $x \in \Omega$ as

$$x = {}^T(x', x_n), \quad x' = {}^T(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

Partial derivatives of a function u in x, x', x_n and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u;\ |\alpha| = k)$.

We denote the $k \times k$ identity matrix by I_k. In particular, when $k = n+1$, we simply write I for I_{n+1}. We also define $(n+1)\times(n+1)$ diagonal matrices Q_j, Q' and $\widetilde{Q}$ by

$$Q_j = \mathrm{diag}\,(0, \dots, 0, \underbrace{1}_{j-th}, 0, \dots, 0), \quad j = 0, 1, \dots, n,$$

and

$$Q' = \mathrm{diag}\,(0, 1, \dots, 1, 0), \qquad \widetilde{Q} = \mathrm{diag}\,(0, 1, \dots, 1).$$

We then have, for $u = {}^T(\phi, w) \in \mathbb{R}^{n+1}$, $w = {}^T(w^1, \dots, w^n) = {}^T(w', w^n)$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad Q_j u = \begin{pmatrix} 0 \\ w^j \\ 0 \end{pmatrix}, \quad Q_n u = \begin{pmatrix} 0 \\ 0 \\ w^n \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ w' \\ 0 \end{pmatrix}, \quad \widetilde{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

We note that

$$[Q_0 u] = [\phi] \quad \text{for } u = {}^T(\phi, w).$$

For a function $f = f(x')$ ($x' \in \mathbb{R}^{n-1}$), we denote its Fourier transform by $\widehat{f}$ or $\mathscr{F} f$:

$$\widehat{f}(\xi') = (\mathscr{F} f)(\xi') = \int_{\mathbb{R}^{n-1}} f(x') e^{-i\xi' \cdot x'} \, dx'.$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$(\mathscr{F}^{-1} f)(x) = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} \, d\xi'.$$

We will denote the resolvent set of a closed operator B by $\rho(B)$ and the spectrum of B by $\sigma(B)$. For a bounded linear operator B we denote the spectral radius of B by $r(B)$. For $\Lambda \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ we denote:

$$\Sigma(\Lambda, \theta) = \{\lambda \in \mathbb{C}; \, |\arg(\lambda - \Lambda)| \le \theta\}.$$

We denote the set of bounded linear operators from X_1 to X_2 by $L(X_1, X_2)$, and if $X_1 = X_2$, we simply write $L(X_1)$ for $L(X_1, X_1)$. The operator norm is denoted by $|\cdot|_{L(X_1, X_2)}$.

3 Main Results

Let us consider the linearized problem

$$\partial_t u + L(t) u = 0, \quad w|_{x_n = 0,1} = 0, \quad u|_{t=s} = u_0. \tag{3.1}$$

Here $u = {}^T(\phi, w)$ and $L(t)$ is the operator of the form

$$L(t) = \begin{pmatrix} v_p^1(t)\partial_{x_1} & \gamma^2 \mathrm{div}\,(\rho_p \,\cdot\,) \\ \nabla \left(\frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p} \,\cdot \right) & -\frac{\nu}{\rho_p} \Delta I_n - \frac{\nu + \nu'}{\rho_p} \nabla \mathrm{div} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_p^2} (\partial_{x_n}^2 v_p^1(t)) \boldsymbol{e}_1 & v_p^1(t) \partial_{x_1} I_n + (\partial_{x_n} v_p^1(t)) \boldsymbol{e}_1 {}^T \boldsymbol{e}_n \end{pmatrix}.$$

We introduce the space Z_s defined by

$$\begin{aligned} Z_s = \{u = {}^T(\phi, w); \, & \phi \in C_{loc}([s, \infty); H^1), \\ & \partial_{x'}^{\alpha'} w \in C_{loc}([s, \infty); L^2) \cap L^2_{loc}([s, \infty); H^1_0) \ (|\alpha'| \le 1), \\ & w \in C_{loc}((s, \infty); H^1_0)\}, \end{aligned}$$

where the linearized problem can be uniquely solved. We will denote the solution operator for (3.1) by $\mathscr{U}(t, s)$.

Theorem 3.1. *Let $s \ge 0$ be arbitrarily given. For any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in H^1 \times L^2$ with $\partial_{x'} w_0 \in L^2$ there exists a unique solution $u_s(t) = \mathscr{U}(t, s) u_0$ of (3.1) in Z_s.*

Furthermore, $\mathscr{U}(t,s)u_0$ satisfies estimates

$$\|\mathscr{U}(t,s)u_0\|_2 \le C\|u_0\|_2,$$

and

$$\|\partial_x Q_0 \mathscr{U}(t,s)u_0\|_2 + \|\partial_{x'}\widetilde{Q}\mathscr{U}(t,s)u_0\|_2 + (t-s)^{\frac{1}{2}}\|\partial_{x_n}\widetilde{Q}\mathscr{U}(t,s)u_0\|_2$$
$$\le C\left\{\|u_0\|_{H^1\times L^2} + \|\partial_{x'}w_0\|_2\right\},$$

for $0 < t-s \le 4T$, $s \ge 0$.

Proof. The uniqueness in Z_s can be shown by an elementary energy method. As for the existence, it is not difficult to show the unique existence of solution $u \in C_{loc}([0,\infty);H^1) \times (C_{loc}([0,\infty);H_0^1) \cap L^2_{loc}([0,\infty);H^2))$ for $u_0 \in H^1 \times H_0^1$. Existence for $u_0 \in H^1 \times L^2$ with $\partial_{x'}w_0 \in L^2$ then follows from an approximation argument by using Lemma 4.4 and 4.5 below. The proof of the estimate also follows from Lemma 4.4 below. $\square$

Theorem 3.2. *There exist constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \ge \nu_0$ and $\gamma^2/(2\nu+\nu') \ge \gamma_0^2$, then for any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in (H^1\times L^2)\cap L^1(\mathbb{R}^{n-1};H^1(0,1)\times L^2(0,1))$ with $\partial_{x'}w_0 \in L^2$ the solution $u_s(t) = \mathscr{U}(t,s)u_0$ of problem* (3.1) *can be decomposed as*

$$\mathscr{U}(t,s)u_0 = \mathscr{U}^{(0)}(t,s)u_0 + \mathscr{U}^{(\infty)}(t,s)u_0,$$

where each term on the right-hand side has the following properties for $t-s \ge 4T$, $s \ge 0$.

(i)

$$\|\partial_{x'}^k\partial_{x_n}^l \mathscr{U}^{(0)}(t,s)u_0\|_2 \le C(t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{L^1(\mathbb{R}^{n-1};H^1(0,1)\times L^2(0,1))}, \tag{3.2}$$

$$\|\partial_{x'}^k\partial_{x_n}^l(\mathscr{U}^{(0)}(t,s)u_0 - \sigma_{t,s}u^{(0)}(t))\|_2 \le C(t-s)^{-\frac{n-1}{4}-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{L^1(\mathbb{R}^{n-1};H^1(0,1)\times L^2(0,1))}, \tag{3.3}$$

$k,l = 0,1$. Here

$$\sigma_{t,s} = \mathscr{F}^{-1}\left(e^{-(i\kappa_0\xi_1+\kappa_1|\xi'|^2)(t-s)}[\widehat{\phi}_0]\right),$$

$u^{(0)}(t) = u^{(0)}(x_n,t)$ is some T-periodic function (see Lemma 4.6 below), and $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$ are some constants satisfying

$$\kappa_1 = \tfrac{\gamma^2}{\nu}K,\ \ K > 0.$$

(ii)

$$\|\partial_x^l \mathscr{U}^{(\infty)}(t,s)u_0\|_2 \le Ce^{-d(t-s)}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'}w_0\|_2), \tag{3.4}$$

$l = 0,1$, for some positive constant d.

Remark. In both Theorems 3.1 and 3.2 we assume following smoothness for external force and boundary data:

$$\widetilde{g}^n \in C^1[0,1],\ \ \widetilde{g}^1 \in C^2(0,\infty : L^2(0,1)).$$

We combine estimates from Proposition 2.1 with basic assumption (2.9) on $\nu|\widetilde{g}^n|_\infty$ to get

$$|\rho_p - 1|_\infty \le \frac{C}{\gamma^2},\ \ |\partial_{x_n}\rho_p|_\infty \le \frac{C}{\gamma^2},\ \ |\widetilde{P}'(\rho_p) - \gamma^2|_\infty \le \frac{C}{\gamma^2}.$$

Furthermore, without loss of generality, we also assume that

$$|\partial_{x_n}^2 \rho_p|_\infty \leq \frac{C}{\gamma^2},$$

holds true. Therefore, the smallness assumptions on $\widetilde{g}^n$ in Theorems 3.1 and 3.2 are expressed in terms of smallness of $\frac{1}{\gamma^2}$.

Again regarding to Proposition 2.1, in what follows we use bounds on v_p^1 as

$$|\partial_t^k v_p^1|_{\infty,2} + |\partial_{x_n} v_p^1|_{\infty,2} + |\partial_t^k \partial_{x_n}^2 v_p^1|_{\infty,2} \leq C, \quad k \geq 0.$$

The decay rate (3.2) is the same one as that of an $n-1$ dimensional heat kernel. Our result shows that this is an optimal decay rate, and, in fact, estimate (3.3) shows that the asymptotic leading part of solutions is given by an $n-1$ dimensional heat kernel, which moves in x_1 direction with a constant speed, multiplied by time-periodic function.

A proof of Theorem 3.2 will be outlined in Section 4 and Section 5. In Section 4 we prove the exponential decay for the bounded frequency part and high frequency part. In Section 5 we prove the asymptotic estimates (3.2) and (3.3) for low frequency part.

4 Proof of Theorem 3.2 - 1. Exponential decay estimates

In this section we introduce decomposition of $\mathscr{U}(t,s)$ based on size of $|\xi'|$ and prove that bounded frequency and high frequency parts decay exponentially. From now on we simply denote $\nu+\nu'$ by $\widetilde{\nu}$:

$$\widetilde{\nu} = \nu + \nu'.$$

To simplify further calculations we suppose

$$\nu \geq 1, \quad \tilde{\nu} \geq 1, \quad \gamma \geq 1, \quad \frac{\gamma^2}{\nu+\tilde{\nu}} \geq 1.$$

To prove Theorem 3.2, we consider the Fourier transform of (3.1) in x' variable. The Fourier transform of (3.1) is written as

$$\partial_t \widehat{\phi} + i\xi_1 v_p^1 \widehat{\phi} + i\gamma^2 \xi' \cdot (\rho_p \widehat{w}') + \gamma^2 \partial_{x_n}(\rho_p \widehat{w}^n) = 0, \tag{4.1}$$

$$\begin{aligned} &\partial_t \widehat{w}' + \tfrac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)\widehat{w}' - i\tfrac{\widetilde{\nu}}{\rho_p}\xi'(i\xi'\cdot\widehat{w}' + \partial_{x_n}\widehat{w}^n) \\ &\quad + i\xi' \tfrac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\widehat{\phi} + i\xi_1 v_p^1 \widehat{w}' + (\partial_{x_n} v_p^1)\widehat{w}^n \boldsymbol{e}_1' + \tfrac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1)\widehat{\phi}\boldsymbol{e}_1' = 0, \end{aligned} \tag{4.2}$$

$$\partial_t \widehat{w}^n + \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)\widehat{w}^n - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}(i\xi'\cdot\widehat{w}' + \partial_{x_n}\widehat{w}^n) + \partial_{x_n}\left(\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\widehat{\phi}\right) + i\xi_1 v_p^1 \widehat{w}^n = 0, \tag{4.3}$$

$$\widehat{w}|_{x_n=0,1} = 0, \tag{4.4}$$

for $t > s \geq 0$, and

$$\widehat{u}|_{t=s} = \widehat{u}_0 = {}^T(\widehat{\phi}_0, \widehat{w}_0). \tag{4.5}$$

Here $\widehat{\phi} = \widehat{\phi}(\xi', x_n, t)$ and $\widehat{w} = \widehat{w}(\xi', x_n, t)$ are the Fourier transform of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbb{R}^{n-1}$ with $\xi' \in \mathbb{R}^{n-1}$ being the dual variable and $\boldsymbol{e}_1' = (1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$. We thus arrive at the following problem

$$\frac{d}{dt}u + \widehat{L}_{\xi'}(t)u = 0, \quad u|_{t=s} = u_0, \tag{4.6}$$

on $X \equiv H^1(0,1) \times L^2(0,1)$. Here, for each t, $\widehat{L}_{\xi'}(t)$ is an operator on X with domain $D(\widehat{L}_{\xi'}(t)) = D \equiv H^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1))$ with a parameter $\xi' \in \mathbb{R}^{n-1}$. Here $u = {}^T(\phi(x_n, t), w(x_n, t))$ $(x_n \in [0,1], t \geq s \geq 0)$ and $\widehat{L}_{\xi'}(t)$ is the operator of the form

$$\widehat{L}_{\xi'}(t) = \widehat{A}_{\xi'} + \widehat{B}_{\xi'}(t) + \widehat{C}_0(t),$$

where

$$\widehat{A}_{\xi'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu \mathscr{A}_{\xi'} I_{n-1} + \frac{\widetilde{\nu}}{\rho_p} \xi'^T \xi' & -i\frac{\widetilde{\nu}}{\rho_p}\xi' \partial_{x_n} \\ 0 & -i\frac{\widetilde{\nu}}{\rho_p}{}^T\xi' \partial_{x_n} & \nu \mathscr{A}_{\xi'} - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix},$$

with

$$\mathscr{A}_{\xi'} = \frac{1}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2),$$

$$\widehat{B}_{\xi'}(t) = \begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2 \rho_p {}^T\xi' & \gamma^2 \partial_{x_n}(\rho_p \,\cdot\,) \\ i\xi' \frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p} & i\xi_1 v_p^1(t) I_{n-1} & 0 \\ \partial_{x_n}\left(\frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p}\,\cdot\,\right) & 0 & i\xi_1 v_p^1(t) \end{pmatrix},$$

$$\widehat{C}_0(t) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1(t))\boldsymbol{e}_1' & 0 & (\partial_{x_n} v_p^1(t))\boldsymbol{e}_1' \\ 0 & 0 & 0 \end{pmatrix}.$$

For each fixed ξ', $\widehat{A}_{\xi'} + \widehat{B}_0$ is a closed operator on X with domain $D(\widehat{A}_{\xi'} + \widehat{B}_0) = D$; and $-(\widehat{A}_{\xi'} + \widehat{B}_0)$ generates an analytic semigroup in X (see [2]). Since v_p^1 is bounded smooth function of x_n, t, $\widehat{L}_{\xi'}(t)$ can be seen as a lower order perturbation of $\widehat{A}_{\xi'} + \widehat{B}_0$. Therefore, we can show that

$$(|\lambda| + 1)|(\lambda + \widehat{L}_{\xi'}(t))^{-1}|_{L(X)} \leq C,$$

and

$$|(\widehat{L}_{\xi'}(t) - \widehat{L}_{\xi'}(s))(\Lambda + \widehat{L}_{\xi'}(0))^{-1}|_{L(X)} \leq C|t-s|^\alpha,$$

where $\lambda \in \Sigma(\Lambda, \theta)$ with some $\Lambda > 0$, $\theta \in (\frac{\pi}{2}, \pi)$ and $\alpha > 0$. It then follows from [9, Theorem 5.2.1] that, for each $\xi' \in \mathbb{R}^{n-1}$, there exists a unique solution $\widehat{u}_s(t) = \widehat{U}_{\xi'}(t,s)\widehat{u}_0$ of problem

(4.1)–(4.5) satisfying $\widehat{u}_s \in C_{loc}([s,\infty);X)\cap C^1((s,\infty);X)\cap C((s,\infty);D)$. Moreover, $\widehat{U}_{\xi'}(t,s)$ satisfies $|\widehat{U}_{\xi'}(t,s)|_{L(X)} \leq C_{t_1}$ and $|\widehat{L}_{\xi'}(t)\widehat{U}_{\xi'}(t,s)|_{L(X)} \leq C_{t_1}(t-s)^{-1}$ with $0 \leq s \leq t \leq t_1$ for all $t_1 \geq 0$, which implies that $|\partial_{x_n}\widetilde{Q}\widehat{U}_{\xi'}(t,s)u_0|_2 \leq C_{t_1}(t-s)^{-\frac{1}{2}}|u_0|_{H^1\times L^2}$. The solution $\mathscr{U}(t,s)u_0$ of (3.1) is then given by $\mathscr{U}(t,s)u_0 = \mathscr{F}^{-1}(\widehat{U}_{\xi'}(t,s)\widehat{u}_0)$ for all $t-s\geq 0, s\geq 0$.

We decompose $\mathscr{U}(t,s)u_0$ in the following way. Let $0<r<R$. Define $\chi^{(0)}(\xi')$, $\chi^{(1)}(\xi')$ and $\chi^{(\infty)}(\xi')$ by

$$\chi^{(0)}(\xi') = 1 \text{ if } |\xi'|\leq r,\ \chi^{(0)}(\xi') = 0 \text{ if } |\xi'|>r,$$

$$\chi^{(\infty)}(\xi') = 1 \text{ if } |\xi'|\geq R,\ \chi^{(\infty)}(\xi') = 0 \text{ if } |\xi'|<R,$$

and

$$\chi^{(1)} = (1-\chi^{(0)})(1-\chi^{(\infty)}).$$

We decompose $\mathscr{U}(t,s)u_0$ as

$$\mathscr{U}(t,s)u_0 = \mathscr{V}_0(t,s)u_0 + \mathscr{V}_1(t,s)u_0 + \mathscr{V}_\infty(t,s)u_0,$$

where

$$\mathscr{V}_j(t,s)u_0 = \mathscr{F}^{-1}\left(\chi^{(j)}\widehat{U}_{\xi'}(t,s)\widehat{u}_0\right), \quad j=0,1,\infty.$$

In this section we will show that $\mathscr{V}_j(t,s)u_0$ $(j=1,\infty)$ decay exponentially in time. The low frequency part $\mathscr{V}_0(t,s)u_0$ will be investigated in Section 5, which all together gives estimates (3.2) and (3.3).

Proposition 4.1. *There exist constants $R_0>1$, $\nu_0>0$ and $\gamma_0>0$ such that if $\nu\geq\nu_0$ and $\gamma^2/(\nu+\widetilde{\nu})\geq\gamma_0^2$ then there exists a constant $d>0$ such that the estimate*

$$\|\mathscr{V}_\infty(t,s)u_0\|_{H^1} \leq Ce^{-d(t-s-4T)}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'}w_0\|_2),$$

holds uniformly in $t-s\geq 4T$, $s\geq 0$.

Proposition 4.2. *There exist constants $\nu_0>0$ and $\gamma_0>0$ such that if $\nu\geq\nu_0$ and $\gamma^2/(\nu+\widetilde{\nu})\geq\gamma_0^2$ then for any $0<r<R_0$ there exists a constant $d(r)>0$ such that the estimate*

$$\|\mathscr{V}_1(t,s)u_0\|_{H^1} \leq Ce^{-d(t-s-4T)}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'}w_0\|_2),$$

holds uniformly in $t-s\geq 4T$, $s\geq 0$.

Proof of Proposition 4.1. We investigate problem (4.6) for $|\xi'|\geq R>0$ and prove Proposition 4.1. We also give a proof of the estimate given in Theorem 3.1.

We introduce following notation. For $v={}^T(\phi,w)$, or $v=w$ we define $D_{\xi'}(v)$ by

$$D_{\xi'}(v) = |\xi'|^2|v|_2^2 + |\partial_{x_n}v|_2^2.$$

First, we state the exponential decay estimate for large $t-s$.

Lemma 4.3. *There exist constants $R_0 > 1$, $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_0^2$ then there exists a constant $d > 0$ such that the estimate*

$$|\widehat{\phi}(t)|_2^2 + |\widehat{w}(t)|_2^2 + D_{\xi'}(\widehat{u}_s)(t) \leq Ce^{-d(t-s-4T)} \left\{ |\widehat{\phi}(4T)|_2^2 + |\widehat{w}(4T)|_2^2 + D_{\xi'}(\widehat{u}_s)(4T) \right\},$$

holds uniformly for $t - s \geq 4T$, $s \geq 0$ provided that $|\xi'| \geq R_0$.

Proof of Lemma 4.3 is similar to the proof of [3, Proposition 6.1]. In our case in calculations appear new disturbance terms thanks to the $\rho_p \not\equiv 1$ and term $\frac{\nu}{\gamma^2 \rho_p^2}(\partial_{x_n}^2 v_p^1)\widehat{\phi}\boldsymbol{e}_1'$, but those can be handled in standard way by smallness assumptions on Mach and Reynolds numbers, so we omit the proof. See also [4].

Second, we state estimates for $t - s$ near 0.

Lemma 4.4. *There holds the following estimate uniformly for $0 < t - s \leq 4T$, $s \geq 0$ and $\xi' \in \mathbf{R}^{n-1}$:*

$$(1 + |\xi'|^2)(|\widehat{\phi}(t)|_2^2 + |\widehat{w}(t)|_2^2) + |\partial_{x_n}\widehat{\phi}(t)|_2^2 + (t - s)|\partial_{x_n}\widehat{w}(t)|_2^2 \leq C \left\{ (1 + |\xi'|^2)|\widehat{u}_0|_2^2 + |\partial_{x_n}\widehat{\phi}_0|_2^2 \right\}.$$

Proposition 4.1 is an immediate consequence of Lemma 4.3 and 4.4. □

The estimate in Theorem 3.1 also follows from Lemma 4.4. It remains to prove Lemma 4.4. To prove Lemma 4.4 we employ the following estimates.

Lemma 4.5. *For all $0 \leq t - s \leq 4T$, $s \geq 0$ there hold the estimates*

$$(1+|\xi'|^2)(\frac{1}{\gamma^2}|\widehat{\phi}(t)|_2^2 + |\widehat{w}(t)|_2^2) + \nu \int_s^t (1+|\xi'|^2) D_{\xi'}(\widehat{w})\, ds \leq C(1+|\xi'|^2)(\frac{1}{\gamma^2}|\widehat{\phi}_0|_2^2 + |\widehat{w}_0|_2^2) \quad (4.7)$$

and

$$|\partial_{x_n}\widehat{\phi}(t)|_2^2 \leq C \left\{ (1 + |\xi'|^2)|\widehat{u}_0|_2^2 + |\partial_{x_n}\widehat{\phi}_0|_2^2 \right\}. \quad (4.8)$$

Proof of Lemma 4.5 is similar to the proof of [3, Proposition 6.11], so we omit the proof.

Proof of Lemma 4.4. We write (4.2) and (4.3) as

$$\partial_t w + \widehat{T}_{\xi'} w = -\widehat{E}_{\xi'}(t)w - \widehat{F}_{\xi'}(t)\phi.$$

Here $\widehat{T}_{\xi'}$ is the operator on $L^2(0,1)$ of the form

$$\widehat{T}_{\xi'} = \begin{pmatrix} \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\widetilde{\nu}}{\rho_p}\xi'^T\xi' & -i\frac{\widetilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ -i\frac{\widetilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix},$$

with domain $D(\widehat{T}_{\xi'}) = H^2(0,1) \cap H_0^1(0,1)$,

$$\widehat{E}_{\xi'}(t) = \begin{pmatrix} i\xi_1 v_p^1(t)I_{n-1} & (\partial_{x_n} v_p^1(t))\boldsymbol{e}_1' \\ 0 & i\xi_1 v_p^1(t) \end{pmatrix},$$

and

$$\widehat{F}_{\xi'}(t)=\left(\begin{array}{c} i\xi'\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}+\frac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1(t))\boldsymbol{e}_1' \\ \partial_{x_n}\left(\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\ \cdot\right) \end{array}\right).$$

Then w is written as

$$w(t)=e^{-(t-s)\widehat{T}_{\xi'}}w_0-\int_s^t e^{-(t-z)\widehat{T}_{\xi'}}\widehat{E}_{\xi'}(z)w(z)\,dz-\int_s^t e^{-(t-z)\widehat{T}_{\xi'}}\widehat{F}_{\xi'}(z)\phi(z)\,dz. \tag{4.9}$$

Using the equality

$$(\widehat{T}_{\xi'}w,\rho_p w)=\nu D_{\xi'}(w)+\widetilde{\nu}|i\xi'\cdot w'+\partial_{x_n}w^n|_2^2,$$

one can see that

$$|\partial_{x_n}^l e^{-t\widehat{T}_{\xi'}}w_0|_2\le Ct^{-\frac{l}{2}}|w_0|_2\ \ (l=0,1), \tag{4.10}$$

for $0<t\le 4T$.

We have estimate

$$|\widehat{E}_{\xi'}(z)w(z)|_2\ \le\ C(1+|\xi'|)|w(z)|_2\le C(1+|\xi'|)|u_0|_2, \tag{4.11}$$

and moreover by (4.7) and (4.8) we have

$$\begin{aligned} |\widehat{F}_{\xi'}(z)\phi(z)|_2\ &\le\ C\left\{(1+|\xi'|)|\phi(z)|_2+|\partial_{x_n}\phi(z)|_2\right\} \\ &\le\ C\left\{(1+|\xi'|)|u_0|_2+|\partial_{x_n}\phi_0|_2\right\}. \end{aligned} \tag{4.12}$$

It follows from (4.9)–(4.12) that

$$\begin{aligned} (t-s)^{\frac{1}{2}}|\partial_{x_n}w(t)|_2\ &\le\ C|w_0|_2+C(t-s)^{\frac{1}{2}}\int_s^t(t-z)^{-\frac{1}{2}}(|\widehat{E}_{\xi'}(z)w(z)|_2+|\widehat{F}_{\xi'}(z)\phi(z)|_2)\,dz \\ &\le\ C|w_0|_2+C\left\{(1+|\xi'|)|u_0|_2+|\partial_{x_n}\phi_0|_2\right\}, \end{aligned}$$

for $0<t-s\le 4T$. This, together with (4.7) and (4.8) gives the desired estimate of Lemma 4.4. $\square$

We next prove Proposition 4.2. To prove Proposition 4.2 we decompose $\chi^{(1)}\widehat{U}_{\xi'}(t,s)\widehat{u}_0$ based on a spectral property of $\widehat{L}_{\xi'}(t)$ with $\xi'=0$, namely, $\widehat{L}_0(t)$.

We introduce the formal adjoint operator of $\widehat{L}_{\xi'}(t)$ with respect to the weighted inner product $\langle\cdot,\cdot\rangle$. We define an operator $\widehat{L}_{\xi'}^*(t)$ by

$$\widehat{L}_{\xi'}^*(t)=\widehat{A}_{\xi'}^*+\widehat{B}_{\xi'}^*(t)+\widehat{C}_0^*(t),$$

with domain of definition $D(\widehat{L}_{\xi'}^*(t))=D$, where

$$\widehat{A}_{\xi'}^*=\widehat{A}_{\xi'},\ \ \widehat{B}_{\xi'}^*(t)=-\widehat{B}_{\xi'}(t),$$

and

$$\widehat{C}_0^*(t) = \begin{pmatrix} 0 & \frac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1)^T \boldsymbol{e}_1' & 0 \\ 0 & 0 & 0 \\ 0 & (\partial_{x_n} v_p^1)^T \boldsymbol{e}_1' & 0 \end{pmatrix}.$$

We then have

$$\langle \widehat{A}_{\xi'} u, v\rangle = \langle u, \widehat{A}_{\xi'}^* v\rangle = \langle u, \widehat{A}_{\xi'} v\rangle,$$
$$\langle \widehat{B}_{\xi'}(t) u, v\rangle = \langle u, \widehat{B}_{\xi'}^*(t) v\rangle = -\langle u, \widehat{B}_{\xi'}(t) v\rangle,$$
$$\langle \widehat{C}_0(t) u, v\rangle = \langle u, \widehat{C}_0^*(t) v\rangle,$$

and

$$\langle \widehat{L}_{\xi'}(t) u, v\rangle = \langle u, \widehat{L}_{\xi'}^*(t) v\rangle,$$

for $u, v \in D$.

Lemma 4.6.

(i) *There exists a T-time-periodic solution $u^{(0)}(x_n, t)$ of*

$$\partial_t u + \widehat{L}_0(t) u = 0, \tag{4.13}$$

$$w|_{x_n=0,1} = 0, \tag{4.14}$$

with following properties:

$$u^{(0)}(x_n, t) = {}^T(\phi^{(0)}, \frac{1}{\gamma^2} w^{(0),1}(x_n, t)\boldsymbol{e}_1', 0),$$

where

$$\phi^{(0)}(x_n) = \alpha_0 \frac{\gamma^2 \rho_p(x_n)}{\widetilde{P}'(\rho_p(x_n))}, \qquad \alpha_0 = \left[\frac{\gamma^2 \rho_p}{\widetilde{P}'(\rho_p)}\right]^{-1},$$

$$w^{(0),1}(x_n, t) = -\int_{-\infty}^t e^{-(t-s)\nu A} \nu \frac{\alpha_0 \gamma^2}{\widetilde{P}'(\rho_p)\rho_p} (\partial_{x_n}^2 v_p^1(s))\, ds.$$

We also have following estimates

$$|\partial_t^k w^{(0),1}(t)|_{L^2} \le C |\partial_t^k \partial_{x_n}^2 v_p^1(t)|_{L^2}, \quad k \ge 0.$$

(ii) *Let $u_0 \in X$. Then there holds*

$$[Q_0 \widehat{U}_0(t, s) u_0] = [Q_0 u_0], \quad \textit{for all } t \ge s \ge 0,$$

where $\widehat{U}_0(t, s)$ denotes the solution operator for (4.13) and (4.14).

Proof. Fact that $u^{(0)}(x_n, t)$ is T-time-periodic solution can be shown by direct computation. Estimates on $\partial_t^k w^{(0),1}$ are obtained in similar way as those for v_p^1 in from Proposition 2.1. Let $\widehat{U}_0(t, s) u_0 = {}^T(\phi(t), w(t))$. The first row of equation (4.13) is written as

$$\partial_t \phi + \gamma^2 \partial_{x_n}(\rho_p w^n) = 0.$$

Taking mean value of this over $(0,1)$ we get

$$\partial_t[\phi] + \gamma^2[\partial_{x_n}(\rho_p w^n)] = 0.$$

Using integration by parts and boundary condition (4.14) we get $[\partial_{x_n}(\rho_p w^n)] = 0$, and hence,

$$\partial_t[\phi] = 0.$$

This gives $[\phi(t)] = [\phi_0]$ for all $t \geq 0$. □

Definition 4.7.

(i) *Let us define projections* $\widehat{\Pi}^{(0)}(t)$ *and* $\widehat{\Pi}^{(0)*}(t)$ *as*

$$\widehat{\Pi}^{(0)}(t)u = \langle u, u^{(0)*}\rangle u^{(0)}(t) = [Q_0 u]u^{(0)}(t),$$

$$\widehat{\Pi}^{(0)*}(t)u = \langle u, u^{(0)}(t)\rangle u^{(0)*},$$

respectively, with

$$u^{(0)*} = {}^T(\frac{\gamma^2}{\alpha_0}\phi^{(0)}, 0, 0).$$

We also define $\widehat{\Pi}_c^{(0)}(t)$ *by*

$$\widehat{\Pi}_c^{(0)}(t) = I - \widehat{\Pi}^{(0)}(t).$$

(ii) *Let* $u^{(0)}(t) = u_0^{(0)} + u_1^{(0)}(t)$, *where*

$$u_0^{(0)} = {}^T(\phi^{(0)}, 0, 0), \quad u_1^{(0)}(t) = {}^T(0, \frac{1}{\gamma^2}w^{(0),1}(t)\boldsymbol{e}_1', 0).$$

We make a simple observation which will be useful in the argument below.

Lemma 4.8. *Let* $u \in X$.

(i) $\frac{d}{dt}Q_0\widehat{\Pi}^{(0)}(t)u = 0$, *i.e.*, ϕ*-component of* $\widehat{\Pi}^{(0)}(t)u$ *is independent of* t.

(ii) *Let* $u_1(t) = {}^T(\phi_1, w_1(t)) = \widehat{\Pi}_c^{(0)}(t)u$. *Then*

$$|\phi_1|_2 \leq |\partial_{x_n}\phi_1|_2.$$

Furthermore, if $\widetilde{Q}u \in H_0^1(0,1)$ *then*

$$|w_1(t)|_2 \leq |\partial_{x_n}w_1(t)|_2.$$

Proof. It is easy to see (i). As for (ii), since $[\phi_1] = 0$, the Poincaré inequality gives $|\phi_1|_2 \leq |\partial_{x_n}\phi_1|_2$. Let $\widetilde{Q}u \in H_0^1(0,1)$. Since $w^{(0),1}(t) \in H_0^1(0,1)$ for each t, we have $\widetilde{Q}\widehat{\Pi}^{(0)}(t)u \in H_0^1(0,1)$. Therefore, $w_1(t) \in H_0^1(0,1)$ for each t, and hence, the Poincaré inequality gives $|w_1(t)|_2 \leq |\partial_{x_n}w_1(t)|_2$ for each t. This concludes the proof. □

Based on Definition 4.7 for each t we decompose $u(t)$ into the parts of $\widehat{\Pi}^{(0)}(t)$ and $\widehat{\Pi}_c^{(0)}(t)$.

Let $u(t)$ be a solution of (4.6). We decompose $u(t)$ as

$$u(t) = \sigma(t)u^{(0)}(t) + u_1(t),$$

where

$$\sigma(t) = [Q_0 u(t)] = \langle u(t), u^{(0)*}\rangle,$$
$$u_1(t) = \widehat{\Pi}_c^{(0)}(t)u(t).$$

Using this decomposition we rewrite problem (4.6). To do so, we define some notation. We write

$$\widetilde{M_{\xi'}}(t) = \widehat{L}_{\xi'}(t) - \widehat{L}_0(t) = \widetilde{A}_{\xi'} + \widetilde{B}_{\xi'}(t),$$

where

$$\begin{aligned}\widetilde{A}_{\xi'} &= \widehat{A}_{\xi'} - \widehat{A}_0 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_p}|\xi'|^2 I_{n-1} + \frac{\widetilde{\nu}}{\rho_p}\xi'^T\xi' & -i\frac{\widetilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ 0 & -i\frac{\widetilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}|\xi'|^2 \end{pmatrix},\end{aligned}$$

$$\widetilde{B}_{\xi'}(t) = \widehat{B}_{\xi'}(t) - \widehat{B}_0(t) = \begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2\rho_p{}^T\xi' & 0 \\ i\frac{\widetilde{P'}(\rho_p)}{\gamma^2\rho_p}\xi' & i\xi_1 v_p^1(t) I_{n-1} & 0 \\ 0 & 0 & i\xi_1 v_p^1(t) \end{pmatrix}.$$

Substituting $u(t) = \sigma(t)u^{(0)}(t) + u_1(t)$ into (4.6), we have

$$\frac{d}{dt}(\sigma(t)u^{(0)}(t) + u_1(t)) + \widehat{L}_0(t)(\sigma(t)u^{(0)}(t) + u_1(t)) + \widetilde{M_{\xi'}}(t)(\sigma(t)u^{(0)}(t) + u_1(t)) = 0. \quad (4.15)$$

Since $Q_0\frac{d}{dt}u^{(0)}(t) = 0$, we have $\widehat{\Pi}^{(0)}(t)\frac{d}{dt}u^{(0)}(t) = [Q_0\frac{d}{dt}u^{(0)}(t)]u^{(0)}(t) = 0$. Therefore, applying $\widehat{\Pi}^{(0)}(t)$ and $\widehat{\Pi}_c^{(0)}(t)$ to this equation, we have

$$\begin{cases} \left(\frac{d}{dt}\sigma(t)\right)u^{(0)}(t) + \widehat{\Pi}^{(0)}(t)\widetilde{M_{\xi'}}(t)(\sigma(t)u^{(0)}(t) + u_1(t)) = 0, \\ \sigma(t)\left(\frac{d}{dt}u^{(0)}(t) + \widehat{L}_0(t)u^{(0)}(t)\right) + \frac{d}{dt}u_1(t) + \widehat{L}_0(t)u_1(t) \\ \qquad\qquad + \widehat{\Pi}_c^{(0)}(t)\widetilde{M_{\xi'}}(t)(\sigma(t)u^{(0)}(t) + u_1(t)) = 0. \end{cases}$$

Since $\widehat{\Pi}^{(0)}(t)\widetilde{M_{\xi'}}(t)u = [Q_0\widetilde{M_{\xi'}}(t)u]u^{(0)}(t)$, $Q_0\widetilde{M_{\xi'}}(t) = Q_0\widetilde{B}_{\xi'}(t)$ and fact that $u^{(0)}(t)$ is solution to (4.13), (4.14) we arrive at

$$\frac{d}{dt}\sigma(t) + [Q_0\widetilde{B}_{\xi'}(t)(\sigma(t)u^{(0)}(t) + u_1(t))] = 0, \quad (4.16)$$

$$\frac{d}{dt}u_1(t) + \widehat{L}_0(t)u_1(t) + \widetilde{M}_{\xi'}(t)(\sigma(t)u^{(0)}(t) + u_1(t)) - [Q_0\widetilde{B}_{\xi'}(t)(\sigma(t)u^{(0)}(t) + u_1(t))]u^{(0)}(t) = 0. \tag{4.17}$$

Proposition 4.2 can be proved by estimating solutions of (4.16)–(4.17). We will employ an energy method to obtain the necessary estimates on solutions of (4.16)–(4.17).

We introduce some notations. For $u = {}^T(\phi, w)$ we define $E_0(u)$ by

$$E_0(u) = \tfrac{1}{\gamma^2}\left|\sqrt{\tfrac{\tilde{P}'(\rho_p)}{\gamma^2\rho_p}}\phi\right|_2^2 + \left|\sqrt{\rho_p}w\right|_2^2.$$

For $v = \phi$, $v = w = {}^T(w_1, \dots, w^n)$ or $v = {}^T(\phi, w)$, we define $D_{\xi'}(v)$ by

$$D_{\xi'}(v) = |\xi'|^2|v|_2^2 + |\partial_{x_n}v|_2^2,$$

and, for $w = {}^T(w_1, \dots, w^n)$, we define $\widetilde{D}_{\xi'}(w)$ by

$$\widetilde{D}_{\xi'}(w) = \nu D_{\xi'}(w) + \tilde{\nu}|i\xi' \cdot w' + \partial_{x_n}w^n|_2^2.$$

We define $J(t)(u)$ by

$$J(t)(u) = -2\mathrm{Re}\,\langle \sigma u^{(0)}(t) + \widehat{\Pi}_c^{(0)}(t)u, \widehat{B}_{\xi'}(t)\widetilde{Q}\widehat{\Pi}_c^{(0)}(t)u\rangle,$$

where $\sigma = [Q_0u]$. We note that there exists a constant $b_0 > 0$ such that

$$|J(t)(u)| \le \frac{b_0\gamma^2}{2\nu}\left(\frac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0\left(\widehat{\Pi}_c^{(0)}(t)u\right)\right) + \frac{1}{2}\widetilde{D}_{\xi'}\left(\widetilde{Q}\widehat{\Pi}_c^{(0)}(t)u\right).$$

Let b_1 be a positive constant (to be determined in Proposition 4.10 (ii) below) and define $E_1(t)(u)$ by

$$E_1(t)(u) = \left(1 + \tfrac{b_1\gamma^2}{\nu}\right)\left(\tfrac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0\left(\widehat{\Pi}_c^{(0)}(t)u\right)\right) + \widetilde{D}_{\xi'}\left(\widetilde{Q}\widehat{\Pi}_c^{(0)}(t)u\right) + J(t)(u),$$

where $\sigma = [Q_0u]$.

In what follows we denote the solution $u(t)$ of (4.6) by $u(t) = \sigma(t)u^{(0)}(t) + u_1(t)$ with

$$\sigma(t) = [Q_0u(t)],\ u_1(t) = {}^T(\phi_1(t), w_1(t)) = \widehat{\Pi}_c^{(0)}(t)u;$$

and we often omit "t" of $u(t)$, $\sigma(t)$ and $u_1(t) = {}^T(\phi_1(t), w_1(t))$ if no confusion occurs.

Proposition 4.9. *There exist constants $\gamma_0 > 0$ and $\nu_0 > 0$ such that if $\gamma \ge \gamma_0$ and $\nu \ge \nu_0$ following estimates hold true:*

(i)

$$\tfrac{d}{dt}(\tfrac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0(u_1)) + \widetilde{D}_{\xi'}(w_1) \le C\left\{\left(\tfrac{\nu}{\gamma^4} + \tfrac{1}{\gamma^2}\right)|\phi_1|_2^2 + \left(\tfrac{\nu+\tilde{\nu}}{\gamma^4} + \tfrac{1}{\gamma^2}\right)|\xi'|^2|\sigma|^2\right\}, \tag{4.18}$$

(ii) *there exists constant $b_1 \ge b_0$ such that*

$$\tfrac{d}{dt}E_1(u) + \left(1 + \tfrac{b_1\gamma^2}{\nu}\right)\widetilde{D}_{\xi'}(w_1) + \left|\sqrt{\rho_p}\partial_t w_1\right|_2^2$$
$$\le C\left\{\left(\tfrac{1}{\gamma^2} + \tfrac{1}{\nu} + \tfrac{\nu^2}{\gamma^4}\right)|\phi_1|_2^2 + \tfrac{1}{\gamma^2}|\xi'|^2|\phi_1|_2^2 + \left(\tfrac{\nu+\tilde{\nu}}{\gamma^2} + \tfrac{1}{\nu} + \tfrac{1}{\gamma^2}\right)|\xi'|^2|\sigma|^2 + \tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}|\xi'|^4|\sigma|^2\right\},$$

(iii)

$$\frac{d}{dt}\frac{1}{\gamma^2}\left|\sqrt{\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}}\partial_{x_n}\phi_1\right|_2^2 + \frac{1}{\nu+\tilde{\nu}}\left|\frac{\widetilde{P}'(\rho_p)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2$$
$$\le C\left\{\left(\frac{1}{\nu}\left(\frac{1}{\gamma^2}+\frac{1}{(\nu+\tilde{\nu})}\right)+|\xi'|^2\right)\widetilde{D}_{\xi'}(w_1) + \frac{1}{\nu+\tilde{\nu}}|\sqrt{\rho_p}\partial_t w_1|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^4}|\xi'|^2(|\sigma|^2+|\phi_1|_2^2)\right\},$$

(iv) *if* $\frac{\nu+\tilde{\nu}}{\gamma^2}\le\frac{1}{4}$ *then*

$$\frac{d}{dt}|\sigma|^2 + \frac{\tilde{\alpha}_0\gamma^2}{\nu}\min\{1,|\xi'|^2\}|\sigma|^2$$
$$\le C\left\{\frac{\nu+\tilde{\nu}}{\gamma^2}|\sigma|^2|\xi'|^2 + \left(\frac{\nu}{\gamma^2}+\frac{\gamma^2}{\nu}\right)(1+|\xi'|^2)|\phi_1|_2^2 + \frac{\gamma^2}{\nu}|\sqrt{\rho_p}\partial_t w_1|_2^2 + \frac{\gamma^2}{\nu}\left(\frac{1}{\nu}+\tilde{\nu}\right)\widetilde{D}_{\xi'}(w_1)\right\},$$

with some constant $\tilde{\alpha}_0>0$.

Proof. Proofs of (i), (iii) and (iv) are same as proofs of [6, Proposition 4.7, 4.9 and 4.10], respectively, so we omit them. We prove (ii) for reader's convenience. In the proof we will often use following lemma.

Lemma 4.10. *For each t there hold the following assertions with $C>0$ independent of t.*

(i) $\langle u^{(0)}(t), u_1(t)\rangle = \langle u_1^{(0)}(t), u_1(t)\rangle$,

(ii) $\left|[Q_0\widetilde{B}_{\xi'}(t)u_0^{(0)}]\right| + \left|[Q_0\widetilde{B}_{\xi'}(t)u_1^{(0)}(t)]\right| \le C|\xi'|$,

(iii) $\left|[Q_0\widetilde{B}_{\xi'}(t)u_1(t)]\right| \le C|\xi'|(|\phi_1(t)|_2+\gamma^2|w_1'(t)|_2)$.

The proof of Lemma 4.10 is straightforward, so we omit it.

Let us proof (ii) from Proposition 4.9. We recall that $u(t)=\sigma(t)u^{(0)}(t)+u_1(t)$ satisfies

$$\frac{d}{dt}u + \widehat{L}_{\xi'}(t)u = 0,$$

which implies

$$\langle\partial_t u, \partial_t\widetilde{Q}u_1\rangle + \langle\widehat{L}_{\xi'}(t)u, \partial_t\widetilde{Q}u_1\rangle = 0. \tag{4.19}$$

Since

$$\partial_t\sigma = -[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)}(t)+u_1)]$$

and

$$\langle u^{(0)}(t), \partial_t\widetilde{Q}u_1\rangle = \langle u_1^{(0)}(t), \partial_t\widetilde{Q}u_1\rangle,$$

by using Lemma 4.10, we have

$$\begin{aligned}
&\mathrm{Re}\,\langle\partial_t u, \partial_t\widetilde{Q}u_1\rangle\\
&= \mathrm{Re}\left\{\langle\partial_t(\sigma u^{(0)}(t)), \partial_t\widetilde{Q}u_1\rangle + \langle\partial_t u_1, \partial_t\widetilde{Q}u_1\rangle\right\}\\
&= \mathrm{Re}\left\{-[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)}(t)+u_1)]\langle u_1^{(0)}(t), \partial_t\widetilde{Q}u_1\rangle + \left|\sqrt{\rho_p}\partial_t w_1\right|_2^2 + \langle\sigma\partial_t u^{(0)}(t), \partial_t\widetilde{Q}u_1\rangle\right\}\\
&\ge \frac{3}{4}\left|\sqrt{\rho_p}\partial_t w_1\right|_2^2 - C\left\{\frac{|\xi'|^2}{\gamma^4}(|\sigma|^2+|\phi_1|_2^2) + \frac{1}{\nu}\widetilde{D}_{\xi'}(w_1)\right\} + \mathrm{Re}\langle\sigma\partial_t u^{(0)}(t), \partial_t\widetilde{Q}u_1\rangle.
\end{aligned} \tag{4.20}$$

Since $\widehat{L}_0(t)u^{(0)} = -\partial_t u^{(0)}$ and $\widehat{B}_0(t)u^{(0)} = 0$, we see that

$$
\begin{aligned}
&\langle \widehat{L}_{\xi'}(t)u, \partial_t \widetilde{Q}u_1\rangle \\
&\quad = \langle \widetilde{M}_{\xi'}(t)(\sigma u^{(0)}(t)), \partial_t \widetilde{Q}u_1\rangle + \langle \widehat{L}_{\xi'}(t)u_1, \partial_t \widetilde{Q}u_1\rangle + \langle \sigma \widehat{L}_0(t)u^{(0)}(t), \partial_t \widetilde{Q}u_1\rangle \\
&\quad = \langle \widetilde{A}_{\xi'}(\sigma u^{(0)}(t)), \partial_t \widetilde{Q}u_1\rangle + \langle \widetilde{B}_{\xi'}(t)(\sigma u^{(0)}(t)), \partial_t \widetilde{Q}u_1\rangle \\
&\qquad + \langle \widehat{A}_{\xi'}u_1, \partial_t \widetilde{Q}u_1\rangle + \langle \widehat{B}_{\xi'}(t)u_1, \partial_t \widetilde{Q}u_1\rangle + \langle \widehat{C}_0(t)u_1, \partial_t \widetilde{Q}u_1\rangle + \langle \sigma \widehat{L}_0(t)u^{(0)}(t), \partial_t \widetilde{Q}u_1\rangle \\
&\quad = \langle \widetilde{A}_{\xi'}(\sigma u^{(0)}(t)), \partial_t \widetilde{Q}u_1\rangle + \langle \widehat{B}_{\xi'}(t)(\sigma u^{(0)}(t) + u_1), \partial_t \widetilde{Q}u_1\rangle \\
&\qquad + \langle \widehat{A}_{\xi'}u_1, \partial_t \widetilde{Q}u_1\rangle + \langle \widehat{C}_0(t)u_1, \partial_t \widetilde{Q}u_1\rangle - \langle \sigma \partial_t u^{(0)}(t), \partial_t \widetilde{Q}u_1\rangle.
\end{aligned}
\tag{4.21}
$$

We will show

$$
\begin{aligned}
&\mathrm{Re}\left\{ \langle \widehat{B}_{\xi'}(t)(\sigma u^{(0)}(t) + u_1), \partial_t \widetilde{Q}u_1\rangle + \langle \widehat{A}_{\xi'}u_1, \partial_t \widetilde{Q}u_1\rangle \right\} \\
&\quad \geq \tfrac{1}{2}\tfrac{d}{dt}\left(\widetilde{D}_{\xi'}(w_1) + J(t)(u) \right) - \varepsilon |\sqrt{\rho_p}\partial_t w_1|_2^2 \\
&\qquad - C\left\{ \tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \tfrac{\gamma^2}{\nu}\widetilde{D}_{\xi'}(w_1) + \tfrac{1}{\varepsilon\nu}\widetilde{D}_{\xi'}(w_1) \right\}
\end{aligned}
\tag{4.22}
$$

for any $\varepsilon > 0$.

It is easy to see that

$$
\mathrm{Re}\, \langle \widehat{A}_{\xi'}u_1, \partial_t \widetilde{Q}u_1\rangle = \frac{1}{2}\frac{d}{dt}\widetilde{D}_{\xi'}(w_1). \tag{4.23}
$$

Since $\widehat{B}_{\xi'}^*(t) = -\widehat{B}_{\xi'}(t)$, we have

$$
\begin{aligned}
\langle \widehat{B}_{\xi'}(t)(\sigma u^{(0)}(t)), \partial_t \widetilde{Q}u_1\rangle &= -\tfrac{d}{dt}\langle \sigma u^{(0)}(t), \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle + \langle \partial_t \sigma u^{(0)}(t), \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle \\
&\quad + \langle \sigma \partial_t u_1^{(0)}(t), \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle + \langle \sigma u^{(0)}(t), (\partial_t \widehat{B}_{\xi'}(t))\widetilde{Q}u_1\rangle.
\end{aligned}
\tag{4.24}
$$

By (4.16), we have

$$
\partial_t \sigma = -[Q_0 \widetilde{B}_{\xi'}(t)(\sigma u^{(0)}(t) + u_1)].
$$

Lemma 4.10 then implies

$$
\begin{aligned}
&\left| \langle \partial_t \sigma u^{(0)}(t), \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle \right| \\
&\quad \leq \left| [Q_0 \widetilde{B}_{\xi'}(t)(\sigma u^{(0)}(t) + u_1)] \right| \left| \langle u^{(0)}(t), \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle \right| \leq C\left\{ \tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \tfrac{\gamma^2}{\nu}\widetilde{D}_{\xi'}(w_1) \right\}.
\end{aligned}
\tag{4.25}
$$

In the same way we get

$$
|\langle \sigma \partial_t u_1^{(0)}(t), \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle| \leq C\left\{ \frac{|\xi'|^2}{\gamma^2}|\sigma|^2 + \frac{1}{\gamma^2\nu}\widetilde{D}_{\xi'}(w_1) \right\}, \tag{4.26}
$$

and

$$
|\langle \sigma u^{(0)}(t), (\partial_t \widehat{B}_{\xi'}(t))\widetilde{Q}u_1\rangle| \leq C\left\{ \frac{|\xi'|^2}{\gamma^2}|\sigma|^2 + \frac{1}{\gamma^2\nu}\widetilde{D}_{\xi'}(w_1) \right\}. \tag{4.27}
$$

Similarly,

$$
\begin{aligned}
&\langle \widehat{B}_{\xi'}(t)u_1, \partial_t \widetilde{Q}u_1\rangle \\
&\quad = -\tfrac{d}{dt}\langle u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle + \langle \partial_t u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle + \langle u_1, (\partial_t \widehat{B}_{\xi'}(t))\widetilde{Q}u_1\rangle \\
&\quad = -\tfrac{d}{dt}\langle u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle + \langle \partial_t Q_0 u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle + \langle \partial_t \widetilde{Q}u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle + \langle u_1, (\partial_t \widehat{B}_{\xi'}(t))\widetilde{Q}u_1\rangle.
\end{aligned}
\tag{4.28}
$$

We first estimate the second term on the right of (4.28). By (4.17), we have

$$
\begin{aligned}
\partial_t Q_0 u_1 &= -Q_0\left\{\widehat{L}_{\xi'}(t)u_1 + \widetilde{M}_{\xi'}(t)(\sigma u^{(0)}(t)) - [Q_0\widetilde{B}_{\xi'}(t)(\sigma u^{(0)}(t)+u_1)]u^{(0)}\right\} \\
&= -\left\{Q_0\widehat{B}_{\xi'}(t)u_1 + Q_0\widetilde{B}_{\xi'}(t)(\sigma u^{(0)}(t)) - [Q_0\widetilde{B}_{\xi'}(t)(\sigma u^{(0)}(t)+u_1)]u_0^{(0)}\right\}.
\end{aligned}
$$

Since $\langle \partial_t Q_0 u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle = \langle \partial_t Q_0 u_1, Q_0\widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle$, we see from Lemma 4.10 that

$$
\begin{aligned}
&\left|\langle \partial_t Q_0 u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle\right| \\
&\quad \le C\left\{|Q_0\widehat{B}_{\xi'}(t)u_1|_2 + |Q_0\widetilde{B}_{\xi'}(t)(\sigma u^{(0)}(t))|_2\right. \\
&\qquad \left.+\left|[Q_0\widetilde{B}_{\xi'}(t)(\sigma u^{(0)}(t)+u_1)]\right||u_0^{(0)}|_2\right\} \times \tfrac{1}{\gamma^2}|Q_0\widehat{B}_{\xi'}(t)\widetilde{Q}u_1|_2 \\
&\quad \le C\left\{\tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \tfrac{\gamma^2}{\nu}\widetilde{D}_{\xi'}(w_1)\right\}.
\end{aligned}
\tag{4.29}
$$

The third term on the right of (4.28) is estimated as

$$
\left|\langle \partial_t \widetilde{Q}u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle\right| \le C|\partial_t w_1|_2|\xi'||w_1|_2 \le \varepsilon|\sqrt{\rho_p}\partial_t w_1|_2^2 + \tfrac{C}{\varepsilon\nu}\widetilde{D}_{\xi'}(w_1), \tag{4.30}
$$

for any $\varepsilon > 0$. The fourth term on the right of (4.28) is estimated as

$$
|\langle u_1, (\partial_t \widehat{B}_{\xi'}(t))\widetilde{Q}u_1\rangle| \le C|\xi'||w_1|_2^2 \le C\frac{1}{\nu}\widetilde{D}_{\xi'}(w_1).
$$

This, together with (4.29) and (4.30), gives

$$
\begin{aligned}
&\mathrm{Re}\,\langle \widehat{B}_{\xi'}(t)u_1, \partial_t \widetilde{Q}u_1\rangle \\
&\qquad \ge -\tfrac{d}{dt}\langle u_1, \widehat{B}_{\xi'}(t)\widetilde{Q}u_1\rangle - \varepsilon|\sqrt{\rho_p}\partial_t w_1|_2^2 - C\left\{\tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \left(\tfrac{\gamma^2}{\nu} + \tfrac{1}{\varepsilon\nu}\right)\widetilde{D}_{\xi'}(w_1)\right\},
\end{aligned}
\tag{4.31}
$$

for any $\varepsilon > 0$. We thus obtain (4.22) from (4.23)–(4.27) and (4.31).

A straightforward computation gives

$$
\left|\langle \widetilde{A}_{\xi'}(\sigma u^{(0)}(t)), \partial_t \widetilde{Q}u_1\rangle\right| \le \varepsilon|\sqrt{\rho_p}\partial_t w_1|_2^2 + \tfrac{C}{\varepsilon}\tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}|\xi'|^2(1+|\xi'|^2)|\sigma|^2, \tag{4.32}
$$

and

$$
\left|\langle \widehat{C}_0(t)u_1, \partial_t \widetilde{Q}u_1\rangle\right| \le \varepsilon|\sqrt{\rho_p}\partial_t w_1|_2^2 + \tfrac{C}{\varepsilon}\left\{\tfrac{\nu^2}{\gamma^4}|\phi_1|_2^2 + \tfrac{1}{\nu}\widetilde{D}_{\xi'}(w_1)\right\}, \tag{4.33}
$$

for any $\varepsilon > 0$.

Taking $\varepsilon > 0$ suitably small, we see from (4.19)–(4.22), (4.32) and (4.33) that

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}(\widetilde{D}_{\xi'}(w_1) + J(t)(u)) + \tfrac{1}{2}|\sqrt{\rho_p}\partial_t w_1|_2^2 \\ &\quad \le C\left\{\tfrac{\nu^2}{\gamma^4}|\phi_1|_2^2 + \tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}|\xi'|^2(1+|\xi'|^2)|\sigma|^2 + \left(\tfrac{\gamma^2}{\nu} + \tfrac{1}{\nu}\right)\widetilde{D}_{\xi'}(w_1)\right\}. \end{aligned} \tag{4.34}$$

Adding $2 \times (4.34)$ to $\left(1 + \frac{b_1\gamma^2}{\nu}\right) \times (4.18)$ with suitably large $b_1 \ge b_0$, we obtain the desired estimate. This completes the proof. □

Proof of Proposition 4.2. We assume $|\xi'| \le R_0$. We fix b_1 in $E_1(t)(u)$ as in Proposition 4.9 (ii). As in the proof of [6, Proposition 4.2], by making a suitable linear combination of inequalities (i)-(iv) of Proposition 4.9 one can show, that there exist constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \ge \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \ge \gamma_0^2$ then

$$\frac{d}{dt}E_3(u) + c_1\widetilde{D}_{\xi'}(w_1) + c_2|\sqrt{\rho_p}\partial_t w_1|_2^2 + c_3\left|\frac{\widetilde{P}'(\rho_p)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2 + c_4|\xi'|^2|\sigma|_2^2 \le 0, \tag{4.35}$$

for ξ' with $|\xi'| \le R_0$, where $c_1,\ c_2,\ c_3,\ c_4$ are some positive numbers and $E_3(u)$ is a quantity equivalent to $E_0(u) + D_{\xi'}(u)$.

Since $|\xi'| \le R_0$, we see that

$$\widetilde{D}_{\xi'}(w_1) + \left|\frac{\widetilde{P}'(\rho_p)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2 + |\xi'|^2|\sigma|_2^2 \ge c_0|\xi'|^2E_3(u), \tag{4.36}$$

for some constant $c_0 > 0$. It then follows from (4.35) and (4.36) that

$$E_3(u(t)) \le e^{-c_0|\xi'|^2(t-s-4T)}E_3(u(4T)), \tag{4.37}$$

for $t - s \ge 4T,\ |\xi'| \le R_0$.

Let $\widehat{u}_s = {}^T(\widehat{\phi}, \widehat{w})$ be the solution of problem (4.1)–(4.5). There exist constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \ge \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \ge \gamma_0^2$ then we deduce from (4.37) that

$$\begin{aligned} &|\xi'|^{2k}|\widehat{u}_s(\xi', \cdot, t)|_{H^1}^2 \\ &\quad \le C|\xi'|^{2k}e^{-c_0|\xi'|^2(t-s-4T)}E_3(\widehat{u}_s(\xi', \cdot, s+4T)), \end{aligned} \tag{4.38}$$

for ξ' with $|\xi'| \le R_0$.

If we integrate (4.38) over $0 < r \le |\xi'| < R_0$, then

$$\|\mathscr{V}_1(t,s)u_0\|_{H^1} \le Ce^{-d(t-s-4T)}\|\mathscr{U}(s+4T,s)u_0\|_{H^1},$$

with $d = \frac{1}{2}c_0r^2 > 0$. Applying Theorem 3.1 to estimate $\|\mathscr{U}(s+4T,s)u_0\|_{H^1}$, we have the desired estimate in Proposition 4.2. □

5 Proof of Theorem 3.2 - 2. Asymptotic behavior

In this section we prove the asymptotic behaviour as given in (3.2) and (3.3).

Theorem 5.1. *There exist $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$ then there exists $r_0 > 0$ such that $\mathscr{V}_0(t,s)u_0$ can be written as*

$$\mathscr{V}_0(t,s)u_0 = \mathscr{U}^{(0)}(t,s)u_0 + \mathscr{R}^{(0)}(t,s)u_0,$$

where $\mathscr{U}^{(0)}(t,s)u_0$ has the properties in Theorem 3.2 (i) *and $\mathscr{R}^{(0)}(t,s)u_0$ satisfies the estimate (3.4) in Theorem* 3.2 (ii) *with $\mathscr{U}^{(\infty)}(t,s)u_0$ replaced by $\mathscr{R}^{(0)}(t,s)u_0$.*

Theorem 3.2 immediately follows from Proposition 4.1, Proposition 4.2 and Theorem 5.1 with $r = r_0$ and $R = R_0$ by setting $\mathscr{U}^{(\infty)}(t,s)u_0 = \mathscr{R}^{(0)}(t,s)u_0 + \mathscr{V}_1(t,s)u_0 + \mathscr{V}_\infty(t,s)u_0$.

To prove Theorem 5.1 we will investigate spectral properties of the operator $\widehat{L}_{\xi'}(t)$ in more detail. Since $\widehat{L}_{\xi'}(t)$ is periodic in t, we will use theory of periodic solutions, namely monodromy operators. Let us introduce some notation. We decompose $\widehat{L}_{\xi'}(t)$ as

$$\widehat{L}_{\xi'}(t) = \widehat{L}_1 + \widehat{M}_{\xi'}(t),$$

where $\xi' = {}^T(\xi_1, \cdots, \xi_{n-1})$, and

$$\widehat{L}_1 = \widehat{L}_0(t) - \widehat{C}_0(t) = \begin{pmatrix} 0 & 0 & \gamma^2 \partial_{x_n}(\rho_p \,\cdot\,) \\ 0 & -\frac{\nu}{\rho_p}\partial_{x_n}^2 I_{n-1} & 0 \\ \partial_{x_n}\left(\frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p}\,\cdot\right) & 0 & -\frac{\nu+\tilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix},$$

$\widehat{M}_{\xi'}(t) =$

$$\begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2\rho_p\xi' & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1(t))\boldsymbol{e}_1' + i\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\xi' & iv_p^1(t)\xi_1 I_{n-1} + \frac{\nu}{\rho_p}|\xi'|^2 I_{n-1} + \frac{\tilde{\nu}}{\rho_p}\xi'^T\xi' & (\partial_{x_n}v_p^1(t))\boldsymbol{e}_1' - i\frac{\tilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ 0 & -i\frac{\tilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & i\xi_1 v_p^1(t) + \frac{\nu}{\rho_p}|\xi'|^2 \end{pmatrix}.$$

We note that

$$\widehat{M}_0(t) = \widehat{C}_0(t) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1(t))\boldsymbol{e}_1' & 0 & (\partial_{x_n}v_p^1(t))\boldsymbol{e}_1' \\ 0 & 0 & 0 \end{pmatrix}.$$

First we investigate properties of solution operator $\widehat{U}_0(t,s)$ for

$$\begin{aligned} &\partial_t u + \widehat{L}_0(t)u = 0, \\ &w|_{x_n=0,1} = 0, \ u|_{t=s} = u_0, \end{aligned} \tag{5.1}$$

and, then, by perturbation argument for $|\xi'| \ll 1$, we analyze properties of solution operator $\widehat{U}_{\xi'}(t,s)$ for

$$\begin{aligned} &\partial_t u + \widehat{L}_{\xi'}(t)u = 0, \\ &w|_{x_n=0,1} = 0, \ u|_{t=s} = u_0, \end{aligned} \tag{5.2}$$

where $u = {}^T(\phi, w', w^n)$ is an unknown function of $x_n \in [0,1]$ and $t - s \geq 0, s \geq 0$, with $w = (w', w^n)$. Let us first focus on the case $\xi' = 0$. Since (5.1) is also written as

$$\partial_t u + \widehat{L}_1 u = -\widehat{M}_0(t)u,$$

the solution operator $\widehat{U}_0(t,s)$ for (5.1) can be written as

$$\widehat{U}_0(t,s)u_0 = e^{-(t-s)L_1}u_0 - \int_s^t e^{-(t-z)L_1}\widehat{M}_0(z)\widehat{U}_0(z,s)u_0\,dz. \tag{5.3}$$

We state simple, but useful lemma.

Lemma 5.2. *Let $u^{(0)}$ be a function defined as*

$$u^{(0)} = u^{(0)}(t)|_{t=0}.$$

Then

$$u^{(0)}(t) = \widehat{U}_0(t,0)u^{(0)}, \ \textit{for all } t \geq 0.$$

Proof. Proof follows easily from Lemma 4.6 and the uniqueness of the solution. □

In what follows we will denote $\widehat{U}_0(T) = \widehat{U}_0(T,0)$ corresponding monodromy operator.

Definition 5.3. *Let us define projections $\widehat{\Pi}^{(0)}$, $\widehat{\Pi}_c^{(0)}$ and $\widehat{\Pi}^{(0)*}$ as*

$$\begin{aligned} \widehat{\Pi}^{(0)}u &= \langle u, u^{(0)*}\rangle u^{(0)} = [Q_0 u]u^{(0)}, \ u \in X, \\ \widehat{\Pi}_c^{(0)}u &= (I - \widehat{\Pi}^{(0)})u, \ u \in X, \\ \widehat{\Pi}^{(0)*}u &= \langle u, u^{(0)}\rangle u^{(0)*}, \ u \in X, \end{aligned}$$

respectively, with $u^{(0)}$ given in Lemma 5.2 and

$$u^{(0)*} = {}^T(\frac{\gamma^2}{\alpha_0}\phi^{(0)}, 0, 0).$$

Proposition 5.4. *Operator $\widehat{\Pi}^{(0)}$ is an eigen-projection for monodromy operator $\widehat{U}_0(T)$ for eigenvalue 1 with commuting property*

$$\widehat{U}_0(T)\widehat{\Pi}^{(0)}u_0 = \widehat{\Pi}^{(0)}\widehat{U}_0(T)u_0 = \widehat{\Pi}^{(0)}u_0, \ u_0 \in X.$$

Moreover there exists constant $\gamma_0 > 0$ such that if $\gamma \geq \gamma_0$ then eigenvalue 1 is simple and spectrum of monodromy operator $\widehat{U}_0(T)$ can be decomposed as

$$\sigma(\widehat{U}_0(T)) = \{1\} \cup \{\lambda : |\lambda| \leq \delta_0\},$$

where δ_0 *is a constant satisfying* $0 < \delta_0 < 1$.

To prove Proposition 5.4 we first obtain decay estimates on $\widehat{U}_0(t,0)\widehat{\Pi}_c^{(0)}u_0$.

Lemma 5.5. *Let* $u_0 \in X$*. Then* $\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0$ *which is solution of equation (5.1), is also a solution of*

$$\partial_t \widehat{\Pi}_c^{(0)} v + \widehat{L}_1 \widehat{\Pi}_c^{(0)} v = -\widehat{M}_0(t)\widehat{\Pi}_c^{(0)} v, \qquad (5.4)$$
$$\widehat{\Pi}_c^{(0)} v|_{t=s} = \widehat{\Pi}_c^{(0)} u_0.$$

Proof. Lemma 5.5 is easily concluded out of following facts:

$$\widehat{L}_1 : \widehat{\Pi}_c^{(0)} X \to \widehat{\Pi}_c^{(0)} X,$$

$$\widehat{\Pi}_c^{(0)}\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0 = \widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0,$$
$$\widehat{\Pi}_c^{(0)}\widehat{U}_0(t,s)\widehat{\Pi}^{(0)}u_0 = 0.$$

$\square$

In light of Lemma 5.5 we see that to obtain decay estimates on $\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0$ it is enough to obtain decay estimates on solution of equation (5.4). To do so, we investigate spectral properties of $\widehat{L}_1$ on $\widehat{\Pi}_c^{(0)}X$.

Lemma 5.6. *There exists a constant* $\gamma_0 > 0$ *such that if* $\gamma \geq \gamma_0$ *then there exist positive numbers* η_0 *and* θ_0 *with* $\theta_0 \in (\frac{\pi}{2}, \pi)$ *such that the following estimates hold uniformly for* $\lambda \in \rho(-\widehat{L}_1) \cap \Sigma(-\eta_0, \theta_0)$*:*

$$\left|(\lambda + \widehat{L}_1)^{-1} f\right|_{H^l \times L^2} \leq \tfrac{C}{1+|\lambda|}|f|_{H^1 \times L^2}, \ l = 0, 1,$$

$$\left|\partial_{x_n}^l \widetilde{Q}(\lambda + \widehat{L}_1)^{-1} f\right|_2 \leq \tfrac{C}{(|\lambda|+1)^{1-\frac{l}{2}}}|f|_{H^1 \times L^2}, \ l = 1, 2,$$

for $f \in \widehat{\Pi}_c^{(0)} X$.

Proof. Lemma 5.6 is proved in similar way as [6, Lemma 5.2].

We can see that $\widehat{L}_1$ depends on $\frac{1}{\gamma^2}$ through ρ_p (recall $|\rho_p - 1| \leq C\frac{1}{\gamma^2}$). Let us introduce operator $\widehat{L}_{1,0}$,

$$\widehat{L}_{1,0} := \begin{pmatrix} 0 & 0 & \gamma^2 \partial_{x_n} \\ 0 & -\nu\partial_{x_n}^2 I_{n-1} & 0 \\ \partial_{x_n} & 0 & -(\nu + \tilde{\nu})\partial_{x_n}^2 \end{pmatrix},$$

which naturally arises in case $\gamma = \infty$. We regard $\widehat{L}_1$ as a perturbation from $\widehat{L}_{1,0}$ to estimate $(\lambda + \widehat{L}_1)^{-1}$. As for $\widehat{L}_{1,0}$ we have following result.

Lemma 5.7. *There exist positive numbers η_0 and θ_0 with $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that the following estimates hold uniformly for $\lambda \in \rho(-\widehat{L}_{1,0}) \cap \Sigma(-\eta_0, \theta_0)$:*

$$\left|(\lambda + \widehat{L}_{1,0})^{-1} f\right|_{H^l \times L^2} \leq \tfrac{C}{1+|\lambda|} |f|_{H^l \times L^2}, \ l = 0, 1,$$

$$\left|\partial^l_{x_n} \widetilde{Q}(\lambda + \widehat{L}_{1,0})^{-1} f\right|_2 \leq \tfrac{C}{(|\lambda|+1)^{1-\frac{l}{2}}} |f|_{H^{l-1} \times L^2}, \ l = 1, 2,$$

for $f \in \widehat{\Pi}_c^{(0)} X$.

Proof of Lemma 5.7 is similar to the proof of [2, Lemma 3.1 (ii)], so we omit the proof.

We continue the proof of Lemma 5.6. Since ρ_p is smooth, strictly positive and $\rho_p = 1 + O(\frac{1}{\gamma^2})$, we have

$$|(\widehat{L}_1 - \widehat{L}_{1,0})f|_{H^\ell \times L^2} \leq C \left(1 + \frac{\nu + \widetilde{\nu}}{\gamma^2}\right) \frac{1}{\gamma^2} |f|_{H^1 \times H^2}, \quad \ell = 0, 1.$$

This, together with Lemma 5.7, implies that if $\frac{1}{\gamma^2} \ll 1$, then $\Sigma(-\frac{\eta_0}{2}, \theta_0) \subset \rho(-\widehat{L}_1)$ and we get the desired estimates. $\square$

From Lemma 5.6 we can conclude that for $\gamma \geq \gamma_0$ the $-\widehat{L}_1$ is sectorial operator on $\widehat{\Pi}_c^{(0)} X$ and we can write

$$e^{-(t-s)\widehat{L}_1} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda(t-s)} (\lambda + \widehat{L}_1)^{-1} \, d\lambda \text{ on } \widehat{\Pi}_c^{(0)} X,$$

where contour Γ is in $\rho(-\widehat{L}_1|_{\widehat{\Pi}_c^{(0)} X}) \cap \Sigma(-\eta_0, \theta_0)$. Using this identity and estimates in Lemma 5.6 we get by standard calculation that

$$|e^{-(t-s)\widehat{L}_1} \widehat{\Pi}_c^{(0)} u_0|_{H^\ell \times L^2} \leq C e^{-d(t-s)} |\widehat{\Pi}_c^{(0)} u_0|_{H^1 \times L^2}, \ \ell = 0, 1, \tag{5.5}$$

$$|\partial_{x_n} \widetilde{Q} e^{-(t-s)\widehat{L}_1} \widehat{\Pi}_c^{(0)} u_0|_{L^2} \leq \frac{C}{(t-s)^{\frac{1}{2}}} e^{-d(t-s)} |\widehat{\Pi}_c^{(0)} u_0|_{H^1 \times L^2}, \tag{5.6}$$

for some d positive number. Finally we get following decay estimates on $\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)} u_0$.

Lemma 5.8. *There exists a constant $\gamma_0 > 0$ such that if $\gamma \geq \gamma_0$ then there exists positive number d such that following decay estimates hold true,*

$$|\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)} u_0|_{H^\ell \times L^2} \leq C e^{-d(t-s)} |\widehat{\Pi}_c^{(0)} u_0|_{H^1 \times L^2}, \ \ell = 0, 1, \tag{5.7}$$

$$|\partial_{x_n} \widetilde{Q} \widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)} u_0|_{L^2} \leq C((t-s)^{-\frac{1}{2}} + (t-s)^{\frac{1}{2}}) e^{-d(t-s)} |\widehat{\Pi}_c^{(0)} u_0|_{H^1 \times L^2}.$$

Proof. Using identity (5.3)

$$\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)} u_0 = e^{-(t-s)L_1} \widehat{\Pi}_c^{(0)} u_0 - \int_s^t e^{-(t-z)L_1} \widehat{M_0}(z) \widehat{U}_0(z,s) \widehat{\Pi}_c^{(0)} u_0 \, dz,$$

we have

$$|\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0|_{H^\ell\times L^2} \le |e^{-(t-s)L_1}\widehat{\Pi}_c^{(0)}u_0|_{H^\ell\times L^2} + |\int_s^t e^{-(t-z)L_1}\widehat{M_0}(z)\widehat{U}_0(z,s)\widehat{\Pi}_c^{(0)}u_0\,dz|_{H^\ell\times L^2}.$$

From the form of $\widehat{M_0}(z)$ we see that $\widehat{M_0}(z) = Q'\widehat{M_0}(z)(I-Q')$ and we can rewrite above right-hand side as

$$= |e^{-(t-s)L_1}\widehat{\Pi}_c^{(0)}u_0|_{H^\ell\times L^2} + |\int_s^t e^{-(t-z)L_1}Q'\widehat{M_0}(z)(I-Q')\widehat{U}_0(z,s)\widehat{\Pi}_c^{(0)}u_0\,dz|_{H^\ell\times L^2}.$$

It is easy to observe that

$$e^{-(t-z)L_1}Q' = e^{-(t-z)\nu A}Q',$$

and from (5.3) and form of $\widehat{M_0}(z)$ we see that

$$(I-Q')\widehat{U}_0(z,s) = (I-Q')e^{-(z-s)L_1}.$$

Using these two equations and (5.5) we get

$$\begin{aligned}
&|\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0|_{H^\ell\times L^2} \\
&\qquad \le |e^{-(t-s)L_1}\widehat{\Pi}_c^{(0)}u_0|_{H^\ell\times L^2} + |\int_s^t e^{-(t-z)\nu A}\widehat{M_0}(z)(I-Q')e^{-(z-s)\widehat{L}_1}\widehat{\Pi}_c^{(0)}u_0\,dz|_{L^2} \\
&\qquad \le Ce^{-d(t-s)}|\widehat{\Pi}_c^{(0)}u_0|_{H^1\times L^2} + C\int_s^t e^{-(t-z)\frac{\nu}{2}}\widehat{M_0^\infty}|e^{-(z-s)\widehat{L}_1}\widehat{\Pi}_c^{(0)}u_0|_{L^2\times L^2}\,dz,
\end{aligned}$$

where $\widehat{M_0^\infty} := \sup_z|\widehat{M_0}(z)|_{L^2} \le C(1+\frac{\nu}{\gamma^2})$. Thanks to (5.5), we have

$$\int_s^t e^{-(t-z)\frac{\nu}{2}}|e^{-(z-s)\widehat{L}_1}\widehat{\Pi}_c^{(0)}u_0|_{L^2\times L^2}\,dz \le \int_s^t e^{-[(t-z)\frac{\nu}{2}+(z-s)d]}\,dz|\widehat{\Pi}_c^{(0)}u_0|_{H^1\times L^2}.$$

Therefore, with $\widetilde{d} = \min\{d,\frac{\nu}{2}\}$, we finally get

$$|\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0|_{H^\ell\times L^2} \le Ce^{-\widetilde{d}(t-s)}|\widehat{\Pi}_c^{(0)}u_0|_{H^1\times L^2}.$$

As for second inequality, we follow calculations above using (5.6) to get

$$\begin{aligned}
&|\partial_{x_n}\widetilde{Q}\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0|_{L^2} \\
&\qquad \le |\partial_{x_n}\widetilde{Q}e^{-(t-s)L_1}\widehat{\Pi}_c^{(0)}u_0|_{L^2} + |\int_s^t \partial_{x_n}\widetilde{Q}e^{-(t-z)L_1}\widehat{M_0}(z)\widehat{U}_0(z,s)\widehat{\Pi}_c^{(0)}u_0\,dz|_{L^2} \\
&\qquad \le \frac{C}{(t-s)^{\frac{1}{2}}}e^{-d(t-s)}|\widehat{\Pi}_c^{(0)}u_0|_{H^1\times L^2} + C\int_s^t \frac{1}{(t-z)^{\frac{1}{2}}}e^{-(t-z)\frac{\nu}{2}}\widehat{M_0^\infty}|e^{-(z-s)\widehat{L}_1}\widehat{\Pi}_c^{(0)}u_0|_{L^2}\,dz \\
&\qquad \le C((t-s)^{-\frac{1}{2}} + (t-s)^{\frac{1}{2}})e^{-\widetilde{d}(t-s)}|\widehat{\Pi}_c^{(0)}u_0|_{H^1\times L^2}.
\end{aligned}$$

$\square$

Proof of Proposition 5.4. We already know that $\widehat{U}_0(t,0)\widehat{\Pi}^{(0)}u_0$ is a periodic solution of (5.1) which is equivalent to the fact that $\widehat{U}_0(T)\widehat{\Pi}^{(0)}u_0 = \widehat{\Pi}^{(0)}u_0$ which is equivalent to the fact that 1 is an eigenvalue of monodromy operator $\widehat{U}_0(T)$. To prove that 1 is simple we first show that if

$$(\widehat{U}_0(T)-1)u = 0, \quad u \neq 0, \tag{5.8}$$

then $u = Cu^{(0)}$ for some $C \in \mathbb{C}$. Let us decompose u using projections $\widehat{\Pi}^{(0)}$ and $\widehat{\Pi}_c^{(0)}$,

$$u = Cu^{(0)} + u_1, \quad \widehat{\Pi}_c^{(0)}u_1 = u_1.$$

Since (5.8) is equivalent to the fact that $\widehat{U}_0(t,0)u$ is a periodic solution of (5.1), $\widehat{U}_0(t,0)u_1$ must also be a periodic solution of (5.1). But combining the decay estimate (5.7) and periodicity of $\widehat{U}_0(t,0)u_1$ we get $u_1 = 0$.

Furthermore, we apply $\widehat{\Pi}^{(0)}$ to

$$(\widehat{U}_0(T)-1)u = u^{(0)}, \tag{5.9}$$

and we get

$$[Q_0(\widehat{U}_0(T)-1)u] = [Q_0u^{(0)}] = 1,$$

which yields $0 = [\phi(T)] - [\phi_0] = 1$. We conclude that u satisfying (5.9) cannot exist. So 1 is simple.

Let us compute a spectral radius of $\widehat{U}_0(T)$ on complementary space $\widehat{\Pi}_c^{(0)}X$. Using the decay estimate (5.7) we have

$$\begin{aligned} r(\widehat{U}_0(T)|_{\widehat{\Pi}_c^{(0)}X}) &= r(\widehat{\Pi}_c^{(0)}\widehat{U}_0(T)\widehat{\Pi}_c^{(0)}) = \lim_{n\to\infty} |(\widehat{\Pi}_c^{(0)}\widehat{U}_0(T,0)\widehat{\Pi}_c^{(0)})^n|_{L(X)}^{\frac{1}{n}} \\ &= \lim_{n\to\infty} |\widehat{\Pi}_c^{(0)}\widehat{U}_0(nT,0)\widehat{\Pi}_c^{(0)}|_{L(X)}^{\frac{1}{n}} \le \lim_{n\to\infty} C^{\frac{1}{n}}e^{-dT} \le e^{-dT} < 1. \end{aligned}$$

Thus we proved the decomposition for $\sigma(\widehat{U}_0(T))$. □

So we can see that solution u to (5.1) with $u|_{t=s} = u_0$ can be decomposed into $\widehat{U}_0(t,s)\widehat{\Pi}^{(0)}u_0$ and $\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0$, where $\widehat{\Pi}^{(0)}u_0$ gives periodic solution $\widehat{U}_0(t,s)\widehat{\Pi}^{(0)}u_0$ to (5.1) and $\widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}u_0$ decays exponentially.

In what follows we will need estimates on the whole solution of (5.1).

Lemma 5.9. *There exists a constant $\gamma_0 > 0$ such that if $\gamma \ge \gamma_0$ then there exists positive number d such that following estimates hold true for $\widehat{U}_0(t,s)\widehat{u}_0$,*

$$|\widehat{U}_0(t,s)\widehat{u}_0|_{H^\ell \times L^2} \le Ce^{-d(t-s)}|\widehat{u}_0|_{H^1\times L^2} + C|\widehat{\phi}_0|_{L^2}, \ l = 0, 1,$$

$$|\partial_{x_n}\widetilde{Q}\widehat{U}_0(t,s)\widehat{u}_0|_{L^2} \le Ce^{-d(t-s)}((t-s)^{-\frac{1}{2}} + (t-s)^{\frac{1}{2}})|\widehat{u}_0|_{H^1\times L^2} + C|\widehat{\phi}_0|_{L^2}.$$

Using decomposition

$$\widehat{U}_0(t,s)\widehat{u}_0 = [\widehat{\phi}_0]u^{(0)} + \widehat{U}_0(t,s)\widehat{\Pi}_c^{(0)}\widehat{u}_0,$$

we can obtain estimates in Lemma 5.9 by straightforward calculation, so we omit the proof.

We are in position to start investigation of $\widehat{U}_{\xi'}(t,s)$. Since (5.2) can be written as

$$\partial_t u + \widehat{L}_1 u = -\widehat{M}_{\xi'}(t)u,$$

$$u|_{t=s} = u_0,$$

we can write $\widehat{U}_{\xi'}(t,s)$ in a form of perturbation of $e^{-(t-s)L_1}$ as

$$\widehat{U}_{\xi'}(t,s)u_0 = e^{-(t-s)L_1}u_0 - \int_s^t e^{-(t-z)L_1}\widehat{M}_{\xi'}(z)\widehat{U}_{\xi'}(z,s)u_0\,dz.$$

By (5.3) we already have

$$\widehat{U}_0(t,s)u_0 = e^{-(t-s)L_1}u_0 - \int_s^t e^{-(t-z)L_1}\widehat{M}_0(z)\widehat{U}_0(z,s)u_0\,dz.$$

Let us define difference operator $\widehat{V}_{\xi'}$ as

$$\widehat{V}_{\xi'}(t,s) = \widehat{U}_{\xi'}(t,s) - \widehat{U}_0(t,s).$$

Then we can write

$$\widehat{V}_{\xi'}(t,s)u_0 = -\int_s^t e^{-(t-z)L_1}\left\{\left(\widehat{M}_{\xi'}(z) - \widehat{M}_0(z)\right)\widehat{U}_{\xi'}(z,s) + \widehat{M}_0(z)\widehat{V}_{\xi'}(z,s)\right\}u_0\,dz.$$

It follows that $v = \widehat{V}_{\xi'}(t,s)u_0$ is a solution of the following problem:

$$\partial_t v + \widehat{L}_1 v = -\widehat{M}_0(t)v - (\widehat{M}_{\xi'}(t) - \widehat{M}_0(t))u, \quad v|_{t=s} = 0,$$

with $u = \widehat{U}_{\xi'}(t,s)u_0$, namely,

$$\partial_t v + \widehat{L}_0(t)v = -(\widehat{M}_{\xi'}(t) - \widehat{M}_0(t))u, \quad v|_{t=s} = 0. \tag{5.10}$$

Therefore, in terms of $\widehat{U}_0(t,s)$, a solution of (5.10) can be written as

$$v(t,s) = \widehat{V}_{\xi'}(t,s)u_0 = -\int_s^t \widehat{U}_0(t,z)\left(\widehat{M}_{\xi'}(z) - \widehat{M}_0(z)\right)\widehat{U}_{\xi'}(z,s)u_0\,dz. \tag{5.11}$$

Finally we get following equation for $\widehat{V}_{\xi'}$,

$$\begin{aligned}\widehat{V}_{\xi'}(t,s)u_0 &= -\int_s^t \widehat{U}_0(t,z)\left(\widehat{M}_{\xi'}(z) - \widehat{M}_0(z)\right)\widehat{U}_{\xi'}(z,s)u_0\,dz \\ &= -\int_s^t \widehat{U}_0(t,z)\left(\widehat{M}_{\xi'}(z) - \widehat{M}_0(z)\right)\widehat{U}_0(z,s)u_0\,dz \\ &\quad - \int_s^t \widehat{U}_0(t,z)\left(\widehat{M}_{\xi'}(z) - \widehat{M}_0(z)\right)\widehat{V}_{\xi'}(z,s)u_0\,dz.\end{aligned} \tag{5.12}$$

Let us define operator $\widehat{S}_{\xi'}$ as

$$(\widehat{S}_{\xi'}V)(t)u_0 = -\int_0^t \widehat{U}_0(t,s)\left(\widehat{M}_{\xi'}(s) - \widehat{M}_0(s)\right)V(s)u_0\,ds,$$

where $V \in L(X,Y)$ is defined as

$$V : u_0 \in X \mapsto V(s)u_0 \in Y,$$

with

$$Y = \{u = {}^T(\phi, w) \in C([0,T] : X) : \partial_{x_n} w \in C((0,T] : L^2(0,1)), |u|_Y < \infty\},$$

and

$$|u|_Y = \sup_{z\in[0,T]} (|u(z)|_{H^1\times L^2} + z^{\frac{1}{2}}|\partial_{x_n} w(z)|_{L^2}).$$

We see that

$$\widehat{S}_{\xi'} : L(X,Y) \to L(X,Y).$$

We note that norm on $L(X,Y)$ is defined as

$$|V|_{L(X,Y)} = \sup_{u_0\in X,\ |u_0|_X\le 1} |V(\cdot)u_0|_Y.$$

Denoting $\widehat{V}_{\xi'}(\cdot,0)$ and $\widehat{U}_{\xi'}(\cdot,0)$ as $\widehat{V}_{\xi'}$ and $\widehat{U}_{\xi'}$, respectively, we immediately see that

$$\widehat{V}_{\xi'}, \widehat{U}_{\xi'} \in L(X,Y).$$

We rewrite (5.12) in operator form as

$$(I - \widehat{S}_{\xi'})\widehat{V}_{\xi'} = \widehat{S}_{\xi'}\widehat{U}_0.$$

Now we see that if we have $|\widehat{S}_{\xi'}|_{L(L(X,Y))} < 1$ for Banach space $L(X,Y)$ we get equation

$$\widehat{V}_{\xi'} = (I - \widehat{S}_{\xi'})^{-1}\widehat{S}_{\xi'}\widehat{U}_0 = \sum_{n=1}^{\infty} \widehat{S}^n_{\xi'}\widehat{U}_0,$$

which immediately implies formula for $\widehat{U}_{\xi'}$,

$$\widehat{U}_{\xi'} = \widehat{U}_0 + \sum_{n=1}^{\infty} \widehat{S}^n_{\xi'}\widehat{U}_0.$$

Proposition 5.10. *There exists a constant $\gamma_0 > 0$ such that if $\gamma \ge \gamma_0$ then there exists $r_0 > 0$ ($r_0 = \frac{C}{\gamma^2+\nu+\widetilde{\nu}}$) such that if $|\xi'| < r_0$ then $|\widehat{S}_{\xi'}|_{L(L(X,Y))} < 1$ and therefore $(I - \widehat{S}_{\xi'})^{-1}$ exists and associated Neumann series converges, so we can write*

$$\widehat{U}_{\xi'} = \widehat{U}_0 + \sum_{n=1}^{\infty} \widehat{S}^n_{\xi'}\widehat{U}_0.$$

Proof. Let $u_0 \in X$. We expand $\widehat{S}_{\xi'}$ as

$$\widehat{S}_{\xi'} = \sum_{j=1}^{n-1} \xi_j \widehat{S}_1^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k \widehat{S}_{jk}^{(2)}, \tag{5.13}$$

with

$$(\widehat{S}_j^{(1)} V)(t)u_0 = -\int_0^t \widehat{U}_0(t,s)\widehat{M}_j^{(1)}(s)V(s)u_0\,ds,$$
$$(\widehat{S}_{jk}^{(2)} V)(t)u_0 = -\int_0^t \widehat{U}_0(t,s)\widehat{M}_{jk}^{(2)}(s)V(s)u_0\,ds,$$

where

$$\widehat{M}_1^{(1)}(s) = \begin{pmatrix} iv_p^1(s) & i\gamma^2\rho_p \boldsymbol{e}_1' & 0 \\ i\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\boldsymbol{e}_1' & iv_p^1(s)I_{n-1} & -i\frac{\tilde{\nu}}{\rho_p}\partial_{x_n}\boldsymbol{e}_1' \\ 0 & -i\frac{\tilde{\nu}}{\rho_p}\partial_{x_n}{}^T\boldsymbol{e}_1' & iv_p^1(s) \end{pmatrix},$$

$$\widehat{M}_j^{(1)}(s) = \begin{pmatrix} 0 & i\gamma^2\rho_p \boldsymbol{e}_j' & 0 \\ i\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\boldsymbol{e}_j' & 0 & -i\frac{\tilde{\nu}}{\rho_p}\partial_{x_n}\boldsymbol{e}_j' \\ 0 & -i\frac{\tilde{\nu}}{\rho_p}\partial_{x_n}{}^T\boldsymbol{e}_j' & 0 \end{pmatrix}, \text{ for } j = 2,\dots,n-1,$$

$$\widehat{M}_{jk}^{(2)}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_p}\delta_{jk}I_{n-1} + \frac{\tilde{\nu}}{\rho_p}{}^T\boldsymbol{e}_j'\boldsymbol{e}_k' & 0 \\ 0 & 0 & \frac{\nu}{\rho_p}\delta_{jk} \end{pmatrix}, \text{ for } j,k = 1,\dots,n-1,$$

and $\boldsymbol{e}_j' = (0,\dots,0,\underbrace{1}_{j-th},0,\dots,0)$, $j = 1,\dots,n-1$.

We compute separately estimates on $(\widehat{S}_j^{(1)}V)(t)u_0$ and $(\widehat{S}_{jk}^{(2)}V)(t)u_0$. We denote components of $V(\cdot)u_0$ as $V(\cdot)u_0 = {}^T(\phi, w)$. For $j = 2,\dots,n-1$ we have

$$|\widehat{S}_j^{(1)}V(t)u_0|_{H^1\times L^2} \le \int_0^t |\widehat{U}_0(t,s) \begin{pmatrix} i\gamma^2\rho_p w^j \\ i(\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\phi - \frac{\tilde{\nu}}{\rho_p}\partial_{x_n}w^n)\boldsymbol{e}_j' \\ -i\frac{\tilde{\nu}}{\rho_p}\partial_{x_n}w^j \end{pmatrix} |_{H^1\times L^2}\,ds.$$

Using estimates from Lemma 5.9 we get

$$|\widehat{S}_j^{(1)}V(t)u_0|_{H^1\times L^2} \le \int_0^t Ce^{-d(t-s)}\left[\gamma^2|w^j|_{H^1} + |\phi|_{L^2} + \widetilde{\nu}(|\partial_{x_n}w^n|_{L^2} + |\partial_{x_n}w^j|_{L^2})\right] + C\gamma^2|w^j|_{L^2}\,ds$$

$$\le \int_0^t Ce^{-d(t-s)}\left[\gamma^2(1+s^{-\frac{1}{2}}) + 1 + \widetilde{\nu}s^{-\frac{1}{2}}\right]|V(\cdot)u_0|_Y + C\gamma^2|V(\cdot)u_0|_Y\,ds$$

$$\leq C|V(\cdot)u_0|_Y(\gamma^2+\widetilde{\nu})(1+t).$$

By similar computation we obtain estimates on derivative

$$t^{\frac{1}{2}}|\partial_{x_n}\widetilde{Q}\widehat{S}_j^{(1)}V(t)u_0|_{L^2} \leq t^{\frac{1}{2}}\int_0^t |\partial_{x_n}\widetilde{Q}\widehat{U}_0(t,s)\begin{pmatrix} i\gamma^2\rho_p w^j \\ i(\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\phi - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}w^n)\boldsymbol{e}'_j \\ -i\frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}w^j \end{pmatrix}|_{L^2}\, ds$$

$$\leq C|V(\cdot)u_0|_Y(\gamma^2+\widetilde{\nu})(1+t^2).$$

In case $j=1$ we get

$$|\widehat{S}_1^{(1)}V(t)u_0|_{H^1\times L^2} \leq \int_0^t |\widehat{U}_0(t,s)\begin{pmatrix} iv_p^1\phi + i\gamma^2\rho_p w^1 \\ i(\frac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\phi - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}w^n)\boldsymbol{e}'_1 + iv_p^1 w' \\ -i\frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}w^1 + iv_p^1 w^n \end{pmatrix}|_{H^1\times L^2}\, ds$$

$$\leq C|V(\cdot)u_0|_Y(\gamma^2+\widetilde{\nu})(1+t),$$

and for derivative we have

$$t^{\frac{1}{2}}|\partial_{x_n}\widetilde{Q}\widehat{S}_1^{(1)}V(t)u_0|_{L^2} \leq C|V(\cdot)u_0|_Y(\gamma^2+\widetilde{\nu})(1+t^2).$$

For $j,k=1,\ldots,n-1$ we have

$$|\widehat{S}_{jk}^{(2)}V(t)u_0|_{H^1\times L^2} \leq \int_0^t |\widehat{U}_0(t,s)\begin{pmatrix} 0 \\ \frac{\nu}{\rho_p}\delta_{jk}w' + \frac{\widetilde{\nu}}{\rho_p}{}^T\boldsymbol{e}'_j w^k \\ \frac{\nu}{\rho_p}\delta_{jk}w^n \end{pmatrix}|_{H^1\times L^2}\, ds \leq C|V(\cdot)u_0|_Y(\nu\delta_{jk}+\widetilde{\nu}),$$

and for derivative we obtain

$$t^{\frac{1}{2}}|\partial_{x_n}\widetilde{Q}\widehat{S}_{jk}^{(2)}V(t)u_0|_{H^1\times L^2} \leq C|V(\cdot)u_0|_Y(\nu\delta_{jk}+\widetilde{\nu})t(1+t).$$

Now we put all estimates together to get

$$|\widehat{S}_{\xi'}V(\cdot)u_0|_Y \leq \sum_{j=1}^{n-1}|\xi_j|\ |\widehat{S}_1^{(1)}V(\cdot)u_0|_Y + \sum_{j,k=1}^{n-1}|\xi_j\xi_k|\ |\widehat{S}_{jk}^{(2)}V(\cdot)u_0|_Y$$

$$\leq C(T)|V(\cdot)u_0|_Y\left(\sum_{j=1}^{n-1}|\xi_j| + \sum_{j,k=1}^{n-1}|\xi_j\xi_k|\right).$$

Thus we conclude that there exist $r_0>0$ and $0<q<1$ such that if $|\xi'|\leq r_0$ then $|\widehat{S}_{\xi'}V(\cdot)u_0|_Y < q|V(\cdot)u_0|_Y$. Therefore, we have

$$|\widehat{S}_{\xi'}V|_{L(X,Y)} = \sup_{u_0\in X,\ |u_0|_X\leq 1}|\widehat{S}_{\xi'}V(\cdot)u_0|_Y \leq \sup_{u_0\in X,\ |u_0|_X\leq 1} q|V(\cdot)u_0|_Y = q|V|_{L(X,Y)},$$

which concludes the proof. $\square$

Assuming that $\gamma \geq \gamma_0$ and $|\xi'| < r_0$ from Proposition 5.10 we can use the expansion (5.13) and rewrite $\widehat{U}_{\xi'}(T,0) = \widehat{U}_{\xi'}(T)$ more precisely as

$$\widehat{U}_{\xi'}(T) = \widehat{U}_0(T) + \sum_{j=1}^{n-1} \xi_j \widehat{S}_j^{(1)} \widehat{U}_0(T) + \sum_{j,k=1}^{n-1} \xi_j \xi_k \left(\widehat{S}_{jk}^{(2)} + \widehat{S}_j^{(1)} \widehat{S}_k^{(1)} \right) \widehat{U}_0(T) + o(|\xi'|^2).$$

Theorem 5.11. *There exist positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$ then there exists $r_0 > 0$ such that for each ξ' with $|\xi'| \leq r_0$ it holds that*

$$\sigma(\widehat{U}_{\xi'}(T)) \cap \{\lambda; |\lambda - 1| < \frac{\eta_0}{2}\} = \{\widehat{\mu}(\xi')\},$$

where $\widehat{\mu}(\xi')$ is a simple eigenvalue of $\widehat{U}_{\xi'}(T)$ that has the form

$$\widehat{\mu}(\xi') = 1 - i\kappa_0 \xi_1 T - \kappa_1 |\xi'|^2 T + O(|\xi'|^3) T,$$

as $|\xi'| \to 0$. Here $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$ are some constants having the properties given in Theorem 3.2 (i).

Proof. Theorem 5.11 is proved by applying the analytic perturbation theory from Kato [7]. We here derive the asymptotic of $\widehat{\mu}(\xi')$ only.

We proceed as in [3, Theorem 5.3]. Since $\widehat{\mu}^{(0)} = 1$ is simple, we can see that $\widehat{\mu}(\xi')$ is simple and $\widehat{\mu}(\xi')$ is expanded as

$$\widehat{\mu}_0(\xi') = \widehat{\mu}^{(0)} + \sum_{j=1}^{n-1} \xi_j \widehat{\mu}_j^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k \widehat{\mu}_{jk}^{(2)} + O(|\xi'|^3),$$

with

$$\begin{aligned} \widehat{\mu}_j^{(1)} &= \langle \widehat{T}_j^{(1)} u^{(0)}, u^{(0)*} \rangle, \\ \widehat{\mu}_{jk}^{(2)} &= \langle (\widehat{T}_{jk}^{(2)} + \widehat{T}_{jk}^{(3)}) u^{(0)}, u^{(0)*} \rangle - \langle \widehat{T}_j^{(1)} \widehat{S} \widehat{T}_k^{(1)} u^{(0)}, u^{(0)*} \rangle, \end{aligned}$$

where

$$\begin{aligned} \widehat{T}_j^{(1)} &= \widehat{S}_j^{(1)}(\widehat{U}_0)(T) = -\int_0^T \widehat{U}_0(T,s) \widehat{M}_j^{(1)}(s) \widehat{U}_0(s,0)\, ds, \\ \widehat{T}_{jk}^{(2)} &= \widehat{S}_{jk}^{(2)}(\widehat{U}_0)(T) = -\int_0^T \widehat{U}_0(T,s) \widehat{M}_{jk}^{(2)}(s) \widehat{U}_0(s,0)\, ds, \end{aligned}$$

$$\widehat{T}_{jk}^{(3)} = \widehat{S}_j^{(1)}(\widehat{S}_k^{(1)}(\widehat{U}_0))(T) = \int_0^T \int_0^s \widehat{U}_0(T,s) \widehat{M}_j^{(1)}(s) \widehat{U}_0(s,s_1) \widehat{M}_k^{(1)}(s_1) \widehat{U}_0(s_1,0)\, ds_1\, ds,$$

$$\widehat{S} = \widehat{\Pi}_c^{(0)} \left((\widehat{U}_0 - I)\Big|_{\widehat{\Pi}_c^{(0)} X} \right)^{-1} \widehat{\Pi}_c^{(0)}.$$

Proposition 5.12.

$$\widehat{\mu}_j^{(1)} = \begin{cases} 0 & \text{for} \quad j \neq 1, \\ -i \displaystyle\int_0^T \int_0^1 \phi^{(0)}(x_n) v_p^1(x_n, s) + \rho_p(x_n) w^{(0),1}(x_n, s)\, dx_n\, ds & \text{for} \quad j = 1. \end{cases}$$

Proof of Proposition 5.12. Since

$$\widehat{U}_0(t,0)u^{(0)} = \begin{pmatrix} \phi^{(0)} \\ -\frac{1}{\gamma^2} \displaystyle\int_{-\infty}^t e^{-(t-s)\nu A} \nu \frac{\gamma^2 \alpha_0}{\widetilde{P}'(\rho_p)\rho_p} (\partial_{x_n}^2 v_p^1(s))\, ds \boldsymbol{e}_1' \\ 0 \end{pmatrix} = \begin{pmatrix} \phi^{(0)} \\ \frac{1}{\gamma^2} w^{(0),1}(x_n, t) \boldsymbol{e}_1' \\ 0 \end{pmatrix},$$

and

$$\widehat{T}_j^{(1)} u^{(0)} = -i \int_0^T \widehat{U}_0(T,s) \begin{pmatrix} (\phi^{(0)} v_p^1(s) + \rho_p w^{(0),1}(s))\delta_{1j} \\ \alpha_0 \boldsymbol{e}_j' + v_p^1(s) \frac{1}{\gamma^2} w^{(0),1}(s) \boldsymbol{e}_1' \delta_{1j} \\ -\frac{\widetilde{\nu}}{\gamma^2 \rho_p} \partial_{x_n} w^{(0),1}(s) \delta_{1j} \end{pmatrix} ds, \tag{5.14}$$

we can immediately see the relation for $\widehat{\mu}_j^{(1)}$. $\square$

Let us next compute $\widehat{\mu}_{jk}^{(2)}$. One can easily see that

$$\langle \widehat{T}_{jk}^{(2)} u^{(0)}, u^{(0)*} \rangle = 0 \text{ for all } j, k = 1, \dots, n-1,$$

$$\langle \widehat{T}_{jk}^{(3)} u^{(0)}, u^{(0)*} \rangle = 0 \text{ for } j \neq k,$$

$$\langle \widehat{T}_j^{(1)} \widehat{S} \widehat{T}_k^{(1)} u^{(0)}, u^{(0)*} \rangle = 0 \text{ for } j \neq k.$$

We thus obtain the following proposition.

Proposition 5.13. *For $j \neq k$,*

$$\widehat{\mu}_{jk}^{(2)} = 0.$$

We next consider $\widehat{\mu}_{jj}^{(2)}$ for $j = 2, \dots, n-1$.

Proposition 5.14. *For $j = 2, \dots, n-1$, $\widehat{\mu}_{jj}^{(2)}$ satisfies*

$$\widehat{\mu}_{jj}^{(2)} = -\frac{\gamma^2}{\nu} T \left(K + O\left(\frac{1}{\nu}\right)\right),$$

with some positive constant K.

Proof of Proposition 5.14. For $j = 2, \dots, n-1$ we get

$$\langle \widehat{T}_{jj}^{(3)} u^{(0)}, u^{(0)*} \rangle = \alpha_0 \langle \int_0^T \int_0^s \widehat{U}_0(T,s) \begin{pmatrix} -\gamma^2 \rho_p e^{-(s-s_1)\nu A} \cdot 1 \\ 0 \\ \frac{\widetilde{\nu}}{\rho_p} \partial_{x_n} e^{-(s-s_1)\nu A} \cdot 1 \end{pmatrix} ds_1 ds, u^{(0)*} \rangle. \tag{5.15}$$

Let us estimate $\langle \widehat{T}_{jj}^{(3)} u^{(0)}, u^{(0)*}\rangle$ for $j = 2, \ldots, n-1$. We first recall that $[Q_0 \widehat{U}_0(t,s)u_0] = [\phi_0]$. Using this fact we can see from (5.15)

$$\langle \widehat{T}_{jj}^{(3)} u^{(0)}, u^{(0)*}\rangle = -\gamma^2 \int_0^T \int_0^s \int_0^1 \alpha_0 \rho_p e^{-(s-s_1)\nu A} \cdot 1 \, dx_n \, ds_1 \, ds.$$

First we estimate from above,

$$\int_0^T \int_0^s \int_0^1 \rho_p e^{-(s-s_1)\nu A} \cdot 1 \, dx_n \, ds_1 \, ds \le \int_0^T \int_0^s |\rho_p|_\infty C e^{-(s-s_1)\frac{\nu}{2}} \, ds_1 \, ds \ \le \frac{1}{\nu} TC.$$

Second we estimate from below,

$$\int_0^T \int_0^s \int_0^1 \rho_p e^{-(s-s_1)\nu A} \cdot 1 \, dx_n \, ds_1 \, ds \overset{\nu s = s}{=} \frac{1}{\nu} \int_0^{\nu T} \int_0^{\frac{s}{\nu}} \int_0^1 \rho_p e^{-(s-\nu s_1)A} \cdot 1 \, dx_n \, ds_1 \, ds$$

$$\overset{\nu s_1 = s_1}{=} \frac{1}{\nu^2} \int_0^{\nu T} \int_0^s \int_0^1 \rho_p e^{-(s-s_1)A} \cdot 1 \, dx_n \, ds_1 \, ds \overset{\text{Fubini}}{=} \frac{1}{\nu^2} \int_0^{\nu T} \int_{s_1}^{\nu T} \int_0^1 \rho_p e^{-(s-s_1)A} \cdot 1 \, dx_n \, ds \, ds_1$$

$$\overset{s - s_1 = \widetilde{s}}{=} \frac{1}{\nu^2} \int_0^{\nu T} \int_0^{\nu T - s_1} \int_0^1 \rho_p e^{-\widetilde{s}A} \cdot 1 \, dx_n \, d\widetilde{s} \, ds_1.$$

By the maximum principle, we have $e^{-\widetilde{s}A} \cdot 1 \ \ge 0$. It then follows

$$\frac{1}{\nu^2} \int_0^{\nu T} \int_0^{\nu T - s_1} \int_0^1 \underbrace{\rho_p e^{-\widetilde{s}A} \cdot 1}_{\ge 0} \, dx_n \, d\widetilde{s} \, ds_1 \ge \frac{1}{\nu^2} \int_0^{\frac{\nu T}{2}} \int_0^{\frac{\nu T}{2}} \int_0^1 \rho_p e^{-\widetilde{s}A} \cdot 1 \, dx_n \, d\widetilde{s} \, ds_1.$$

Supposing $\nu T > 2$, we obtain

$$\frac{1}{\nu^2} \int_0^{\frac{\nu T}{2}} \int_0^{\frac{\nu T}{2}} \int_0^1 \rho_p e^{-\widetilde{s}A} \cdot 1 \, dx_n \, d\widetilde{s} \, ds_1 \ge \frac{1}{\nu^2} \int_0^{\frac{\nu T}{2}} \int_0^1 \int_0^1 \rho_p e^{-\widetilde{s}A} \cdot 1 \, dx_n \, d\widetilde{s} \, ds_1$$

$$= \frac{T}{2\nu} \int_0^1 \int_0^1 \rho_p e^{-\widetilde{s}A} \cdot 1 \, dx_n \, d\widetilde{s}.$$

Since $\int_0^1 \int_0^1 \rho_p e^{-\widetilde{s}A} \cdot 1 \, dx_n \, d\widetilde{s} > 0$, we conclude that

$$-\frac{\gamma^2}{\nu} TC_1 \le \langle \widehat{T}_{jj}^{(3)} u^{(0)}, u^{(0)*}\rangle \le -\frac{\gamma^2}{\nu} TC_2,$$

for some positive constants C_1 and C_2.

Let us estimate $\langle \widehat{T}_j^{(1)} \widehat{S} \widehat{T}_j^{(1)} u^{(0)}, u^{(0)*}\rangle$ for $j = 2, \ldots, n-1$. We see from (5.14)

$$\widehat{S}\widehat{T}_j^{(1)} u^{(0)} = -i\alpha_0 \begin{pmatrix} 0 \\ (e^{-T\nu A} - 1)^{-1} \int_0^T e^{-(T-s)\nu A} \cdot 1 \, ds \, \boldsymbol{e}_j' \\ 0 \end{pmatrix},$$

for $j = 2, \ldots, n-1$. Therefore, we get

$$\langle \widehat{T}_j^{(1)} \widehat{S} \widehat{T}_j^{(1)} u^{(0)}, u^{(0)*} \rangle$$

$$= \alpha_0 \langle \int_0^T \widehat{U}_0(T,s) \begin{pmatrix} -\gamma^2 \rho_p e^{-s\nu A} (e^{-T\nu A} - 1)^{-1} \int_0^T e^{-(T-s_1)\nu A} \cdot 1 \, ds_1 \\ 0 \\ \frac{\widetilde{\nu}}{\rho_p} \partial_{x_n} e^{-s\nu A} (e^{-T\nu A} - 1)^{-1} \int_0^T e^{-(T-s_1)\nu A} \cdot 1 \, ds_1 \end{pmatrix} ds, u^{(0)*} \rangle. \tag{5.16}$$

From (5.16) we can easily see that $\langle \widehat{T}_j^{(1)} \widehat{S} \widehat{T}_j^{(1)} u^{(0)}, u^{(0)*} \rangle$ is of order $\frac{\gamma^2}{\nu^2}$.

□

We finally consider $\widehat{\mu}_{11}^{(2)}$.

Proposition 5.15. *There holds*

$$\widehat{\mu}_{11}^{(2)} = -\frac{\gamma^2}{\nu} T \left(K + O(\frac{1}{\gamma}) + O(\frac{1}{\nu}) + O(\frac{\nu + \widetilde{\nu}}{\gamma^2}) \right).$$

Proof of Proposition 5.15. We get

$$\langle \widehat{T}_{11}^{(3)} u^{(0)}, u^{(0)*} \rangle$$

$$= \langle \int_0^T \int_0^s \widehat{U}_0(T,s) \widehat{M}_1^{(1)}(s) \widehat{U}_0(s,s_1) i \begin{pmatrix} \phi^{(0)} v_p^1(s_1) + \rho_p w^{(0),1}(s_1) \\ (\alpha_0 + v_p^1(s_1) \frac{1}{\gamma^2} w^{(0),1}(s_1)) \boldsymbol{e}_1' \\ -\frac{\widetilde{\nu}}{\gamma^2 \rho_p} \partial_{x_n} w^{(0),1}(s_1) \end{pmatrix} ds_1 ds, u^{(0)*} \rangle, \tag{5.17}$$

and

$$\langle \widehat{T}_1^{(1)} \widehat{S} \widehat{T}_1^{(1)} u^{(0)}, u^{(0)*} \rangle = \langle \widehat{T}_1^{(1)} \widehat{S} \int_0^T \widehat{U}_0(T,s) i \begin{pmatrix} \phi^{(0)} v_p^1(s) + \rho_p w^{(0),1}(s) \\ (\alpha_0 + v_p^1(s) \frac{1}{\gamma^2} w^{(0),1}(s)) \boldsymbol{e}_1' \\ -\frac{\widetilde{\nu}}{\gamma^2 \rho_p} \partial_{x_n} w^{(0),1}(s) \end{pmatrix} ds, u^{(0)*} \rangle. \tag{5.18}$$

To estimate the right-hand side of (5.17) and (5.18) it is convenient to transform the whole problem and consider newly obtained monodromy operator $\widetilde{U}_{\xi'}(T)$

$$\widehat{U}_{\xi'}(T) = Q_\gamma \widetilde{U}_{\xi'}(T) Q_\gamma^{-1},$$

where

$$Q_\gamma = \begin{pmatrix} \gamma & 0 \\ 0 & I_n \end{pmatrix},$$

which leads to the fact that spectrum of these operators is the same. Furthermore, we have

$$\widehat{U}_0(T) = Q_\gamma \widetilde{U}_0(T) Q_\gamma^{-1}.$$

We set $\widetilde{u} = Q_\gamma^{-1} u$. Then $\widetilde{u} = {}^T(\widetilde{\phi}, \widetilde{w}) = \widetilde{U}_0(t,s)\widetilde{u}_0$ is a solution of the problem

$$\begin{gathered}
\partial_t \widetilde{\phi} + \gamma \partial_{x_n}(\rho_p \widetilde{w}^n) = 0, \\
\partial_t \widetilde{w}' - \frac{\nu}{\rho_p}\partial_{x_n}^2 \widetilde{w}' + (\partial_{x_n} v_p^1)\widetilde{w}^n \boldsymbol{e}_1' + \frac{\nu}{\gamma \rho_p^2}(\partial_{x_n}^2 v_p^1)\widetilde{\phi}\boldsymbol{e}_1' = 0, \\
\partial_t \widetilde{w}^n - \frac{\nu + \widetilde{\nu}}{\rho_p}\partial_{x_n}^2 \widetilde{w}^n + \gamma \partial_{x_n}\left(\frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p}\widetilde{\phi}\right) = 0, \\
\widetilde{w}|_{x_n = 0,1} = 0,
\end{gathered}$$

for $t > s \geq 0$, and

$$\widetilde{u}|_{t=s} = \widetilde{u}_0 = {}^T(\widetilde{\phi}_0, \widetilde{w}_0).$$

One can show that

$$\begin{gathered}
|\widetilde{\phi}(t)|_2 + |\widetilde{w}^n(t)|_2 \leq C\left(|\widetilde{\phi}_0|_2 + |\widetilde{w}_0^n|_2\right), \\
\int_0^t |\widetilde{w}^n|_2\, ds \leq \int_0^t |\partial_{x_n}\widetilde{w}^n|_2\, ds \leq C\sqrt{t}\frac{1}{\sqrt{\nu + \widetilde{\nu}}}\left(|\widetilde{\phi}_0|_2 + |\widetilde{w}_0^n|_2\right), \\
|\widetilde{\phi}(t)|_{H^1} \leq C|\widetilde{\phi}_0|_{H^1} + C\left(\tfrac{\gamma}{\nu + \widetilde{\nu}} + \tfrac{1}{\gamma^2} + \sqrt{t}\tfrac{1}{\gamma\sqrt{\nu + \widetilde{\nu}}}\right)(|\widetilde{\phi}_0|_2 + |\widetilde{w}_0^n|_2), \\
|\widetilde{S}|_{L(X)} = |Q_\gamma^{-1}\widehat{S}Q_\gamma|_{L(X)} \leq C.
\end{gathered} \tag{5.19}$$

Furthermore, (5.17) can be written as

$$\begin{aligned}
&\langle \widehat{T}_{11}^{(3)} u^{(0)}, u^{(0)*}\rangle \\
&= \langle \int_0^T \int_0^s \widetilde{U}_0(T,s)\widetilde{M}_1^{(1)}(s)\widetilde{U}_0(s,s_1) i \begin{pmatrix} \phi^{(0)} v_p^1(s_1) + \rho_p w^{(0),1}(s_1) \\ \gamma(\alpha_0 + v_p^1(s_1)\frac{1}{\gamma^2} w^{(0),1}(s_1))\boldsymbol{e}_1' \\ -\frac{\tilde{\nu}}{\gamma \rho_p}\partial_{x_n} w^{(0),1}(s_1) \end{pmatrix} ds_1 ds, u^{(0)*}\rangle,
\end{aligned} \tag{5.20}$$

where

$$\widetilde{M}_1^{(1)}(s) = Q_\gamma^{-1}\widehat{M}_1^{(1)}(s) Q_\gamma = \begin{pmatrix} i v_p^1(s) & i\gamma \rho_p \boldsymbol{e}_1' & 0 \\ i\gamma \frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p}\boldsymbol{e}_1' & i v_p^1(s) I_{n-1} & -i\frac{\tilde{\nu}}{\rho_p}\partial_{x_n}\boldsymbol{e}_1' \\ 0 & -i\frac{\tilde{\nu}}{\rho_p}\partial_{x_n}{}^T\boldsymbol{e}_1' & i v_p^1(s) \end{pmatrix}.$$

From (5.15) and (5.20) it is easy to see that for $j \neq 1$

$$\langle \widehat{T}_{11}^{(3)} u^{(0)}, u^{(0)*} \rangle = \langle \widehat{T}_{jj}^{(3)} u^{(0)}, u^{(0)*} \rangle$$

$$+\langle \int_0^T \int_0^s \widetilde{U}_0(T,s)\widetilde{M}_1^{(1)}(s)\widetilde{U}_0(s,s_1) i \begin{pmatrix} \phi^{(0)} v_p^1(s_1) + \rho_p w^{(0),1}(s_1) \\ v_p^1(s_1)\frac{1}{\gamma} w^{(0),1}(s_1)\boldsymbol{e}_1' \\ -\frac{\widetilde{\nu}}{\gamma\rho_p}\partial_{x_n} w^{(0),1}(s_1) \end{pmatrix} ds_1 ds, u^{(0)*} \rangle.$$

Using the estimates (5.19) we get

$$|\langle \widehat{T}_{11}^{(3)} u^{(0)}, u^{(0)*} \rangle - \langle \widehat{T}_{jj}^{(3)} u^{(0)}, u^{(0)*} \rangle| \leq C\frac{\gamma^2}{\nu}\left(\frac{1}{\gamma} + \frac{\nu + \widetilde{\nu}}{\gamma^2}\right).$$

Similarly, we can estimate the right-hand side of (5.18) as

$$|\langle \widehat{T}_1^{(1)} \widehat{S} \widehat{T}_1^{(1)} u^{(0)}, u^{(0)*} \rangle| \leq C\frac{\gamma^2}{\nu}\left(\frac{1}{\gamma} + \frac{1}{\nu} + \frac{\nu + \widetilde{\nu}}{\gamma^2}\right).$$

We thus obtain

$$\widehat{\mu}_{11}^{(2)} = -\frac{\gamma^2}{\nu} T \left(K + O(\frac{1}{\gamma}) + O(\frac{1}{\nu}) + O(\frac{\nu + \widetilde{\nu}}{\gamma^2})\right).$$

This completes the proof. $\square$

We now turn to the proof of Theorem 5.11. One can see from Propositions 5.13, 5.14 and 5.15 that $\widehat{\mu}_{jj}^{(2)} < 0$ if $\frac{\nu+\tilde{\nu}}{\gamma^2}$, $\frac{1}{\nu}$ and $\frac{1}{\gamma}$ are sufficiently small. Therefore, the desired asymptotic of $\widehat{\mu}(\xi')$ is obtained from Propositions 5.12–5.15. This completes the proof of Theorem 5.11. $\square$

Theorem 5.16. *There exist constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_0^2$ then the following statements hold true:*

(i) *Let $\widehat{\Pi}(\xi')$ be the eigen-projection associated with $\widehat{\mu}(\xi')$. Then there exists a positive number r_0 such that for any ξ' with $|\xi'| \leq r_0$ the projection $\widehat{\Pi}(\xi')$ is written in the form*

$$\widehat{\Pi}(\xi') = \widehat{\Pi}^{(0)} + \widehat{\Pi}^{(1)}(\xi'), \tag{5.21}$$

where

$$\widehat{\Pi}^{(1)}(\xi') = \sum_{j=1}^{n-1} \xi_j \widehat{\Pi}_j^{(1)} + \widehat{\Pi}^{(2)}(\xi') \quad \text{and} \quad \widehat{\Pi}_j^{(1)} = \widehat{\Pi}^{(0)} \widehat{T}_j^{(1)} \widehat{S} + \widehat{S}\widehat{T}_j^{(1)} \widehat{\Pi}^{(0)}.$$

Furthermore, we have estimates

$$|\widehat{\Pi}_j^{(1)} u_0|_{H^1 \times L^2} \leq C|u_0|_{H^1 \times L^2}, \tag{5.22}$$

$$|\widehat{\Pi}^{(2)}(\xi') u_0|_{H^1 \times L^2} \leq C|\xi'|^2 |u_0|_{H^1 \times L^2}. \tag{5.23}$$

(ii) *There exists a positive number r_0 such that for any ξ' with $|\xi'| \le r_0$ the spectral radius of $\widehat{U}_{\xi'}(T)$ on $(I - \widehat{\Pi}(\xi'))X$ satisfies estimate*

$$r(\widehat{U}_{\xi'}(T)|_{(I-\widehat{\Pi}(\xi'))X}) \le \delta_0 < 1.$$

To prove Theorem 5.16 and Theorem 5.1 we use the following estimates.

Lemma 5.17. *There exists a constant $\gamma_0 > 0$ such that if $\gamma \ge \gamma_0$ then for $0 < t - s \le 2T$ and $|\xi'| < 1$ hold true the following estimates:*

$$\begin{aligned} |\widehat{U}_{\xi'}(t,s)\widehat{u}_0|_{H^1 \times L^2} &\le C|\widehat{u}_0|_{H^1 \times L^2}, \\ |\partial_{x_n}\widetilde{Q}\widehat{U}_{\xi'}(t,s)\widehat{u}_0|_{L^2} &\le C(t-s)^{-\frac{1}{2}}|\widehat{u}_0|_{H^1 \times L^2}. \end{aligned} \tag{5.24}$$

Proof of Lemma 5.17. Since $\widehat{U}_{\xi'}(t,s) = \widehat{U}_0(t,s) + \widehat{V}_{\xi'}(t,s)$, we see from (5.11)

$$\widehat{U}_{\xi'}(t,s)u_0 = \widehat{U}_0(t,s)u_0 - \int_s^t \widehat{U}_0(t,z)\left(\widehat{M}_{\xi'} - \widehat{M}_0\right)(z)\widehat{U}_{\xi'}(z,s)u_0\,dz. \tag{5.25}$$

First let us estimate $|\widehat{U}_{\xi'}(t,s)u_0|_{H^1 \times L^2}$. We have

$$|\widehat{U}_{\xi'}(t,s)u_0|_{H^1 \times L^2} \le |\widehat{U}_0(t,s)u_0|_{H^1 \times L^2} + \int_s^t |\widehat{U}_0(t,z)\left(\widehat{M}_{\xi'} - \widehat{M}_0\right)(z)\widehat{U}_{\xi'}(z,s)u_0|_{H^1 \times L^2}\,dz.$$

Using estimate from Lemma 5.9 we have

$$\begin{aligned} |\widehat{U}_{\xi'}(t,s)u_0|_{H^1 \times L^2} &\le C|u_0|_{H^1 \times L^2} + C\int_s^t |(\widehat{M}_{\xi'} - \widehat{M}_0)(z)\widehat{U}_{\xi'}(z,s)u_0|_{H^1 \times L^2}\,dz \\ &\le C|u_0|_{H^1 \times L^2} + C\int_s^t |\xi'|(|\widehat{U}_{\xi'}(z,s)u_0|_{H^1 \times L^2} + |\partial_{x_n}\widetilde{Q}\widehat{U}_{\xi'}(z,s)u_0|_{L^2})\,dz. \end{aligned}$$

Using Gronwall inequality we obtain

$$|\widehat{U}_{\xi'}(t,s)u_0|_{H^1 \times L^2} \le Ce^{c_1(t-s)}\left(|u_0|_{H^1 \times L^2} + \int_s^t |\partial_{x_n}\widetilde{Q}\widehat{U}_{\xi'}(z,s)u_0|_2\,dz\right). \tag{5.26}$$

Now we estimate $|\partial_{x_n}\widetilde{Q}\widehat{U}_{\xi'}(t,s)u_0|_2$ analogously to above estimates by using Lemma 5.9 and (5.26), to obtain

$$|\partial_{x_n}\widetilde{Q}\widehat{U}_{\xi'}(t,s)u_0|_2 \le C(t-s)^{-\frac{1}{2}}|u_0|_{H^1 \times L^2},$$

for $0 < t - s \le 2T$. This together with (5.26) concludes the proof. □

Proof of Theorem 5.16. Expansion of $\widehat{\Pi}(\xi')$ and (5.23) are obtained using results in [7]. Estimate (5.22) is obtained as follows. If $j \ne 1$, then

$$\widehat{\Pi}^{(0)}\widehat{T}_j^{(1)}\widehat{S}u_0 = [Q_0\widehat{T}_j^{(1)}\widehat{S}u_0]u^{(0)} = \int_0^1\int_0^T Q_0\widehat{U}_0(T,s)\widehat{M}_j^{(1)}(s)\widehat{U}_0(s,0)\widehat{S}u_0\,ds\,dx_n u^{(0)}$$

$$= \int_0^T\int_0^1 i\gamma^2\rho_p Q_j\widehat{U}_0(s,0)\widehat{S}u_0\,dx_n\,ds\ u^{(0)} = \int_0^T\int_0^1 i\gamma^2\rho_p e^{-s\nu A}(e^{-T\nu A}-1)^{-1}w_0^j\,dx_n\,ds\ u^{(0)},$$

and therefore,

$$|\widehat{\Pi}^{(0)}\widehat{T}_j^{(1)}\widehat{S}u_0|_{H^1\times L^2} \le C\frac{\gamma^2}{\nu}|w_0^j|_2 \ \text{ for } j\neq 1.$$

If $j=1$, then

$$\widehat{\Pi}^{(0)}\widehat{T}_1^{(1)}\widehat{S}u_0 = \int_0^T\int_0^1 iv_p^1 Q_0\widehat{U}_0(s,0)\widehat{S}u_0 + i\gamma^2\rho_p Q_1\widehat{U}_0(s,0)\widehat{S}u_0\,dx_n\,ds\ u^{(0)}.$$

Therefore, we get

$$|\widehat{\Pi}^{(0)}\widehat{T}_1^{(1)}\widehat{S}u_0|_{H^1\times L^2} \le C(1+\gamma^2)|u_0|_{H^1\times L^2}.$$

We next estimate $\widehat{S}\widehat{T}_j^{(1)}\widehat{\Pi}^{(0)}u_0$. If $j\neq 1$, then

$$\widehat{S}\widehat{T}_j^{(1)}\widehat{\Pi}^{(0)}u_0 = \widehat{S}\widehat{T}_j^{(1)}[\phi_0]u^{(0)} = [\phi_0]\widehat{S}\int_0^T \widehat{U}_0(T,s)\widehat{M}_j^{(1)}(s)\widehat{U}_0(s,0)u^{(0)}\,ds$$

$$= [\phi_0]\widehat{S}\int_0^T \widehat{U}_0(T,s)\widehat{M}_j^{(1)}(s)\begin{pmatrix}\phi^{(0)}\\ \frac{1}{\gamma^2}w^{(0),1}(x_n,t)\boldsymbol{e}_1'\\ 0\end{pmatrix}ds$$

$$= [\phi_0]\widehat{S}\int_0^T \widehat{U}_0(T,s)\begin{pmatrix}0\\ i\alpha_0\boldsymbol{e}_j'\\ 0\end{pmatrix}ds = i[\phi_0]\int_0^T (e^{-T\nu A}-1)^{-1}e^{-(T-s)\nu A}\,ds\begin{pmatrix}0\\ \alpha_0\boldsymbol{e}_j'\\ 0\end{pmatrix}.$$

Therefore, we have

$$|\widehat{S}\widehat{T}_j^{(1)}\widehat{\Pi}^{(0)}u_0|_{H^1\times L^2} \le \frac{C}{\nu}|\phi_0|_1 \ \text{ for } j\neq 1.$$

In case $j=1$,

$$\widehat{S}\widehat{T}_1^{(1)}\widehat{\Pi}^{(0)}u_0 = i[\phi_0]\widehat{S}\int_0^T \widehat{U}_0(T,s)\begin{pmatrix}\phi^{(0)}v_p^1+\rho_p w^{(0),1}(x_n,t)\\ (\alpha_0+v_p^1\frac{1}{\gamma^2}w^{(0),1}(x_n,t))\boldsymbol{e}_1'\\ -\frac{\widetilde{\nu}}{\gamma^2\rho_p}\partial_{x_n}w^{(0),1}(x_n,t)\end{pmatrix}ds.$$

Therefore, we get following estimate

$$|\widehat{S}\widehat{T}_1^{(1)}\widehat{\Pi}^{(0)}u_0|_{H^1\times L^2} \le C(1+\frac{1}{\gamma^2}+\frac{\widetilde{\nu}}{\gamma^2})|\phi_0|_1.$$

Thus we proved (5.22). From (5.22) and (5.23) we immediately get following estimate

$$\left|\left(I-\widehat{\Pi}(\xi')\right)u_0\right|_{H^1\times L^2} \le C\left[1+|\xi'|\left(1+\gamma^2\right)+O(|\xi'|^2)\right]|u_0|_{H^1\times L^2}. \tag{5.27}$$

Now we concentrate on proving the estimate on spectral radius. From (4.38) we see that there exist constants $\nu_0>0$ and $\gamma_0>0$ such that if $\nu\ge\nu_0$ and $\gamma^2/(\nu+\widetilde{\nu})\ge\gamma_0^2$ then we have

$$|\widehat{U}_{\xi'}(t,s)u_0|_{H^1}^2 \le Ce^{-c_0|\xi'|^2(t-s-4T)}E_3(\widehat{u}_s(\xi',\cdot,s+4T)),$$

for $|\xi'|\le R_0$ and $t-s\ge 4T$, $s\ge 0$. Using Lemma 4.4 on $E_3(\widehat{u}_s(\xi',\cdot,s+4T))$ we get following estimate,

$$|\widehat{U}_{\xi'}(t,s)u_0|_{H^1\times L^2}^2 \le C|u_0|_{H^1\times L^2}^2. \tag{5.28}$$

Using (5.21), (5.22), (5.23) and (5.25) we get

$$\left|\left\{\left(I-\widehat{\Pi}(\xi')\right)\widehat{U}_{\xi'}(T)\right\}^m\right|_{L(X)} = \left|\left(I-\widehat{\Pi}(\xi')\right)\widehat{U}_{\xi'}(mT,0)\right|_{L(X)}$$

$$\le \left|\widehat{\Pi}_c^{(0)}\widehat{U}_{\xi'}(mT,0)\right|_{L(X)} + \left|\widehat{\Pi}^{(1)}(\xi')\widehat{U}_{\xi'}(mT,0)\right|_{L(X)} \le \left|\widehat{\Pi}_c^{(0)}\widehat{U}_0(mT,0)\right|_{L(X)}$$

$$+\left|\int_0^{mT}\widehat{\Pi}_c^{(0)}\widehat{U}_0(mT,z)\left(\widehat{M_{\xi'}}-\widehat{M_0}\right)(z)\widehat{U}_{\xi'}(z,0)\,dz\right|_{L(X)} + C|\xi'|\left|\widehat{U}_{\xi'}(mT,0)\right|_{L(X)}.$$

Combining this with (5.7) and (5.28) we obtain

$$\left|\left\{\left(I-\widehat{\Pi}(\xi')\right)\widehat{U}_{\xi'}(T)\right\}^m\right|_{L(X)} \le Ce^{-dmT}+C|\xi'|, \tag{5.29}$$

for any $m\in N$ and some $d>0$. Let us compute a spectral radius of $\widehat{U}_{\xi'}(T,0)$ on complementary space $(I-\widehat{\Pi}(\xi'))X$,

$$r\left(\widehat{U}_{\xi'}(T)|_{(I-\widehat{\Pi}(\xi'))X}\right) = r\left((I-\widehat{\Pi}(\xi'))\widehat{U}_{\xi'}(T)(I-\widehat{\Pi}(\xi'))\right)$$

$$\le \left|\left((I-\widehat{\Pi}(\xi'))\widehat{U}_{\xi'}(T)(I-\widehat{\Pi}(\xi'))\right)^m\right|_{L(X)}^{\frac{1}{m}}, \text{ for any } m\in N.$$

Fix $m_0\in N$ large and $r_0>0$ small such that for $|\xi'|\le r_0$ it holds for (5.29) that

$$Ce^{-dmT}+C|\xi'|\le e^{-T}.$$

Then we immediately see that

$$r\left(\widehat{U}_{\xi'}(T)|_{(I-\widehat{\Pi}(\xi'))X}\right)\le e^{-\frac{T}{m_0}}<1,$$

for all $|\xi'|\le r_0$. □

Proof of Theorem 5.1. Theorem 5.1 now follows from Lemma 5.9, Theorem 5.11, 5.16 and Lemma 5.17 in the following way. Let us fix positive numbers ν_0, γ_0 and $r_0<1$ such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ then for each ξ' with $|\xi'| \leq r_0$ all above mentioned statements 5.9–5.17 holds true. We also assume that $t - s \geq 4T$ and $k, l = 0, 1$.

As an immediate consequence of Theorem 5.11 we have following decomposition,

$$\widehat{U}_{\xi'}(T) = \widehat{U}_{\xi'}(T)\widehat{\Pi}(\xi') + \widehat{U}_{\xi'}(T)(I - \widehat{\Pi}(\xi')) = \widehat{\mu}(\xi')\widehat{\Pi}(\xi') + \widehat{U}_{\xi'}(T)(I - \widehat{\Pi}(\xi')), \tag{5.30}$$

where

$$\widehat{\mu}(\xi') = 1 + \lambda_0(\xi')T = e^{\lambda_0(\xi')T}.$$

We further write $\mathscr{V}_0(t,s)u_0$ as

$$\begin{aligned}\mathscr{V}_0(t,s)u_0 &= \sigma_{t,s}u^{(0)}(t) + \mathscr{U}_1^{(0)}(t,s)u_0 + \mathscr{U}_2^{(0)}(t,s)u_0 \\ &\quad + \mathscr{U}_3^{(0)}(t,s)u_0 + \mathscr{R}^{(0)}(t,s)u_0,\end{aligned}$$

where

$$\begin{aligned}
\sigma_{t,s}u^{(0)}(t) &= \mathscr{F}^{-1}\left(e^{-(i\kappa_0\xi_1 + \kappa_1|\xi'|^2)(t-s)}\widehat{\Pi}^{(0)}(t)\widehat{u}_0\right), \\
\mathscr{U}_1^{(0)}(t,s)u_0 &= \mathscr{F}^{-1}\left((\chi^{(0)} - 1)e^{-(i\kappa_0\xi_1 + \kappa_1|\xi'|^2)(t-s)}\widehat{\Pi}^{(0)}(t)\widehat{u}_0\right), \\
\mathscr{U}_2^{(0)}(t,s)u_0 &= \mathscr{F}^{-1}\left(\chi^{(0)}(e^{\lambda_0(\xi')(t-s)} - e^{-(i\kappa_0\xi_1 + \kappa_1|\xi'|^2)(t-s)})\widehat{\Pi}^{(0)}(t)\widehat{u}_0\right), \\
\mathscr{U}_3^{(0)}(t,s)u_0 &= \mathscr{F}^{-1}\left(\chi^{(0)}(\widehat{U}_{\xi'}(t, t-\tau_1)\widehat{\Pi}(\xi')\widehat{U}_{\xi'}(T)^m\widehat{U}_{\xi'}(s+\tau_2, s) - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t))\widehat{u}_0\right), \\
\mathscr{R}^{(0)}(t,s)u_0 &= \mathscr{F}^{-1}\left(\chi^{(0)}\widehat{U}_{\xi'}(t, t-\tau_1)(I - \widehat{\Pi}(\xi'))\widehat{U}_{\xi'}(T)^m\widehat{U}_{\xi'}(s+\tau_2, s)\widehat{u}_0\right).
\end{aligned}$$

Here $\mathscr{U}_3^{(0)}(t,s)u_0$ and $\mathscr{R}^{(0)}(t,s)\widehat{u}_0$ are given as follows. Since

$$\widehat{U}_{\xi'}(t+T, s+T) = \widehat{U}_{\xi'}(t,s),$$

if τ_1, τ_2, m is defined in such a way that $t - s = \tau_1 + mT + \tau_2$, where $t - \tau_1, s + \tau_2$ are integer multiples of T and $\tau_1, \tau_2 \in [T, 2T)$, we have

$$\begin{aligned}
&\mathscr{F}^{-1}\left(\chi^{(0)}(\widehat{U}_{\xi'}(t,s) - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t))\widehat{u}_0\right) \\
&= \mathscr{F}^{-1}\left(\chi^{(0)}(\widehat{U}_{\xi'}(t, t-\tau_1)\widehat{U}_{\xi'}(T)^m\widehat{U}_{\xi'}(s+\tau_2, s) - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t))\widehat{u}_0\right) \\
&= \mathscr{F}^{-1}\left(\chi^{(0)}(\widehat{U}_{\xi'}(t, t-\tau_1)\widehat{\Pi}(\xi')\widehat{U}_{\xi'}(T)^m\widehat{U}_{\xi'}(s+\tau_2, s) - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t))\widehat{u}_0\right) \\
&+ \mathscr{F}^{-1}\left(\chi^{(0)}\widehat{U}_{\xi'}(t, t-\tau_1)(I - \widehat{\Pi}(\xi'))\widehat{U}_{\xi'}(T)^m\widehat{U}_{\xi'}(s+\tau_2, s)\widehat{u}_0\right) = \mathscr{U}_3^{(0)}(t,s)u_0 + \mathscr{R}^{(0)}(t,s)u_0.
\end{aligned}$$

As for $\|\partial_{x'}^k\partial_{x_n}^l\mathscr{U}_3^{(0)}(t,s)u_0\|_2^2$, we have following estimates. Let us first introduce estimates in x_n. Let

$$I(\xi'; t, s) = \widehat{U}_{\xi'}(t, t-\tau_1)\widehat{\Pi}(\xi')\widehat{U}_{\xi'}(T)^m\widehat{U}_{\xi'}(s+\tau_2, s)\widehat{u}_0 - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t)\widehat{u}_0.$$

Since $\widehat{\Pi}(\xi')\widehat{U}_{\xi'}(T) = e^{\lambda_0(\xi')T}\widehat{\Pi}(\xi')$, we have

$$\begin{aligned} I(\xi';t,s) &= \widehat{U}_{\xi'}(t,t-\tau_1)e^{\lambda_0(\xi')mT}\widehat{\Pi}(\xi')\widehat{U}_{\xi'}(s+\tau_2,s)\widehat{u}_0 - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t)\widehat{u}_0 \\ &= I_1(\xi';t,s) + I_2(\xi';t,s), \end{aligned}$$

where

$$I_1(\xi';t,s) = \left(\widehat{U}_{\xi'}(t,t-\tau_1)e^{\lambda_0(\xi')mT}\widehat{\Pi}^{(0)}\widehat{U}_{\xi'}(s+\tau_2,s) - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t)\right)\widehat{u}_0,$$

$$I_2(\xi';t,s) = \widehat{U}_{\xi'}(t,t-\tau_1)e^{\lambda_0(\xi')mT}\widehat{\Pi}^{(1)}(\xi')\widehat{U}_{\xi'}(s+\tau_2,s)\widehat{u}_0.$$

Since $|(M_{\xi'} - M_0)(z)u|_{H^1\times L^2} \le C|\xi'|(|u|_{H^1\times L^2} + |\partial_{x_n} w|_2)$, applying (5.25) for $\widehat{U}_{\xi'}(t,t-\tau_1)$ and $\widehat{U}_{\xi'}(s+\tau_2,s)$, we have

$$\begin{aligned} |I_1(\xi';t,s)|_{H^1} \le{}& \left|\left(\widehat{U}_0(t,t-\tau_1)e^{\lambda_0(\xi')mT}\widehat{\Pi}^{(0)}\widehat{U}_0(s+\tau_2,s) - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t)\right)\widehat{u}_0\right|_{H^1} \\ &+C|e^{\lambda_0(\xi')mT}||\xi'||\widehat{u}_0|_{H^1\times L^2}. \end{aligned} \tag{5.31}$$

Let us consider the first term on the right of (5.31). By Lemma 4.6 (ii) we have

$$\widehat{\Pi}^{(0)}\widehat{U}_0(s+\tau_2,s)\widehat{u}_0 = [Q_0\widehat{U}_0(s+\tau_2,s)\widehat{u}_0]u^{(0)} = [\widehat{\phi}_0]u^{(0)}.$$

Since $t-\tau_1 = mT + s + \tau_2 = m'T$ for some $m' \in \mathbb{N}$, we see from Lemma 5.2

$$\widehat{U}_0(t,t-\tau_1)u^{(0)} = \widehat{U}_0(\tau_1,0)u^{(0)} = u^{(0)}(\tau_1).$$

On the other hand, since $u^{(0)}(t) = u^{(0)}(\tau_1)$, we have

$$e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t)\widehat{u}_0 = e^{\lambda_0(\xi')(t-s)}[\widehat{\phi}_0]u^{(0)}(t) = e^{\lambda_0(\xi')(t-s)}[\widehat{\phi}_0]u^{(0)}(\tau_1).$$

We thus obtain

$$\begin{aligned} &\left(\widehat{U}_0(t,t-\tau_1)e^{\lambda_0(\xi')mT}\widehat{\Pi}^{(0)}\widehat{U}_0(s+\tau_2,s) - e^{\lambda_0(\xi')(t-s)}\widehat{\Pi}^{(0)}(t)\right)\widehat{u}_0 \\ &\qquad = (e^{\lambda_0(\xi')mT} - e^{\lambda_0(\xi')(t-s)})[\widehat{\phi}_0]u^{(0)}(t). \end{aligned} \tag{5.32}$$

Since

$$\begin{aligned} e^{\lambda_0(\xi')mT} - e^{\lambda_0(\xi')(t-s)} &= e^{\lambda_0(\xi')mT}(1 - e^{\lambda_0(\xi')(\tau_1+\tau_2)}) \\ &= e^{\lambda_0(\xi')mT}\int_0^1 e^{\theta\lambda_0(\xi')(\tau_1+\tau_2)}\,d\theta\lambda_0(\xi')(\tau_1+\tau_2), \end{aligned}$$

we see from (5.31) and (5.32)

$$|I_1(\xi';t,s)|_{H^1} \le C|\xi'||e^{\lambda_0(\xi')mT}|(|\widehat{\phi}_0|_1 + |\widehat{u}_0|_{H^1\times L^2}).$$

Using estimates (5.22), (5.23) and (5.24) we obtain

$$|I_2(\xi';t,s)|_{H^1} \le C|\xi'|\ |e^{\lambda_0(\xi')mT}|\ |\widehat{u}_0|_{H^1\times L^2}.$$

Therefore we get

$$\|\partial_{x'}^k\partial_{x_n}^l \mathscr{U}_3^{(0)}(t,s)u_0\|_2^2 \le C(t-s)^{-\frac{n+1}{2}-k}(\|\phi_0\|_{L^1}^2 + \|u_0\|^2_{L^1(\mathbb{R}^{n-1};H^1(0,1)\times L^2(0,1))}).$$

As for $\|\partial_{x'}^k\partial_{x_n}^l \mathscr{U}_1^{(0)}(t,s)u_0\|_2^2$, since $\mathrm{supp}(\chi^{(0)}-1)\subset\{|\xi'|\ge r_0\}$, it is easy to see that

$$\|\partial_{x'}^k\partial_{x_n}^l \mathscr{U}_1^{(0)}(t,s)u_0\|_2^2 \le Ce^{-\kappa_1 r_0^2(t-s)}(t-s)^{-\frac{n-1}{2}-k}\|\phi_0\|^2_{L^1(\mathbb{R}^n)}.$$

As for $\|\partial_{x'}^k\partial_{x_n}^l \mathscr{U}_2^{(0)}(t,s)u_0\|_2^2$, using the mean value theorem $|1-e^f| = \int_0^1 |e^{\theta f}|\,d\theta|f|$, we have

$$\|\partial_{x'}^k\partial_{x_n}^l \mathscr{U}_2^{(0)}(t,s)u_0\|_2^2 \le \left\|\partial_{x'}^k\partial_{x_n}^l\chi^{(0)}e^{-(i\kappa_0\xi_1+\kappa_1|\xi'|^2)(t-s)}(e^{(\lambda_0(\xi')+i\kappa_0\xi_1+\kappa_1|\xi'|^2)(t-s)}-1)\widehat{\Pi}^{(0)}(t)\widehat{u}_0\right\|_2^2$$

$$\le C\int_{|\xi'|\le r_0} e^{-\kappa_1|\xi'|^2(t-s)}(t-s)^2|\xi'|^{6+2k}\,d\xi'\|\phi_0\|^2_{L^1(R^n)} \le C(t-s)^{-\frac{n+1}{2}-k}\|\phi_0\|^2_{L^1(\mathbb{R}^n)}.$$

As for $\|\partial_{x'}^k\partial_{x_n}^l\sigma_{t,s}u^{(0)}(t)\|_2^2$, it is straightforward to see that

$$\|\partial_{x'}^k\partial_{x_n}^l\sigma_{t,s}u^{(0)}(t)\|_2^2 \le C(t-s)^{-\frac{n-1}{2}-k}\|\phi_0\|^2_{L^1(\mathbb{R}^n)}.$$

So we have that

$$\mathscr{U}^{(0)}(t,s)u_0 = \sigma_{t,s}u^{(0)}(t) + \mathscr{U}_1^{(0)}(t,s)u_0 + \mathscr{U}_2^{(0)}(t,s)u_0 + \mathscr{U}_3^{(0)}(t,s)u_0.$$

As for $\|\mathscr{R}^{(0)}(t,s)u_0\|_{H^1}^2$, it is estimated as follows. Let

$$J(\xi';t,s) = \widehat{U}_{\xi'}(t,t-\tau_1)(I-\widehat{\Pi}(\xi'))\widehat{U}_{\xi'}(T,0)^m\widehat{U}_{\xi'}(s+\tau_2,s)\widehat{u}_0.$$

Using (5.24), we have

$$|J(\xi';t,s)|_{H^1} \le C\left|\Big((I-\widehat{\Pi}(\xi'))\widehat{U}_{\xi'}(T,0)\Big)^m\widehat{U}_{\xi'}(s+\tau_2,s)\widehat{u}_0\right|_{H^1\times L^2}.$$

Applying (5.29) with r_0 and m_0 from Theorem 5.16 we get

$$|J(\xi';t,s)|_{H^1} \le C\left|\Big((I-\widehat{\Pi}(\xi'))\widehat{U}_{\xi'}(T,0)\Big)^{m_0\frac{m}{m_0}}\right|_{L(X)}|\widehat{U}_{\xi'}(s+\tau_2,s)\widehat{u}_0|_{H^1\times L^2}$$

$$\le C\left|\Big((I-\widehat{\Pi}(\xi'))\widehat{U}_{\xi'}(T,0)\Big)^{m_0}\right|^{\frac{m}{m_0}}_{L(X)}|\widehat{U}_{\xi'}(s+\tau_2,s)\widehat{u}_0|_{H^1\times L^2}$$

$$\le Ce^{-\frac{1}{m_0}mT}|\widehat{U}_{\xi'}(s+\tau_2,s)\widehat{u}_0|_{H^1\times L^2}.$$

We employ (5.24) again to conclude

$$|J(\xi';t,s)|_{H^1} \le Ce^{-\frac{1}{m_0}mT}|\widehat{u}_0|_{H^1\times L^2} \le Ce^{-\frac{1}{m_0}(t-s)}|\widehat{u}_0|_{H^1\times L^2}.$$

Finally integrating in ξ' we get

$$\|\mathscr{R}^{(0)}(t,s)u_0\|_{H^1}^2 \le Ce^{-2d(t-s)}\int_{R^{n-1}}|\widehat{u}_0|^2_{H^1\times L^2}\,d\xi' = Ce^{-2d(t-s)}\|u_0\|^2_{H^1\times L^2},$$

where $d>0$, which concludes the proof. $\square$

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