Lie symmetry analysis of time fractional differential equations

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Lie symmetry analysis of time fractional differential equations

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Abstract

We study time fractional linear and nonlinear evolution systems with variable coefficients via Lie symmetry analysis. For both classes of the system, we give complete group classification and for linear time fractional evolutions systems, we give exact solutions corresponding to infinitesimal symmetries of optimal systems of Lie algebras generated by infinitesimal symmetries. For fractional nonlinear evolution system, we give explicit invariant solutions in some particular cases. The group invariant solutions are expressed in terms of special functions. More concretely, with the help of the infinitesimal symmetries we reduce the system of time fractional partial differential equations into a system of fractional ordinary differential equations which have Euler-type integer order differential operator up to second order. Even though finding exact solutions to fractional differential equations is not easy, we are able to give solutions to fractional differential equations with Euler-type integer order differential operator up to arbitrary high order.

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Summary

A symmetry of differential equation is a transformation that preserves the form of the equation. Although determining symmetries can be computationally intensive, Lie symmetry analysis is an efficient, algorithmic method for solving various types of differential equations. For example through Lie symmetry analysis, we can find invariant solutions to two dimensional partial differential equations by reducing them into the ordinary differential equations. Lie symmetry analysis is well developed for partial and ordinary differential equations. In last few decades, fractional differential equations have emerged in applications. The fractional derivatives are also known as integro-differential operators, and they characterize and model processes with long term memory rather than integer order derivatives which characterize local properties. Gazizov R. K. et. al. [Vest. UGATU **9** 3(21): 125-135 (2007)] presented formulas for extended infinitesimals of fractional ordinary and partial differential equations, and developed Lie symmetry analysis for fractional differential equations. Huang Q. et. al. [J. Math. Phys. **56** 123504 (2015)] first generalized Lie symmetry analysis of fractional differential equation into a system of fractional differential equations. Then, Singla K. et. al. [J. Math. Phys. **57** 101504 (2016)] tried to correct the formula for extended infinitesimals which were obtained in Huang's work.

We study time fractional linear and nonlinear evolution systems with variable coefficients via Lie symmetry analysis. To study the fractional systems we have obtained formulas for extended infinitesimals that match with the ones obtained neither by Huang nor by Singla. The linear time fractional evolution systems of our interest were studied before by Huang. Huang obtained only elementary monomial solutions and determined that there are three types of functions for the variable coefficient of time fractional linear evolution systems, so that the fractional linear evolution system admits symmetries. Here, we not only give complete group classification of invariant solutions but also exact solutions corresponding to infinitesimal symmetries of optimal systems of Lie algebras of infinitesimal symmetries. The group invariant solutions are expressed in terms of three kinds of special functions: the Mittag-Leffler functions, the

generalized Wright functions and the Fox H-functions. Which of these special functions we use in any given case depends on order of the fractional derivative and right hand sides of the equations in the fractional linear system. For fractional nonlinear evolution system, we give complete group classification along with explicit invariant solutions in some particular cases.

In briefly, our method to study fractional evolution systems is as follows: We reduce the system of time fractional partial differential equations into a system of fractional ordinary differential equations with the help of the infinitesimal symmetries, which were obtained by solving the equation of invariance surface condition. Then, we obtain group invariant solutions to the original system by finding solutions to the reduced systems. The reduced systems of fractional linear evolution systems become systems of fractional differential equations with Euler-type integer order differential operator. Even though finding exact solutions to fractional differential equations is not easy, we are able to give solutions to these reduced systems by showing some contiguity relations of special functions. More concretely, we reduce the problem of finding non-trivial solutions to Euler-type, linear fractional differential equations with higher order integer derivatives into a problem of finding roots of algebraic polynomials.

The linear and nonlinear fractional systems of our interest correspond to generalized time fractional diffusion-wave equations. In other words, for special values of order of fractional derivative the time fractional evolution systems correspond to the well-known diffusion or wave equations. Moreover, we show that the invariant solutions correspond to known solutions of diffusion and wave equations after applying some transformations. Also, Buckwar E. et. al. [J. Math. Anal. Appl. **227** (1998) etc.] studied time fractional diffusion-wave equations with constant coefficient via scaling symmetry and when α is between 0 and 1, Metzler R. et. al. [Physica A **211** (1994)] gave solutions to time fractional diffusion-wave equations with variable coefficient using fractional Laplacian transformations. For particular values of parameters, the solutions that we obtained correspond to these solutions as well.

Chapter 1

Introduction

1.1 Fractional derivative

Here we explain the derivatives of arbitrary real order, which unify and generalize the notions of integer order differentiation. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [24].

The following definitions and elementary properties are taken from [20]. The functions $f(t)$ that we can take fractional derivatives are defined on the closed interval $0 \le t \le T$, bounded everywhere in the half open interval $0 < t \le T$ and have better behavior at the lower limit 0 than has t^{-1}, which means:

$$\lim_{t \to 0} t f(t) = 0.$$

We can also take fractional derivatives from so-called differintegrable series:

$$f(t) = t^p \sum_{j=0}^{\infty} a_j t^{\frac{j}{n}}, \quad a_0 \neq 0, \quad p > -1, \quad n \in \mathbb{Z}_+.$$

Notice that p has been chosen to ensure that the leading coefficient is nonzero. Most of the special functions of mathematical physics are differintegrable series according to this definition.

1.1.1 Definitions of fractional derivatives

In the literature, there exist many approaches and various definitions. But the most common ones are: the Grünwald-Letnikov derivative, the Caputo derivative and the

Riemann-Liouville derivative [24]. From here on, the value of $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is determined by

$$n - 1 < \alpha < n.$$

The Grünwald-Letnikov derivative:

We know the higher order integer derivatives are determined as

$$\frac{d^n}{dt^n} f(t) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - kh).$$

So, we generalize the order n to positive real α :

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha}_{GL} f(t) &:= \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t - kh) \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0) t^{k-\alpha}}{\Gamma(k - \alpha + 1)} + \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \end{aligned}$$

where $f^{(k)}(t) \in C[0, t]$ for $k = 1, 2, \ldots, n$ and

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}$$

The Riemann-Liouville derivative:

$$\frac{d^\alpha}{dt^\alpha}_{RL} f(t) := \begin{cases} \frac{d^n}{dt^n} f(t), & \text{for } \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau, & \text{for } \alpha \in (n - 1, n) \text{ with } n \in \mathbb{N}. \end{cases} \tag{1.1}$$

The Caputo derivative:

$$\frac{d^\alpha}{dt^\alpha}_{C} f(t) := \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau.$$

These definitions correspond to each other in certain cases. We can see that if $f^{(k)}(0) = 0$ for $k = 0, \ldots, n - 1$ then the Grünwald-Letnikov derivative corresponds to the Caputo derivative. Also, if we take an assumption that the function $f(t)$ is n times continuously differentiable, then integrating by parts and differentiating (1.1) we come at the definition of Grünwald-Letnikov derivative:

$$\begin{aligned}
\frac{d^\alpha}{dt^\alpha}_{RL} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t f(\tau)(t-\tau)^{n-\alpha-1} d\tau \\
&= \frac{1}{\Gamma(n-\alpha+1)} \frac{d^n}{dt^n} \int_0^t -f(\tau) d(t-\tau)^{n-\alpha} \\
&= \frac{1}{\Gamma(n-\alpha+1)} \frac{d^n}{dt^n} \left(-f(\tau)(t-\tau)^{n-\alpha} \Big|_0^t + \int_0^t f'(\tau)(t-\tau)^{n-\alpha} d\tau \right) \\
&= \frac{f(0)}{\Gamma(n-\alpha+1)} \frac{d^n}{dt^n} t^{n-\alpha} + \frac{1}{\Gamma(n-\alpha+1)} \frac{d^n}{dt^n} \int_0^t f'(\tau)(t-\tau)^{n-\alpha} d\tau \\
&= \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(n-\alpha+1)} \frac{d^n}{dt^n} \int_0^t f'(\tau)(t-\tau)^{n-\alpha} d\tau. \qquad (1.2)
\end{aligned}$$

Interchanging the order of differentiation and integration [24]

$$\frac{d}{dt} \int_0^t F(t,\tau) d\tau = \int_0^t \frac{\partial F(t,\tau)}{\partial t} d\tau + F(t, t-0)$$

in (1.2), it equals to

$$\begin{aligned}
\frac{d^\alpha}{dt^\alpha}_{RL} f(t) &= \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(n-\alpha)} \frac{d^{n-1}}{dt^{n-1}} \int_0^t f'(\tau)(t-\tau)^{n-\alpha-1} d\tau \\
&= \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{f'(0)}{\Gamma(n-\alpha+1)} \frac{d^{n-1}}{dt^{n-1}} t^{n-\alpha} \\
&\quad + \frac{1}{\Gamma(n-\alpha+1)} \frac{d^{n-1}}{dt^{n-1}} \int_0^t f'(\tau)(t-\tau)^{n-\alpha} d\tau \\
&= \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{f'(0)t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(n-\alpha)} \frac{d^{n-2}}{dt^{n-2}} \int_0^t f''(\tau)(t-\tau)^{n-\alpha-1} d\tau \\
&\vdots \\
&= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)t^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \\
&= \frac{d^\alpha}{dt^\alpha}_{GL} f(t),
\end{aligned}$$

since $\lim_{\tau \to t-0} (t-\tau)^{n-\alpha} = 0$.

In this work, we adopt the Riemann-Liouville derivative

$$\frac{d^\alpha}{dt^\alpha} \equiv \frac{d^\alpha}{dt^\alpha}_{RL}.$$

A major difference between the Riemann-Liouville derivative and the Caputo derivative is that the Caputo derivative of a constant is 0, whereas the Riemann-Liouville derivative of a constant is non-zero, as will be seen in the next section.

1.1.2 Fractional derivative of power functions

Let us take fractional derivative of function t^p :

$$\frac{d^\alpha}{dt^\alpha}t^p = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}\tau^p d\tau.$$

Substituting $\tau = tu$ into the above expression, we obtain

$$\begin{aligned}\frac{d^\alpha}{dt^\alpha}t^p &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^1 (t-tu)^{n-\alpha-1}t^{p+1}u^p du \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left[t^{n-\alpha-1+p+1}\int_0^1 (1-u)^{n-\alpha-1}u^p du\right].\end{aligned}$$

We can use the beta function formula for $p > -1$:

$$\begin{aligned}\frac{d^\alpha}{dt^\alpha}t^p &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left[t^{n-\alpha+p}B(p+1,n-\alpha)\right] \\ &= \frac{d^n}{dt^n}\frac{t^{n-\alpha+p}}{\Gamma(n-\alpha)}\frac{\Gamma(p+1)\Gamma(n-\alpha)}{\Gamma(n-\alpha+p+1)} \\ &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha+p+1)}\frac{d^n}{dt^n}t^{n-\alpha+p} \\ &= \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}t^{p-\alpha}. \qquad (1.3)\end{aligned}$$

As a corollary, for $p = 0$, we have a fractional derivative of the unit function:

$$\frac{d^\alpha}{dt^\alpha}1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},$$

then for any constant function including zero, we have:

$$\frac{d^\alpha}{dt^\alpha}C = C\frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

1.1.3 Elementary properties of fractional derivative

The following properties are useful in our calculation.

Linearity

The linearity of the fractional derivative follows directly from the definition:

$$\frac{d^\alpha}{dt^\alpha}\sum_{j=0}^{N} C_j f_j(t) = \sum_{j=0}^{N} C_j \frac{d^\alpha}{dt^\alpha} f_j(t) \quad \text{for } N \in \mathbb{N}.$$

Taking derivative from a series

If the series $\sum f_j(t)$ as well as the series $\sum \frac{d^\alpha}{dt^\alpha} f_j(t)$ converge uniformly in $0 < t < T$, then

$$\frac{d^\alpha}{dt^\alpha}\sum_{j=0}^{\infty} f_j(t) = \sum_{j=0}^{\infty} \frac{d^\alpha}{dt^\alpha} f_j(t) \text{ for } 0 < t < T.$$

Scale change of the Riemann-Liouville derivative

For $\beta \in \mathbb{R}$, we have the following property:

$$\begin{aligned}
\frac{d^\alpha}{dt^\alpha} f(\beta t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\beta\tau)d\tau \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^{\beta t}\left(t-\frac{T}{\beta}\right)^{n-\alpha-1} f(T)\frac{1}{\beta}dT \\
&= \frac{\beta^{\alpha-n}}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^{\beta t}(\beta t - T)^{n-\alpha-1} f(T)dT \\
&= \frac{\beta^{\alpha}}{\Gamma(n-\alpha)}\frac{d^n}{d(\beta t)^n}\int_0^{\beta t}(\beta t - T)^{n-\alpha-1} f(T)dT \\
&= \beta^\alpha \frac{d^\alpha}{d(\beta t)^\alpha} f(\beta t).
\end{aligned}$$

Generalized Leibniz's rule

The fractional differentiation rule for product of two functions is

$$\frac{d^\alpha}{dt^\alpha}(f(t)g(t)) = \sum_{j=0}^{\infty}\binom{\alpha}{j}\frac{d^{\alpha-j}}{dt^{\alpha-j}} f(t)\frac{d^j}{dt^j} g(t). \tag{1.4}$$

Composition rule or Sequential derivative

$$\frac{\partial^\alpha}{\partial t^\beta}\frac{\partial^\beta}{\partial t^\beta} f(t) = \frac{\partial^{\alpha+\beta}}{\partial t^{\alpha+\beta}} f(t) - \sum_{j=1}^{n}\left[\frac{\partial^{\beta-j}}{\partial t^{\beta-j}} f(t)\right]\Bigg|_{t=0}\frac{t^{m-\alpha-j}}{\Gamma(1+m-\alpha-j)} \tag{1.5}$$

here $m-1 \leq \alpha < m$, $n-1 \leq \beta < n$.

1.2 Special functions

The calculus of Newton and Leibniz and the analytic functions that solve the differential equations were seen necessary and sufficient to provide a proper and complete mechanical description of the physical world. However the increased sensitivity of experimental tools, vast amount of data and high computational capabilities lead to complex systems which are not fully described by classical mathematical models [31]. To describe and study those models we need the so-called special functions, some of which are introduced below.

The Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}$$

is defined for $z \in \mathbb{C}$ and for $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ [8]. The fractional derivative of product of Mittag-Leffler function and power function is

$$\frac{d^\alpha}{dt^\alpha} t^{\beta-1} E_{\mu,\beta}(\lambda t^\mu) = t^{\beta-\alpha-1} E_{\mu,\beta-\alpha}(\lambda t^\mu)$$

for $\beta > 0$ and $\mu > 0$ [24]. Furthermore, the fractional derivative of exponential function is expressed in Mittag-Leffler function

$$\frac{d^\alpha}{dt^\alpha} e^{\lambda t} = t^{-\alpha} E_{1,1-\alpha}(\lambda t), \text{ for } \lambda \in \mathbb{R}.$$

The Gauss hypergeometric function

$${}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+k)} \frac{z^k}{k!}$$

is defined for $|z| < 1$. The fractional derivative of product of Gauss hypergeometric function and power function is

$$\frac{d^\alpha}{dt^\alpha} t^{\gamma-1} 2F_1\left(\begin{matrix} \mu, \beta \\ \gamma \end{matrix} ; \lambda t \right) = \frac{\Gamma(\gamma) t^{\gamma-\alpha+1}}{\Gamma(\gamma-\alpha)} {}_2F_1\left(\begin{matrix} \mu, \beta \\ \gamma - \alpha \end{matrix} ; \lambda t \right)$$

for $\Re(\gamma) > 0$.

The Wright function

$$\Psi(z;\alpha,\beta)=\sum_{i=0}^{\infty}\frac{z^i}{i!\Gamma(\alpha i+\beta)}$$

is defined for $z\in\mathbb{C}$ and for real α satisfying $\alpha>-1$ and $\beta\in\mathbb{C}$ [9].

The generalized Wright function

$${}_p\Psi_q\left[z\left|\begin{array}{l}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right.\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(A_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(B_j+\beta_j k)}\frac{z^k}{k!}$$

is defined for $z\in\mathbb{C}$, $p,q\in\mathbb{N}_0=\{0,1,2,\ldots\}$, $A_i,B_j\in\mathbb{C}$ and $\alpha_i,\beta_j\in\mathbb{R}\setminus\{0\}$ $(i=1,\ldots,p; j=1,\ldots,q)$. The generalized Wright function is absolutely convergent, and thus it is an entire function for $\Delta=\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1$ [15].

The Fox H-function

$$H^{m,l}_{p,q}\left[z\left|\begin{array}{l}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(B_j-\beta_j s)\prod_{i=1}^{l}\Gamma(1-A_i+\alpha_i s)}{\prod_{i=l+1}^{p}\Gamma(A_i-\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-B_j+\beta_j s)}z^s ds,$$

is defined for $z\in\mathbb{C}\backslash\{0\}$, $m,l,p,q\in\mathbb{N}_0$ with $(m,l)\neq(0,0)$, $\alpha_i,\beta_j\in\mathbb{R}_+$ and $A_i,B_j\in\mathbb{R}$ $(i=1,\ldots,p; j=1,\ldots,q)$. If there exists any empty product in the above expression, then it is taken to be 1. The contour L separates the poles of the gamma functions $\Gamma(B_j-\beta_j s)$ $(j=1,\ldots,m)$ from the poles of the gamma functions $\Gamma(1-A_i+\alpha_i s)$ $(i=1,\ldots,l)$. In this work, we take L as $L_{\gamma+i\infty}$, a contour that extends from the point $\gamma-i\infty$ to the point $\gamma+i\infty$, where γ is chosen such that L separates the poles as stated above (see Figure 1.1). The above integral converges under the conditions [18]

$$\mu=\sum_{i=1}^{l}\alpha_i-\sum_{i=l+1}^{p}\alpha_i+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j>0 \text{ and } |\arg z|<\frac{\pi\mu}{2}.$$

With regard to expressions for solutions of fractional differential equations, we are particularly interested in the case $l=0$ of the H-function. In this case, the H-function vanishes exponentially for large z as [17]

$$H^{m,0}_{p,q}[z]\approx O\left(\exp\left(-\nu z^{\frac{1}{\nu}}\epsilon^{\frac{1}{\nu}}\right)z^{\frac{2\delta+1}{2\nu}}\right), \tag{1.6}$$

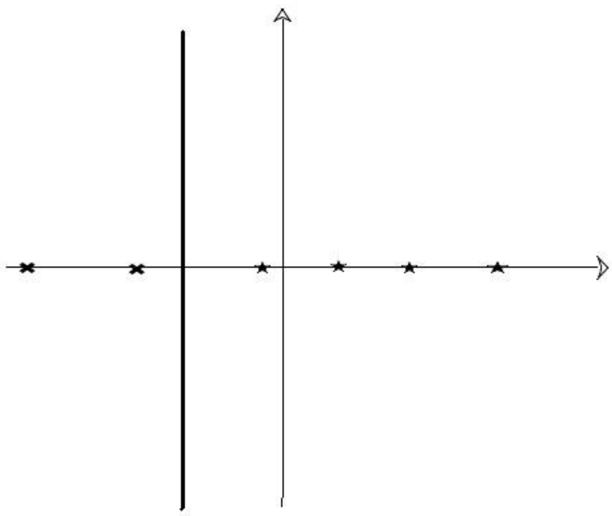


Fig. 1.1 The Fox H-function contour

where

$$\epsilon = \prod_{i=1}^{p} (\alpha_i)^{\alpha_i} \prod_{j=1}^{q} (\beta_j)^{-\beta_j}, \qquad \delta = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i + \frac{p-q}{2}$$

and

$$\nu = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > 0. \tag{1.7}$$

The following identities for H-functions are known to hold for $z > 0$ [18]:

$$H_{p,q}^{m,l}\left[z \middle| \begin{matrix} (A_i, \alpha_i)_{1,p} \\ (B_j, \beta_j)_{1,q} \end{matrix}\right] = H_{q,p}^{l,m}\left[\frac{1}{z} \middle| \begin{matrix} (1-B_j, \beta_j)_{1,q} \\ (1-A_i, \alpha_i)_{1,p} \end{matrix}\right], \tag{1.8}$$

$$H_{p,q}^{m,l}\left[z \middle| \begin{matrix} (A_i, \alpha_i)_{1,p} \\ (B_j, \beta_j)_{1,q} \end{matrix}\right] = kH_{p,q}^{m,l}\left[z^k \middle| \begin{matrix} (A_i, k\alpha_i)_{1,p} \\ (B_j, k\beta_j)_{1,q} \end{matrix}\right] \text{ for } k > 0, \tag{1.9}$$

$$z^{\sigma} H_{p,q}^{m,l}\left[z \middle| \begin{matrix} (A_i, \alpha_i)_{1,p} \\ (B_j, \beta_j)_{1,q} \end{matrix}\right] = H_{p,q}^{m,l}\left[z \middle| \begin{matrix} (A_i + \sigma\alpha_i, \alpha_i)_{1,p} \\ (B_j + \sigma\beta_j, \beta_j)_{1,q} \end{matrix}\right] \text{ for } \sigma \in \mathbb{C}. \tag{1.10}$$

The higher order derivative of product of power and H-function is also known [18]:

$$\frac{d^N}{dz^N}\left(z^{\rho-1}H^{m,l}_{p,q}\left[az^{\sigma}\middle|\begin{array}{c}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right]\right)$$
$$= z^{\rho-N-1}H^{m,l+1}_{p+1,q+1}\left[az^{\sigma}\middle|\begin{array}{c}(1-\rho,\sigma),(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q},(1-\rho+N,\sigma)\end{array}\right], \quad (1.11)$$

where $N\in\mathbb{N},\ a,\rho,\sigma\in\mathbb{C}$ and $\Re(\sigma)>0$.

Moreover, the following relations hold among the above special functions:

$$E_{1,1}(z) = e^z, \quad (1.12)$$

$${}_2F_1\left(\begin{array}{c}a,1\\1\end{array};z\right) = (1-z)^{-a}, \qquad \text{for } |z|<1, \quad (1.13)$$

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[z\middle|\begin{array}{c}(1,1)\\(\beta,\alpha)\end{array}\right], \quad (1.14)$$

$$E_{2,1}(z^2) = \frac{e^z+e^{-z}}{2} \quad (1.15)$$

$$E_{2,2}(z^2) = \frac{e^z-e^{-z}}{2z} \quad (1.16)$$

$$\Psi\left(z;\alpha,\beta\right) = {}_0\Psi_1\left[z\middle|\begin{array}{c}-\\(\beta,\alpha)\end{array}\right], \quad (1.17)$$

$${}_2\Psi_1\left[z\middle|\begin{array}{c}(A_1,1),(A_2,1)\\(B_1,1)\end{array}\right] = \frac{\Gamma(A_1)\Gamma(A_2)}{\Gamma(B_1)}{}_2F_1\left(\begin{array}{c}A_1,A_2\\B_1\end{array};z\right) \text{ for } |z|<1, \quad (1.18)$$

$${}_3\Psi_1\left[z\middle|\begin{array}{c}(A_1,1),(A_2,1),(1,1)\\(1,2)\end{array}\right] = \Gamma(A_1)\Gamma(A_2){}_2F_1\left(\begin{array}{c}A_1,A_2\\\frac{1}{2}\end{array};\frac{z}{4}\right), \text{ for } |z|<4 \quad (1.19)$$

$${}_3\Psi_1\left[z\middle|\begin{array}{c}(A_1,1),(A_2,1),(1,1)\\(2,2)\end{array}\right] = \Gamma(A_1)\Gamma(A_2){}_2F_1\left(\begin{array}{c}A_1,A_2\\\frac{3}{2}\end{array};\frac{z}{4}\right), \text{ for } |z|<4 \quad (1.20)$$

Also, the generalized Wright functions can be expressed in terms of Fox H-functions as

$${}_p\Psi_q\left[z\middle|\begin{array}{c}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right] = \begin{cases} H^{1,p}_{p,q+1}\left[-z\middle|\begin{array}{c}(1-A_i,\alpha_i)_{1,p}\\(0,1),(1-B_1,\beta_1),(1-B_j,\beta_j)_{2,q}\end{array}\right], \text{ for } \beta_1>0\\ H^{1,p}_{p+1,q}\left[-z\middle|\begin{array}{c}(1-A_i,\alpha_i)_{1,p},(B_1,-\beta_1)\\(0,1),(1-B_j,\beta_j)_{2,q}\end{array}\right], \text{ for } -1<\beta_1<0,\end{cases} \quad (1.21)$$

where α_i $(i = 1, \ldots, p)$ and β_j $(j = 2, \ldots, q)$ are positive real numbers [16].

Note: Here we use conventional notations for Wright functions and generalized Wright functions, without considering the consistency of this work.

1.3 Basics of Lie symmetry analysis for differential equations

In last five decades, there appear many works using Lie point symmetry methods to exploit the invariance property of differential equations. The Lie symmetry method is algorithmic and can be used to various types of differential equations. Here we briefly present some main points about the Lie symmetry analysis for partial differential equations (PDEs) [12, 13, 22], then generalize it to analyze fractional partial differential equations (FPDEs) [6, 7] and systems thereof [10, 26, 27].

1.3.1 Main concepts of Lie symmetry analysis of PDE

The transformation group G is a collection of invertible transformations $\bar{y} = T(y)$, $y \in \mathbb{R}^n$, satisfying the following conditions:

1. G contains an identity transformation $I : I(y) = y$.
2. G contains the inverse transformation of any transformation $T \in G$.
3. G contains the product T_2T_1 of any $T_1, T_2 \in G$.

Let us consider a one-parameter transformation group G of transformations T_a :

$$\bar{x} = T^1(x, t, u, a), \quad \bar{t} = T^2(x, t, u, a), \quad \bar{u} = T^3(x, t, u, a),$$

where the functions $T^i(x, t, u, a)$ are defined in a neighborhood of $a = 0$ and satisfy the conditions

$$T^1(x, t, u, 0) = x, \quad T^2(x, t, u, 0) = t, \quad T^3(x, t, u, 0) = u.$$

Expanding the functions $T^i(x, t, u, a)$ into the Taylor series in the group parameter a in a neighborhood of $a = 0$ and neglecting the terms of order $O(a^2)$ then using the above initial condition, we arrive at the following infinitesimal transformation of the group G :

$$\bar{x} \approx x + a\xi(x,t,u), \quad \bar{t} \approx t + a\tau(x,t,u), \quad \bar{u} \approx u + a\eta(x,t,u), \tag{1.22}$$

where

$$\begin{aligned} \xi(x,t,u) &= \left.\frac{\partial T^1(x,t,u,a)}{\partial a}\right|_{a=0}, \\ \tau(x,t,u) &= \left.\frac{\partial T^2(x,t,u,a)}{\partial a}\right|_{a=0}, \\ \eta(x,t,u) &= \left.\frac{\partial T^3(x,t,u,a)}{\partial a}\right|_{a=0}. \end{aligned}$$

The functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ are called infinitesimals and can serve as tangent vector field of the group G. This tangent vector field is often written as a first-order linear differential operator

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}.$$

The operator X is called the infinitesimal generator of the one-parameter group G.

The transformations (1.22) of the group G generated by X are found by solving the Lie equations

$$\frac{d\bar{x}}{da} = \xi(\bar{x},\bar{t},\bar{u}), \quad \frac{d\bar{t}}{da} = \tau(\bar{x},\bar{t},\bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{x},\bar{t},\bar{u}),$$

with the initial conditions

$$\bar{x}|_{a=0} = x, \quad \bar{t}|_{a=0} = t, \quad \bar{u}|_{a=0} = u.$$

Definition 1 ([12])**.** *A function $F(x,t,u)$ is an invariant of the group G of transformations T_a if $F(\bar{x},\bar{t},\bar{u}) = F(x,t,u)$.*

Theorem 1 ([12])**.** *A function $F(x,t,u)$ is an invariant of the group G generated by infinitesimal X if and only if it solves the following first-order linear PDE:*

$$XF \equiv \xi(x,t,u)\frac{\partial F}{\partial x} + \tau(x,t,u)\frac{\partial F}{\partial t} + \eta(x,t,u)\frac{\partial F}{\partial u} = 0.$$

We can also talk about the transformation of derivatives

$$\begin{aligned}
\bar{u}_{\bar{t}} &\approx u_t + \epsilon\mu^{(0)}(x,t,u,u_t,u_x),\\
\bar{u}_{\bar{x}} &\approx u_x + \epsilon\mu^{(1)}(x,t,u,u_t,u_x),\\
\bar{u}_{\bar{x}\bar{x}} &\approx u_{xx} + \epsilon\mu^{(2)}(x,t,u,u_t,u_x,u_{xt},u_{xx}),\\
&\vdots
\end{aligned} \tag{1.23}$$

where subscripts denote partial derivatives. The $\mu^{(i)}$ $(i = 0, 1, \dots)$ are extended infinitesimals and are well known

$$\begin{aligned}
\mu^{(0)} &= D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau),\\
\mu^{(1)} &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
\mu^{(2)} &= D_x(\mu^{(1)}) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau),\\
&\vdots
\end{aligned} \tag{1.24}$$

Here D_x and D_t are the total derivative operators defined as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + \dots\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \dots.
\end{aligned}$$

Starting from the group G of transformations (1.22) and then adding the transformations (1.23), one obtains the prolonged group $G_{(n)}$, which acts on the space of $n+4$ variables $(x,t,u,u_t,u_x,u_{xx},\dots,u_{x^n})$. The generators of prolonged groups are

$$\begin{aligned}
X_{(1)} &= X + \mu^{(0)}\frac{\partial}{\partial u_t} + \mu^{(1)}\frac{\partial}{\partial u_x},\\
X_{(2)} &= X^{(1)} + \mu^{(2)}\frac{\partial}{\partial u_{xx}},\\
&\vdots
\end{aligned}$$

Definition 2 ([12])**.** *A group $G_{(n)}$ of transformations (1.22), (1.23) is a symmetry group of n-th order PDE*

$$u_t = F(x,t,u,u_x,u_{xx},\dots,u_{x^n}), \tag{1.25}$$

if it conserves the form of the equation (1.25).

From this definition we can see that the transformations of the symmetry group G map every solution of (1.25) into a solution of the same equation. To determine the infinitesimals, we need to solve the following determining equation

$$X_{(n)}(u_t - F(x, t, u, u_x, \dots, u_{x^n}))|_{(1.25)} = 0, \tag{1.26}$$

which is derived from

$$\bar{u}_{\bar{t}} - F(\bar{x}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \dots, \bar{u}_{\bar{x}^n}) = u_t - F(x, t, u, u_x, \dots, u_{x^n}).$$

Since the equation (1.26) contains the derivatives u_t, $u_x, \dots, u_{x^n}$ of function $u(x,t)$ considered as an independent variable along with x and t, the determining equation is split into several independent equations becoming an overdetermined system of differential equations for the infinitesimals and extended infinitesimals. The set of all solutions to the determining equation is a Lie algebra, i.e., it is closed with respect to the commutator. In other words, if X, X' are solutions to the determining equation, then the commutator

$$[X, X'] = X(X') - X'(X)$$

is also a solution to (1.26).

If a group transformation maps a solution into itself, we arrive at group invariant solutions. Given a group that leaves a PDE invariant, one desires to minimize the search for group-invariant solutions to that of finding inequivalent branches of solutions, that is to say to give them a classification, which leads to the concept of the optimal systems. Consequently, the problem of determining the optimal system of subgroups is reduced to the corresponding problem for subalgebras. In applications, one usually constructs the optimal system of subalgebras, from which the optimal systems of subgroup and group invariant solutions are reconstructed. The invariant solutions of (1.25) corresponding to any infinitesimal symmetry can be obtained using Lie symmetry transformations applied to the invariant solutions corresponding to the infinitesimal symmetries of any optimal system of one-dimensional subalgebras of infinitesimal symmetries [22]. The optimal systems of low-dimensional Lie algebras are determined in [23]. For this reason, we are only interested in the invariant solutions corresponding to the infinitesimal symmetries of the optimal system.

One can find more information on Lie methods and its application to differential equation in [2, 12, 13, 22].

1.3.2 Lie symmetry analysis of FPDE

In [5–7] were generalized Lie symmetry analysis methods to fractional differential equations. Here we carry out the basic formulas for Lie symmetry analysis of FPDEs analogously to the original work of [6].

The time fractional PDE with two independent variables in a general form is

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F(x,t,u,u_x,u_{xx},\ldots), \qquad \alpha > 0. \tag{1.27}$$

Consider a one-parameter Lie group of infinitesimal transformations (1.22) along with

$$\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\mu^{(\alpha)} + O(\epsilon^2), \tag{1.28}$$

where $\mu^{(\alpha)}$ is also an extended infinitesimal. The transformation (1.22), (1.23) with (1.28) conserves the structure of (1.27), hence the invariance condition is

$$\tau(x,t,u)|_{t=0} = 0. \tag{1.29}$$

In the following calculation we use the notation

$$\bar{u}_{\bar{t}^n} \approx u_{t^n} + \epsilon\mu_t^{(n)}.$$

The formula of αth order extended infinitesimal $\mu^{(\alpha)}$ was obtained in [6] for FODEs in detail and the formula for FPDEs was presented. Here, we show the detailed computation of the αth order extended infinitesimal for FPDE in detail. Using the generalized Leibniz rule (1.4), we have

$$\frac{\partial^\alpha \bar{u}(\bar{x},\bar{t})}{\partial \bar{t}^\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\bar{t}^{n-\alpha}}{\Gamma(n-\alpha+1)} \bar{u}_{\bar{t}^n}(\bar{x},\bar{t}).$$

We get the αth order extended infinitesimal in the following manner

$$\begin{aligned} \mu^{(\alpha)} &= \frac{d}{d\epsilon}\left(\frac{\partial^\alpha \bar{u}(\bar{x},\bar{t})}{\partial \bar{t}^\alpha}\right)\Bigg|_{\epsilon=0} \\ &= \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)} \mu_t^{(n)} + \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{n-\alpha}{\Gamma(n-\alpha+1)} t^{n-\alpha-1} \tau u_{t^n}. \end{aligned}$$

Substituting the formula (8.25) of [11]

$$\mu_t^{(n)} = D_t^n(\eta - \xi u_x - \tau u_t) + \tau u_{t^{n+1}} + \xi u_{xt^n}$$

into the last expression, we get

$$\begin{aligned}
\mu^{(\alpha)} &= \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}D_t^n(\eta-\xi u_x-\tau u_t)}{\Gamma(n-\alpha+1)}+\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}\tau u_{t^{n+1}}}{\Gamma(n-\alpha+1)}\\
&\quad+\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}\xi u_{xt^n}}{\Gamma(n-\alpha+1)}+\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{(n-\alpha)t^{n-\alpha-1}\tau u_{t^n}}{\Gamma(n-\alpha+1)}\\
&= D_t^{\alpha}(\eta-\xi u_x-\tau u_t)+\sum_{n=1}^{\infty}\binom{\alpha}{n-1}\frac{t^{n-\alpha-1}\tau u_{t^n}}{\Gamma(n-\alpha)}\\
&\quad+\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}\xi u_{xt^n}}{\Gamma(n-\alpha+1)}+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{(n-\alpha)t^{n-\alpha-1}\tau u_{t^n}}{\Gamma(n-\alpha+1)}-\frac{\alpha t^{-\alpha-1}\tau u}{\Gamma(1-\alpha)}.
\end{aligned}$$

Using the identity

$$\binom{\alpha}{n-1}+\binom{\alpha}{n}=\binom{\alpha+1}{n}$$

into the last expression, we get

$$\begin{aligned}
\mu^{(\alpha)} &= D_t^{\alpha}(\eta-\xi u_x-\tau u_t)+\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}\xi u_{xt^n}}{\Gamma(n-\alpha+1)}+\sum_{n=1}^{\infty}\binom{\alpha+1}{n}\frac{t^{n-\alpha-1}\tau u_{t^n}}{\Gamma(n-\alpha+1)}\\
&= D_t^{\alpha}(\eta-\xi u_x-\tau u_t)+\xi D_t^{\alpha}(u_x)+\tau D_t^{\alpha+1}(u)\\
&= D_t^{\alpha}(\eta)-D_t^{\alpha}(\xi u_x)-D_t^{\alpha}(\tau u_t)+\xi D_t^{\alpha}(u_x)+\tau D_t^{\alpha+1}(u).
\end{aligned}$$

Hence, the αth order extended infinitesimal $\mu^{(\alpha)}$ has the following form [5–7]

$$\mu^{(\alpha)}=D_t^{\alpha}(\eta)+\xi D_t^{\alpha}(u_x)-D_t^{\alpha}(\xi u_x)+D_t^{\alpha}(D_t(\tau)u)-D_t^{\alpha+1}(\tau u)+\tau D_t^{\alpha+1}(u), \quad (1.30)$$

where D_t^{α} is the total fractional derivative operator. By using the generalized Leibniz rule (1.4) as following

$$\begin{aligned}
D_t^{\alpha}(\xi u_x) &= \xi D_t^{\alpha}(u_x)+\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x),\\
D_t^{\alpha}(D_t(\tau)u) &= D_t(\tau)D_t^{\alpha}(u)+\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(D_t(\tau))D_t^{\alpha-n}(u),\\
D_t^{\alpha+1}(\tau u) &= \tau D_t^{\alpha+1}(u)+\sum_{n=1}^{\infty}\binom{\alpha+1}{n}D_t^{\alpha+1-n}(u)D_t^n(\tau),
\end{aligned}$$

the infinitesimal (1.30) can be rewritten as [25]

$$\mu^{(\alpha)} = D_t^\alpha(\eta) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}D_t^{n+1}(\tau)D_t^{\alpha-n}(u). \tag{1.31}$$

In a view of the generalization of chain rule for composite functions we have

$$\frac{d^m g(y(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-y(t)]^r\frac{d^m}{dt^m}[(y(t))^{k-r}]\frac{d^k g(y)}{dy^k} \tag{1.32}$$

and using the generalized Leibniz rule (1.4), we get the following explicit expression for $\mu^{(\alpha)}$ [5–7, 25, 29]

$$\begin{aligned}\mu^{(\alpha)} = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) \\ + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) + \mu,\end{aligned} \tag{1.33}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

It should be noted that we get $\mu = 0$ when the infinitesimal η is linear on u.

According to the Lie symmetry theory, the infinitesimal generator of (1.22), (1.23) and (1.28) is given by

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \mu^{(\alpha)}\frac{\partial}{\partial u_{t^\alpha}} + \mu^{(1)}\frac{\partial}{\partial u_x} + \mu^{(2)}\frac{\partial}{\partial u_{xx}} + \cdots. \tag{1.34}$$

The infinitesimal invariance criterion for (1.27) or determining equation can be written as

$$X(u_{t^\alpha} - F(t, x, u, u_x, u_{xx}, \cdots))|_{\frac{\partial^\alpha u}{\partial t^\alpha}=F} = 0. \tag{1.35}$$

1.3.3 Lie symmetry analysis of the system of FPDE

In [10, 26] Lie symmetry analysis was generalized to study the systems of FPDEs. But we found that the extended infinitesimals derived in both of these two papers do not match with the ones that we obtained below.

So, we carry out the derivation of extended infinitesimals for a system of FPDEs. The general form of a system of time fractional PDEs with two independent variables

x and t is as follows:

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F_1(x,t,u,u_x,u_{xx},\ldots,v,v_x,v_{xx},\ldots), \\ \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = F_2(x,t,u,u_x,u_{xx},\ldots,v,v_x,v_{xx},\ldots). \end{cases} \tag{1.36}$$

The infinitesimal generator of (1.36) is given by

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial v},$$

and the corresponding extended infinitesimal generator is

$$\tilde{X} = X + \mu^{(\alpha)}\frac{\partial}{\partial u_{t^\alpha}} + \mu^{(1)}\frac{\partial}{\partial u_x} + \cdots + \phi^{(\alpha)}\frac{\partial}{\partial v_{t^\alpha}} + \phi^{(1)}\frac{\partial}{\partial v_x} + \cdots, \tag{1.37}$$

where τ, ξ, μ and ϕ are infinitesimals and $\mu^{(\alpha)}$, $\mu^{(n)}$, $\phi^{(\alpha)}$ and $\phi^{(n)}$ $(n = 1,2,\ldots)$ are extended infinitesimals. Explicitly, $\mu^{(n)}$ and $\phi^{(n)}$ are given by

$$\begin{aligned} \mu^{(1)} &= D_x(\mu) - u_x D_x(\xi) - u_t D_x(\tau), & \phi^{(1)} &= D_x(\phi) - v_x D_x(\xi) - v_t D_x(\tau), \\ \mu^{(2)} &= D_x(\mu^{(1)}) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau), & \phi^{(2)} &= D_x(\phi^{(1)}) - v_{xx} D_x(\xi) - v_{xt} D_x(\tau), \\ &\vdots & &\vdots \end{aligned} \tag{1.38}$$

where D_x is now the total derivative operator defined as

$$D_x := \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \cdots + v_x\frac{\partial}{\partial v} + v_{xx}\frac{\partial}{\partial v_x} + \cdots.$$

The αth order extended infinitesimals have the following forms [10, 26]:

$$\begin{aligned} \mu^{(\alpha)} &= D_t^\alpha(\mu) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}D_t^{n+1}(\tau)D_t^{\alpha-n}(u), \\ \phi^{(\alpha)} &= D_t^\alpha(\phi) - \alpha D_t(\tau)\frac{\partial^\alpha v}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(v_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}D_t^{n+1}(\tau)D_t^{\alpha-n}(v), \end{aligned} \tag{1.39}$$

which are derived analogously to the case of FPDE. Here, D_t is the total derivative operator defined as

$$D_t := \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + \cdots + v_t\frac{\partial}{\partial v} + v_{xt}\frac{\partial}{\partial v_x} + \cdots.$$

Because the lower limit of the integral in (1.1) is fixed, it should be invariant with respect to point transformations. We thus arrive at the following initial condition

$$\tau(x,t,u,v)|_{t=0} = 0. \tag{1.40}$$

We should note that the last three terms on the right-hand side of each equation in (1.39) are already in factored forms with respect to the partial derivatives of u and v. Hence, we only need to consider the first terms, $D_t^\alpha(\mu)$ and $D_t^\alpha(\phi)$.

In the following lemma, we present explicit forms of the extended infinitesimals $\mu^{(\alpha)}$, $\phi^{(\alpha)}$ that are readily computed.

Lemma 2. *The extended infinitesimals $\mu^{(\alpha)}$ and $\phi^{(\alpha)}$ in (1.39) can be re-written as*

$$\begin{aligned}\mu^{(\alpha)} \;=\;& \frac{\partial^\alpha \mu}{\partial t^\alpha} - u\frac{\partial^\alpha \mu_u}{\partial t^\alpha} - v\frac{\partial^\alpha \mu_v}{\partial t^\alpha} + (\mu_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} + \mu_v\frac{\partial^\alpha v}{\partial t^\alpha} \\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(u) \\ & + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \mu_v}{\partial t^n} D_t^{\alpha-n}(v) + \mu_1,\end{aligned} \tag{1.41}$$

$$\begin{aligned}\phi^{(\alpha)} \;=\;& \frac{\partial^\alpha \phi}{\partial t^\alpha} - v\frac{\partial^\alpha \phi_v}{\partial t^\alpha} - u\frac{\partial^\alpha \phi_u}{\partial t^\alpha} + (\phi_v - \alpha D_t(\tau))\frac{\partial^\alpha v}{\partial t^\alpha} + \phi_u\frac{\partial^\alpha u}{\partial t^\alpha} \\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(v_x) + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \phi_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(v) \\ & + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \phi_u}{\partial t^n} D_t^{\alpha-n}(u) + \phi_1,\end{aligned} \tag{1.42}$$

where

$$\begin{aligned}\mu_1 \;=\;& \sum_{n=2}^{\infty}\sum_{m_1+m_2=2}^{n}\sum_{\substack{k_1=0,..,m_1\\ k_2=0,..,m_2\\ k_1+k_2\geq 2}}\sum_{r_1=0}^{k_1}\sum_{r_2=0}^{k_2}\binom{\alpha}{n}\binom{n}{m_1}\binom{n-m_1}{m_2}\binom{k_1}{r_1}\binom{k_2}{r_2}\frac{1}{k_1!k_2!} \\ & \times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^{r_1}(-v)^{r_2}\frac{\partial^{m_1} u^{k_1-r_1}}{\partial t^{m_1}}\frac{\partial^{m_2} v^{k_2-r_2}}{\partial t^{m_2}}\frac{\partial^{n-m_1-m_2+k_1+k_2}\mu}{\partial t^{n-m_1-m_2}\partial u^{k_1}\partial v^{k_2}},\end{aligned}$$

and

$$\phi_1 = \sum_{n=2}^{\infty} \sum_{m_1+m_2=2}^{n} \sum_{\substack{k_1=0,..,m_1 \\ k_2=0,..,m_2 \\ k_1+k_2\geq 2}} \sum_{r_1=0}^{k_1} \sum_{r_2=0}^{k_2} \binom{\alpha}{n}\binom{n}{m_1}\binom{n-m_1}{m_2}\binom{k_1}{r_1}\binom{k_2}{r_2}\frac{1}{k_1!k_2!} \times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^{r_1}(-v)^{r_2}\frac{\partial^{m_1}u^{k_1-r_1}}{\partial t^{m_1}}\frac{\partial^{m_2}v^{k_2-r_2}}{\partial t^{m_2}}\frac{\partial^{m-m_1-m_2+k_1+k_2}\phi}{\partial t^{n-m_1-m_2}\partial u^{k_1}\partial v^{k_2}}.$$

Proof. It is sufficient to prove the formula for $\mu^{(\alpha)}$. This can be done by carrying out an expansion of the first term $D_t^\alpha(\mu)$ in (1.39) by applying a generalized Leibniz rule (1.4) and a generalized chain rule (1.32) [20, 24], as follows:

$$\begin{aligned} D_t^\alpha(\mu) &= \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}D_t^n(\mu) \\ &= \sum_{n=0}^{\infty}\sum_{m_1+m_2=0}^{n}\binom{\alpha}{n}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\binom{n}{m_1}\binom{n-m_1}{m_2} \\ &\quad\times\left[\frac{\partial^{n-m_1-m_2}\partial^{m_1}\partial^{m_2}\mu(x,t,u(x,t_1),v(x,t_2))}{\partial t^{n-m_1-m_2}\partial t_1^{m_1}\partial t_2^{m_2}}\right]\Bigg|_{\substack{t_1=t\\t_2=t}} \\ &= \sum_{n=0}^{\infty}\sum_{m_1+m_2=0}^{n}\sum_{k_1=0}^{m_1}\sum_{r_1=0}^{k_1}\sum_{k_2=0}^{m_2}\sum_{r_2=0}^{k_2}\binom{\alpha}{n}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\binom{n}{m_1}\binom{n-m_1}{m_2}\binom{k_1}{r_1}\binom{k_2}{r_2} \\ &\quad\times\frac{1}{k_1!k_2!}(-u)^{r_1}(-v)^{r_2}\frac{\partial^{m_1}u^{k_1-r_1}}{\partial t^{m_1}}\frac{\partial^{m_2}v^{k_2-r_2}}{\partial t^{m_2}}\frac{\partial^{n-m_1-m_2+k_1+k_2}\mu}{\partial t^{n-m_1-m_2}\partial u^{k_1}\partial v^{k_2}}. \end{aligned} \tag{1.43}$$

Because μ_1 is equal to the partial sum obtained from (1.43) by retaining the terms for which the sum of k_1 and k_2 is greater than 1, we need to examine only the case in which $k_1+k_2\leq 1$. All possible summations over values of $(m_1,m_2,k_1,k_2,r_1,r_2)$ satisfying this inequality can be divided into five cases, which we index by i. We write the resulting sum $D_t^\alpha(\mu)_{i)}$. These quantities are listed in the following table. The explicit

Subcase i	$(m_1,m_2,k_1,k_2,r_1,r_2)$	$D_t^\alpha(\mu)_{i)}$
1)	$(0,0,0,0,0,0)$	$D_t^\alpha(\mu)_{1)} = \sum_{n=0}^{\infty}\binom{\alpha}{n}D_t^{\alpha-n}1\cdot\frac{\partial^n\mu}{\partial t^n} = \frac{\partial^\alpha\mu}{\partial t^\alpha}$
2)	$(0,0,1,0,1,0)$	$D_t^\alpha(\mu)_{2)} = \sum_{n=0}^{\infty}\binom{\alpha}{n}D_t^{\alpha-n}1(-u)\frac{\partial^{n+1}\mu}{\partial t^n\partial u} = -u\frac{\partial^\alpha\mu_u}{\partial t^\alpha}$
3)	$(0,0,0,1,0,1)$	$D_t^\alpha(\mu)_{3)} = -v\frac{\partial^\alpha\mu_v}{\partial t^\alpha}$
4)	$(m_1,0,1,0,0,0)$	$D_t^\alpha(\mu)_{4)} = \mu_u\frac{\partial^\alpha u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\mu_u}{\partial t^n}D_t^{\alpha-n}u$
5)	$(0,m_2,0,1,0,0)$	$D_t^\alpha(\mu)_{5)} = \mu_v\frac{\partial^\alpha v}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\mu_v}{\partial t^n}D_t^{\alpha-n}v$

form of $\mu^{(\alpha)}$ given in the statement of the lemma can be obtained by substituting the sum of $D_t^\alpha(\mu)_{i)}$ in the above five cases and μ_1 into (1.39). The formula for $\varphi^{(\alpha)}$ can be obtained similarly. □

Note: If $\mu^{(\alpha)}$ or $\phi^{(\alpha)}$ is linear in u and v, then $\mu_1 = 0$ and $\phi_1 = 0$, respectively.

The infinitesimal invariance criterion in the Lie symmetry analysis for the system given in (1.27) is

$$\begin{cases} \tilde{X}(u_{t^\alpha} - F_1(x,t,u,u_x,u_{xx},\ldots,v,v_x,v_{xx},\ldots))\Big|_{(1.27)} = 0, \\ \tilde{X}(v_{t^\alpha} - F_2(x,t,u,u_x,u_{xx},\ldots,v,v_x,v_{xx},\ldots))\Big|_{(1.27)} = 0, \end{cases} \tag{1.44}$$

where $\tilde{X}$ is given by (1.37), (1.38), (1.41) and (1.42).

Chapter 2

Solutions of linear fractional differential equations

As the applications of fractional ordinary differential equations (FODEs) emerge in diverse fields, various methods for studying the FODEs are appearing rapidly, such as power series method, method of integral transformations, Green's function method, Adomian decomposition method and variety of numerical methods. Among these methods, the indicial polynomial method has a similar idea to one that we are introducing in this chapter. But in [28], the indicial polynomial method is only applied to constant coefficient fractional ordinary differential equation with rational order of fractional derivative and the solutions are expressed in Mittag-Leffler functions.

In this chapter, we derive exact solutions expressed in terms of well-known special functions for FODEs of the form:

$$\frac{d^\alpha}{dz^\alpha}\varphi(z) = \frac{a_m}{\alpha^m} z^m \frac{d^m}{dz^m}\varphi(z) + \frac{a_{m-1}}{\alpha^{m-1}} z^{m-1} \frac{d^{m-1}}{dz^{m-1}}\varphi(z) + \cdots + \frac{a_1}{\alpha} z \frac{d}{dz}\varphi(z) + a_0\varphi(z), \quad (2.1)$$

where $\alpha \in \mathbb{R}_+$, $a_i \in \mathbb{R}$ $(i = 0, \ldots, m)$ and $a_m \neq 0$, and for systems of the form:

$$\begin{cases} \frac{d^\alpha}{dz^\alpha}\varphi(z) = \frac{a_{m_1}}{\alpha^{m_1}} z^{m_1} \frac{d^{m_1}}{dz^{m_1}}\psi(z) + \frac{a_{m_1-1}}{\alpha^{m_1-1}} z^{m_1-1} \frac{d^{m_1-1}}{dz^{m_1-1}}\psi(z) + \cdots + \frac{a_1}{\alpha} z \frac{d}{dz}\psi(z) + a_0\psi(z), \\ \frac{d^\alpha}{dz^\alpha}\psi(z) = \frac{b_{m_2}}{\alpha^{m_2}} z^{m_2} \frac{d^{m_2}}{dz^{m_2}}\varphi(z) + \frac{b_{m_2-1}}{\alpha^{m_2-1}} z^{m_2-1} \frac{d^{m_2-1}}{dz^{m_2-1}}\varphi(z) + \cdots + \frac{b_1}{\alpha} z \frac{d}{dz}\varphi(z) + b_0\varphi(z), \end{cases} \quad (2.2)$$

where $\alpha \in \mathbb{R}_+$, $a_i, b_j \in \mathbb{R}$ $(i = 0, \ldots, m_1; j = 0, \ldots, m_2)$ and $a_{m_1} b_{m_2} \neq 0$.

It is interesting to consider the forms taken by (2.1) and (2.2) in the particular cases that $m = 2$ and $m_1 = m_2 = 1$, because these are the cases most commonly considered in scientific and engineering fields. In these cases, we obtain the FODE

$$\frac{d^\alpha}{dz^\alpha}\varphi(z) = a\varphi(z) + \frac{b}{\alpha} z \frac{d}{dz}\varphi(z) + \frac{c}{\alpha^2} z^2 \frac{d^2}{dz^2}\varphi(z), \text{ where } a, b, c \in \mathbb{R} \tag{2.3}$$

and the system of FODEs

$$\begin{cases} \frac{d^\alpha}{dz^\alpha}\varphi(z) = a_1\psi(z) + \frac{b_1}{\alpha} z \frac{d}{dz}\psi(z), \\ \frac{d^\alpha}{dz^\alpha}\psi(z) = a_2\varphi(z) + \frac{b_2}{\alpha} z \frac{d}{dz}\varphi(z), \end{cases} \quad \text{where } a_1, a_2, b_1, b_2 \in \mathbb{R}. \tag{2.4}$$

In the case that $\alpha = 1$ and $\varphi(z)$ takes the form $\varphi(z) = z^r e^{-\frac{1}{cz}}\phi(\frac{1}{cz})$, (2.3) reduces to Kummer's equation,

$$z^2 \frac{d^2}{dz^2}\phi(z) + \left(2r - \frac{b}{c} + 2 - z\right)\frac{d}{dz}\phi(z) + \left(\frac{b}{c} - r - 2\right)\phi(z) = 0,$$

where $r = -\frac{b-c+\sqrt{(b-c)^2-4ac}}{2c}$. Further, in the case that $\alpha = 2$ and $\varphi(z)$ takes the form $\varphi(z) = (cz^2 - 1)^{-\frac{b-2c}{4c}}\phi(\sqrt{c}z)$, (2.3) reduces to the associated Legendre differential equation,

$$(1 - z^2)\frac{d^2}{dz^2}\phi(z) - 2z\frac{d}{dz}\phi(z) + \left(l(l+1) - \frac{s^2}{1 - z^2}\right)\phi(z) = 0,$$

where $l = \frac{-c+\sqrt{b^2+c^2-4ac-2bc}}{2c}$ and $s = \frac{b}{2c} - 1$. The solutions of the above equations can be expressed in terms of Kummer's function and associated Legendre functions, respectively.

2.1 Construction of exact solutions

We express solutions to (2.3) and (2.4) using three kinds of special functions: Mittag-Leffler functions, generalized Wright functions and Fox H-functions. Which of these functions we use in any given case depends on the right-hand side and order of the fractional derivative of (2.3) or (2.4).

2.1.1 Solutions expressed in terms of Mittag-Leffler functions

The following lemma concerns fractional derivatives of products of Mittag-Leffler functions and power functions.

Lemma 3. *For arbitrary positive values of* α, β *and* B, *and for any real* a, *the following equality holds:*

$$\frac{d^\alpha}{dz^\alpha}\left(z^{B-1}E_{\beta,B}(az^\beta)\right) = a^m z^{B+m\beta-\alpha-1}E_{\beta,B+m\beta-\alpha}(az^\beta).$$

Here, m *is the smallest non-negative integer such that* $B+m\beta-\alpha-1$ *is not a negative integer.*

Proof. This lemma can be proven straightforwardly by taking the fractional derivative term by term in a series representation of the Mittag-Leffler function via (1.3) as following

$$\begin{aligned}\frac{d^\alpha}{dz^\alpha}\left(z^{B-1}E_{\beta,B}(az^\beta)\right) &= \frac{d^\alpha}{dz^\alpha}\sum_{i=0}^{\infty}\frac{a^i z^{B-1+\beta i}}{\Gamma(B+\beta i)} \\ &= \sum_{i=0}\frac{a^i z^{B-1+\beta i-\alpha}}{\Gamma(B+\beta i-\alpha)} \\ &= \sum_{i=0}\frac{a^{i+m} z^{B-1+\beta i+\beta m-\alpha}}{\Gamma(B+\beta i+\beta m-\alpha)}.\end{aligned}$$

The last equality follows from considering the condition on m, *ensuring the definition of gamma function.* □

For the cases specified below, we construct solutions of (2.3) and (2.4) in terms of Mittag-Leffler functions.

Proposition 1. *For arbitrary* $\alpha > 0$, *we have the following solutions expressed in terms of Mittag-Leffler functions.*

1. *For* $a \in \mathbb{R}$, *the equation*

$$\frac{d^\alpha \varphi}{dz^\alpha} = a\varphi, \quad z \in \mathbb{R} \tag{2.5}$$

   has a solution $\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k}E_{\alpha,1+\alpha-k}(az^\alpha)$, *where* c_k $(k = 1,\dots,n)$ *are constants.*

2. *For* $a_1, a_2 \in \mathbb{R}$, *the system*

$$\begin{cases}\frac{d^\alpha\varphi}{dz^\alpha} = a_1\psi, \\ \frac{d^\alpha\psi}{dz^\alpha} = a_2\varphi,\end{cases} \quad z \in \mathbb{R} \tag{2.6}$$

has a solution

$$\begin{cases}\varphi(z) = \sum\limits_{k=1}^{n} c_{k,1} z^{\alpha-k} E_{2\alpha,1+\alpha-k}(a_1 a_2 z^{2\alpha}) + a_1 \sum\limits_{k=1}^{n} c_{k,2} z^{2\alpha-k} E_{2\alpha,1+2\alpha-k}(a_1 a_2 z^{2\alpha}),\\ \psi(z) = a_2 \sum\limits_{k=1}^{n} c_{k,1} z^{2\alpha-k} E_{2\alpha,1+2\alpha-k}(a_1 a_2 z^{2\alpha}) + \sum\limits_{k=1}^{n} c_{k,2} z^{\alpha-k} E_{2\alpha,1+\alpha-k}(a_1 a_2 z^{2\alpha}),\end{cases}$$

where $c_{k,1}, c_{k,2}$ *(*$k = 1, \dots, n$*) are constants.*

Proof. From the linearity of (2.5) and (2.6), it is sufficient to show that the single terms

$$\varphi_k(z) = c_k z^{\alpha-k} E_{\alpha,1+\alpha-k}(a z^{\alpha})$$

and

$$\begin{cases}\varphi_k(z) = z^{\alpha-k} E_{2\alpha,1+\alpha-k}(a_1 a_2 z^{2\alpha})\\ \psi_k(z) = a_2 z^{2\alpha-k} E_{2\alpha,1+2\alpha-k}(a_1 a_2 z^{2\alpha})\end{cases}$$

satisfy (2.5) and (2.6), respectively, for $k = 1, \dots, n$. This is easily done using Lemma 3. □

Although the first assertion of Proposition 1 was demonsrated in [24], we included it here for completeness.

2.1.2 Solutions expressed in terms of generalized Wright functions

Let us formulate the following contiguous relations for the generalized Wright functions. These are used below to obtain solutions of (2.3) and (2.4).

Lemma 4. *Let us assume that the generalized Wright function is absolutely convergent, i.e., that* $\Delta = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1$. *Then the following equalities hold for* $\alpha \in \mathbb{R}_+$ *and* $a \in \mathbb{R}$.

1. *If* $\beta_1 > 0$ *and* $B_1 > 0$, *then we have*

$$\frac{d^{\alpha}}{dz^{\alpha}}\left(z^{B_1-1}{}_p\Psi_q\left[az^{\beta_1}\middle|\begin{matrix}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{matrix}\right]\right) = a^m z^{B_1+m\beta_1-1-\alpha}$$
$$\times{}_{p+1}\Psi_{q+1}\left[az^{\beta_1}\middle|\begin{matrix}(1,1),(A_i+m\alpha_i,\alpha_i)_{1,p}\\(1+m,1),(B_1+m\beta_1-\alpha,\beta_1),(B_j+m\beta_j,\beta_j)_{2,q}\end{matrix}\right], \quad z \in \mathbb{R},$$

where m *is the smallest non-negative integer such that* $B_1 + m\beta_1 - \alpha - 1$ *is not a negative integer.*

2. *For $\sigma \in \mathbb{R}\setminus\{0\}$ and $R \in \mathbb{R}$, the following equality holds*

$$\left(\frac{1}{\alpha}z\frac{d}{dz}+R\right)\left(z^{\frac{A_1\sigma}{\alpha_1}-\alpha R}{}_p\Psi_q\left[az^{\sigma}\middle|\begin{array}{c}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right]\right)$$
$$=\frac{\sigma}{\alpha_1\alpha}z^{\frac{A_1\sigma}{\alpha_1}-\alpha R}{}_p\Psi_q\left[az^{\sigma}\middle|\begin{array}{c}(A_1+1,\alpha_1),(A_i,\alpha_i)_{2,p}\\(B_j,\beta_j)_{1,q}\end{array}\right].$$

Proof. To prove the first assertion, let us write the function whose fractional derivative we are taking, $\varphi(z)$, as

$$\varphi(z)=z^{B_1-1}{}_p\Psi_q\left[az^{\beta_1}\middle|\begin{array}{c}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right].$$

Then, taking the Riemann-Liouville derivative (1.1), we obtain

$$\frac{d^\alpha\varphi}{dz^\alpha}=\sum_{i=0}^{\infty}\frac{\Gamma(A_1+\alpha_1 i)\cdots\Gamma(A_p+\alpha_p i)a^i}{\Gamma(B_1-\alpha+\beta_1 i)\Gamma(B_2+\beta_2 i)\cdots\Gamma(B_q+\beta_q i)i!}z^{B_1-1+\beta_1 i-\alpha},$$

which equals

$$\frac{d^\alpha\varphi}{dz^\alpha}=\sum_{i=0}^{\infty}\frac{\Gamma(A_1+\alpha_1(i+m))\cdots\Gamma(A_p+\alpha_p(i+m))a^{i+m}z^{B_1-1+\beta_1 i+\beta_1 m-\alpha}}{\Gamma(B_1-\alpha+\beta_1 m+\beta_1 i)\Gamma(B_2+\beta_2 m+\beta_2 i)\cdots\Gamma(B_q+\beta_q m+\beta_q i)(i+m)!},$$

where we have employed the condition on the integer m. The first assertion of the lemma can be proved through multiplying both the numerator and the denominator of the last expression by $i!$.

The second assertion of the lemma can be proved through straightforward calculation with the help of the identity $(A_1+\alpha_1 i)\Gamma(A_1+\alpha_1 i)=\Gamma(A_1+1+\alpha_1 i)$ for $i=0,1,2,\dots$ as following

$$\left(\frac{1}{\alpha}z\frac{d}{dz}+R\right)\left(z^{\frac{A_1\sigma}{\alpha_1}-\alpha R}{}_p\Psi_q\left[az^{\sigma}\middle|\begin{array}{c}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right]\right)$$
$$=\left(\frac{1}{\alpha}z\frac{d}{dz}+R\right)\sum_{i=0}^{\infty}\frac{\prod_{k=1}^{p}\Gamma(A_k+\alpha_k i)a^i}{\prod_{j=1}^{q}\Gamma(B_j+\beta_j i)}z^{\frac{A_1\sigma}{\alpha_1}-\alpha R+\sigma i}$$
$$=\sum_{i=0}^{\infty}\frac{\prod_{k=1}^{p}\Gamma(A_k+\alpha_k i)a^i}{\prod_{j=1}^{q}\Gamma(B_j+\beta_j i)}\left(\frac{1}{\alpha}\left(\frac{A_1}{\alpha_1}\sigma-\alpha R+\sigma i\right)+R\right)z^{\frac{A_1\sigma}{\alpha_1}-\alpha R}.$$

□

We obtain the following corollary from the first assertion of Lemma 4 in the case $A_1 = \alpha_1 = 1$.

Corollary 1. *When $B_1 > 0$, $\beta_1 > 0$ and $A_1 = \alpha_1 = 1$, the fractional derivative of product of power function and generalized Wright function ${}_p\Psi_q$ is*

$$\begin{aligned} &\frac{d^\alpha}{dz^\alpha}\left(z^{B_1-1}{}_p\Psi_q\left[az^{\beta_1}\middle|\begin{array}{c}(1,1),(A_i,\alpha_i)_{2,p}\\(B_j,\beta_j)_{1,q}\end{array}\right]\right)\\ = \;& a^m z^{B_1+m\beta_1-1-\alpha}{}_p\Psi_q\left[az^{\beta_1}\middle|\begin{array}{c}(1,1),(A_i+m\alpha_i,\alpha_i)_{2,p}\\(B_1+m\beta_1-\alpha,\beta_1),(B_j+m\beta_j,\beta_j)_{2,q}\end{array}\right], \end{aligned}$$

where m is the smallest non-negative integer such that $B_1 + m\beta_1 - \alpha - 1$ is not a negative integer.

Before moving on to the formulation of exact solutions of (2.3) and (2.4), we introduce the following notation.

Let us consider the case that $c = 0$ and $b \neq 0$ in (2.3). Then writing $-\frac{a}{b}$ as $\bar{s}$, we can rewrite the right-hand side of (2.3) as

$$a\varphi + \frac{b}{\alpha}z\frac{d\varphi}{dz} = b\left(\frac{1}{\alpha}z\frac{d}{dz} - \bar{s}\right)\varphi.$$

Then, in the case of (2.4), assuming $b_1b_2 \neq 0$ and introducing the quantities

$$\tilde{s}_1 = -\frac{a_1}{b_1}, \quad \tilde{s}_2 = -\frac{a_2}{b_2},$$

we rewrite the right-hand side of (2.4) as follows:

$$\begin{aligned} a_1\psi + \frac{b_1}{\alpha}z\frac{d\psi}{dz} &= b_1\left(\frac{1}{\alpha}z\frac{d}{dz} - \tilde{s}_1\right)\psi,\\ a_2\varphi + \frac{b_2}{\alpha}z\frac{d\varphi}{dz} &= b_2\left(\frac{1}{\alpha}z\frac{d}{dz} - \tilde{s}_2\right)\varphi. \end{aligned}$$

Now, let us assume $c \neq 0$. Then the characteristic equation of the right-hand side of (2.3) is

$$s^2 + \left(\frac{b}{c} - \frac{1}{\alpha}\right)s + \frac{a}{c} = 0. \tag{2.7}$$

We write the determinant and roots of (2.7) as $D = \frac{1}{\alpha^2} - \frac{2b}{\alpha c} + \frac{b^2}{c^2} - \frac{4a}{c}$ and $s_{1,2} = \frac{1}{2}\left(\frac{1}{\alpha} - \frac{b}{c} \pm \sqrt{D}\right)$, respectively. Then we can rewrite the right-hand side of (2.3) in the factorized differential form

$$a\varphi + \frac{b}{\alpha} z \frac{d\varphi}{dz} + \frac{c}{\alpha^2} z^2 \frac{d^2\varphi}{dz^2} = c\left(\frac{1}{\alpha} z \frac{d}{dz} - s_1\right)\left(\frac{1}{\alpha} z \frac{d}{dz} - s_2\right)\varphi. \tag{2.8}$$

This notation is useful for at least two reasons. First, it reveals the uniformity in given solutions of (2.3) and (2.4) with different orders of fractional derivatives. In particular, in the case $c \neq 0$, we can avoid a tedious computation by simply rewriting the right-hand side of (2.3) in factorized operator form. Second, using this notation, we can easily generalize (2.3) and (2.4) into cases with higher-order derivatives and obtain solutions thereof. We will discuss this generalization in the next section.

We now formulate the solutions of (2.3) and (2.4) expressed in terms of generalized Wright function as follows.

Proposition 2. *We have the following solutions expressed in terms of the generalized Wright function.*

1. *For $\alpha > 1$ and $a, b \in \mathbb{R}$ with $b \neq 0$, the equation*

$$\frac{d^\alpha \varphi}{dz^\alpha} = a\varphi + \frac{b}{\alpha} z \frac{d\varphi}{dz}, \quad z \in \mathbb{R} \tag{2.9}$$

has as a solution

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} {}_2\Psi_1 \left[bz^\alpha \middle| \begin{array}{c} \left(1 - \frac{k}{\alpha} - \bar{s}, 1\right), (1,1) \\ (1+\alpha-k, \alpha) \end{array} \right],$$

where $\bar{s} = -\frac{a}{b}$, and c_k $(k = 1, \ldots, n)$ are constants.

2. *For $\alpha > 2$ and $a, b, c \in \mathbb{R}$ with $c \neq 0$, the equation*

$$\frac{d^\alpha \varphi}{dz^\alpha} = a\varphi + \frac{b}{\alpha} z \frac{d\varphi}{dz} + \frac{c}{\alpha^2} z^2 \frac{d^2\varphi}{dz^2}, \quad z \in \mathbb{R} \tag{2.10}$$

has as a solution

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} {}_3\Psi_1 \left[cz^\alpha \middle| \begin{array}{c} \left(1 - \frac{k}{\alpha} - s_1, 1\right), \left(1 - \frac{k}{\alpha} - s_2, 1\right), (1,1) \\ (1+\alpha-k, \alpha) \end{array} \right],$$

where $D = \frac{1}{\alpha^2} - \frac{2b}{\alpha c} + \frac{b^2}{c^2} - \frac{4a}{c}$, $s_{1,2} = \frac{1}{2}\left(\frac{1}{\alpha} - \frac{b}{c} \pm \sqrt{D}\right)$, and c_k $(k = 1, \ldots, n)$ are constants.

3. *For $\alpha > 1$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $b_1 b_2 \neq 0$, the system*

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = a_1 \psi + \frac{b_1}{\alpha} z \frac{d\psi}{dz}, \\ \frac{d^\alpha \psi}{dz^\alpha} = a_2 \varphi + \frac{b_2}{\alpha} z \frac{d\varphi}{dz}, \end{cases} \qquad z \in \mathbb{R} \tag{2.11}$$

has as a solution

$$\varphi(z) = \sum_{k=1}^{n} c_{k,1} z^{\alpha-k} {}_3\Psi_1 \left[4b_1 b_2 z^{2\alpha} \middle| \begin{matrix} \left(1 - \frac{k}{2\alpha} - \frac{\tilde{s}_1}{2}, 1\right), \left(\frac{1}{2} - \frac{k}{2\alpha} - \frac{\tilde{s}_2}{2}, 1\right), (1,1) \\ (1+\alpha-k, 2\alpha) \end{matrix} \right]$$
$$+ 2b_1 \sum_{k=1}^{n} c_{k,2} z^{2\alpha-k} {}_3\Psi_1 \left[4b_1 b_2 z^{2\alpha} \middle| \begin{matrix} \left(\frac{3}{2} - \frac{k}{2\alpha} - \frac{\tilde{s}_1}{2}, 1\right), \left(1 - \frac{k}{2\alpha} - \frac{\tilde{s}_2}{2}, 1\right), (1,1) \\ (1+2\alpha-k, 2\alpha) \end{matrix} \right],$$

$$\psi(z) = 2b_2 \sum_{k=1}^{n} c_{k,1} z^{2\alpha-k} {}_3\Psi_1 \left[4b_1 b_2 z^{2\alpha} \middle| \begin{matrix} \left(1 - \frac{k}{2\alpha} - \frac{\tilde{s}_1}{2}, 1\right), \left(\frac{3}{2} - \frac{k}{2\alpha} - \frac{\tilde{s}_2}{2}, 1\right), (1,1) \\ (1+2\alpha-k, 2\alpha) \end{matrix} \right]$$
$$+ \sum_{k=1}^{n} c_{k,2} z^{\alpha-k} {}_3\Psi_1 \left[4b_1 b_2 z^{2\alpha} \middle| \begin{matrix} \left(\frac{1}{2} - \frac{k}{2\alpha} - \frac{\tilde{s}_1}{2}, 1\right), \left(1 - \frac{k}{2\alpha} - \frac{\tilde{s}_2}{2}, 1\right), (1,1) \\ (1+\alpha-k, 2\alpha) \end{matrix} \right],$$

where $\tilde{s}_1 = -\frac{a_1}{b_1}$, $\tilde{s}_2 = -\frac{a_2}{b_2}$, and $c_{k,1}$, $c_{k,2}$ $(k = 1, \dots, n)$ are constants.

Proof. The proof can be carried out similarly in all three cases using Lemma 4 and Corollary 1. For this reason, we present proofs only for the second and third cases. As in Proposition 1, from the linearity of (2.10), it is sufficient to show that a single summand,

$$\varphi_k(z) = z^{\alpha-k} {}_3\Psi_1 \left[cz^\alpha \middle| \begin{matrix} \left(1 - \frac{k}{\alpha} - s_1, 1\right), \left(1 - \frac{k}{\alpha} - s_2, 1\right), (1,1) \\ (1+\alpha-k, \alpha) \end{matrix} \right], \text{ where } 1 \leq k \leq n,$$

of the solution $\varphi(z)$ satisfies (2.10). Because $1 + \alpha - k > 0$ for any k, by Corollary 1, we have the following identity for the left-hand side of (2.10):

$$\frac{d^\alpha \varphi_k}{dz^\alpha} = cz^{\alpha-k} {}_3\Psi_1 \left[cz^\alpha \middle| \begin{matrix} (2 - \frac{k}{\alpha} - s_1, 1), (2 - \frac{k}{\alpha} - s_2, 1), (1,1) \\ (1+\alpha-k, \alpha) \end{matrix} \right]. \tag{2.12}$$

Then, by virtue of (2.8) and the second assertion of Lemma 4, the right-hand side of (2.10) becomes

$$
\begin{aligned}
&c\left(\frac{1}{\alpha}z\frac{d}{dz}-s_1\right)\left(\frac{1}{\alpha}z\frac{d}{dz}-s_2\right)\varphi_k \\
&\quad = c\left(\frac{1}{\alpha}z\frac{d}{dz}-s_1\right)\left(z^{\alpha-k}{}_3\Psi_1\left[cz^\alpha\left|\begin{matrix}\left(1-\frac{k}{\alpha}-s_1,1\right),\left(2-\frac{k}{\alpha}-s_2,1\right),(1,1)\\(1+\alpha-k,\alpha)\end{matrix}\right.\right]\right),
\end{aligned}
$$

which is equal to the left-hand side of (2.12).

In the third assertion, we only need to show that the summands

$$
\begin{cases}
\varphi_{k,1}(z)=z^{\alpha-k}{}_3\Psi_1\left[4b_1b_2z^{2\alpha}\left|\begin{matrix}\left(1-\frac{k}{2\alpha}-\frac{\tilde{s}_1}{2},1\right),\left(\frac{1}{2}-\frac{k}{2\alpha}-\frac{\tilde{s}_2}{2},1\right),(1,1)\\(1+\alpha-k,2\alpha)\end{matrix}\right.\right],\\
\psi_{k,1}(z)=2b_2z^{2\alpha-k}{}_3\Psi_1\left[4b_1b_2z^{2\alpha}\left|\begin{matrix}\left(1-\frac{k}{2\alpha}-\frac{\tilde{s}_1}{2},1\right),\left(\frac{3}{2}-\frac{k}{2\alpha}-\frac{\tilde{s}_2}{2},1\right),(1,1)\\(1+2\alpha-k,2\alpha)\end{matrix}\right.\right]
\end{cases}
$$

and

$$
\begin{cases}
\varphi_{k,2}(z)=2b_1z^{2\alpha-k}{}_3\Psi_1\left[4b_1b_2z^{2\alpha}\left|\begin{matrix}\left(\frac{3}{2}-\frac{k}{2\alpha}-\frac{\tilde{s}_1}{2},1\right),\left(1-\frac{k}{2\alpha}-\frac{\tilde{s}_2}{2},1\right),(1,1)\\(1+2\alpha-k,2\alpha)\end{matrix}\right.\right],\\
\psi_{k,2}(z)=z^{\alpha-k}{}_3\Psi_1\left[4b_1b_2z^{2\alpha}\left|\begin{matrix}\left(\frac{1}{2}-\frac{k}{2\alpha}-\frac{\tilde{s}_1}{2},1\right),\left(1-\frac{k}{2\alpha}-\frac{\tilde{s}_2}{2},1\right),(1,1)\\(1+\alpha-k,2\alpha)\end{matrix}\right.\right]
\end{cases}
$$

satisfy (2.11) for $k=1,\dots,n$. For this purpose, it is sufficient to show that $(\varphi_{k,1}(z),\psi_{k,1}(z))$ satisfies (2.11). Then, the proof that $(\varphi_{k,2}(z),\psi_{k,2}(z))$ satisfies (2.11) follows from the symmetry property of the system. After applying Corollary 1 with $m=1$ for $\varphi_{k,1}$ and $m=0$ for $\psi_{k,1}$, we obtain the following expressions for the left-hand side of (2.11):

$$
\begin{cases}
\dfrac{d^\alpha\varphi_{k,1}}{dz^\alpha}=4b_1b_2z^{2\alpha-k}{}_3\Psi_1\left[4b_1b_2z^{\alpha-k}\left|\begin{matrix}(1,1),\left(2-\frac{k}{2\alpha}-\frac{\tilde{s}_1}{2},1\right),\left(\frac{3}{2}-\frac{k}{2\alpha}-\frac{\tilde{s}_2}{2},1\right)\\(1+2\alpha-k,2\alpha)\end{matrix}\right.\right],\\
\dfrac{d^\alpha\psi_{k,1}}{dz^\alpha}=2b_2z^{2\alpha-k}{}_3\Psi_1\left[4b_1b_2z^{\alpha-k}\left|\begin{matrix}(1,1),\left(1-\frac{k}{2\alpha}-\frac{\tilde{s}_1}{2},1\right),\left(\frac{3}{2}-\frac{k}{2\alpha}-\frac{\tilde{s}_2}{2},1\right)\\(1+2\alpha-k,2\alpha)\end{matrix}\right.\right].
\end{cases}
\tag{2.13}
$$

Finally, applying the second assertion of Lemma 4 to the right-hand side of (2.11), with $R=\frac{a_1}{b_1}$ and $\sigma=2\alpha$ for the first equation and $R=\frac{a_2}{b_2}$ and $\sigma=2\alpha$ for the second equation, we obtain identically (2.13). □

2.1.3 Solutions expressed in terms of Fox H-functions

Let us first present the following technical lemma on the fractional differentiation of Fox H-functions with an argument raised to negative power.

Lemma 5. *Let* $\nu = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > 0$ *and* $\mu = \sum\limits_{j=1}^{m} \beta_j - \sum\limits_{j=m+1}^{q} \beta_j - \sum\limits_{i=1}^{p} \alpha_i > 0$. *Then the following equalities hold.*

1. *For* $a > 0$,

$$\frac{d^\alpha}{dz^\alpha} H_{p,q}^{m,0}\left[az^{-\alpha_p} \middle| \begin{matrix} (A_i, \alpha_i)_{1,p-1}, (1, \alpha_p) \\ (B_j, \beta_j)_{1,q} \end{matrix} \right] = z^{-\alpha} H_{p,q}^{m,0}\left[az^{-\alpha_p} \middle| \begin{matrix} (A_i, \alpha_i)_{1,p-1}, (1-\alpha, \alpha_p) \\ (B_j, \beta_j)_{1,q} \end{matrix} \right], \quad z > 0.$$

2. *If* $m \geq 1$ *then for* $a \in \mathbb{R} \setminus \{0\}$

$$\left(\frac{\beta_1}{\alpha_p} z \frac{d}{dz} + B_1 \right) H_{p,q}^{m,0}\left[az^{-\alpha_p} \middle| \begin{matrix} (A_i, \alpha_i)_{1,p-1}, (1, \alpha_p) \\ (B_j, \beta_j)_{1,q} \end{matrix} \right] = H_{p,q}^{m,0}\left[az^{-\alpha_p} \middle| \begin{matrix} (A_i, \alpha_i)_{1,p-1}, (1, \alpha_p) \\ (B_1+1, \beta_1), (B_j, \beta_j)_{2,q} \end{matrix} \right].$$

Proof. By virtue of the asymptotic expression (1.6), the fractional derivative of the function

$$\varphi(z) = H_{p,q}^{m,0}\left[az^{-\alpha_p} \middle| \begin{matrix} (A_i, \alpha_i)_{1,p-1}, (1, \alpha_p) \\ (B_j, \beta_j)_{1,q} \end{matrix} \right]$$

is well defined. Substituting $s = zu$ into the definition of the Riemann-Liouville derivative (1.1), we obtain

$$\frac{d^\alpha \varphi(z)}{dz^\alpha} = \frac{d^n}{dz^n} \frac{z^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-u)^{n-\alpha-1} \frac{1}{2\pi i} \times \int_{L_{\gamma+i\infty}} \frac{\prod\limits_{j=1}^{m} \Gamma(B_j - \beta_j s)}{\prod\limits_{i=1}^{p-1} \Gamma(A_i - \alpha_i s) \Gamma(1 - \alpha_p s) \prod\limits_{j=m+1}^{q} \Gamma(1 - B_j + \beta_j s)} \left(a(zu)^{-\alpha_p} \right)^s ds du. \quad (2.14)$$

The well-known formula for the beta function expressed in terms of the gamma function

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \text{ where } \Re(a), \Re(b) > 0,$$

applies to our case, in which it becomes

$$\int_0^1 (1-u)^{n-\alpha-1}u^{-\alpha_p s}du = \frac{\Gamma(n-\alpha)\Gamma(1-\alpha_p s)}{\Gamma(n-\alpha-\alpha_p s+1)}, \tag{2.15}$$

choosing suitable $\gamma < 0$ for the contour $L_{\gamma+i\infty}$ in (2.14). Next, interchanging the order of the integrals in (2.14) and using (2.15) we obtain as following

$$\begin{aligned}
\frac{d^\alpha \varphi(z)}{dz^\alpha} &= \frac{d^n}{dz^n}\frac{z^{n-\alpha}}{\Gamma(n-\alpha)}\int_{L_{\gamma+i\infty}} \frac{\prod_{j=1}^{m}\Gamma(B_j-\beta_j s)a^s z^{-\alpha_p s}}{\prod_{i=1}^{p-1}\Gamma(A_i-\alpha_i s)\Gamma(1-\alpha_p s)\prod_{j=m+1}^{q}\Gamma(1-B_j+\beta_j s)} \\
&\quad \times \frac{1}{2\pi i}\int_0^1 (1-u)^{n-\alpha-1}(u)^{-\alpha_p s}\,duds \\
&= \frac{d^n}{dz^n} z^{n-\alpha}\int_{L_{\gamma+i\infty}} \frac{\prod_{j=1}^{m}\Gamma(B_j-\beta_j s)a^s z^{-\alpha_p s}}{\prod_{i=1}^{p-1}\Gamma(A_i-\alpha_i s)\Gamma(n-\alpha+1-\alpha_p s)\prod_{j=m+1}^{q}\Gamma(1-B_j+\beta_j s)} \\
&= \frac{d^n}{dz^n} z^{n-\alpha} H^{m,0}_{p,q}\left[az^{-\alpha_p}\middle| \begin{matrix}(A_i,\alpha_i)_{1,p-1},(n-\alpha+1,\alpha_p)\\ (B_j,\beta_j)_{1,q}\end{matrix}\right].
\end{aligned} \tag{2.16}$$

The first assertion is then proved by applying (1.8) and (1.11) to (2.16).

To prove the second assertion, let us take the derivative of the H-function using (1.11) with $N = 1$. This yields

$$\begin{aligned}
\frac{d}{dz}H^{m,0}_{p,q}\left[az^{-\alpha_p}\middle| \begin{matrix}(A_i,\alpha_i)_{1,p-1},(1,\alpha_p)\\ (B_j,\beta_j)_{1,q}\end{matrix}\right] &= z\frac{d}{dz}H^{0,m}_{q,p}\left[\frac{z^{\alpha_p}}{a}\middle| \begin{matrix}(1-B_j,\beta_j)_{1,q}\\ (1-A_i,\alpha_i)_{1,p-1},(0,\alpha_p)\end{matrix}\right] \\
&= H^{0,m+1}_{q+1,p+1}\left[\frac{z^{\alpha_p}}{a}\middle| \begin{matrix}(0,\alpha_p),(1-B_j,\beta_j)_{1,q}\\ (1-A_i,\alpha_i)_{1,p-1},(0,\alpha_p),(1,\alpha_p)\end{matrix}\right].
\end{aligned}$$

Here, note that the common term $(0, \alpha_p)$ is canceled out in the last expression, and thus, by virtue of (1.8), we have

$$z\frac{d}{dz}H^{m,0}_{p,q}\left[az^{-\alpha_p}\middle| \begin{matrix}(A_i,\alpha_i)_{1,p-1},(1,\alpha_p)\\ (B_j,\beta_j)_{1,q}\end{matrix}\right] = H^{m,0}_{p,q}\left[az^{-\alpha_p}\middle| \begin{matrix}(A_i,\alpha_i)_{1,p-1},(0,\alpha_p)\\ (B_j,\beta_j)_{1,q}\end{matrix}\right].$$

The quantity in the second assertion is calculated as follows:

$$\begin{aligned}
&\left(\frac{\beta_1}{\alpha_p} z\frac{d}{dz} + B_1\right) H^{m,0}_{p,q}\left[az^{-\alpha_p}\middle|\begin{matrix}(A_i,\alpha_i)_{1,p-1},(1,\alpha_p)\\(B_j,\beta_j)_{1,q}\end{matrix}\right]\\
&= \frac{1}{2\pi i}\int_{L_{\gamma+i\infty}}\left[\frac{\beta_1}{\Gamma(-\alpha_p s)\alpha_p}+\frac{B_1}{\Gamma(1-\alpha_p s)}\right]\frac{\Gamma\left(B_1-\beta_1 s\right)\prod_{j=2}^{m}\Gamma(B_j-\beta_j s)}{\prod_{j=1}^{p-1}\Gamma(A_j-\alpha_j s)\prod_{j=m+1}^{q}\Gamma(1-B_j+\beta_j s)}\\
&\quad\times(az^{-\alpha_p})^s ds\\
&= \frac{1}{2\pi i}\int_{L_{\gamma+i\infty}}\frac{(B_1-\beta_1 s)\Gamma\left(B_1-\beta_1 s\right)\prod_{j=2}^{m}\Gamma(B_j-\beta_j s)}{\prod_{j=1}^{p-1}\Gamma(A_j-\alpha_j s)\Gamma(1-\alpha_p s)\prod_{j=m+1}^{q}\Gamma(1-B_j+\beta_j s)}(az^{-\alpha_p})^s ds,
\end{aligned}$$

which completes the proof.

An alternative proof of the second assertion can be obtained by using the formulas for particular derivatives of Fox-H functions given in [18]. □

Unlike the previously presented solutions expressed in terms of Mittag-Leffler functions and generalized Wright functions, we present the solutions expressed in terms of Fox-H functions for $z > 0$.

Henceforth, we use sign function for $x \in \mathbb{R}$ defined as

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0,\\ -1, & x < 0.\end{cases}$$

Proposition 3. *For the following cases, we have solutions expressed in terms of Fox H-functions.*

1. *For $0 < \alpha < 1$ and $a, b \in \mathbb{R}$ with $b > 0$, the equation*

$$\frac{d^\alpha\varphi}{dz^\alpha} = a\varphi + \frac{b}{\alpha} z\frac{d\varphi}{dz}, \quad z > 0 \tag{2.17}$$

has as a solution

$$\varphi(z) = c_1 H^{1,0}_{1,1}\left[\frac{z^{-\alpha}}{b}\middle|\begin{matrix}(1,\alpha)\\(-\bar{s},1)\end{matrix}\right],$$

where $\bar{s} = -\frac{a}{b}$, and c_1 is a constant.

2. *For* $0<\alpha<2$ *and* $a,b,c\in\mathbb{R}$ *with* $c>0$, *the equation*

$$\frac{d^\alpha\varphi}{dz^\alpha}=a\varphi+\frac{b}{\alpha}z\frac{d\varphi}{dz}+\frac{c}{\alpha^2}z^2\frac{d^2\varphi}{dz^2},\quad z>0 \tag{2.18}$$

has as a solution

$$\varphi(z)=c_1H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{c}\middle|\begin{array}{c}(1,\alpha)\\(-s_1,1),(-s_2,1)\end{array}\right],$$

where $s_{1,2}=\frac{1}{2}\left(\frac{1}{\alpha}-\frac{b}{c}\pm\sqrt{D}\right)$, $D=\frac{1}{\alpha^2}-\frac{2b}{\alpha c}+\frac{b^2}{c^2}-\frac{4a}{c}$, *and* c_1 *is a constant.*

3. *For* $0<\alpha<1$ *and* $a_1,a_2,b_1,b_2\in\mathbb{R}$ *with* $b_1b_2>0$, *the system*

$$\begin{cases}\frac{d^\alpha\varphi}{dz^\alpha}=a_1\psi+\frac{b_1}{\alpha}z\frac{d\psi}{dz},\\ \frac{d^\alpha\psi}{dz^\alpha}=a_2\varphi+\frac{b_2}{\alpha}z\frac{d\varphi}{dz},\end{cases}\quad z>0 \tag{2.19}$$

has as a solution

$$\begin{cases}\varphi(z)=c_1\,\mathrm{sgn}(b_1)H^{2,0}_{1,2}\left[\frac{z^{-2\alpha}}{4b_1b_2}\middle|\begin{array}{c}(1,2\alpha)\\\left(\frac{1}{2}-\frac{\tilde{s}_1}{2},1\right),\left(-\frac{\tilde{s}_2}{2},1\right)\end{array}\right],\\ \psi(z)=c_1\sqrt{\frac{b_2}{b_1}}H^{2,0}_{1,2}\left[\frac{z^{-2\alpha}}{4b_1b_2}\middle|\begin{array}{c}(1,2\alpha)\\\left(-\frac{\tilde{s}_1}{2},1\right),\left(\frac{1}{2}-\frac{\tilde{s}_2}{2},1\right)\end{array}\right],\end{cases}$$

where $\tilde{s}_1=-\frac{a_1}{b_1}$, $\tilde{s}_2=-\frac{a_2}{b_2}$, *and* c_1 *is a constant.*

Proof. Analogously to the proof of Proposition 2, the three assertions of this proposition can be proved in a similar manner by using Lemma 5. For this reason, we consider only the second and third assertions with $c_1=1$, without loss of generality.

Let us consider the second assertion of the proposition. Because the convergence condition of H-functions holds (i.e. $\mu=2-\alpha>0$), we can apply the first assertion of Lemma 5. We thereby obtain

$$\frac{d^\alpha}{dz^\alpha}H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{c}\middle|\begin{array}{c}(1,\alpha)\\(-s_1,1),(-s_2,1)\end{array}\right]=z^{-\alpha}H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{c}\middle|\begin{array}{c}(1-\alpha,\alpha)\\(-s_1,1),(-s_2,1)\end{array}\right]$$

for the left-hand side of (2.18), which is further simplified by virtue of (1.9) into the form

$$\frac{d^\alpha}{dz^\alpha} H_{1,2}^{2,0}\left[\frac{z^{-\alpha}}{c}\middle|\begin{matrix}(1,\alpha)\\(-s_1,1),(-s_2,1)\end{matrix}\right] = cH_{1,2}^{2,0}\left[\frac{z^{-\alpha}}{c}\middle|\begin{matrix}(1,\alpha)\\(-s_1+1,1),(-s_2+1,1)\end{matrix}\right].$$

The right-hand side of (2.18) is obtained in analogy to the second assertion of Proposition 2, by using (2.8) and interchanging the parameters $(-s_1, 1)$ and $(-s_2, 1)$ in the solution $\varphi(z)$ in accordance with the second assertion of Lemma 5 as following

$$\begin{aligned}
&a\varphi + \frac{b}{\alpha}z\frac{d\varphi}{dz} + \frac{c}{\alpha^2}z^2\frac{d^2\varphi}{dz^2}\\
&= c\left(\frac{1}{\alpha}z\frac{d}{dz} - s_1\right)\left(\frac{1}{\alpha}z\frac{d}{dz} - s_2\right) H_{1,2}^{2,0}\left[\frac{z^{-\alpha}}{c}\middle|\begin{matrix}(1,\alpha)\\(-s_2,1),(-s_1,1)\end{matrix}\right]\\
&= c\left(\frac{1}{\alpha}z\frac{d}{dz} - s_1\right) H_{1,2}^{2,0}\left[\frac{z^{-\alpha}}{c}\middle|\begin{matrix}(1,\alpha)\\(-s_1,1),(-s_2+1,1)\end{matrix}\right]\\
&= cH_{1,2}^{2,0}\left[\frac{z^{-\alpha}}{c}\middle|\begin{matrix}(1,\alpha)\\(-s_1+1,1),(-s_2+1,1)\end{matrix}\right].
\end{aligned}$$

Next, considering the third assertion, for the sake of compatibility with the second assertion of Lemma 5, we rewrite the right-hand side of (2.19) as follows:

$$\begin{cases}a_1\psi + \frac{b_1}{\alpha}z\frac{d\psi}{dz} = 2b_1\left(\frac{1}{2\alpha}z\frac{d}{dz} + \frac{a_1}{2b_1}\right)\psi,\\ a_2\varphi + \frac{b_2}{\alpha}z\frac{d\varphi}{dz} = 2b_2\left(\frac{1}{2\alpha}z\frac{d}{dz} + \frac{a_2}{2b_2}\right)\varphi.\end{cases}$$

Because the convergence condition $\mu = 1 - \alpha > 0$ holds for both $\varphi(z)$ and $\psi(z)$, we can apply the first assertion of Lemma 5 and (1.9). We thereby obtain

$$\begin{cases}\dfrac{d^\alpha\varphi}{dz^\alpha} = 2\,\mathrm{sgn}(b_1)\sqrt{b_1b_2}H_{1,2}^{2,0}\left[\dfrac{z^{-2\alpha}}{4b_1b_2}\middle|\begin{matrix}(1,2\alpha)\\\left(1-\frac{\tilde{s}_1}{2},1\right),\left(\frac{1}{2}-\frac{\tilde{s}_2}{2},1\right)\end{matrix}\right],\\ \dfrac{d^\alpha\psi}{dz^\alpha} = 2\,\mathrm{sgn}(b_2)b_2H_{1,2}^{2,0}\left[\dfrac{z^{-2\alpha}}{4b_1b_2}\middle|\begin{matrix}(1,2\alpha)\\\left(\frac{1}{2}-\frac{\tilde{s}_1}{2},1\right),\left(1-\frac{\tilde{s}_2}{2},1\right)\end{matrix}\right].\end{cases}$$

Applying the second assertion of Lemma 5 with $B_1 = -\frac{\tilde{s}_1}{2}$ for the first equation and $B_1 = -\frac{\tilde{s}_2}{2}$ for the second equation, we obtain the desired form. □

We now show that for a special case of (2.18), we have solutions expressed in terms of Wright functions.

Corollary 2. *Let the determinant of* (2.7) *be* $D = \frac{1}{\alpha^2} - \frac{2b}{\alpha c} + \frac{b^2}{c^2} - \frac{4a}{c} = \frac{1}{4}$, *and suppose* $c \neq 0$.

1. *For* $0 < \alpha < 2$ *and* $c > 0$, (2.18) *has a solution of the following form:*

$$\varphi(z) = c_1 z^{\frac{1}{2}\left(\frac{1}{\alpha} - \frac{b}{c} + \frac{1}{2}\right)\alpha} \Psi\left(-\frac{2z^{-\frac{\alpha}{2}}}{\sqrt{c}}; -\frac{\alpha}{2}, \frac{1}{2}\left(\frac{3}{\alpha} - \frac{b}{c} + \frac{1}{2}\right)\alpha\right).$$

2. *For* $\alpha > 2$, (2.10) *has a solution of the following form:*

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} {}_2\Psi_1\left(\frac{cz^\alpha}{4} \middle| \begin{array}{c} \left(\frac{3}{2} - \frac{2k+1}{\alpha} + \frac{b}{c}, 2\right), (1,1) \\ (1+\alpha-k, \alpha) \end{array}\right).$$

Proof. To prove the first assertion, we need to show that $\varphi(z)$ corresponds to the solution given in the second assertion of Proposition 3. When $D = \frac{1}{4}$ and $0 < \alpha < 2$, the roots of the characteristic equation (2.7) become $s_1 = \frac{1}{2}\left(\frac{1}{\alpha} - \frac{b}{c} + \frac{1}{2}\right)$ and $s_2 = s_1 - \frac{1}{2}$. In this case, the solution given in the second assertion of Proposition 3 is

$$\tilde{\varphi}(z) = c_1 H_{1,2}^{2,0}\left[\frac{z^{-\alpha}}{c} \middle| \begin{array}{c} (1, \alpha) \\ (-s_1, 1), \left(-s_1 - \frac{1}{2}, 1\right) \end{array}\right].$$

Then, applying the duplication formula for the gamma function

$$\Gamma(-s_1 - s)\,\Gamma\left(-s_1 - s + \frac{1}{2}\right) = \sqrt{\pi} 2^{1+2(s_1+s)} \Gamma(-2s_1 - 2s)$$

to $\tilde{\varphi}(z)$, it becomes

$$\begin{aligned}
\tilde{\varphi}(z) &= c_1\sqrt{\pi}2^{1+2s_1}H_{1,1}^{1,0}\left[\frac{4z^{-\alpha}}{c}\middle|\begin{array}{c}(1,\alpha)\\(-2s_1,2)\end{array}\right]\\
&= c_1\sqrt{\pi}4^{s_1}H_{1,1}^{1,0}\left[\frac{2z^{-\frac{\alpha}{2}}}{\sqrt{c}}\middle|\begin{array}{c}\left(1,\frac{\alpha}{2}\right)\\(-2s_1,1)\end{array}\right]\\
&= c_1\sqrt{\pi}c^{s_1}z^{\frac{\alpha}{2}\left(\frac{1}{\alpha}-\frac{b}{c}+\frac{1}{2}\right)}H_{1,1}^{1,0}\left[\frac{2z^{-\frac{\alpha}{2}}}{\sqrt{c}}\middle|\begin{array}{c}\left(\frac{\alpha}{2}\left(\frac{3}{\alpha}-\frac{b}{c}+\frac{1}{2}\right),\frac{\alpha}{2}\right)\\(0,1)\end{array}\right]\\
&= c_1\sqrt{\pi}c^{s_1}z^{\frac{\alpha}{2}\left(\frac{1}{\alpha}-\frac{b}{c}+\frac{1}{2}\right)}{}_0\Psi_1\left[-\frac{2z^{-\frac{\alpha}{2}}}{\sqrt{c}}\middle|\begin{array}{c}-\\\left(\frac{\alpha}{2}\left(\frac{3}{\alpha}-\frac{b}{c}+\frac{1}{2}\right),-\frac{\alpha}{2}\right)\end{array}\right]\\
&= c_1\sqrt{\pi}c^{s_1}z^{\frac{\alpha}{2}\left(\frac{1}{\alpha}-\frac{b}{c}+\frac{1}{2}\right)}\Psi\left[-\frac{2z^{-\frac{\alpha}{2}}}{\sqrt{c}};-\frac{\alpha}{2},\frac{\alpha}{2}\left(\frac{3}{\alpha}-\frac{b}{c}+\frac{1}{2}\right)\right].
\end{aligned}$$

The second through fifth equalities here follow from (1.9), (1.10), (1.21) and (1.17), respectively. It is thus seen that $\tilde{\varphi}(z)=\varphi(z)$.

We can prove the second assertion by applying the duplication formula for the gamma function

$$\Gamma\left(1-\frac{k}{\alpha}-s_1+i\right)\Gamma\left(1-\frac{k}{\alpha}-s_1+i+\frac{1}{2}\right)=\sqrt{\pi}2^{\frac{2k+1}{\alpha}-\frac{1}{2}-i}\Gamma\left(2-\frac{2k}{\alpha}-2s_1+2i\right)$$

to the solution given in the second assertion of Proposition 2 as following

$$\begin{aligned}
\varphi(z) &= \sum_{k=1}^{n}c_kz^{\alpha-k}{}_3\Psi_1\left[cz^{\alpha}\middle|\begin{array}{c}\left(1-\frac{k}{\alpha}-s_1,1\right),\left(1-\frac{k}{\alpha}-s_2,1\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right]\\
&= \sum_{k=1}^{n}c_k\sum_{i=0}^{\infty}\frac{c^iz^{\alpha-k+\alpha i}\Gamma\left(1-\frac{k}{\alpha}-s_1+i\right)\Gamma\left(1-\frac{k}{\alpha}-s_2+i\right)\Gamma(1+i)}{\Gamma\left(1+\alpha-k+\alpha i\right)i!}\\
&= \sum_{k=1}^{n}c_k\sum_{i=0}^{\infty}\frac{\sqrt{\pi}2^{\frac{2k+1}{\alpha}-\frac{1}{2}-i}c^iz^{\alpha-k+\alpha i}\Gamma\left(2-\frac{2k}{\alpha}-2s_1+2i\right)\Gamma(1+i)}{\Gamma\left(1+\alpha-k+\alpha i\right)i!}\\
&= \sum_{k=1}^{n}\bar{c}_kz^{\alpha-k}{}_2\Psi_1\left[\frac{cz^{\alpha}}{2}\middle|\begin{array}{c}\left(2-\frac{2k}{\alpha}-2s_1,2\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right].
\end{aligned}$$

□

To this point, we have presented several exact solutions of (2.3) and (2.4). These solutions are classified according to the kind of special functions used to express them. Now, we discuss the second advantage of rewriting the right-hand sides of (2.3) and

(2.4) in factorized differential operator form. Specifically, we show that utilizing this form, we are able to generalize our treatment of (2.3) and (2.4) to the cases of (2.1) and (2.2) with arbitrary integers m, m_1 and m_2. As a result, we find that the solutions to (2.3) and (2.4) of Proposition 2 and the solutions to (2.17) and (2.18) of Proposition 3 can all be represented in a unified manner by a single general formula.

2.2 Solutions of FODE with higher order derivatives

We seek a solution of the following FODE with an mth-order Cauchy-Euler differential operator on the right-hand side:

$$\frac{d^\alpha \varphi}{dz^\alpha} = \frac{a_m}{\alpha^m} z^m \frac{d^m \varphi}{dz^m} + \frac{a_{m-1}}{\alpha^{m-1}} z^{m-1} \frac{d^{m-1}\varphi}{dz^{m-1}} + \cdots + \frac{a_1}{\alpha} z \frac{d\varphi}{dz} + a_0 \varphi, \quad z > 0, \tag{2.20}$$

where a_i $(i = 0, \ldots, m)$ are real numbers and $a_m \neq 0$. We represent the right-hand side of (2.20) by $P(\varphi)$. Then, from

$$P(z^s) = \left(a_0 + \sum_{i=1}^{m} a_i \prod_{j=0}^{i-1} \left(\frac{s}{\alpha} - \frac{j}{\alpha} \right) \right) z^s,$$

we see that the characteristic polynomial of P is

$$\tilde{P}(s) = a_0 + \sum_{i=1}^{m} a_i \prod_{j=0}^{i-1} \left(s - \frac{j}{\alpha} \right).$$

Let $s_1, s_2, \ldots, s_m$ be the roots of the characteristic polynomial $\tilde{P}(s)$. Then, we can rewrite the right-hand side of (2.20) as

$$P(\varphi) = a_m \prod_{i=1}^{m} \left(\frac{1}{\alpha} z \frac{d}{dz} - s_i \right) \varphi. \tag{2.21}$$

Now, generalizing the results of the previous section, we formulate the following theorem.

Theorem 6. *The equation (2.20) has the following solutions:*

1. For $0 < \alpha < m$ *and* $a_m > 0$,

$$\varphi(z) = c_1 H^{m,0}_{1,m}\left[\frac{z^{-\alpha}}{a_m}\middle| \begin{matrix} (1,\alpha) \\ (-s_1,1),(-s_2,1),\ldots,(-s_m,1) \end{matrix}\right];$$

2. For $\alpha > m$,

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k}{}_{m+1}\Psi_1\left[a_m z^{\alpha}\middle| \begin{matrix} \left(1-\frac{k}{\alpha}-s_1,1\right),\ldots,\left(1-\frac{k}{\alpha}-s_m,1\right),(1,1) \\ (1+\alpha-k,\alpha) \end{matrix}\right],$$

where c_k $(k = 1,\ldots,n)$ *are arbitrary constants.*

Proof. Similarly to the proof of Proposition 3, by applying the first assertion of Lemma 5 and the relationship (1.10) with $\sigma = 1$, we obtain

$$\begin{aligned}\frac{d^{\alpha}\varphi(z)}{dz^{\alpha}} &= z^{-\alpha}H^{m,0}_{1,m}\left[\frac{z^{-\alpha}}{a_m}\middle| \begin{matrix} (1-\alpha,\alpha) \\ (-s_1,1),\ldots,(-s_m,1) \end{matrix}\right] \\ &= a_m H^{m,0}_{1,m}\left[\frac{z^{-\alpha}}{a_m}\middle| \begin{matrix} (1-\alpha+\alpha,\alpha) \\ (1-s_1,1),\ldots,(1-s_m,1) \end{matrix}\right]\end{aligned} \tag{2.22}$$

for the left-hand side of (2.20). Using the second assertion of Lemma 5 repetitively, we obtain the right hand side of (2.21) as

$$a_m \prod_{i=1}^{m}\left(\frac{1}{\alpha}z\frac{d}{dz} - s_i\right)\varphi = a_m H^{m,0}_{1,m}\left[\frac{z^{-\alpha}}{a_m}\middle| \begin{matrix} (1,\alpha) \\ (1-s_1,1),\ldots,(1-s_m,1) \end{matrix}\right],$$

which equals to (2.22).

Now let us prove the second assertion. From the linearity of (2.20), it is sufficient to show that a single summand

$$\varphi_k(z) = z^{\alpha-k}{}_{m+1}\Psi_1\left[a_m z^{\alpha}\middle| \begin{matrix} (1-\frac{k}{\alpha}-s_1,1),\ldots,(1-\frac{k}{\alpha}-s_m,1),(1,1) \\ (1+\alpha-k,\alpha) \end{matrix}\right] \text{ for } 1 \le k \le n$$

of the solution $\varphi(z)$ satisfies (2.20). By Corollary 1 with $m = 1$, we have the following identity for the left-hand side of (2.20):

$$\frac{d^{\alpha}\varphi_k}{dz^{\alpha}} = a_m z^{\alpha-k}{}_{m+1}\Psi_1\left[a_m z^{\alpha}\middle| \begin{matrix} (1,1),(2-\frac{k}{\alpha}-s_1,1),\ldots,(2-\frac{k}{\alpha}-s_m,1) \\ (1+\alpha-k,\alpha) \end{matrix}\right]. \tag{2.23}$$

Then, by virtue of (2.21) and the second assertion of Lemma 4 with $\sigma = \alpha$, $R = -s_i$, the right-hand side of (2.20) becomes

$$a_m \prod_{i=1}^{m} \left(\frac{1}{\alpha} z \frac{d}{dz} - s_i \right) \varphi_k$$
$$= a_m z^{\alpha-k}{}_{m+1}\Psi_1 \left[a_m z^\alpha \middle| \begin{matrix} (1,1),(2-\frac{k}{\alpha}-s_1,1),\ldots,(2-\frac{k}{\alpha}-s_m,1) \\ (1-k+\alpha,\alpha) \end{matrix} \right],$$

which equals to the corresponding left-hand side (2.23). □

It is now clear that the solutions given in the first and second assertions of Proposition 2 can be expressed by using only the first assertion of Theorem 6 and that the solutions of the first and second assertion of Proposition 3 can be expressed by using only the second assertion of Theorem 6 by taking $m = 1$ and $m = 2$, respectively.

In a similar manner, we can generalize the system (2.4) as

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{a_{m_1}}{\alpha^{m_1}} z^{m_1} \frac{d^{m_1}\psi}{dz^{m_1}} + \frac{a_{m_1-1}}{\alpha^{m_1-1}} z^{m_1-1} \frac{d^{m_1-1}\psi}{dz^{m_1-1}} + \cdots + \frac{a_1}{\alpha} z \frac{d\psi}{dz} + a_0\psi, \\ \frac{d^\alpha \psi}{dz^\alpha} = \frac{b_{m_2}}{\alpha^{m_2}} z^{m_2} \frac{d^{m_2}\varphi}{dz^{m_2}} + \frac{b_{m_2-1}}{\alpha^{m_2-1}} z^{m_2-1} \frac{d^{m_2-1}\varphi}{dz^{m_2-1}} + \cdots + \frac{b_1}{\alpha} z \frac{d\varphi}{dz} + b_0\varphi, \end{cases} \quad z > 0, \qquad (2.24)$$

where a_i $(i = 1, \ldots, m_1)$ and b_j $(j = 1, \ldots, m_2)$ are real numbers and $a_{m_1} b_{m_2} \neq 0$. The characteristic polynomials of the right-hand sides of the first and second equations of this system are

$$P_1(s) = a_0 + \sum_{i=1}^{m_1} a_i \prod_{j=0}^{i-1} \left(s - \frac{j}{\alpha} \right), \qquad P_2(s) = b_0 + \sum_{i=1}^{m_2} b_i \prod_{j=0}^{i-1} \left(s - \frac{j}{\alpha} \right). \qquad (2.25)$$

We write the roots of the characteristic polynomials $P_1(s)$ and $P_2(s)$ as $s_1, s_2, \ldots, s_{m_1}$ and $s_{m_1+1}, s_{m_1+2}, \ldots, s_{m_1+m_2}$, respectively. Then, we can rewrite the right-hand sides of the equations in (2.24) as

$$\sum_{i=0}^{m_1} \frac{a_i}{\alpha^i} z^i \frac{d^i \psi}{dz^i} = 2^{m_1} a_{m_1} \prod_{i=1}^{m_1} \left(\frac{1}{2\alpha} z \frac{d}{dz} - \frac{s_i}{2} \right) \psi(z)$$

and

$$\sum_{i=0}^{m_2} \frac{b_i}{\alpha^i} z^i \frac{d^i \varphi}{dz^i} = 2^{m_2} b_{m_2} \prod_{i=m_1+1}^{m_1+m_2} \left(\frac{1}{2\alpha} z \frac{d}{dz} - \frac{s_i}{2} \right) \varphi(z).$$

The following result concerns solutions of (2.24).

Theorem 7. *The system (2.24) has the following solutions. Here, we use* m *to represent* $m_1 + m_2$ *and* A *to represent* $2^{m_1+m_2} a_{m_1} b_{m_2}$.

1. *For* $0 < \alpha < \frac{m}{2}$ *and* $a_{m_1} b_{m_2} > 0$,

$$\begin{aligned}\varphi(z) &= c_1 \operatorname{sgn}(b_{m_2}) H^{m,0}_{1,m}\left[\frac{z^{-2\alpha}}{A}\middle|\begin{matrix}(1,2\alpha)\\ \left(-\frac{s_i}{2}+\frac{1}{2},1\right)_{1,m_1},\left(-\frac{s_i}{2},1\right)_{m_1+1,m}\end{matrix}\right],\\ \psi(z) &= c_1 2^{\frac{m_2-m_1}{2}}\sqrt{\frac{b_{m_2}}{a_{m_1}}} H^{m,0}_{1,m}\left[\frac{z^{-2\alpha}}{A}\middle|\begin{matrix}(1,2\alpha)\\ \left(-\frac{s_i}{2},1\right)_{1,m_1},\left(-\frac{s_i}{2}+\frac{1}{2},1\right)_{m_1+1,m}\end{matrix}\right].\end{aligned}$$

2. *For* $\alpha > \frac{m}{2}$,

$$\begin{aligned}\varphi(z) &= \sum_{k=1}^{n} c_{k,1} z^{\alpha-k}\varphi_{k1}(z) + 2^{m_1} a_{m_1}\sum_{k=1}^{n} c_{k,2} z^{2\alpha-k}\varphi_{k2}(z),\\ \psi(z) &= 2^{m_2} b_{m_2}\sum_{k=1}^{n} c_{k,1} z^{2\alpha-k}\psi_{k1}(z) + \sum_{k=1}^{n} c_{k,2} z^{\alpha-m_2}\psi_{k2}(z),\end{aligned}$$

where

$$\begin{aligned}\varphi_{k1}(z) &= {}_{m+1}\Psi_1\left[Az^{2\alpha}\middle|\begin{matrix}\left(1-\frac{k}{2\alpha}-\frac{s_i}{2},1\right)_{1,m_1},\left(\frac{1}{2}-\frac{k}{2\alpha}-\frac{s_i}{2},1\right)_{m_1+1,m},(1,1)\\ (1+\alpha-k,2\alpha)\end{matrix}\right],\\ \varphi_{k2}(z) &= {}_{m+1}\Psi_1\left[Az^{2\alpha}\middle|\begin{matrix}\left(\frac{3}{2}-\frac{k}{2\alpha}-\frac{s_i}{2},1\right)_{1,m_1},\left(1-\frac{k}{2\alpha}-\frac{s_i}{2},1\right)_{m_1+1,m},(1,1)\\ (1+2\alpha-k,2\alpha)\end{matrix}\right],\\ \psi_{k1}(z) &= {}_{m+1}\Psi_1\left[Az^{2\alpha}\middle|\begin{matrix}\left(1-\frac{k}{2\alpha}-\frac{s_i}{2},1\right)_{1,m_1},\left(\frac{3}{2}-\frac{k}{2\alpha}-\frac{s_i}{2},1\right)_{m_1+1,m},(1,1)\\ (1+2\alpha-k,2\alpha)\end{matrix}\right],\\ \psi_{k2}(z) &= {}_{m+1}\Psi_1\left[Az^{2\alpha}\middle|\begin{matrix}\left(\frac{1}{2}-\frac{k}{2\alpha}-\frac{s_i}{2},1\right)_{1,m_1},\left(1-\frac{k}{2\alpha}-\frac{s_i}{2},1\right)_{m_1+1,m},(1,1)\\ (1+\alpha-k,2\alpha)\end{matrix}\right],\end{aligned}$$

and c_1, $c_{k,1}$ *and* $c_{k,2}$ $(k = 1, \ldots, n)$ *are constants.*

Proof. Let us prove the first assertion. The left-hand side of the first equation in (2.24) is obtained as following using first assertion of Lemma 5

$$\begin{aligned}\frac{d^\alpha\varphi}{dz^\alpha} &= \operatorname{sgn}(b_k) z^{-\alpha} H^{m+k,0}_{1,m+k}\left[\frac{z^{-2\alpha}}{2^{m+k}a_m b_k}\middle|\begin{matrix}(1-\alpha,2\alpha)\\ (-\frac{s_i}{2}+\frac{1}{2},1)_{1,m},(-\frac{s_i}{2},1)_{m+1,m+k}\end{matrix}\right]\\ &= \operatorname{sgn}(b_k)\sqrt{2^{m+k}a_m b_k} H^{m+k,0}_{1,m+k}\left[\frac{z^{-2\alpha}}{2^{m+k}a_m b_k}\middle|\begin{matrix}(1,2\alpha)\\ (-\frac{s_i}{2}+1,1)_{1,m},(-\frac{s_i}{2}+\frac{1}{2},1)_{m+1,m+k}\end{matrix}\right]\end{aligned}$$

and by the second assertion of Lemma 5 the right-hand side is

$$2^m a_m \prod_{i=1}^{m} \left(\frac{1}{2\alpha} z \frac{d}{dz} - \frac{s_i}{2}\right) \psi = \operatorname{sgn}(a_m) \sqrt{2^{m+k} a_m b_k} \times$$
$$H^{m+k,0}_{1,m+k} \left[\frac{z^{-2\alpha}}{2^{m+k} a_m b_k} \middle| \begin{matrix} (1, 2\alpha) \\ (-\frac{s_i}{2}+1, 1)_{1,m}, (-\frac{s_i}{2}+\frac{1}{2}, 1)_{m+1,m+k} \end{matrix} \right].$$

Since $\operatorname{sgn}(b_k) = \operatorname{sgn}(a_m)$, the above two quantities are equal. Now let us check the second equation. By the first assertion of Lemma 5 and (1.10), the left hand side is

$$\begin{aligned} \frac{d^\alpha \psi}{dz^\alpha} &= 2^{\frac{k-m}{2}} \sqrt{\frac{b_k}{a_m}} z^{-\alpha} H^{m+k,0}_{1,m+k} \left[\frac{z^{-2\alpha}}{2^{m+k} a_m b_k} \middle| \begin{matrix} (1-\alpha, 2\alpha) \\ (-\frac{s_i}{2}, 1)_{1,m}, (-\frac{s_i}{2}+\frac{1}{2}, 1)_{m+1,m+k} \end{matrix} \right] \\ &= \sqrt{\frac{2^{k-m} b_k}{a_m}} \sqrt{2^{m+k} a_m b_k} H^{m+k,0}_{1,m+k} \left[\frac{z^{-2\alpha}}{2^{m+k} a_m b_k} \middle| \begin{matrix} (1, 2\alpha) \\ (-\frac{s_i}{2}+\frac{1}{2}, 1)_{1,m}, (-\frac{s_i}{2}+1, 1)_{m+1,m+k} \end{matrix} \right] \\ &= 2^k |b_k| H^{m+k,0}_{1,m+k} \left[\frac{z^{-2\alpha}}{2^{m+k} a_m b_k} \middle| \begin{matrix} (1, 2\alpha) \\ (-\frac{s_i}{2}+\frac{1}{2}, 1)_{1,m}, (-\frac{s_i}{2}+1, 1)_{m+1,m+k} \end{matrix} \right] \end{aligned}$$

and by the second assertion of Lemma 5 the right-hand side is

$$2^k b_k \prod_{i=m+1}^{m+k} \left(\frac{1}{2\alpha} z \frac{d}{dz} - \frac{s_i}{2}\right) \varphi = 2^k \operatorname{sgn}(b_k) \times$$
$$H^{m+k,0}_{1,m+k} \left[\frac{z^{-2\alpha}}{2^{m+k} a_m b_k} \middle| \begin{matrix} (1, 2\alpha) \\ (-\frac{s_i}{2}+\frac{1}{2}, 1)_{1,m}, (-\frac{s_i}{2}+1, 1)_{m+1,m+k} \end{matrix} \right].$$

Since $|b_k| = \operatorname{sgn}(b_k) b_k$, the above two quantities are equal to each other.

Now let $\alpha \geq \frac{m}{2}$. Then, we only need to check that the summand

$$\begin{aligned} \varphi_{1,j}(z) &= z^{\alpha-j} \\ &\times {}_{m+k+1}\Psi_1 \left[2^{m+k} a_m b_k z^{2\alpha} \middle| \begin{matrix} \left(1 - \frac{j}{2\alpha} - \frac{s_i}{2}, 1\right)_{1,m}, \left(\frac{1}{2} - \frac{j}{2\alpha} - \frac{s_i}{2}, 1\right)_{m+1,m+k}, (1,1) \\ (1+\alpha-j, 2\alpha) \end{matrix} \right], \end{aligned}$$

$$\begin{aligned} \psi_{1,j}(z) &= 2^k b_k z^{2\alpha-j} \\ &\times {}_{m+k+1}\Psi_1 \left[2^{m+k} a_m b_k z^{2\alpha} \middle| \begin{matrix} \left(1 - \frac{j}{2\alpha} - \frac{s_i}{2}, 1\right)_{1,m}, \left(\frac{3}{2} - \frac{j}{2\alpha} - \frac{s_i}{2}, 1\right)_{m+1,m+k}, (1,1) \\ (1+2\alpha-j, 2\alpha) \end{matrix} \right] \end{aligned}$$

of the solution satisfies (2.21). Let us check the first equation. The left hand side follows from Corollary 1 by taking $1+\alpha-j+m2\alpha-\alpha-1\geq 0 \Rightarrow m=1$

$$\frac{d^\alpha \varphi_{1,j}}{dz^\alpha} = 2^{m+k}a_m b_k \\ \times z^{2\alpha-j}{}_{m+k+1}\Psi_1\left[2^{m+k}a_m b_k z^{2\alpha}\middle| \begin{matrix}(1,1),(2-\frac{j}{2\alpha}-\frac{s_i}{2})_{1,m},(\frac{3}{2}-\frac{j}{2\alpha}-\frac{s_i}{2},1)_{m+1,m+k}\\(1+2\alpha-j,2\alpha)\end{matrix}\right].$$

For the right hand side, we use the second assertion of Lemma 4 with $R=-\frac{s_i}{2}$ and $\sigma=2\alpha$

$$2^m a_m \prod_{i=1}^{m}\left(\frac{1}{2\alpha}z\frac{d}{dz}-\frac{s_i}{2}\right)\psi_{1,j}(z) = 2^{m+k}a_m b_k z^{2\alpha-j} \\ \times {}_{m+k+1}\Psi_1\left[2^{m+k}a_m b_k z^{2\alpha}\middle| \begin{matrix}(2-\frac{j}{2\alpha}-\frac{s_i}{2},1)_{1,m},(\frac{3}{2}-\frac{j}{2\alpha}-\frac{s_i}{2},1)_{m+1,m+k},(1,1)\\(1+2\alpha-j,2\alpha)\end{matrix}\right],$$

which equals to the corresponding left hand side of the first equation in the system. For the left-hand side of the second equation in (2.21), we use Corollary 1 with $1+2\alpha-j+m2\alpha-\alpha-1\geq 0\Rightarrow m=0$

$$\frac{d^\alpha \psi_{1,j}}{dz^\alpha} = 2^k b_k z^{\alpha-j} \\ \times {}_{m+k+1}\Psi_1\left[2^{m+k}a_m b_k z^{2\alpha}\middle| \begin{matrix}(1,1),(1-\frac{j}{2\alpha}-\frac{s_i}{2},1)_{1,m},(\frac{3}{2}-\frac{j}{2\alpha}-\frac{s_i}{2},1)_{m+1,m+k}\\(1+\alpha-j,2\alpha)\end{matrix}\right].$$

For the right hand side, we use the second assertion of Lemma 4 with $R=-\frac{s_i}{2}$, $\sigma=2\alpha$

$$2^k b_k \prod_{i=m+1}^{m+k}\left(\frac{1}{2\alpha}z\frac{d}{dz}-\frac{s_i}{2}\right)\varphi_{1,j}(z) = 2^k b_k z^{\alpha-j} \\ \times {}_{m+k+1}\Psi_1\left[2^{m+k}a_m b_k z^{2\alpha}\middle| \begin{matrix}(\frac{3}{2}-\frac{j}{2\alpha}-\frac{s_i}{2},1)_{m+1,m+k},(1-\frac{j}{2\alpha}-\frac{s_i}{2},1)_{1,m},(1,1)\\(1+\alpha-j,2\alpha)\end{matrix}\right],$$

which equals to the left hand side. $\square$

We, thus, see that the third assertion of Proposition 2 and the third assertion of Proposition 3 correspond to a particular case $m_1=m_2=1$ of Theorem 7.

Chapter 3

Lie symmetry analysis of a class of time fractional diffusion-wave systems

In 1987, G.W. Bluman et al. [3] gave a complete group classification and some invariant solutions of variable coefficient wave equation $u_{tt} = c^2(x)u_{xx}$ and its corresponding system $u_t = c^2(x)v_x$, $v_t = u_x$. In 2015, Q. Huang et al. [10] studied Lie symmetries of the systems of the following form:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = c^2(x)v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = u_x, \end{cases} \tag{3.1}$$

which can be considered as a time fractional generalization of the corresponding systems of wave equations. Here, the fractional derivative is defined in Riemann-Liouville's manner and $c(x)$ is a sufficiently differentiable function.

In [10], the admitted symmetries of (3.1) are determined and the reduced systems of fractional ordinary differential equations (FODEs), even some explicit invariant solutions, are presented for non-constant case of $c(x)$. In [27], the Lie symmetries of (3.1) are found for the case $c(x) \equiv \text{const}$. In both [10] and [27], there were studied Lie symmetries and reduced systems, but the invariant solutions were not obtained explicitly. Here, we explicitly give invariant solutions corresponding to each symmetries in the optimal system in terms of special functions that were introduced in Chapter 1.

The importance of finding group invariant solutions of (3.1) lies in the fact that if $(u(x,t), v(x,t))$ solves (3.1), then $u(x,t)$ solves the sequential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha}{\partial t^\alpha}u = c^2(x)u_{xx}, \tag{3.2}$$

and $v(x,t)$ solves

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha}{\partial t^\alpha}v = (c^2(x)v_x)_x. \tag{3.3}$$

However, it should be noted that, in general, the Lie group of point transformations that leaves the system in (3.1) invariant, does not necessarily correspond to a Lie group of point transformations that leaves the sequential equations invariant. These sequential equations can be considered as generalizations of wave equations into time fractional case. Another generalization of the wave equation into time fractional equation with the Riemann-Liouville derivative was studied via Lie symmetry analysis by R. Gorenflo et al. [9] for the case $c(x) \equiv c$, where c is a constant. From this discussion, we can see that (3.1) interpolates between the corresponding systems of heat equations and wave equations when α varies from $\frac{1}{2}$ to 1 by virtue of the formula (1.5). The behavior of invariant solutions corresponding to the order of fractional derivative is seen from the graphs of the solutions.

3.1 Classification of group invariant solutions

In [10], there were obtained classification of group invariant solutions with regard to the function $c(x)$. It was determined that there are three types of function $c(x)$ so that the system (3.1) admits infinitesimal symmetries. In the following subsections, we determine the optimal systems and corresponding reduced systems for three cases of the function $c(x)$.

3.1.1 Invariant solutions of (3.1) with $c(x) = m_1(x+m_2)^{m_3}$

If $c(x) = m_1(x+m_2)^{m_3}$, here m_1, m_2, m_3 are constants and $m_3 \neq 0$, then the symmetries of (3.1) are known [10]:

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_2 = (x+m_2)\frac{\partial}{\partial x} + \frac{1-m_3}{\alpha}t\frac{\partial}{\partial t} + m_3 u\frac{\partial}{\partial u}.$$

We assume, without loss of generality, that $c(x) = x^m$, where m is a non-zero constant. Then, the symmetries become

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \quad \text{and} \quad X_2 = x\frac{\partial}{\partial x} + \frac{1-m}{\alpha}t\frac{\partial}{\partial t} + mu\frac{\partial}{\partial u}.$$

The Lie algebra generated by X_1 and X_2 is Abelian, thus the optimal system consists of

$$\begin{aligned} U_1 &= X_1, \\ U_2 &= X_2 + aX_1 = x\frac{\partial}{\partial x} + \frac{1-m}{\alpha}t\frac{\partial}{\partial t} + (a+m)u\frac{\partial}{\partial u} + av\frac{\partial}{\partial v}, \qquad \text{here } a \in \mathbb{R}. \end{aligned}$$

The element U_1 does not yield any invariant solutions. The characteristic equation of U_2 reads

$$\frac{dx}{x} = \frac{\alpha dt}{(1-m)t} = \frac{du}{(a+m)u} = \frac{dv}{av},$$

which gives the similarity variable $z = x^{\frac{m-1}{\alpha}}t$. The similarity transformation, in some fields known as ansatz, is

$$\begin{cases} u = x^{a+m}\varphi(z), \\ v = x^a\psi(z). \end{cases} \tag{3.4}$$

Let us substitute (3.4) into (3.1) with $c(x) = x^m$. Then the left hand side the first equation becomes

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}x^{a+m}\varphi(x^{\frac{m-1}{\alpha}}\tau)d\tau \\ &= \begin{bmatrix} \bar{\tau} = x^{\frac{m-1}{\alpha}}\tau, & d\bar{\tau} = x^{\frac{m-1}{\alpha}}d\tau \\ t = x^{\frac{1-m}{\alpha}}z, & dt^n = x^{\frac{1-m}{\alpha}n}dz^n \end{bmatrix} \\ &= x^{\frac{m-1}{\alpha}n + \frac{1-m}{\alpha}(n-\alpha-1)+a+m+\frac{1-m}{\alpha}}\frac{1}{\Gamma(n-\alpha)}\int_0^z (z-\bar{\tau})^{n-\alpha-1}\varphi(\bar{\tau})d\bar{\tau} \\ &= x^{2m+a-1}\frac{\partial^\alpha\varphi}{\partial z^\alpha}, \end{aligned}$$

and the right hand side of the first equation in the system becomes

$$\begin{aligned} x^{2m}v_x &= x^{2m}\left(ax^{a-1}\psi + x^a\frac{m-1}{\alpha}x^{\frac{m-1}{\alpha}-1}t\psi\right) \\ &= x^{2m+a-1}\left(a\psi + \frac{m-1}{\alpha}z\psi'\right). \end{aligned}$$

Then the right and left hand sides of the second equation are calculated as

$$
\begin{aligned}
\frac{\partial^\alpha v}{\partial t^\alpha} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}x^a\psi(x^{\frac{m-1}{\alpha}}\tau)d\tau \\
&= \begin{bmatrix} \bar\tau = x^{\frac{m-1}{\alpha}}\tau, & d\bar\tau = x^{\frac{m-1}{\alpha}}d\tau \\ t = x^{\frac{1-m}{\alpha}}z, & dt^n = x^{\frac{1-m}{\alpha}n}dz^n \end{bmatrix} \\
&= x^{\frac{m-1}{\alpha}n+\frac{1-m}{\alpha}(n-\alpha-1)+a+\frac{1-m}{\alpha}}\frac{1}{\Gamma(n-\alpha)}\int_0^z (z-\bar\tau)^{n-\alpha-1}\psi(\bar\tau)d\bar\tau \\
&= x^{a+m-1}\frac{\partial^\alpha\varphi}{\partial z^\alpha}\psi \quad \text{and} \\
u_x &= (a+m)x^{a+m-1}\varphi + x^{(a+1)m}\varphi'(z)\left(\frac{m-1}{\alpha}\right)x^{\frac{m-1}{\alpha}-1}t \\
&= x^{a+m-1}\left[(a+m)\varphi + \frac{m-1}{\alpha}z\varphi'\right].
\end{aligned}
$$

Thus, the reduced system becomes

$$
\begin{cases} \dfrac{d^\alpha\varphi}{dz^\alpha} = am\psi + \dfrac{m-1}{\alpha}z\psi', \\ \dfrac{d^\alpha\psi}{dz^\alpha} = (a+1)m\varphi + \dfrac{m-1}{\alpha}z\varphi'. \end{cases} \tag{3.5}
$$

The problem of finding invariant solutions of (3.1) with $c(x) = x^m$ is, thus, reduced into the problem of finding the solutions of (2.4).

Hence, we give invariant solutions of (3.1) with $c(x) = x^m$ as follows:

1. If $m = 1$, then using the second assertion of Proposition 1 with $a_1 = a$ and $a_2 = a + 1$ we obtain the following via (3.4):

$$
\begin{aligned}
u(x,t) &= x^{a+1}\left(\sum_{k=1}^n c_{k,1}t^{\alpha-k}E_{2\alpha,1+\alpha-k}(a(a+1)t^{2\alpha})\right. \\
&\quad \left. + a\sum_{k=1}^n c_{k,2}t^{2\alpha-k}E_{2\alpha,1+2\alpha-k}(a(a+1)t^{2\alpha})\right), \\
v(x,t) &= x^a\left((a+1)\sum_{k=1}^n c_{k,1}t^{2\alpha-k}E_{2\alpha,1+2\alpha-k}(a(a+1)t^{2\alpha})\right. \\
&\quad \left. + \sum_{k=1}^n c_{k,2}t^{\alpha-k}E_{2\alpha,1+\alpha-k}(a(a+1)t^{2\alpha})\right).
\end{aligned} \tag{3.6}
$$

In [10], there were explicitly obtained two solutions of (3.1) with $c(x) = x$:

i) $u(x,t) = \mu x t^{\alpha-1}$, $v(x,t) \equiv v(t) = \mu\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}$,

ii) $u(x,t) \equiv u(t) = -\mu\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}$, $v(x,t) = \mu\frac{t^{\alpha-1}}{x}$, where μ is an arbitrary constant. These solutions can be obtained from (3.6) for suitably chosen parameters. If we set $a = 0$, $c_{1,1} = \mu\Gamma(\alpha)$ and $c_{i,1} = c_{k,2} = 0$ $(i = 2, \ldots, n;\ k = 1, \ldots, n)$ in (3.6), then it equals to the solution i), and if $a = -1$, $c_{1,2} = \mu\Gamma(\alpha)$ and $c_{k,1} = c_{i,2} = 0$ $(k = 1, \ldots, n;\ i = 2, \ldots, n)$, then (3.6) equals to the solution ii).

2. If $m \neq 1$ and $0 < \alpha < 1$, then using the third assertion of Proposition 3 with $a_1 = a$, $a_2 = a + m$, $b_1 = b_2 = m - 1$, we obtain the following via (3.4):

$$\begin{aligned} u(x,t) &= c\,\mathrm{sgn}(m-1)x^{a+m}H^{2,0}_{1,2}\left[\frac{1}{4(m-1)^2}\frac{x^{2(1-m)}}{t^{2\alpha}}\left| \begin{matrix} (1,2\alpha) \\ (\frac{a+m-1}{2(m-1)},1),(\frac{a+m}{2(m-1)},1) \end{matrix}\right.\right], \\ v(x,t) &= cx^{a}H^{2,0}_{1,2}\left[\frac{1}{4(m-1)^2}\frac{x^{2(1-m)}}{t^{2\alpha}}\left| \begin{matrix} (1,2\alpha) \\ (\frac{a}{2(m-1)},1),(\frac{a+2m-1}{2(m-1)},1) \end{matrix}\right.\right], \end{aligned} \tag{3.7}$$

here $x > 0$, $t > 0$.

3. If $m \neq 1$ and $\alpha > 1$, then using the third assertion of Proposition 2 with $a_1 = a$, $a_2 = a + m$, $b_1 = b_2 = m - 1$, we obtain the following via (3.4):

$$\begin{aligned} &u(x,t) = x^{a+2m-1}t^{\alpha} \\ &\times \sum_{k=1}^{n} c_{k,1}\frac{x^{(1-m)\frac{k}{\alpha}}}{t^k}\,{}_3\Psi_1\left[\omega^2 x^{2(m-1)}t^{2\alpha}\left| \begin{matrix} \left(\omega_1 - \frac{k}{2\alpha},1\right),\left(\omega_2 - \frac{k}{2\alpha},1\right),(1,1) \\ (1+\alpha-k,2\alpha) \end{matrix}\right.\right] \\ &+ \omega x^{a+3m-2}t^{2\alpha} \\ &\times \sum_{k=1}^{n} c_{k,2}\frac{x^{(1-m)\frac{k}{\alpha}}}{t^k}\,{}_3\Psi_1\left[\omega^2 x^{2(m-1)}t^{2\alpha}\left| \begin{matrix} \left(\omega_1 + \frac{1}{2} - \frac{k}{2\alpha},1\right),\left(\omega_2 + \frac{1}{2} - \frac{k}{2\alpha},1\right),(1,1) \\ (1+2\alpha-k,2\alpha) \end{matrix}\right.\right], \\ &v(x,t) = \omega x^{a+2m-2}t^{2\alpha} \\ &\sum_{k=1}^{n} c_{k,1}\frac{x^{(1-m)\frac{k}{\alpha}}}{t^k}\,{}_3\Psi_1\left[\omega^2 x^{2(m-1)}t^{2\alpha}\left| \begin{matrix} \left(\omega_1 - \frac{k}{2\alpha},1\right),\left(\omega_2 + 1 - \frac{k}{2\alpha},1\right),(1,1) \\ (1+2\alpha-k,2\alpha) \end{matrix}\right.\right] \\ &+ x^{a+m-1}t^{\alpha} \\ &\times \sum_{k=1}^{n} c_{k,2}\frac{x^{(1-m)\frac{k}{\alpha}}}{t^k}\,{}_3\Psi_1\left[\omega^2 x^{2(m-1)}t^{2\alpha}\left| \begin{matrix} \left(\omega_1 - \frac{1}{2} - \frac{k}{2\alpha},1\right),\left(\omega_2 + \frac{1}{2} - \frac{k}{2\alpha},1\right),(1,1) \\ (1+\alpha-k,2\alpha) \end{matrix}\right.\right] \end{aligned} \tag{3.8}$$

here $\omega = 2(m-1)$, $\omega_1 = \frac{a}{2(m-1)} + 1$, $\omega_2 = \frac{a+1}{2(m-1)} + 1$, and $c_{k,1}$, $c_{k,2}$ $(k = 1, \ldots, n)$ are constants.

3.1.2 Invariant solutions of (3.1) with $c(x) = m_1 e^{m_2 x}$

If $c(x) = m_1 e^{m_2 x}$, here m_1, m_2 are nonzero constants, then the symmetries are [10]:

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_2 = \frac{1}{m_2}\frac{\partial}{\partial x} - \frac{1}{\alpha}t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

We assume, without loss of generality, that $c(x) = e^{-\frac{x}{2}}$. Then, the Lie symmetries become

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \quad \text{and} \quad X_2 = -2\frac{\partial}{\partial x} - \frac{1}{\alpha}t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

As in Case 1, the commutator of symmetries is zero, i.e. $[X_1, X_2] = 0$, thus the one-dimensional optimal system is

$$U_1 = X_1, \qquad U_2 = X_2 - aX_1, \quad \text{here } a \in \mathbb{R}.$$

Following the characteristic method, we get the invariant solutions:

$$\begin{cases} u = e^{\frac{a-1}{2}x}\varphi(z), \\ v = e^{\frac{a}{2}x}\psi(z) \end{cases} \tag{3.9}$$

with the similarity variable $z = e^{-\frac{x}{2\alpha}}t$. Consequently, the reduced system is obtained as

$$\begin{cases} \dfrac{d^\alpha \varphi}{dz^\alpha} = \dfrac{a}{2}\psi - \dfrac{1}{2\alpha}z\psi', \\ \dfrac{d^\alpha \psi}{dz^\alpha} = \dfrac{a-1}{2}\varphi - \dfrac{1}{2\alpha}z\varphi'. \end{cases} \tag{3.10}$$

Similarly solving the reduced system (3.10), the invariant solutions of (3.1) with $c(x) = e^{-\frac{x}{2}}$ are obtained as follows:

1. For $0 < \alpha < 1$, if we apply the third assertion of Proposition 3 with $a_1 = \frac{a}{2}$, $a_2 = \frac{a-1}{2}$ and $b_1 = b_2 = -\frac{1}{2}$, we obtain the following via (3.9):

$$\begin{aligned} u(x,t) &= ce^{\frac{a-1}{2}x} H^{2,0}_{1,2}\left[\frac{e^x}{t^{2\alpha}}\middle|\begin{matrix}(1, 2\alpha) \\ (\frac{1-a}{2}, 1), (\frac{1-a}{2}, 1)\end{matrix}\right], \\ v(x,t) &= -ce^{\frac{a}{2}x} H^{2,0}_{1,2}\left[\frac{e^x}{t^{2\alpha}}\middle|\begin{matrix}(1, 2\alpha) \\ (-\frac{a}{2}, 1), (\frac{2-a}{2}, 1)\end{matrix}\right] \end{aligned} \tag{3.11}$$

here $x > 0$, $t > 0$.

2. For $\alpha > 1$, applying the third assertion of Proposition 2 with the same parameters as in item 1, we obtain the following via (3.9):

$$\begin{aligned} u(x,t) &= e^{\frac{a-2}{2}x}t^{\alpha}\sum_{k=1}^{n} c_{k,1}\left(\frac{e^{\frac{x}{2\alpha}}}{t}\right)^{k} {}_3\Psi_1\left[\frac{t^{2\alpha}}{e^{x}}\middle| \begin{matrix}\left(1-\frac{a}{2}-\frac{k}{2\alpha},1\right),\left(1-\frac{a}{2}-\frac{k}{2\alpha},1\right),(1,1)\\ (1+\alpha-k,2\alpha)\end{matrix}\right] \\ &+ e^{\frac{a-3}{2}x}t^{2\alpha}\sum_{k=1}^{n} c_{k,2}\left(\frac{e^{\frac{x}{2\alpha}}}{t}\right)^{k} {}_3\Psi_1\left[\frac{t^{2\alpha}}{e^{x}}\middle| \begin{matrix}\left(\frac{3}{2}-\frac{a}{2}-\frac{k}{2\alpha},1\right),\left(\frac{3}{2}-\frac{a}{2}-\frac{k}{2\alpha},1\right),(1,1)\\ (1+2\alpha-k,2\alpha)\end{matrix}\right], \\ v(x,t) &= -e^{\frac{a-2}{2}x}t^{2\alpha}\sum_{k=1}^{n} c_{k,1}\left(\frac{e^{\frac{x}{2\alpha}}}{t}\right)^{k} {}_3\Psi_1\left[\frac{t^{2\alpha}}{e^{x}}\middle| \begin{matrix}\left(1-\frac{a}{2}-\frac{k}{2\alpha},1\right),\left(2-\frac{a}{2}-\frac{k}{2\alpha},1\right),(1,1)\\ (1+2\alpha-k,2\alpha)\end{matrix}\right] \\ &- e^{\frac{a-1}{2}x}t^{\alpha}\sum_{k=1}^{n} c_{k,2}\left(\frac{e^{\frac{x}{2\alpha}}}{t}\right)^{k} {}_3\Psi_1\left[\frac{t^{2\alpha}}{e^{x}}\middle| \begin{matrix}\left(\frac{1}{2}-\frac{a}{2}-\frac{k}{2\alpha},1\right),\left(\frac{3}{2}-\frac{a}{2}-\frac{k}{2\alpha},1\right),(1,1)\\ (1+\alpha-k,2\alpha)\end{matrix}\right], \end{aligned} \tag{3.12}$$

here $c_{k,1}$ and $c_{k,2}$ $(k = 1, \ldots, n)$ are constants.

3.1.3 Invariant solutions of (3.1) with $c(x) \equiv c$

If $c(x) \equiv c$, here c is a constant, then the symmetries of (3.1) are [27]:

$$X_1 = -x\frac{\partial}{\partial x} - \frac{t}{\alpha}\frac{\partial}{\partial t}, \quad X_2 = -\frac{\partial}{\partial x}, \quad X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_4 = c^2 v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.$$

We can assume, without loss of generality, that $c(x) \equiv 1$, then the Lie symmetries become

$$X_1 = -x\frac{\partial}{\partial x} - \frac{t}{\alpha}\frac{\partial}{\partial t}, \quad X_2 = -\frac{\partial}{\partial x}, \quad X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \text{ and } X_4 = v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.$$

The commutator table, where i and j index the row and column, for the Lie algebra generated by these infinitesimal symmetries is given by the Lie bracket operation $[X_i, X_j] = X_i(X_j) - X_j(X_i)$.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	X_2	0	0
X_2	$-X_2$	0	0	0
X_3	0	0	0	0
X_4	0	0	0	0

Table 3.1 Commutator table for case $c(x) \equiv c$.

We see from this table that the Lie algebra is identical to the Lie algebra $A_2 \oplus 2A_1$ given in [23]. The one-dimensional optimal system of the Lie algebra generated by X_1, X_2, X_3 and X_4 is , thus, obtained in [23] as

$$\begin{aligned} U_1 &= X_1 - a_1 X_3 - a_2 X_4, \text{ here } a_1, a_2 \in \mathbb{R}, \\ U_2 &= X_2 - a_1 X_3 - a_2 X_4, \text{ here } (a_1, a_2) \in \{(\pm 1, a), (0, \pm 1), (0, 0) | a \in \mathbb{R}\}, \\ U_3 &= X_3 + a_1 X_4, \text{ here } a_1 \in \mathbb{R}, \\ U_4 &= X_4. \end{aligned}$$

We tabulate the invariant solutions $(u_j(x,t), v_j(x,t))$ expressed as solutions $(\varphi_j(z), \psi_j(z))$ of the reduced system and the reduced system of FODEs corresponding to the symmetry U_j in Table 3.2. Unlike the case of non-constant $c(x)$, the invariant solutions of

OS	Invariant solutions $(u_j(x,t), v_j(x,t))$	Reduced systems of ODE$_j$
U_1	$\begin{cases} u(x,t) = x^{a_1+a_2}\varphi(z) + x^{a_1-a_2}\psi(z), \\ v(x,t) = x^{a_1+a_2}\varphi(z) - x^{a_1-a_2}\psi(z), \end{cases}$ with $z = x^{-\frac{1}{\alpha}}t$	$\begin{cases} \dfrac{d^\alpha \varphi}{dz^\alpha} = (a_1+a_2)\varphi - \dfrac{1}{\alpha} z\varphi', \\ \dfrac{d^\alpha \psi}{dz^\alpha} = -(a_1-a_2)\psi + \dfrac{1}{\alpha} z\psi', \end{cases}$ here $a_1, a_2 \in \mathbb{R}$
U_2	$\begin{cases} u(x,t) = e^{(a_1+a_2)x}\varphi(t) + e^{(a_1-a_2)x}\psi(t), \\ v(x,t) = e^{(a_1+a_2)x}\varphi(t) - e^{(a_1-a_2)x}\psi(t), \end{cases}$	$\begin{cases} \dfrac{d^\alpha \varphi(t)}{dt^\alpha} = (a_1+a_2)\varphi(t), \\ \dfrac{d^\alpha \psi(t)}{dt^\alpha} = -(a_1-a_2)\psi(t), \end{cases}$ here $(a_1, a_2) \in \{(\pm 1, a), (0, \pm 1), (0, 0) \| a \in \mathbb{R}\}$

Table 3.2 The invariant solutions and reduced systems of FODEs of (3.1) with $c(x) \equiv 1$.

(3.1) with $c(x) \equiv 1$ are expressed as sums of solutions of two individual FODEs of the general form (2.3). The invariant solutions of (3.1) with $c(x) \equiv 1$ corresponding to U_1 are given as follows:

1. If $0 < \alpha < 1$, then using the first assertion of Proposition 3 with $b = 1$ for the second summand of the $u(x,t)$ and $v(x,t)$, and using the expression (1.17), we obtain

$$\begin{aligned} u(x,t) &= ct^{(a_1-a_2)\alpha}\Psi\left(-\frac{x}{t^\alpha}; -\alpha, 1 + (a_1 - a_2)\alpha\right), \\ v(x,t) &= -ct^{(a_1-a_2)\alpha}\Psi\left(-\frac{x}{t^\alpha}; -\alpha, 1 + (a_1 - a_2)\alpha\right). \end{aligned} \tag{3.13}$$

The above representation can solve (3.1) because of the linearity of (3.1).

2. If $\alpha > 1$, then using the first assertion of Proposition 2 with $a = a_1 + a_2$, $b = -1$ for the first summands and $a = -a_1 + a_2$, $b = 1$ for the second summands of $u(x,t)$ and $v(x,t)$, we obtain

$$u(x,t) = x^{a_1+a_2-1}t^{\alpha}\sum_{k=1}^{n} c_{k,1}\frac{x^{\frac{k}{\alpha}}}{t^k}\varphi_k\left(\frac{t}{x^{\frac{1}{\alpha}}}\right) + x^{a_1-a_2-1}t^{\alpha}\sum_{k=1}^{n} c_{k,2}\frac{x^{\frac{k}{\alpha}}}{t^k}\psi_k\left(\frac{t}{x^{\frac{1}{\alpha}}}\right),$$
$$v(x,t) = x^{a_1+a_2-1}t^{\alpha}\sum_{k=1}^{n} c_{k,1}\frac{x^{\frac{k}{\alpha}}}{t^k}\varphi_k\left(\frac{t}{x^{\frac{1}{\alpha}}}\right) - x^{a_1-a_2-1}t^{\alpha}\sum_{k=1}^{n} c_{k,2}\frac{x^{\frac{k}{\alpha}}}{t^k}\psi_k\left(\frac{t}{x^{\frac{1}{\alpha}}}\right), \tag{3.14}$$

where

$$\varphi_k(z) = {}_2\Psi_1\left[-z^{\alpha}\middle|\begin{matrix}\left(-a_1 - a_2 - \frac{k}{\alpha} + 1, 1\right), (1,1)\\ (1+\alpha-k, \alpha)\end{matrix}\right],$$
$$\psi_k(z) = {}_2\Psi_1\left[z^{\alpha}\middle|\begin{matrix}\left(-a_1 + a_2 - \frac{k}{\alpha} + 1, 1\right), (1,1)\\ (1+\alpha-k, \alpha)\end{matrix}\right].$$

Since $b = 0$ in both equations of the reduced system corresponding to U_2, we obtain the following invariant solution for arbitrary α by using the first assertion of Proposition 1:

$$u(x,t) = e^{(a_1+a_2)x}t^{\alpha}\sum_{k=1}^{n} c_{k,1}t^{-k}\varphi_k(t) + e^{(a_1-a_2)x}t^{\alpha}\sum_{k=1}^{n} c_{k,2}t^{-k}\psi_k(t),$$
$$v(x,t) = e^{(a_1+a_2)x}t^{\alpha}\sum_{k=1}^{n} c_{k,1}t^{-k}\varphi_k(t) - e^{(a_1-a_2)x}t^{\alpha}\sum_{k=1}^{n} c_{k,2}t^{-k}\psi_k(t), \tag{3.15}$$

where

$$\varphi_k(t) = E_{\alpha,1+\alpha-k}((a_1+a_2)t^{\alpha}), \qquad \psi_k(t) = E_{\alpha,1+\alpha-k}((a_2-a_1)t^{\alpha})$$

and $(a_1, a_2) \in \{(\pm 1, a), (0, \pm 1), (0,0) | a \in \mathbb{R}\}$.

There are no invariant solutions corresponding to U_3 and U_4.

In this section, we explicitly expressed all invariant solutions corresponding to the optimal system for three types of function $c(x)$. As mentioned earlier, other invariant solutions corresponding to any other Lie symmetries can be obtained by symmetry transformation on invariant solutions corresponding to the optimal system.

3.2 Some properties of the solutions

Here, we show that the solution (3.7) corresponds to solutions of anomalous diffusion equation and illustrates some plots of the solutions for various values of parameter α. To do so, let us formulate the following lemma.

Lemma 8. *We have the following formula for composition of fractional derivatives of Fox H- functions*

$$\frac{d^\alpha}{dz^\alpha}\frac{d^\alpha}{dz^\alpha}H^{m,0}_{p,q}\left[az^{-\alpha_p}\left|\begin{matrix}(A_i,\alpha_i)_{1,p-1},(1,\alpha_p)\\(B_j,\beta_j)_{1,q}\end{matrix}\right.\right]=\frac{d^{2\alpha}}{dz^{2\alpha}}H^{m,0}_{p,q}\left[az^{-\alpha_p}\left|\begin{matrix}(A_i,\alpha_i)_{1,p-1},(1,\alpha_p)\\(B_j,\beta_j)_{1,q}\end{matrix}\right.\right].$$

The proof of lemma can be easily done by virtue of Lemma 5.

Using Lemma 8, (3.2) and putting a instead of am, we get the following $u(x,t)$ of (3.7):

$$u(x,t)=cx^{a+m}H^{2,0}_{1,2}\left[\frac{1}{4(m-1)^2}\frac{x^{2(1-m)}}{t^{2\alpha}}\left|\begin{matrix}(1,2\alpha)\\\left(\frac{a}{2(m-1)}+\frac{1}{2},1\right),\left(\frac{a+1}{2(m-1)}+\frac{1}{2},1\right)\end{matrix}\right.\right],\quad(3.16)$$

which solves the anomalous diffusion equation with variable coefficient:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}=x^{2m}\frac{\partial^2 u}{\partial x^2}\qquad\text{for } x>0,t>0,$$

and using Lemma 8, (3.3) and putting a instead of am, we get the following $v(x,t)$ of (3.7):

$$v(x,t)=cx^{a}H^{2,0}_{1,2}\left[\frac{1}{4(m-1)^2}\frac{x^{2(1-m)}}{t^{2\alpha}}\left|\begin{matrix}(1,2\alpha)\\\left(\frac{a}{2(m-1)},1\right),\left(\frac{a+1}{2(m-1)}+1,1\right)\end{matrix}\right.\right],\quad(3.17)$$

which solves the following one:

$$\frac{\partial^{2\alpha}v}{\partial t^{2\alpha}}=\frac{\partial}{\partial x}\left(x^{2m}\frac{\partial v}{\partial x}\right)\qquad\text{for } x>0,t>0.$$

If we take $m=0$ in (3.16), then it can be expressed in Wright function and corresponds to solution (9) of [4]. Also, R. Metzler et al. [19] studied anomalous diffusion equation via Laplace transformation method and found solutions expressed in Fox H-functions. If we take $D=1$, $\theta=-2m$ and $D=1-2m$, $\theta=-2m$ in (14) of [19], then (3.16) and (3.17) correspond to the solutions obtained in [19], respectively.

Furthermore, if we put $a = m - 1$ and $a = -1$ in (3.16) and (3.17) respectively, we get

$$u(x,t) = c\,\mathrm{sgn}(m-1)x^{2m-1}H_{1,2}^{2,0}\left[\frac{1}{4(m-1)^2}\frac{x^{2(1-m)}}{t^{2\alpha}}\left|\begin{matrix}(1,2\alpha)\\(1,1),\left(1+\frac{1}{2(m-1)},1\right)\end{matrix}\right.\right] \tag{3.18}$$

$$v(x,t) = cx^{-1}H_{1,2}^{2,0}\left[\frac{1}{4(m-1)^2}\frac{x^{2(1-m)}}{t^{2\alpha}}\left|\begin{matrix}(1,2\alpha)\\\left(-\frac{1}{2(m-1)},1\right),(1,1)\end{matrix}\right.\right]. \tag{3.19}$$

Furthermore, if we put $\alpha = \frac{1}{2}$ in the above solution, then it equals to

$$u(x,t) = \bar{c}_1 t^{-1-\frac{1}{2(m-1)}} e^{-\frac{x^{2(1-m)}}{4(m-1)^2}t} \tag{3.20}$$

$$\tag{3.21}$$

$$v(x,t) = \bar{c}_2 t^{\frac{1}{2(m-1)}} e^{-\frac{x^{2(1-m)}}{4(m-1)^2}t}, \tag{3.22}$$

where

$$\bar{c}_1 = c\,\mathrm{sgn}(m-1)(2|m-1|)^{-2-\frac{1}{m-1}}, \qquad \bar{c}_2 = c\,(2|m-1|)^{-2-\frac{1}{m-1}}.$$

The solutions (3.21) and (3.22) are well-known solutions of heat equations $u_t = x^{2m}u_{xx}$ and $v_t = (x^{2m}v_x)_x$, respectively.

If the order α of time fractional derivative is a rational number rather than real, we can transform the solutions expressed in terms of Fox H-function into representations expressed in terms of Meijer G-function using Gauss' multiplication formula for the gamma function:

$$\Gamma(Mz) = (2\pi)^{\frac{1-M}{2}} M^{Mz-\frac{1}{2}} \prod_{k=0}^{M-1} \Gamma\left(z + \frac{k}{M}\right), \tag{3.23}$$

where $z \in \mathbb{R}$ and $M \in \mathbb{N}$. For example, we obtain the following invariant solution of (3.1) with $c(x) = x^m$, if we set $\alpha = \frac{7}{8}$ and $c = 1$ in (3.18) and (3.19):

$$u(x,t) = \mathrm{sgn}(m-1)x^{2m-1}H_{1,2}^{2,0}\left[\frac{1}{4(m-1)^2}\frac{x^{2(1-m)}}{t^{\frac{7}{4}}}\left|\begin{matrix}(1,\frac{7}{4})\\(1,1),\left(1+\frac{1}{2(m-1)},1\right)\end{matrix}\right.\right],$$

$$v(x,t) = x^{-1}H_{1,2}^{2,0}\left[\frac{1}{4(m-1)^2}\frac{x^{2(1-m)}}{t^{\frac{7}{4}}}\left|\begin{matrix}(1,\frac{7}{4})\\\left(-\frac{1}{2(m-1)},1\right),(1,1)\end{matrix}\right.\right]$$

which becomes

$$u(x,t) = \operatorname{sgn}(m-1)4x^{2m-1}H^{2,0}_{1,2}\left[\frac{1}{4^4(m-1)^8}\frac{x^{8(1-m)}}{t^7}\middle| \begin{matrix}(1,7)\\ (1,4),\left(1+\frac{1}{2(m-1)},4\right)\end{matrix}\right], \tag{3.24}$$

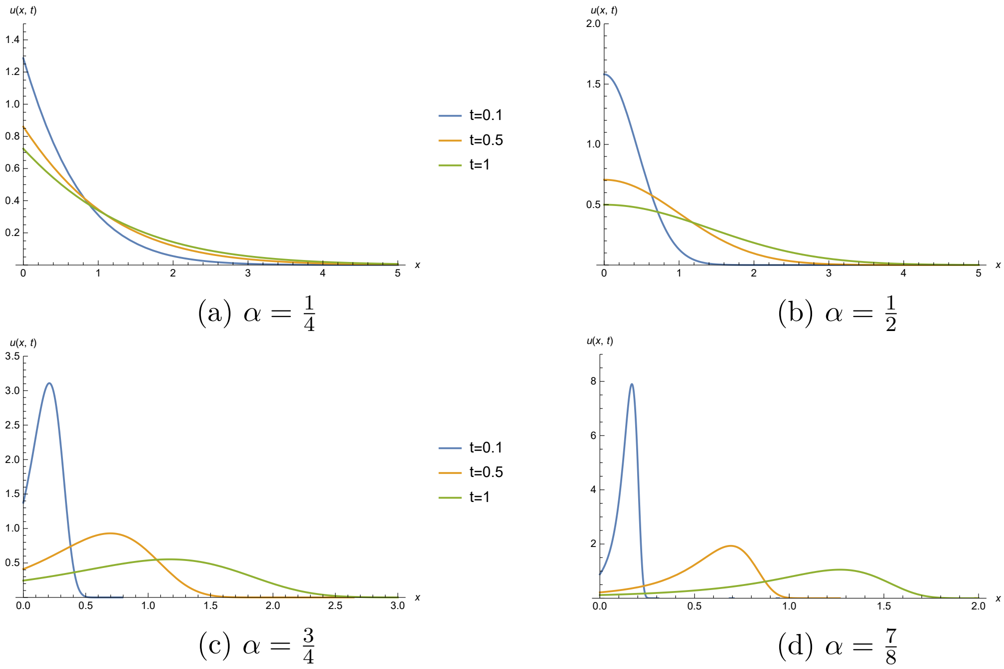
$$v(x,t) = 4x^{-1}H^{2,0}_{1,2}\left[\frac{1}{4^4(m-1)^8}\frac{x^{8(1-m)}}{t^7}\middle| \begin{matrix}(1,7)\\ \left(-\frac{1}{2(m-1)},4\right),(1,4)\end{matrix}\right] \tag{3.25}$$

by virtue of the formula (1.9). Then, using (3.23) for $\Gamma(4(-z))$, $\Gamma\left(4(1/4-z)\right)$ and $\Gamma\left(7(1/7-z)\right)$, we obtain the following expression of the solutions (3.24) and (3.25):

$$u(x,t) = \frac{4^{2+\frac{1}{2(m-1)}}x^{2m-1}}{7^{\frac{1}{2}}}$$
$$\times G^{8,0}_{7,8}\left[\frac{7^7}{4^{12}(m-1)^8}\frac{x^{8(1-m)}}{t^7}\middle| \begin{matrix}\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7},1\\ \frac{1}{4},\frac{2}{4},\frac{3}{4},1,\frac{1}{4}+\frac{1}{8(m-1)},\frac{2}{4}+\frac{1}{8(m-1)},\frac{3}{4}+\frac{1}{8(m-1)},1+\frac{1}{8(m-1)}\end{matrix}\right],$$

$$v(x,t) = \frac{4^{1-\frac{1}{2(m-1)}}}{7^{\frac{1}{2}}}x^{-1}$$
$$\times G^{8,0}_{7,8}\left[\frac{7^7}{4^{12}(m-1)^8}\frac{x^{8(1-m)}}{t^7}\middle| \begin{matrix}\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7},1\\ \frac{1}{4},\frac{2}{4},\frac{3}{4},1,-\frac{1}{8(m-1)},\frac{1}{4}-\frac{1}{8(m-1)},\frac{2}{4}-\frac{1}{8(m-1)},\frac{3}{4}-\frac{1}{8(m-1)}\end{matrix}\right],$$

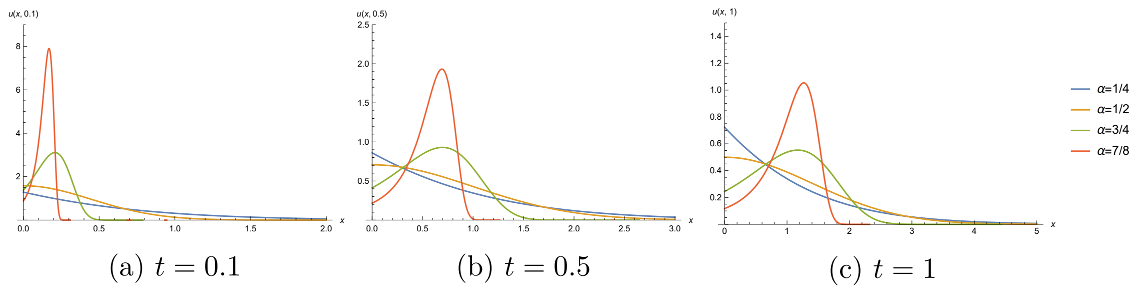
where $G^{m,l}_{p,q}[z]$ is a Meijer G-function. We can see how solutions behave when time variable t is fixed and the order α of fractional derivative varies in Figures 3.2 and 3.4, or when α is fixed and t varies in Figures 3.1 and 3.3. Here, we only provide the illustrations of $u(x,t)$ given in (3.18) by taking $c=-1$, the illustrations of $v(x,t)$ given in (3.19) can be plotted in an analogy. The graphs are plotted using the Mathematica implementation of the special function Meijer G [30]. We can see that the solution graphs alter visibly when α steps over the value $\frac{1}{2}$. For the constant coefficient case $c(x)\equiv 1$, (3.13) behaves more likely to wave-like solution as the fractional order α approaches to 1, which can be seen in the Figure 3.2.



(a) $\alpha = \frac{1}{4}$ (b) $\alpha = \frac{1}{2}$

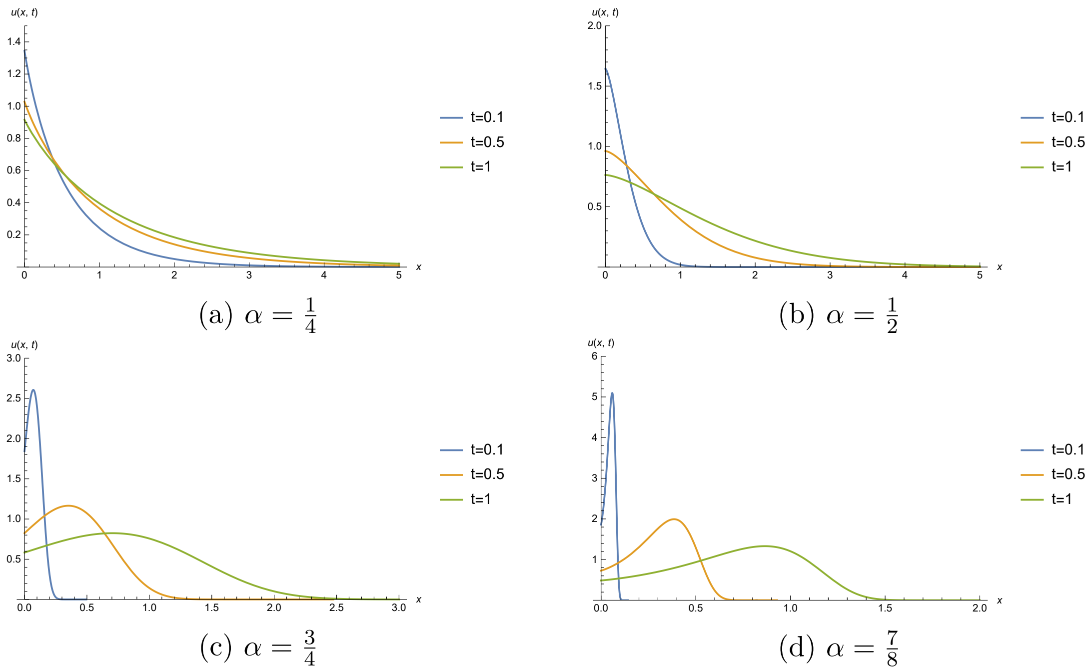
(c) $\alpha = \frac{3}{4}$ (d) $\alpha = \frac{7}{8}$

Fig. 3.1 Plots of solution $u(x,t)$ of (3.1) with $c(x) \equiv 1$ with respect to t.



(a) $t = 0.1$ (b) $t = 0.5$ (c) $t = 1$

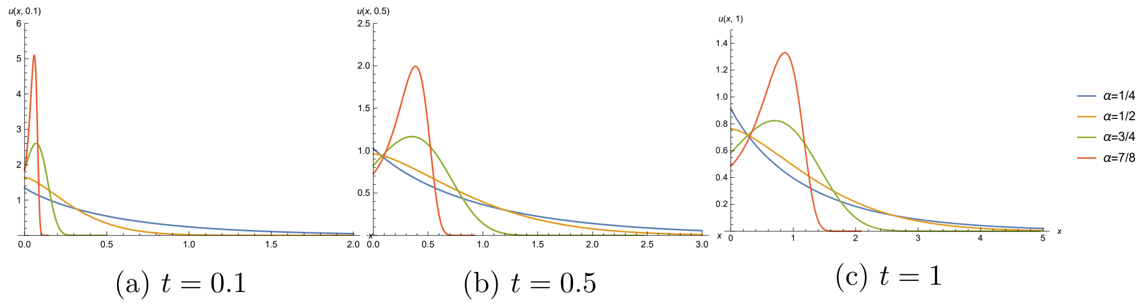
Fig. 3.2 Plots of solution $u(x,t)$ of (3.1) with $c(x) \equiv 1$ for various order α of fractional derivative.



(a) $\alpha = \frac{1}{4}$ (b) $\alpha = \frac{1}{2}$

(c) $\alpha = \frac{3}{4}$ (d) $\alpha = \frac{7}{8}$

Fig. 3.3 Plots of solution $u(x,t)$ of (3.1) with $c(x) = x^{\frac{1}{4}}$ with respect to t.



(a) $t = 0.1$ (b) $t = 0.5$ (c) $t = 1$

Fig. 3.4 Plots of solution $u(x,t)$ of (3.1) with $c(x) = x^{\frac{1}{4}}$ for various order α of fractional derivative.

3.3 Invariant solutions of the special case $\alpha = 1$

In the previous sections, we give invariant solutions of (3.1) considering the order α of fractional derivative satisfying $n-1 < \alpha < n$, where $n \in \mathbb{N}$. Despite this assumption on α, the solutions (3.8), (3.12) still solve (3.1) with $c(x) = x^m$ (here $m \neq 0, 1$) and $c(x) = e^{-\frac{x}{2}}$, respectively, when $\alpha = 1$. The complete group classification and some invariant solutions of system (3.1) for $\alpha = 1$ are found explicitly in G. Bluman, S. Kumei

[3]. Now, let us show the correspondence between the solutions (3.8), (3.12) and the ones obtained in [3] for $c(x) = x^m$ and $c(x) = e^{-\frac{x}{2}}$.

3.3.1 Solutions of (3.1) with $c(x) = x^m$ $(m \neq 0, 1)$ and $\alpha = 1$

If we set $\alpha = 1$ and $n = 1$ in (3.8), then it becomes

$$\begin{aligned} u(x,t) &= c_1 x^{a+m}{}_3\Psi_1\left[4(m-1)^2z^2\middle|\begin{array}{c}\left(\omega_1-\frac{1}{2},1\right),\left(\omega_2-\frac{1}{2},1\right),(1,1)\\(1,2)\end{array}\right]\\ &\quad +2c_2(m-1)x^{a+m}z{}_3\Psi_1\left[4(m-1)^2z^2\middle|\begin{array}{c}(\omega_1,1),(\omega_2,1),(1,1)\\(2,2)\end{array}\right],\\ v(x,t) &= 2c_1(m-1)x^a z{}_3\Psi_1\left[4(m-1)^2z^2\middle|\begin{array}{c}\left(\omega_1-\frac{1}{2},1\right),\left(\omega_2+\frac{1}{2},1\right),(1,1)\\(2,2)\end{array}\right]\\ &\quad +c_2x^a{}_3\Psi_1\left[4(m-1)^2z^2\middle|\begin{array}{c}(\omega_1-1,1),(\omega_2,1),(1,1)\\(1,2)\end{array}\right], \end{aligned}$$

where $z = x^{m-1}t$, $\omega_1 = \frac{a}{2(m-1)} + 1$, $\omega_2 = \frac{a+1}{2(m-1)} + 1$, and c_1, c_2 are constants. By virtue of (1.19) and (1.20), the above solution equals to

$$\begin{aligned} u(x,t) &= \Gamma\left(\omega_1-\frac{1}{2}\right)\Gamma\left(\omega_2-\frac{1}{2}\right)c_1x^{a+m}{}_2F_1\left(\begin{array}{c}\omega_1-\frac{1}{2},\omega_2-\frac{1}{2}\\\frac{1}{2}\end{array};(m-1)^2z^2\right)\\ &\quad +\Gamma(\omega_1-1)\Gamma(\omega_2)c_2ax^{a+m}z{}_2F_1\left(\begin{array}{c}\omega_1,\omega_2\\\frac{3}{2}\end{array};(m-1)^2z^2\right),\\ v(x,t) &= \Gamma\left(\omega_1-\frac{1}{2}\right)\Gamma\left(\omega_2-\frac{1}{2}\right)c_1(a+m)x^az{}_2F_1\left(\begin{array}{c}\omega_1-\frac{1}{2},\omega_2+\frac{1}{2}\\\frac{3}{2}\end{array};(m-1)^2z^2\right)\\ &\quad +\Gamma(\omega_1-1)\Gamma(\omega_2)c_2x^a{}_2F_1\left(\begin{array}{c}\omega_1-1,\omega_2\\\frac{1}{2}\end{array};(m-1)^2z^2\right)\ \text{for } |z|<\frac{1}{|m-1|}. \end{aligned} \tag{3.26}$$

To show the correspondence of the above expression of the solution (3.26) and the one obtained in [3], we will need the formula (15.8.27) of [21]

$$\frac{2\Gamma(\frac{1}{2})\Gamma(a+b+\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{}_2F_1\left(\begin{matrix} a,b \\ \frac{1}{2} \end{matrix};z^2\right)$$
$$= {}_2F_1\left(\begin{matrix} 2a,2b \\ a+b+\frac{1}{2} \end{matrix};\frac{1-z}{2}\right) + {}_2F_1\left(\begin{matrix} 2a,2b \\ a+b+\frac{1}{2} \end{matrix};\frac{1+z}{2}\right) \quad (3.27)$$

and formula (2.3.11) of [1]

$${}_2F_1\left(\begin{matrix} 2a,2b \\ a+b+\frac{1}{2} \end{matrix};\frac{1-z}{2}\right) = \frac{\Gamma(a+b+\frac{1}{2})\Gamma(\frac{1}{2}-a-b)}{\Gamma(a-b+\frac{1}{2})\Gamma(b-a+\frac{1}{2})}{}_2F_1\left(\begin{matrix} 2a,2b \\ a+b+\frac{1}{2} \end{matrix};\frac{1+z}{2}\right)$$
$$+ \frac{\Gamma(a+b-\frac{1}{2})\Gamma(a+b+\frac{1}{2})}{\Gamma(2a)\Gamma(2b)}\left(\frac{1+z}{2}\right)^{\frac{1}{2}-a-b}{}_2F_1\left(\begin{matrix} a-b+\frac{1}{2}, b-a+\frac{1}{2} \\ \frac{3}{2}-a-b \end{matrix};\frac{1+z}{2}\right). \quad (3.28)$$

If we substitute (3.28) into (3.27), we obtain

$$\frac{2\Gamma(\frac{1}{2})\Gamma(a+b+\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{}_2F_1\left(\begin{matrix} a,b \\ \frac{1}{2} \end{matrix};z^2\right)$$
$$= \left(1+\frac{\Gamma(a+b+\frac{1}{2})\Gamma(\frac{1}{2}-a-b)}{\Gamma(a-b+\frac{1}{2})\Gamma(b-a+\frac{1}{2})}\right){}_2F_1\left(\begin{matrix} 2a,2b \\ a+b+\frac{1}{2} \end{matrix};\frac{1+z}{2}\right)$$
$$+ \frac{\Gamma(a+b-\frac{1}{2})\Gamma(a+b+\frac{1}{2})}{\Gamma(2a)\Gamma(2b)}\left(\frac{1+z}{2}\right)^{\frac{1}{2}-a-b}{}_2F_1\left(\begin{matrix} a-b+\frac{1}{2}, b-a+\frac{1}{2} \\ \frac{3}{2}-a-b \end{matrix};\frac{1+z}{2}\right). \quad (3.29)$$

Also, we apply the formulas (15.8.28) of [21]

$$\frac{2\Gamma(-\frac{1}{2})\Gamma(a+b-\frac{1}{2})}{\Gamma(a-\frac{1}{2})\Gamma(b-\frac{1}{2})}{}_2F_1\left(\begin{matrix} a,b \\ \frac{3}{2} \end{matrix};z^2\right)$$
$$= {}_2F_1\left(\begin{matrix} 2a-1,2b-1 \\ a+b-\frac{1}{2} \end{matrix};\frac{1-z}{2}\right) - {}_2F_1\left(\begin{matrix} 2a-1,2b-1 \\ a+b-\frac{1}{2} \end{matrix};\frac{1+z}{2}\right) \quad (3.30)$$

and (2.3.11) of [1]

$$
{}_2F_1\left(\begin{matrix} 2a-1, 2b-1 \\ a+b-\frac{1}{2} \end{matrix}; \frac{1-z}{2}\right) = \frac{\Gamma(a+b-\frac{1}{2})\Gamma(\frac{3}{2}-a-b)}{\Gamma(a-b+\frac{1}{2})\Gamma(b-a+\frac{1}{2})}{}_2F_1\left(\begin{matrix} 2a-1, 2b-1 \\ a+b-\frac{1}{2} \end{matrix}; \frac{1+z}{2}\right)
$$
$$
+\frac{\Gamma(a+b-\frac{3}{2})\Gamma(a+b-\frac{1}{2})}{\Gamma(2a-1)\Gamma(2b-1)}\left(\frac{1+z}{2}\right)^{\frac{3}{2}-a-b}{}_2F_1\left(\begin{matrix} a-b+\frac{1}{2}, b-a+\frac{1}{2} \\ \frac{5}{2}-a-b \end{matrix}; \frac{1+z}{2}\right) \tag{3.31}
$$

If we substitute (3.31) into (3.30), we have

$$
\frac{2\Gamma(-\frac{1}{2})\Gamma(a+b-\frac{1}{2})}{\Gamma(a-\frac{1}{2})\Gamma(b-\frac{1}{2})} z\, {}_2F_1\left(\begin{matrix} a, b \\ \frac{3}{2} \end{matrix}; z^2\right)
$$
$$
= \left(\frac{\Gamma(a+b-\frac{1}{2})\Gamma(\frac{3}{2}-a-b)}{\Gamma(a-b+\frac{1}{2})\Gamma(b-a+\frac{1}{2})} - 1\right){}_2F_1\left(\begin{matrix} 2a-1, 2b-1 \\ a+b-\frac{1}{2} \end{matrix}; \frac{1+z}{2}\right)
$$
$$
+\frac{\Gamma(a+b-\frac{3}{2})\Gamma(a+b-\frac{1}{2})}{\Gamma(2a-1)\Gamma(2b-1)}\left(\frac{1+z}{2}\right)^{\frac{3}{2}-a-b}{}_2F_1\left(\begin{matrix} a-b+\frac{1}{2}, b-a+\frac{1}{2} \\ \frac{5}{2}-a-b \end{matrix}; \frac{1+z}{2}\right) \tag{3.32}
$$

We can apply (3.29), (3.32) for the solution (3.26) as follows

$$
\frac{2\sqrt{\pi}}{\Gamma(\frac{a}{2(m-1)}+1)\Gamma(\frac{a+1}{2(m-1)}+1)}{}_2F_1\left(\begin{matrix} \frac{a}{2(m-1)}+\frac{1}{2}, \frac{a+1}{2(m-1)}+\frac{1}{2} \\ \frac{1}{2} \end{matrix}; (m-1)^2z^2\right)
$$
$$
= \left(\frac{1}{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{3}{2}\right)} + \frac{\Gamma\left(-\frac{2a+1}{2(m-1)}-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2(m-1)}+\frac{1}{2}\right)}\right)
$$
$$
\times\, {}_2F_1\left(\begin{matrix} \frac{a}{m-1}+1, \frac{a+1}{m-1}+1 \\ \frac{2a+1}{2(m-1)}+\frac{3}{2} \end{matrix}; \frac{1+(m-1)z}{2}\right)
$$
$$
+\frac{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{m-1}+1\right)\Gamma\left(\frac{a+1}{m-1}+1\right)}\left(\frac{1+(m-1)z}{2}\right)^{-\frac{2a+1}{2(m-1)}-\frac{1}{2}}
$$
$$
\times\, {}_2F_1\left(\begin{matrix} -\frac{1}{2(m-1)}+\frac{1}{2}, \frac{1}{2(m-1)}+\frac{1}{2} \\ \frac{1}{2}-\frac{2a+1}{2(m-1)} \end{matrix}; \frac{1+(m-1)z}{2}\right) \tag{3.33}
$$

and

$$\begin{aligned}
&\frac{-4\sqrt{\pi}}{\Gamma(\frac{a}{2(m-1)}+\frac{1}{2})\Gamma(\frac{a+1}{2(m-1)}+\frac{1}{2})} z\, {}_2F_1\left(\begin{matrix}\frac{a}{2(m-1)}+1, \frac{a+1}{2(m-1)}+1\\ \frac{3}{2}\end{matrix};(m-1)^2z^2\right)\\
=&\left(\frac{\Gamma\left(-\frac{2a+1}{2(m-1)}-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2(m-1)}+\frac{1}{2}\right)}-\frac{1}{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{3}{2}\right)}\right)\\
&\times {}_2F_1\left(\begin{matrix}\frac{a}{m-1}+1, \frac{a+1}{m-1}+1\\ \frac{2a+1}{2(m-1)}+\frac{3}{2}\end{matrix};\frac{1+(m-1)z}{2}\right)\\
&+\frac{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{m-1}+1\right)\Gamma\left(\frac{a+1}{m-1}+1\right)}\left(\frac{1+(m-1)z}{2}\right)^{-\frac{2a+1}{2(m-1)}-\frac{1}{2}}\\
&\times {}_2F_1\left(\begin{matrix}-\frac{1}{2(m-1)}+\frac{1}{2}, \frac{1}{2(m-1)}+\frac{1}{2}\\ \frac{1}{2}-\frac{2a+1}{2(m-1)}\end{matrix};\frac{1+(m-1)z}{2}\right). \qquad (3.34)
\end{aligned}$$

Substituting (3.33) and (3.34) into $u(x,t)$ of the solution (3.26), we get

$$\begin{aligned}
u(x,t) =&\Gamma\left(\frac{a}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{a+1}{2(m-1)}+\frac{1}{2}\right)c_1x^{a+m}\\
&\times {}_2F_1\left(\begin{matrix}\frac{a}{2(m-1)}+\frac{1}{2}, \frac{a+1}{2(m-1)}+\frac{1}{2}\\ \frac{1}{2}\end{matrix};(m-1)^2z^2\right)\\
&+2\Gamma\left(\frac{a}{2(m-1)}+1\right)\Gamma\left(\frac{a+1}{2(m-1)}+1\right)c_2x^{a+m}(m-1)z\\
&\times {}_2F_1\left(\begin{matrix}\frac{a}{2(m-1)}+1, \frac{a+1}{2(m-1)}+1\\ \frac{3}{2}\end{matrix};(m-1)^2z^2\right),\\
=&\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{a}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2(m-1)}+1\right)\Gamma\left(\frac{a+1}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{a+1}{2(m-1)}+1\right)\\
&\times x^{a+m}\left[\frac{c_1+c_2}{\Gamma(\frac{2a+1}{2(m-1)}+\frac{3}{2})}+\frac{\Gamma\left(-\frac{2a+1}{2(m-1)}-\frac{1}{2}\right)(c_1-c_2)}{\Gamma\left(-\frac{1}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2(m-1)}+\frac{1}{2}\right)}\right]\\
&\times {}_2F_1\left(\begin{matrix}\frac{a}{m-1}+1, \frac{a+1}{m-1}+1\\ \frac{2a+1}{2(m-1)}+\frac{3}{2}\end{matrix};\frac{1+(m-1)z}{2}\right)\\
&+\frac{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{1}{2}\right)(c_1-c_2)}{\Gamma\left(\frac{a}{m-1}+1\right)\Gamma\left(\frac{a+1}{m-1}+1\right)}\left(\frac{1+(m-1)z}{2}\right)^{-\frac{2a+1}{2(m-1)}-\frac{1}{2}}\\
&\times {}_2F_1\left(\begin{matrix}-\frac{1}{2(m-1)}+\frac{1}{2}, \frac{1}{2(m-1)}+\frac{1}{2}\\ -\frac{2a+1}{2(m-1)}+\frac{1}{2}\end{matrix};\frac{1+(m-1)z}{2}\right)
\end{aligned}$$

We can apply (3.29), (3.32) for $v(x,t)$ of the solution (3.26) as follows

$$\frac{2\sqrt{\pi}}{\Gamma(\frac{a}{2(m-1)}+\frac{1}{2})\Gamma(\frac{a+1}{2(m-1)}+\frac{3}{2})}{}_2F_1\left(\begin{matrix}\frac{a}{2(m-1)},\frac{a+1}{2(m-1)}+1\\ \frac{1}{2}\end{matrix};(m-1)^2z^2\right)$$
$$=\left(\frac{1}{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{3}{2}\right)}+\frac{\Gamma\left(-\frac{2a+1}{2(m-1)}-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2(m-1)}-\frac{1}{2}\right)\Gamma\left(\frac{1}{2(m-1)}+\frac{3}{2}\right)}\right)$$
$$\times\,{}_2F_1\left(\begin{matrix}\frac{a}{m-1},\frac{a+1}{m-1}+2\\ \frac{2a+1}{2(m-1)}+\frac{3}{2}\end{matrix};\frac{1+(m-1)z}{2}\right)$$
$$+\frac{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{m-1}\right)\Gamma\left(\frac{a+1}{m-1}+2\right)}\left(\frac{1+(m-1)z}{2}\right)^{-\frac{2a+1}{2(m-1)}-\frac{1}{2}}$$
$$\times\,{}_2F_1\left(\begin{matrix}-\frac{1}{2(m-1)}-\frac{1}{2},\frac{1}{2(m-1)}+\frac{3}{2}\\ \frac{1}{2}-\frac{2a+1}{2(m-1)}\end{matrix};\frac{1+(m-1)z}{2}\right) \tag{3.35}$$

and

$$\frac{-4\sqrt{\pi}(m-1)}{\Gamma(\frac{a}{2(m-1)}+\frac{1}{2})\Gamma(\frac{a+1}{2(m-1)}+\frac{1}{2})}z\,{}_2F_1\left(\begin{matrix}\frac{a}{2(m-1)}+1,\frac{a+1}{2(m-1)}+1\\ \frac{3}{2}\end{matrix};(m-1)^2z^2\right)$$
$$=\left(\frac{\Gamma\left(-\frac{2a+1}{2(m-1)}-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2(m-1)}+\frac{1}{2}\right)}-\frac{1}{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{3}{2}\right)}\right)$$
$$\times\,{}_2F_1\left(\begin{matrix}\frac{a}{m-1}+1,\frac{a+1}{m-1}+1\\ \frac{2a+1}{2(m-1)}+\frac{3}{2}\end{matrix};\frac{1+(m-1)z}{2}\right)$$
$$+\frac{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{m-1}+1\right)\Gamma\left(\frac{a+1}{m-1}+1\right)}\left(\frac{1+(m-1)z}{2}\right)^{-\frac{2a+1}{2(m-1)}-\frac{1}{2}}$$
$$\times\,{}_2F_1\left(\begin{matrix}-\frac{1}{2(m-1)}+\frac{1}{2},\frac{1}{2(m-1)}+\frac{1}{2}\\ \frac{1}{2}-\frac{2a+1}{2(m-1)}\end{matrix};\frac{1+(m-1)z}{2}\right). \tag{3.36}$$

Substituting (3.35) and (3.36) into $v(x,t)$ of the solution (3.26), we get

$$\begin{aligned}
v(x,t) =&2\Gamma\left(\frac{a}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{a+1}{2(m-1)}+\frac{3}{2}\right)(m-1)c_1x^az\\
&\times {}_2F_1\left(\begin{matrix}\frac{a}{2(m-1)}+\frac{1}{2},\frac{a+1}{2(m-1)}+\frac{3}{2}\\ \frac{3}{2}\end{matrix};(m-1)^2z^2\right)\\
&+\Gamma\left(\frac{a}{2(m-1)}\right)\Gamma\left(\frac{a+1}{2(m-1)}+1\right)c_2x^a{}_2F_1\left(\begin{matrix}\frac{a}{2(m-1)},\frac{a+1}{2(m-1)}+1\\ \frac{1}{2}\end{matrix};(m-1)^2z^2\right)\\
=&\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{a}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2(m-1)}\right)\Gamma\left(\frac{a+1}{2(m-1)}+\frac{3}{2}\right)\Gamma\left(\frac{a+1}{2(m-1)}+1\right)x^a\\
&\times\left[\frac{c_1+c_2}{\Gamma(\frac{2a+1}{2(m-1)}+\frac{3}{2})}+\frac{\Gamma\left(-\frac{2a+1}{2(m-1)}-\frac{1}{2}\right)(c_1-c_2)}{\Gamma\left(-\frac{1}{2(m-1)}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2(m-1)}+\frac{1}{2}\right)}\right]\\
&\times {}_2F_1\left(\begin{matrix}\frac{a}{m-1},\frac{a+1}{m-1}+2\\ \frac{2a+1}{2(m-1)}+\frac{3}{2}\end{matrix};\frac{1+(m-1)z}{2}\right)\\
&-\frac{\Gamma\left(\frac{2a+1}{2(m-1)}+\frac{1}{2}\right)(c_1-c_2)}{\Gamma\left(\frac{a}{m-1}\right)\Gamma\left(\frac{a+1}{m-1}+2\right)}\left(\frac{1+(m-1)z}{2}\right)^{-\frac{2a+1}{2(m-1)}-\frac{1}{2}}\\
&\times {}_2F_1\left(\begin{matrix}-\frac{1}{2(m-1)}-\frac{1}{2},\frac{1}{2(m-1)}+\frac{3}{2}\\ -\frac{2a+1}{2(m-1)}+\frac{1}{2}\end{matrix};\frac{1+(m-1)z}{2}\right)
\end{aligned}$$

At the end, we can re-write the solution (3.26) as

$$\begin{aligned}
u(x,t) &= \bar{c}_1x^{a+m}F_1(z)+\bar{c}_2x^{a+m}F_2(z),\\
v(x,t) &= \bar{c}_1x^aG_1(z)-\bar{c}_2x^aG_2(z) \qquad \text{for} \quad |z|<\frac{1}{|m-1|},
\end{aligned}$$

where

$$\begin{aligned}
\bar{c}_1 &= 2^{-\frac{2a+m}{m-1}}\sqrt{\pi}\Gamma\left(\frac{a}{m-1}\right)\Gamma\left(\frac{a+m}{m-1}\right) \\
&\times \left(\frac{c_1+c_2}{\Gamma\left(\frac{2a+3m-2}{2(m-1)}\right)} + \frac{\Gamma\left(-\frac{2a+1}{2(m-1)-\frac{1}{2}}\right)(c_1-c_2)}{\Gamma\left(\frac{m-2}{2(m-1)}\right)\Gamma\left(\frac{m}{2(m-1)}\right)}\right), \\
\bar{c}_2 &= 2^{-\frac{2a+m}{m-1}}\sqrt{\pi}\Gamma\left(\frac{2a+m}{2(m-1)}\right), \\
F_1(z) &= \frac{a+2m-2}{m-1}\,{}_2F_1\left(\begin{matrix}\frac{a+m-1}{m-1},\frac{a+m}{m-1}\\ \frac{2a+3m-2}{2(m-1)}\end{matrix};\frac{1+(m-1)z}{2}\right), \\
F_2(z) &= \left(\frac{1+(m-1)z}{2}\right)^{-\frac{2a+1}{2(m-1)}-\frac{1}{2}}{}_2F_1\left(\begin{matrix}\frac{m-2}{2(m-1)},\frac{m}{2(m-1)}\\ -\frac{2a+1}{2(m-1)}+\frac{1}{2}\end{matrix};\frac{1+(m-1)z}{2}\right), \\
G_1(z) &= \left(\frac{a+m}{m-1}\right){}_2F_1\left(\begin{matrix}\frac{a}{m-1},\frac{a+1}{m-1}+2\\ \frac{2a+3m-2}{2(m-1)}\end{matrix};\frac{1+(m-1)z}{2}\right), \\
G_2(z) &= \left(\frac{1+(m-1)z}{2}\right)^{-\frac{2a+1}{2(m-1)}-\frac{1}{2}}{}_2F_1\left(\begin{matrix}-\frac{m}{2(m-1)},\frac{3m-2}{2(m-1)}\\ -\frac{2a+1}{2(m-1)}+\frac{1}{2}\end{matrix};\frac{1+(m-1)z}{2}\right).
\end{aligned}$$

The functions $F_1(z)$, $F_2(z)$ solve the ODE (2.46) in [3]:

$$(1-(c-1)^2z^2)F''(z) + (1-c)(2s+c-2)zF'(z) + s(1-s)F(z) = 0$$

by substituting $c = m$ and $s = a+m$. Moreover, the functions $F_1(z)$, $F_2(z)$, $G_1(z)$ and $G_2(z)$ solve (3.13) in [3]:

$$(s-c)G_i(z) = (1-(c-1)^2z^2)F_i'(z) + (1-c)szF_i(z) \quad \text{for } i = 1,2.$$

Hence, we can conclude that (3.26) corresponds to the invariant solutions for Case III.A.1 in [3].

3.3.2 Solutions of (3.1) with $c(x) = x$ and $\alpha = 1$

If we set $\alpha = 1$ in (3.6), then it becomes

$$\begin{aligned}
u(x,t) &= x^{a+1}\left(c_1E_{2,1}(a(a+1)t^2) + c_2atE_{2,2}(a(a+1)t^2)\right), \\
v(x,t) &= x^a\left(c_1(a+1)tE_{2,2}(a(a+1)t^2) + c_2E_{2,1}(a(a+1)t^2)\right). \qquad (3.37)
\end{aligned}$$

By virtue of (1.15) and (1.16), the above solution equals to

$$\begin{aligned} u(x,t) &= \bar{c}_1 x^{a+1} e^{\sqrt{a(a+1)}t} + \bar{c}_2 x^{a+1} e^{-\sqrt{a(a+1)}t}, \\ v(x,t) &= \frac{\sqrt{a(a+1)}}{a}\left(\bar{c}_1 x^a e^{\sqrt{a(a+1)}t} - \bar{c}_2 x^a e^{-\sqrt{a(a+1)}t}\right) \quad \text{for } a(a+1) > 0, \end{aligned}$$

where

$$\bar{c}_1 = \frac{1}{2}\left(c_1 + \frac{a}{\sqrt{a(a+1)}}c_2\right) \quad \text{and} \quad \bar{c}_2 = \frac{1}{2}\left(c_1 - \frac{a}{\sqrt{a(a+1)}}c_2\right).$$

If we adopt the following notation in the above solution:

$$F_1(t) = e^{\sqrt{a(a+1)}t}, \qquad F_2(t) = e^{-\sqrt{a(a+1)}t},$$
$$G_1(t) = \frac{\sqrt{a(a+1)}}{a} e^{\sqrt{a(a+1)}t}, \qquad G_2(t) = -\frac{\sqrt{a(a+1)}}{a} e^{-\sqrt{a(a+1)}t},$$

we see that the functions $F_1(t)$ and $F_2(t)$ solve the ODE (2.59) in [3]:

$$F''(t) + s(1-s)F(t) = 0$$

with the substitution $s = a+1$. Also, the functions $G_1(z)$ and $G_2(z)$ are determined by (3.29) in [3]:

$$G_i(t) = (s-1)^{-1} F_i'(t) \quad \text{for } i = 1,2.$$

Thus, (3.37) coincides with the invariant solutions for Case III.A.3 in [3].

3.3.3 Solutions of (3.1) with $c(x) = e^{-\frac{x}{2}}$ and $\alpha = 1$

By setting $n = 1$ and $\alpha = 1$ in (3.12), it becomes

$$\begin{aligned} u(x,t) &= c_1 e^{\frac{a-1}{2}x} {}_3\Psi_1\left[z^2 \middle| \begin{matrix} \left(\frac{1}{2}-\frac{a}{2},1\right), \left(\frac{1}{2}-\frac{a}{2},1\right), (1,1) \\ (1,2) \end{matrix}\right] \\ &\quad + c_2 e^{\frac{a-1}{2}x} z\, {}_3\Psi_1\left[z^2 \middle| \begin{matrix} \left(1-\frac{a}{2},1\right), \left(1-\frac{a}{2},1\right), (1,1) \\ (2,2) \end{matrix}\right], \\ v(x,t) &= -c_1 e^{\frac{a}{2}x} z\, {}_3\Psi_1\left[z^2 \middle| \begin{matrix} \left(\frac{1}{2}-\frac{a}{2},1\right), \left(\frac{3}{2}-\frac{a}{2},1\right), (1,1) \\ (2,2) \end{matrix}\right] \\ &\quad - c_2 e^{\frac{a}{2}x} {}_3\Psi_1\left[z^2 \middle| \begin{matrix} \left(-\frac{a}{2},1\right), \left(1-\frac{a}{2},1\right), (1,1) \\ (1,2) \end{matrix}\right], \end{aligned} \tag{3.38}$$

where $z = e^{-\frac{x}{2}}t$. We can rewrite (3.38) using (1.19) and (1.20) as:

$$\begin{aligned} u(x,t) &= \bar{c}_1 e^{\frac{a-1}{2}x} F_1(z) + \bar{c}_2 e^{\frac{a-1}{2}x} z F_2(z), \\ v(x,t) &= \bar{c}_1 e^{\frac{a}{2}x} z G_1(z) + \bar{c}_2 e^{\frac{a}{2}x} G_2(z) \quad \text{for } |z| < 2, \end{aligned}$$

where

$$\bar{c}_1 = \Gamma\left(\frac{1}{2} - \frac{a}{2}\right)^2 c_1, \qquad \bar{c}_2 = -\Gamma\left(-\frac{a}{2}\right)\Gamma\left(1 - \frac{a}{2}\right) c_2,$$

$$F_1(z) = {}_2F_1\left(\begin{matrix} \frac{1}{2} - \frac{a}{2}, \frac{1}{2} - \frac{a}{2} \\ \frac{1}{2} \end{matrix}; \frac{z^2}{4}\right), \qquad F_2(z) = \frac{a}{2} z\, {}_2F_1\left(\begin{matrix} 1 - \frac{a}{2}, 1 - \frac{a}{2} \\ \frac{3}{2} \end{matrix}; \frac{z^2}{4}\right),$$

$$G_1(z) = \frac{a-1}{2} z\, {}_2F_1\left(\begin{matrix} \frac{1}{2} - \frac{a}{2}, \frac{3}{2} - \frac{a}{2} \\ \frac{3}{2} \end{matrix}; \frac{z^2}{4}\right), \quad G_2(z) = {}_2F_1\left(\begin{matrix} -\frac{a}{2}, 1 - \frac{a}{2} \\ \frac{1}{2} \end{matrix}; \frac{z^2}{4}\right).$$

Using the formulas () and (), we see that $F_1(z)$ and $F_2(z)$ solve the ODE (2.72) in [3]:

$$(4 - z^2)F''(z) + (4s - 1)zF'(z) - 4s^2 F(z) = 0$$

and the functions $F_1(z)$, $F_2(z)$, $G_1(z)$ and $G_2(z)$ solve (3.36) in [3]:

$$G_i(z) = (2s + 1)^{-1}\left((2 - \frac{z^2}{2})F_i'(z) + szF_i(z)\right) - 2 \text{ for } i = 1, 2$$

with the substitution $s = \frac{a-1}{2}$. As a result, (3.38) coincides with the invariant solutions for Case III.A.5 in [3].

3.3.4 Solutions of (3.1) with $c(x) = 1$ and $\alpha = 1$

If we set $\alpha = 1$ and $n = 1$ in (3.14) and (3.15), then they becomes

$$\begin{aligned} u(x,t) &= c_1 x^{a_1+a_2} {}_1\Psi_0\left[-\frac{t}{x}\middle| \begin{matrix} (-a_1 - a_2, 1) \\ - \end{matrix}\right] + c_2 x^{a_1-a_2} {}_1\Psi_0\left[\frac{t}{x}\middle| \begin{matrix} (-a_1 + a_2, 1) \\ - \end{matrix}\right], \\ v(x,t) &= c_1 x^{a_1+a_2} {}_1\Psi_0\left[-\frac{t}{x}\middle| \begin{matrix} (-a_1 - a_2, 1) \\ - \end{matrix}\right] - c_2 x^{a_1-a_2} {}_1\Psi_0\left[\frac{t}{x}\middle| \begin{matrix} (-a_1 + a_2, 1) \\ - \end{matrix}\right], \end{aligned}$$

and

$$\begin{aligned} u(x,t) &= c_1 e^{(a_1+a_2)x} E_{1,1}((a_1 + a_2)t) + c_2 e^{(a_1-a_2)x} E_{1,1}((a_1 - a_2)t), \\ v(x,t) &= c_1 e^{(a_1+a_2)x} E_{1,1}((a_1 + a_2)t) - c_2 e^{(a_1-a_2)x} E_{1,1}((a_1 - a_2)t). \end{aligned}$$

By virtue of (1.12), (1.13) and (1.17), the above solutions equal to

$$\begin{aligned} u(x,t) &= c_1\Gamma\left(-a_1-a_2\right)(x+t)^{a_1+a_2}+c_2\Gamma\left(a_2-a_1\right)(x-t)^{a_1-a_2}, \\ v(x,t) &= c_1\Gamma\left(-a_1-a_2\right)(x+t)^{a_1+a_2}-c_2\Gamma\left(a_2-a_1\right)(x-t)^{a_1-a_2}, \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} u(x,t) &= c_1e^{(a_1+a_2)(x+t)}+c_2e^{(a_1-a_2)(x-t)}, \\ v(x,t) &= c_1e^{(a_1+a_2)(x+t)}-c_2e^{(a_1-a_2)(x-t)}. \end{aligned} \tag{3.40}$$

We see that the solutions (3.39) and (3.40) are the well-known traveling wave solutions. For $n=1$, the solution (3.6) reduces to

$$\begin{aligned} u(x,t) &= \sqrt{a}\bar{c}_1x^{a+1}e^{\sqrt{a(a+1)}t}+\sqrt{a}\bar{c}_2x^{a+1}e^{-\sqrt{a(a+1)}t}, \\ v(x,t) &= \sqrt{a+1}\bar{c}_1x^{a}e^{\sqrt{a(a+1)}t}-\sqrt{a+1}\bar{c}_2x^{a}e^{-\sqrt{a(a+1)}t}. \end{aligned}$$

If we adopt the following notation in the above solution:

$$\begin{aligned} F_1(t) &= \sqrt{a}e^{\sqrt{a(a+1)}t}, \qquad & F_2(t) &= \sqrt{a}e^{-\sqrt{a(a+1)}t}, \\ G_1(t) &= \sqrt{a+1}e^{\sqrt{a(a+1)}t}, \qquad & G_2(t) &= \sqrt{a+1}e^{-\sqrt{a(a+1)}t}, \end{aligned}$$

then we see that the functions $F_1(t)$, $F_2(t)$ solve the ODE (2.59) of [3]:

$$F''(t)+s(1-s)F(t)=0$$

with the substitution $s=a+1$. The functions $F_1(z)$, $F_2(z)$, $G_1(z)$, $G_2(z)$ solve the ODE (3.29):

$$G_i(t)=(s-1)^{-1}F_i'(t)$$

of the same work. So, (3.37) coincides with the invariant solutions of Case III.A.3 of [3].

Chapter 4

Lie symmetry analysis of a class of time fractional nonlinear diffusion-wave systems

In this chapter, we consider the class of time fractional nonlinear systems of the following form:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = b^2(u)u_x, \end{cases} \tag{4.1}$$

where α is a positive non-integer number and $b(u)$ is a sufficiently differentiable non-constant function.

In [14], the nonlinear model of stationary transonic plane-parallel gas flows

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = -uu_x, \end{cases} \tag{4.2}$$

with $0 < \alpha < 1$, was studied using Lie symmetry analysis. The Lie symmetries, some reduced systems of ODEs and some partial solutions of the system (4.2) are obtained in [14]. Substituting $\bar{u}(x,t) = -u(x,t)$ and $\bar{v}(x,t) = -v(x,t)$ into (4.2), we obtain the following equivalent fractional system:

$$\begin{cases} \frac{\partial^\alpha \bar{u}}{\partial t^\alpha} = \bar{v}_x, \\ \frac{\partial^\alpha \bar{v}}{\partial t^\alpha} = \bar{u}\bar{u}_x. \end{cases}$$

This corresponds to the particular case $b(u) = \sqrt{u}$ for the system given in (4.1). It is thus seen that (4.1) can be viewed as a generalization of (4.2) with respect to the above-mentioned substitution. Hence, the results of this paper generalize the results of [14].

The importance of finding exact solutions of (4.1) lies in the fact that if $(u(x,t), v(x,t))$ solves (4.1), then $u(x,t)$ solves the sequential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha}{\partial t^\alpha}u = \Big(b^2(u)u_x\Big)_x. \tag{4.3}$$

For example, in the case $\alpha = 1$, the equation (4.3) becomes the well-known nonlinear wave equation and in the case $\alpha = \frac{1}{2}$, the component $u(x,t)$ of the solutions of (4.1) with $t > 0$ is a solution to the nonlinear heat equation with source

$$u_t = \Big(b^2(u)u_x\Big)_x + \frac{1}{\sqrt{\pi}}\frac{g(x)}{\sqrt{t}}, \qquad g(x) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{u(x,\tau)}{\sqrt{t-\tau}}d\tau\Big|_{t=0}, \quad t > 0,$$

by virtue of the formula (1.5).

We study the system given in (4.1) using Lie symmetry analysis. More explicitly, we present a complete group classification depending on the function $b(u)$ and describe the structure of Lie algebras generated by the infinitesimal symmetries of (4.1). After obtaining the group classification of (4.1), we proceed to finding optimal systems of Lie algebras and the reduced systems of ODEs. Using these optimal systems, we also classify the group invariant solutions corresponding to the infinitesimal symmetries for $0 < \alpha < 1$.

4.1 Lie symmetry analysis of (4.1)

In this section, we study (4.1) using the formulas obtained in Chapter 1. There are two cases regarding the symmetry group of (4.1), as determined by the form of the function $b(u)$, one in which $b(u)$ possesses the form of a power function, and one in which it does not. The only difference between these cases is that in the former case, the symmetry group of (4.1) possesses an additional symmetry that does not exist in the latter case. For each cases we obtain the infinitesimal symmetries.

From (1.44), we obtain the following invariance criterion for (4.1):

$$\begin{cases} \tilde{X}(u_{t^\alpha} - v_x)|_{(4.1)} = 0, \\ \tilde{X}(v_{t^\alpha} - b^2(u)u_x)|_{(4.1)} = 0. \end{cases}$$

In explicit form, this is

$$\begin{cases} \left(\mu^{(\alpha)} - \phi^{(1)}\right)\Big|_{(4.1)} = 0, \\ (\phi^{(\alpha)} - 2\mu b b' u_x - b^2\mu^{(1)})\Big|_{(4.1)} = 0. \end{cases} \tag{4.4}$$

If we substitute (1.38) and (1.39) into (4.4), then we get for the first equation

$$\frac{\partial^\alpha \mu}{\partial t^\alpha} - u\frac{\partial^\alpha \mu_u}{\partial t^\alpha} - v\frac{\partial^\alpha \mu_v}{\partial t^\alpha} + (\mu_u - \alpha D_t(\tau))v_x + \mu_v b^2(u)u_x - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x)$$
$$\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(u) + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \mu_v}{\partial t^n} D_t^{\alpha-n} v + \mu_1$$
$$- \left[\varphi_x + \varphi_u u_x + \varphi_v v_x - \tau_x v_t - \tau_u u_x v_t - \tau_v v_x v_t - \xi_x v_x - \xi_u u_x v_x - \xi_v v_x^2\right] = 0,$$

and for the second equation

$$\frac{\partial^\alpha \varphi}{\partial t^\alpha} - v\frac{\partial^\alpha \varphi_v}{\partial t^\alpha} - u\frac{\partial^\alpha \varphi_u}{\partial t^\alpha} + (\varphi_v - \alpha D_t(\tau))b^2(u)u_x + \varphi_u v_x - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(v_x)$$
$$\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \varphi_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(v) + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \varphi_u}{\partial t^n} D_t^{\alpha-n} u + \varphi_1 - 2b(u)b'(u)\mu u_x$$
$$- b^2(u)\left[\varphi_x + \varphi_u u_x + \varphi_v v_x - \tau_x v_t - \tau_u u_x v_t - \tau_v v_x v_t - \xi_x v_x - \xi_u u_x v_x - \xi_v v_x^2\right] = 0.$$

From above, we obtain the following (overdetermined) system of determining equations by setting the coefficients of the linearly independent partial derivatives $D_t^{\alpha-n}u$, $D_t^{\alpha-n}v$, $D_t^{\alpha-n}u_x$, $D_t^{\alpha-n}v_x$, v_x, u_x, v_t, $u_x v_t$, $v_x v_t$, $u_x v_x$ and v_x^2 equal to 0:

$$\begin{aligned}
&\binom{\alpha}{n}\frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}\tau = 0, \qquad n = 1, 2, \ldots, \\
&\frac{\partial^n \mu_v}{\partial t^n} = 0, \qquad n = 1, 2, \ldots, \\
&D_t^n(\xi) = 0, \qquad n = 1, 2, \ldots, \\
&\mu_u - \alpha D_t(\tau) - \phi_v + \xi_x = 0, \\
&b^2\mu_v - \phi_u = 0, \\
&\frac{\partial^\alpha \mu}{\partial t^\alpha} - v\frac{\partial^\alpha \mu_v}{\partial t^\alpha} - u\frac{\partial^\alpha \mu_u}{\partial t^\alpha} - \phi_x + \mu_1 = 0, \\
&\frac{\partial^n \phi_u}{\partial t^n} = 0, \qquad n = 1, 2, \ldots,
\end{aligned}$$

$$\binom{\alpha}{n}\frac{\partial^n \phi_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}\tau = 0, \qquad n = 1, 2, \ldots,$$
$$b^2\phi_v - \alpha b^2 D_t(\tau) - 2bb'\mu - b^2\mu_u + b^2\xi_x = 0,$$
$$\frac{\partial^\alpha \phi}{\partial t^\alpha} - v\frac{\partial^\alpha \phi_v}{\partial t^\alpha} - u\frac{\partial^\alpha \phi_u}{\partial t^\alpha} - b^2\mu_x + \phi_1 = 0,$$
$$\tau_x = \tau_u = \tau_v = 0,$$
$$\xi_u = \xi_v = 0.$$

Analyzing the above overdetermined system with the initial condition (1.29), we are able to deduce the following infinitesimal symmetries:

Case 1. This is the generic situation, which applies to all forms of $b(u)$, except $b(u) = ku^m$ (with $k, m \neq 0$). In this case, the infinitesimals are

$$\tau = \frac{s_1}{\alpha}t, \quad \xi = s_1 x + s_2, \quad \mu = 0, \quad \phi = s(t),$$

where s_1 and s_2 are arbitrary constants, and $s(t)$ is a solution of the equation $\frac{d^\alpha s(t)}{dt^\alpha} = 0$. With these infinitesimals, we have the following infinitesimal symmetries:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = s(t)\frac{\partial}{\partial v}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t}.$$

Case 2. In the special case that $b(u)$ takes the form ku^m (with $k, m \neq 0$), the infinitesimal are

$$\tau = \frac{s_1}{\alpha}t, \quad \xi = (s_1 + s_3)x + s_2, \quad \mu = \frac{s_3}{m}u, \quad \phi = \frac{(1+m)s_3}{m}v + s(t),$$

where s_1, s_2 and s_3 are arbitrary constants, and $s(t)$ is a solution of the equation $\frac{d^\alpha s(t)}{dt^\alpha} = 0$. Thus, in this case, along with X_1, X_2 and X_3 given above, there is the following additional symmetry:

$$X_4 = x\frac{\partial}{\partial x} + \frac{u}{m}\frac{\partial}{\partial u} + \frac{1+m}{m}v\frac{\partial}{\partial v}.$$

The detailed proof of obtaining above two cases has been omitted here.

Because we now have a complete group classification of (4.1), we are in a position to investigate the one-dimensional optimal systems of Lie algebras of its infinitesimal symmetries and the classification of group invariant solutions. However, before moving on to the next section, we note that the solution of the equation $\frac{d^\alpha s(t)}{dt^\alpha} = 0$ is

$$s(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where c_i are arbitrary constants and n is a positive integer satisfying $n-1 < \alpha < n$.

4.2 Classification of invariant solutions of (4.1)

In this section, we classify the group invariant solutions of (4.1) corresponding to infinitesimal symmetries for the case $0 < \alpha < 1$. We choose optimal systems, which lead us to simpler reduced systems, by using the results given in [23]. In the following two subsections, we determine the optimal systems and corresponding reduced systems for Cases 1 and 2 specified above.

4.2.1 Case 1. For arbitrary $b(u)$, but $b(u) \neq ku^m$

From the discussion above, we know that in this case, for α satisfying $0 < \alpha < 1$, the system in (4.1) possesses the following infinitesimal symmetries:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = t^{\alpha-1}\frac{\partial}{\partial v}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t}.$$

The commutator table for the Lie algebra generated by these infinitesimal symmetries is given below (where i and j index the row and column). We see from this table that

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	$\frac{1-\alpha}{\alpha}X_2$
X_3	$-X_1$	$-\frac{1-\alpha}{\alpha}X_2$	0

Table 4.1 Commutator table for Case 1.

the Lie algebra in this case is identical to the Lie algebra $A_{3,5}$ given in [23]. Thus, the one-dimensional optimal system of the Lie algebra generated by X_1, X_2 and X_3 is that obtained in [23],

$$U_1 = X_1 + aX_2 = \frac{\partial}{\partial x} + at^{\alpha-1}\frac{\partial}{\partial v} \quad (\text{ with } a = 0, 1, -1),$$
$$U_2 = X_3, \quad U_3 = X_2.$$

Then, using the standard characteristic method, we obtain the invariant solutions and reduced systems of ODEs of (4.1) corresponding to each symmetry U_j. These are given below. We consider the reduced systems of ODEs corresponding to U_1 in subsequent

U_j	Invariant solutions $(u_j(x,t), v_j(x,t))$	Reduced systems of ODEs
U_1	$\begin{cases} u(x,t) = \varphi(t), \\ v(x,t) = \psi(t) + at^{\alpha-1}x, \end{cases}$	$\begin{cases} \frac{d^\alpha \varphi}{dt^\alpha} = at^{\alpha-1}, \\ \frac{d^\alpha \psi}{dt^\alpha} = 0, \end{cases}$ $a = 0, 1, -1$
U_2	$\begin{cases} u(x,t) = \varphi(z), \\ v(x,t) = \psi(z), \end{cases}$ with $z = tx^{-\frac{1}{\alpha}}$	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = -\frac{z}{\alpha}\psi', \\ \frac{d^\alpha \psi}{dz^\alpha} = -\frac{z}{\alpha}b^2(\varphi)\varphi', \end{cases}$
U_3	There are no invariant solutions.	

Table 4.2 The optimal systems and reduced systems of (4.1) for Case 1

sections. The reduced system of ODEs corresponding to U_2 depends on $b(u)$, and for this reason, we do not solve it.

4.2.2 Case 2. $b(u) = ku^m$

To obtain the optimal systems for this case, here we construct both the commutator and adjoint tables for the Lie algebras of infinitesimal symmetries. The adjoint table is given by

$$Ad(e^{(\varepsilon Y_i)})Y_j = Y_j - \varepsilon[Y_i, Y_j] + \frac{\varepsilon^2}{2}[Y_i, [Y_i, Y_j]] - \dots, \text{ where } \varepsilon \in \mathbb{R}.$$

In the present case, for α satisfying $0 < \alpha < 1$, the system in (4.1) becomes

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = k^2 u^{2m} u_x, \end{cases} \tag{4.5}$$

and the corresponding infinitesimal symmetries are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = t^{\alpha-1}\frac{\partial}{\partial v}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t}, \quad X_4 = x\frac{\partial}{\partial x} + \frac{u}{m}\frac{\partial}{\partial u} + \frac{m+1}{m}v\frac{\partial}{\partial v}.$$

The optimal systems of a given Lie algebra depend on the structure of that Lie algebra. Because the structure of the Lie algebra generated by the above X_1, X_2, X_3 and X_4 depends on the parameters m and α, we study (4.5) in two subcases characterized by the relations $2m\alpha + \alpha - m \neq 0$ and $2m\alpha + \alpha - m = 0$.

Case 2.1.

Let us consider the case $2m\alpha + \alpha - m \neq 0$. Then, for the Lie algebra generated by the symmetries X_i $(i = 1, \dots, 4)$, we choose a new basis Y_i $(i = 1, \dots, 4)$ such that the Lie algebra consists of the direct sum of two subalgebras L_1 and L_2, where L_1 is generated by Y_1 and Y_2, and L_2 is generated by Y_3 and Y_4. We choose this new basis as follows:

$$Y_1 = -\tfrac{(m+1)\alpha}{2m\alpha+\alpha-m}X_3 - \tfrac{m(\alpha-1)}{2m\alpha+\alpha-m}X_4, \qquad Y_2 = -X_1,$$
$$Y_3 = \tfrac{m\alpha}{2m\alpha+\alpha-m}(X_3 - X_4), \qquad Y_4 = -X_2.$$

The commutator and adjoint tables for Y_i $(i = 1, \dots, 4)$ are given below (where i and j index the row and column). We choose the following optimal system for the Lie

$[Y_i, Y_j]$	Y_1	Y_2	Y_3	Y_4
Y_1	0	Y_2	0	0
Y_2	$-Y_2$	0	0	0
Y_3	0	0	0	Y_4
Y_4	0	0	$-Y_4$	0

Table 4.3 Commutator table for Case 2.1.

$Ad(e^{(\varepsilon Y_i)})Y_j$	Y_1	Y_2	Y_3	Y_4
Y_1	Y_1	$e^{-\epsilon Y_2}$	Y_3	Y_4
Y_2	$Y_1 + \epsilon Y_2$	Y_2	Y_3	Y_4
Y_3	Y_1	Y_2	Y_3	$e^{-\epsilon Y_4}$
Y_4	Y_1	Y_2	$Y_3 + \epsilon Y_4$	Y_4

Table 4.4 Adjoint table for Case 2.1.

algebra generated by Y_i $(i = 1, \dots, 4)$ to simplify the reduced systems of FODEs:

$$\begin{aligned}
U_1 &= Y_2 + aY_4 &&= -X_1 - aX_2, \quad a = 0, 1, -1,\\
U_2 &= Y_2 + \tfrac{(2m\alpha+\alpha-m)a}{m\alpha}Y_3 &&= -X_1 + aX_3 - aX_4, \quad a = 1, -1,\\
U_3 &= (2m\alpha + \alpha - m)Y_1 + aY_4 &&= -aX_2 - (m+1)\alpha X_3 - m(\alpha - 1)X_4, \quad a = 1, -1,\\
U_4 &= Y_1 + \tfrac{(2m\alpha+\alpha-m)a-m\alpha+m}{m\alpha}Y_3 &&= (a-1)X_3 - aX_4, \quad a \in \mathbb{R},\\
U_5 &= \tfrac{2m\alpha+\alpha-m}{m\alpha}Y_3 &&= X_3 - X_4,\\
U_6 &= Y_4 &&= -X_2.
\end{aligned}$$

Remark 1. *We see from Table 4.3 that this Lie algebra is identical to the Lie algebra $2A_2$ given in [23]. If we act on U_2 and U_3 with $Ad(e^{\epsilon Y_1})$ and $Ad(e^{\epsilon Y_3})$, respectively,* *with suitable* ϵ, *we obtain the equivalences* $U_2 \sim Y_1 + aY_3$ *and* $U_3 \sim Y_1 + aY_4$ $(a = \pm 1)$. *This demonstrates the correspondence between the optimal system chosen here and the optimal system in [23].*

In Tables 4.5 and 4.6, we display the similarity variables z_j, invariant solutions $(u_j(x,t), v_j(x,t))$ expressed as solutions $(\varphi(z), \psi(z))$ of the reduced systems of ODEs, and the reduced system of ODEs corresponding to U_j in the optimal system. Note that due to the divergence of the integral in the definition (1.1) of the Riemann-Liouville derivative, $\frac{d^\alpha}{dt^\alpha}(t^p)$ is not defined for $p \leq -1$ [**?**]. For this reason, here we need an additional assumption, which is expressed in Table 4.5, regarding invariant solutions corresponding to the symmetry U_5.

U_j	z_j	Invariant solutions $(u_j(x,t), v_j(x,t))$
U_1	t	$\begin{cases} u(x,t) = \varphi(t), \\ v(x,t) = \psi(t) + axt^{\alpha-1}, \end{cases}$ $a = 0, 1, -1$
U_2	$t\exp\left(\frac{a}{\alpha}x\right)$	$\begin{cases} u(x,t) = \exp\left(\frac{a}{m}x\right)\varphi(z), \\ v(x,t) = a\exp\left(\frac{(m+1)a}{m}x\right)\psi(z), \end{cases}$ $a = 1, -1$
U_3	$tx^{-\frac{m+1}{2m\alpha+\alpha-m}}$	$\begin{cases} u(x,t) = x^{\frac{\alpha-1}{2m\alpha+\alpha-m}}\varphi(z), \\ v(x,t) = x^{\frac{(m+1)(\alpha-1)}{2m\alpha+\alpha-m}}\psi(z) + \frac{a}{2m\alpha+\alpha-m}t^{\alpha-1}\ln(x), \end{cases}$ $a = 1, -1$
U_4	$tx^{\frac{a-1}{\alpha}}$	$\begin{cases} u(x,t) = x^{\frac{a}{m}}\varphi(z), \\ v(x,t) = x^{\frac{(m+1)a}{m}}\psi(z), \end{cases}$ $a \in \mathbb{R}$
U_5	x	$\begin{cases} u(x,t) = t^{-\frac{\alpha}{m}}\varphi(x), \\ v(x,t) = t^{-\frac{(m+1)\alpha}{m}}\psi(x), \end{cases}$ $m < 0$ or $m > \frac{\alpha}{1-\alpha}$
U_6		There are no invariant solutions.

Table 4.5 Similarity variables z_j and invariant solutions (u_j, v_j) for Case 2.1.

Case 2.2.

Next, let us consider the case $2m\alpha + \alpha - m = 0$. In this case, we obtain $m = \frac{\alpha}{1-2\alpha}$ with $\alpha \neq \frac{1}{2}$. Then, for the Lie algebra generated by the symmetries X_i $(i = 1, \ldots, 4)$, we again choose a new basis Y_i $(i = 1, \ldots, 4)$ such that the Lie algebra consists of the direct sum of two subalgebras L_1 and L_2. However, in this case L_1 is generated by Y_1, Y_2 and Y_3, and L_2 is generated by Y_4. The new basis is

$$Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = X_4, \quad Y_4 = X_4 - X_3.$$

U_j	Reduced system of ODEs
U_1	$\begin{cases} \frac{d^\alpha \varphi(t)}{dt^\alpha} = at^{\alpha-1}, \\ \frac{d^\alpha \psi(t)}{dt^\alpha} = 0, \end{cases} \quad a = 0, 1, -1$
U_2	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \left(\frac{m+1}{m}\psi + \frac{1}{\alpha}z\psi'\right), \\ \frac{d^\alpha \psi}{dz^\alpha} = k^2\varphi^{2m}\left(\frac{1}{m}\varphi + \frac{1}{\alpha}z\varphi'\right), \end{cases}$
U_3	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{m+1}{2m\alpha+\alpha-m}\left((\alpha-1)\psi - z\psi'\right) + \frac{a}{2m\alpha+\alpha-m}z^{\alpha-1}, \\ \frac{d^\alpha \psi}{dz^\alpha} = \frac{k^2}{2m\alpha+\alpha-m}\varphi^{2m}\left((\alpha-1)\varphi - (m+1)z\varphi'\right), \end{cases} \quad a = 1, -1$
U_4	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{(m+1)a}{m}\psi + \frac{a-1}{\alpha}z\psi', \\ \frac{d^\alpha \psi}{dz^\alpha} = k^2\varphi^{2m}\left(\frac{a}{m}\varphi + \frac{a-1}{\alpha}z\varphi'\right), \end{cases} \quad a \in \mathbb{R}$
U_5	$\begin{cases} \psi'(x) = \frac{\Gamma\left(1-\frac{\alpha}{m}\right)}{\Gamma\left(1-\frac{(m+1)\alpha}{m}\right)}\varphi(x), \\ k^2\varphi^{2m}\varphi'(x) = \frac{\Gamma\left(1-\frac{(m+1)\alpha}{m}\right)}{\Gamma\left(1-\frac{(2m+1)\alpha}{m}\right)}\psi(x), \end{cases} \quad m < 0 \text{ or } m > \frac{\alpha}{1-\alpha}$

Table 4.6 Reduced systems of (4.1) for Case 2.1.

The commutator and adjoint tables for Y_i $(i = 1, \ldots, 4)$ are given in Tables 4.7 and 4.8: We choose the following optimal system in the new and original bases:

$[Y_i, Y_j]$	Y_1	Y_2	Y_3	Y_4
Y_1	0	0	Y_1	0
Y_2	0	0	$\frac{1-\alpha}{\alpha}Y_2$	0
Y_3	$-Y_1$	$-\frac{1-\alpha}{\alpha}Y_2$	0	0
Y_4	0	0	0	0

Table 4.7 Commutator table for Case 2.2.

$Ad(e^{(\varepsilon Y_i)})Y_j$	Y_1	Y_2	Y_3	Y_4
Y_1	Y_1	Y_2	$Y_3 - \varepsilon Y_1$	Y_4
Y_2	Y_1	Y_2	$Y_3 - \frac{1-\alpha}{\alpha}\varepsilon Y_2$	Y_4
Y_3	$e^{\varepsilon}Y_1$	$e^{\frac{1-\alpha}{\alpha}\varepsilon}Y_2$	Y_3	Y_4
Y_4	Y_1	Y_2	Y_3	Y_4

Table 4.8 Adjoint table for Case 2.2.

$$
\begin{aligned}
U_1 &= Y_1 + aY_2 = X_1 + aX_2, \quad a = 0, 1, -1,\\
U_2 &= Y_1 + a_1Y_2 + a_2Y_4\\
&= X_1 + a_1X_2 - a_2X_3 + a_2X_4, \quad (a_1, a_2) \in \{(a, \pm 1) | a \in \mathbb{R}\},\\
U_3 &= Y_3 + (a-1)Y_4 = (1-a)X_3 + aX_4, \quad a \in \mathbb{R},\\
U_4 &= aY_2 + Y_4 = aX_2 - X_3 + X_4, \quad a = 0, 1, -1,\\
U_5 &= Y_2 = X_2.
\end{aligned}
$$

We see from Table 4.7 that this Lie algebra is identical to the Lie algebra $A_{3,5} \oplus A_1$ given in [23]. Hence, considering the following equivalences regarding the actions of $Ad(e^{\epsilon Y_3})$ on U_2, it is seen that our optimal system of this Lie algebra corresponds bijectively to that given in [23].

a) For any $a > 0$, there exists $b > 0$ such that $Y_1 + aY_2 + Y_4 \sim Y_1 + Y_2 + bY_4$,

b) For any $a > 0$, there exists $b < 0$ such that $Y_1 + aY_2 - Y_4 \sim Y_1 + Y_2 + bY_4$,

c) For any $a < 0$, there exists $b > 0$ such that $Y_1 + aY_2 + Y_4 \sim Y_1 - Y_2 + bY_4$,

d) For any $a < 0$, there exists $b < 0$ such that $Y_1 + aY_2 - Y_4 \sim Y_1 - Y_2 + bY_4$.

In the following Table 4.9, we display the similarity variables z_j and invariant solutions $(u_j(x,t), v_j(x,t))$, which are expressed as solutions of reduced systems. Then, in Table 4.10, we present the reduced systems of ODEs corresponding to the above optimal system. With the above results for the optimal systems in Cases 2.1 and 2.2,

U_j	z_j	Invariant solutions $(u_j(x,t), v_j(x,t))$
U_1	t	$\begin{cases} u(x,t) = \varphi(t), \\ v(x,t) = \psi(t) + axt^{\alpha-1}, \end{cases}$ $a = 0, 1, -1$
U_2	$t\exp(\frac{a_2}{\alpha}x)$	$\begin{cases} u(x,t) = \exp\left(\frac{a_2(1-2\alpha)}{\alpha}x\right)\varphi(z), \\ v(x,t) = a_2\exp\left(\frac{a_2(1-\alpha)}{\alpha}x\right)\psi(z) + a_1xt^{\alpha-1}, \end{cases}$ $(a_1, a_2) \in \{(a, \pm 1) \| a \in \mathbb{R}\}$
U_3	$tx^{\frac{a-1}{\alpha}}$	$\begin{cases} u(x,t) = x^{\frac{a(1-2\alpha)}{\alpha}}\varphi(z), \\ v(x,t) = x^{\frac{a(1-\alpha)}{\alpha}}\psi(z), \end{cases}$ $a \in \mathbb{R}$
U_4	x	$\begin{cases} u(x,t) = t^{2\alpha-1}\varphi(x), \\ v(x,t) = t^{\alpha-1}\psi(x) - a\alpha t^{\alpha-1}\ln(t), \end{cases}$ $a = 0, 1, -1$
U_5		There are no invariant solutions.

Table 4.9 Similarity variables z_j and invariant solutions (u_j, v_j) for Case 2.2.

we arrive at the following three conclusions for the case $m = \frac{\alpha}{1-2\alpha}$.

U_j	Reduced system of ODEs
U_1	$\begin{cases} \frac{d^\alpha \varphi(t)}{dt^\alpha} = at^{\alpha-1}, \\ \frac{d^\alpha \psi(t)}{dt^\alpha} = 0, \end{cases} \quad a = 0, 1, -1$
U_2	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{1}{\alpha}\left((1-\alpha)\psi + z\psi'\right) + a_1 z^{\alpha-1}, \\ \frac{d^\alpha \psi}{dz^\alpha} = \frac{k^2}{\alpha}\varphi^{\frac{2\alpha}{1-2\alpha}}\left((1-2\alpha)\varphi + z\varphi'\right), \end{cases} \quad a_1 \in \mathbb{R}$
U_3	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{1}{\alpha}\left(a(1-\alpha)\psi + (a-1)z\psi'\right), \\ \frac{d^\alpha \psi}{dz^\alpha} = \frac{k^2}{\alpha}\varphi^{\frac{2\alpha}{1-2\alpha}}\left(a(1-2\alpha)\varphi + (a-1)z\varphi'\right), \end{cases} \quad a \in \mathbb{R}$
U_4	$\begin{cases} \psi'(x) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\varphi(x), \\ k^2\varphi^{\frac{2\alpha}{1-2\alpha}}\varphi'(x) = -a\Gamma(\alpha+1), \end{cases} \quad a = 0, 1, -1,$

Table 4.10 Reduced systems of (4.1) for Case 2.2.

1. The two invariant solutions corresponding to U_1 in Cases 2.1 and 2.2 coincide.
2. The invariant solutions corresponding to U_3 in Case 2.1 coincide with the invariant solutions corresponding to U_4 in Case 2.2.
3. The invariant solutions corresponding to U_4 in Case 2.1 coincide with the invariant solutions corresponding to U_3 in Case 2.2.

Even though the elements of the optimal systems in Cases 2.1 and 2.2 correspond to each other, except for the element U_2, the reduced systems of ODEs in Cases 2.1 and 2.2 generally differ, possessing a relationship determined by the choice of the similary variable z.

In the next section, we derive several explicit invariant solutions of the fractional system in (4.1) by solving the reduced systems of ODEs obtained in this section.

4.3 Invariant solutions of (4.1)

In the general case, solving fractional order nonlinear systems of ODEs is a challenging problem. However, here we are able to derive several explicit solutions of the reduced systems of FODEs obtained in the previous section. Then, using these solutions, we obtain several group invariant solutions of (4.1).

4.3.1 Invariant solutions of (4.1) corresponding to U_1

The reduced systems of ODEs corresponding to U_1 are essentially the same in all three cases. The following is the solution to each of these:

$$\begin{cases} \varphi(t) = \frac{a}{\Gamma(\alpha)} t^{2\alpha-1} + c_1 t^{\alpha-1}, \\ \psi(t) = c_2 t^{\alpha-1}, \end{cases}$$

where c_1 and c_2 are arbitrary constants. With the above, we obtain the following for the invariant solution of (4.1):

$$\begin{cases} u(x,t) = \frac{a}{\Gamma(\alpha)} t^{2\alpha-1} + c_1 t^{\alpha-1}, \\ v(x,t) = (c_2 + ax) t^{\alpha-1}. \end{cases} \tag{4.6}$$

4.3.2 Invariant solutions of (4.1) corresponding to U_4 in Case 2.1

The reduced system of ODEs corresponding to U_4 in Case 2.1 has the general form

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = a_1 \psi + a_2 z \psi_z, \\ \frac{d^\alpha \psi}{dz^\alpha} = \varphi^{2m}(b_1 \varphi + b_2 z \varphi_z), \end{cases} \tag{4.7}$$

where a_1, a_2, b_1 and b_2 are constants. We formulate the following lemma with respect to a solution of the system given in (4.7).

Lemma 9. *Let us assume that the parameter m satisfies $m < 0$ or $m > \frac{\alpha}{1-\alpha}$. Then, if the inequalities*

$$m \neq \frac{\alpha}{1-2\alpha}, \quad a_1 - \frac{m+1}{m} a_2 \alpha \neq 0, \quad b_1 - \frac{1}{m} b_2 \alpha \neq 0$$

hold, the system in (4.7) has a solution of the form $\varphi(z) = c_1 z^{\lambda_1}$, $\psi(z) = c_2 z^{\lambda_2}$, where

$$\begin{aligned} \lambda_1 &= -\frac{\alpha}{m}, \qquad \lambda_2 = -\frac{(m+1)\alpha}{m}, \\ c_1^{2m} &= \frac{m^2 \Gamma\left(1 - \frac{\alpha}{m}\right)}{(m a_1 - (m+1) a_2 \alpha)(m b_1 - b_2 \alpha) \Gamma\left(1 - \frac{(2m+1)\alpha}{m}\right)}, \\ c_2 &= \frac{m \Gamma\left(1 - \frac{\alpha}{m}\right)}{(m a_1 - (m+1) a_2 \alpha) \Gamma\left(1 - \frac{m+1}{m}\alpha\right)} c_1. \end{aligned}$$

Proof. Directly substituting $\varphi(z) = c_1 z^{\lambda_1}$ and $\psi(z) = c_2 z^{\lambda_2}$ into (4.7) we obtain

$$\begin{cases} c_1 \frac{\Gamma(1+\lambda_1)}{\Gamma(1+\lambda_1-\alpha)} z^{\lambda_1-\alpha} = c_2(a_1 z^{\lambda_2} + a_2\lambda_2 z^{\lambda_2}), \\ c_2 \frac{\Gamma(1+\lambda_2)}{\Gamma(1+\lambda_2-\alpha)} z^{\lambda_2-\alpha} = c_1^{2m+1} z^{2\lambda_1 m}(b_1 z^{\lambda_1} + b_1\lambda_2 z^{\lambda_1}). \end{cases} \tag{4.8}$$

The powers of z appearing here should be equal in the two equations. From this observation, we have

$$\lambda_1 = -\frac{\alpha}{m}, \qquad \lambda_2 = -\frac{(m+1)\alpha}{m}.$$

Next, by the assumption of the lemma, we see that $\lambda_1 > -1$ and $\lambda_2 > -1$. From these results for λ_1 and λ_2, c_1 and c_2 are obtained as in the statement of the lemma. □

The reduced system of ODEs corresponding to U_4 in Case 2.1 satisfies the conditions of Lemma 9, which implies that there exists a solution $(\varphi(z), \psi(z))$ as in the lemma. Then, from Tables 4.5 and 4.6, we obtain the following explicit invariant solution to (4.1) with the condition that m satisfies either $m < 0$ or $m > \frac{\alpha}{1-\alpha}$:

$$\begin{cases} u(x,t) = & \left[\frac{m^2}{k^2(m+1)} \frac{\Gamma(1-\frac{\alpha}{m})}{\Gamma(1-\frac{(2m+1)\alpha}{m})}\right]^{\frac{1}{2m}} x^{\frac{1}{m}} t^{-\frac{\alpha}{m}}, \\ v(x,t) = & \left[\frac{m^2}{k^2(m+1)} \frac{\Gamma(1-\frac{\alpha}{m})}{\Gamma(1-\frac{(2m+1)\alpha}{m})}\right]^{\frac{1}{2m}} \frac{m\Gamma\left(1-\frac{\alpha}{m}\right)}{(m+1)\Gamma\left(1-\frac{m+1}{m}\alpha\right)} x^{\frac{m+1}{m}} t^{-\frac{(m+1)\alpha}{m}}. \end{cases} \tag{4.9}$$

Finally, note that even though the reduced systems corresponding to U_2 in Case 2.1 and U_3 in Case 2.2 are of the form (4.7), these systems do not satisfy the conditions of Lemma 2.

4.3.3 Invariant solutions of (4.1) corresponding to U_5 in Case 2.1

By direct integration of the reduced system in this case, we obtain the following implicit solution with the condition $m < 0$ or $m > \frac{\alpha}{1-\alpha}$:

$$\begin{cases} x = & \left(\frac{k^2}{(m+1)} \frac{\Gamma\left(1-\frac{(m+1)\alpha}{m}\right)^{2m} \Gamma\left(1-\frac{(2m+1)\alpha}{m}\right)}{\Gamma\left(1-\frac{\alpha}{m}\right)^{2m+1}}\right)^{\frac{1}{2m+2}} \int_{\psi_0}^{\psi} \frac{d\theta}{(\theta^2+c_1)^{\frac{1}{2m+2}}} + c_2, \\ \varphi = & \left(\frac{m+1}{k^2} \frac{\Gamma\left(1-\frac{(m+1)\alpha}{m}\right)^2}{\Gamma\left(1-\frac{\alpha}{m}\right)\Gamma\left(1-\frac{(2m+1)\alpha}{m}\right)} (\psi^2+c_1)\right)^{\frac{1}{2m+2}}. \end{cases} \tag{4.10}$$

Here ψ_0 is an appropriately chosen lower bound, and c_1 and c_2 are constants. The invariant solution $(u(x,t), v(x,t))$ can be obtained from the above implicit solution and the form corresponding to U_5 given in Table 4.5. Integrating the above solution explicitly is difficult, but for some particular values of the parameters c_1, c_2 and m, explicit invariant solutions can be readily obtained. For example, for $c_1 = 0$, we obtain the invariant solution

$$\begin{cases} u(x,t) &= \left(\frac{m^2}{k^2(m+1)(2m+1)}\frac{\Gamma\left(-\frac{\alpha}{m}\right)}{\Gamma\left(-\frac{(2m+1)\alpha}{m}\right)}\right)^{\frac{1}{2m}}(x-c_2)^{\frac{1}{m}}t^{-\frac{\alpha}{m}}, \\ v(x,t) &= \left(\frac{m^{2m+2}}{k^2(m+1)^{4m+1}(2m+1)}\frac{1}{\Gamma\left(-\frac{(m+1)\alpha}{m}\right)^{2m}}\frac{\Gamma\left(-\frac{\alpha}{m}\right)^{2m+1}}{\Gamma\left(-\frac{(2m+1)\alpha}{m}\right)}\right)^{\frac{1}{2m}}(x-c_2)^{\frac{m+1}{m}}t^{-\frac{(m+1)\alpha}{m}}, \end{cases} \tag{4.11}$$

which is identical to that obtained in [14] for $m = \frac{1}{2}$ and $k = 1$. Note that (4.9) is invariant under the transformations corresponding to X_2 and X_4. This implies that (4.11) is an invariant solution not only of the system of ODEs corresponding to U_5 but also of the system of ODEs corresponding to U_4 when $c_2 = 0$. Substituting $m = -\frac{1}{2}$ and $c_1 \neq 0$ into (4.10), we obtain another explicit invariant solution,

$$\begin{cases} u(x,t) = & \frac{c_1}{2k^2}\frac{\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)}t^{2\alpha}\left[\tan^2\left(\frac{\sqrt{c_1}}{2k^2}\Gamma(1+\alpha)(x-c_2)\right)+1\right], \\ v(x,t) = & \sqrt{c_1}t^{\alpha}\tan\left(\frac{\sqrt{c_1}}{2k^2}\Gamma(1+\alpha)(x-c_2)\right). \end{cases} \tag{4.12}$$

4.3.4 Invariant solutions of (4.1) corresponding to U_2 in Case 2.2

It can be easily shown that the reduced system in this case has the solution

$$\begin{cases} \varphi(z) = a_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}z^{2\alpha-1}, \\ \psi(z) = cz^{\alpha-1}. \end{cases}$$

Then, we obtain the following invariant solution of (4.1) using the form of U_2 given in Table 4.9:

$$\begin{cases} u(x,t) = a_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}, \\ v(x,t) = a_2ct^{\alpha-1} + a_1xt^{\alpha-1}, \end{cases} \tag{4.13}$$

where c is a constant.

4.3.5 Invariant solutions of (4.1) corresponding to U_4 in Case 2.2

The solution to the reduced system in this case is

$$\begin{cases} \varphi(x) = \left(-\frac{a\Gamma(\alpha+1)}{k^2(1-2\alpha)}x + c_1\right)^{1-2\alpha}, \\ \psi(x) = \frac{k^2(2\alpha-1)}{2a(1-\alpha)}\frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha+1)}\left(\frac{a\Gamma(\alpha+1)}{k^2(2\alpha-1)}x + c_1\right)^{2(1-\alpha)} + c_2. \end{cases}$$

With this and the form of U_4 given in Table 4.9, we obtain the following explicit invariant solution:

$$\begin{cases} u(x,t) = & \left(-\frac{a\Gamma(\alpha+1)}{k^2(1-2\alpha)}x + c_1\right)^{1-2\alpha} t^{2\alpha-1}, \\ v(x,t) = & \left(\frac{k^2(2\alpha-1)}{2a(1-\alpha)}\frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha+1)}\left(\frac{a\Gamma(\alpha+1)}{k^2(2\alpha-1)}x + c_1\right)^{2(1-\alpha)} - \alpha a \ln(t) + c_2\right)t^{\alpha-1}. \end{cases} \tag{4.14}$$

We have thus found the group invariant solutions in all cases except for those corresponding to U_2 and U_3 in Case 2.1 and U_3 in Case 2.2.

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