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Zeta functions of simple graphs with bounded degree

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Dedicated to my family and ancestors.

Contents

1 Introduction

For mathematical objects (such as graphs, Riemannian manifolds), it is important to study the relationship between shapes (such as geometric properties) and invariants of these objects. These studies have many applications to not only mathematics but also physics. One of the fascinated object of these studies is a graph. Graphs are discrete and simple objects in mathematics. Therefore, they frequently appear in many settings. At least from this stand point, it is important to study the relationship between properties of graphs and their invariants.

All graphs in this thesis are assumed to be connected, countable and simple. Spectral analysis of graphs focuses on the relationship between properties of graphs and the spectrums of operators which are related to graphs. In this thesis, we focus on investigating the relationship between paths and the spectrum of the Laplacian from the view point of number theory.

To explain more precisely, let X be a graph and Δ_X be the Laplacian of X. Especially for a finite graph X, it is well-known that closed geodesics are deeply related to the spectrum of Δ_X. The relationship describes as the so-called Ihara formula explicitly. The Ihara zeta function of X is defined by

$$Z_X(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} u^m\right).$$

Here, N_m stands for the number of closed geodesics of length m in X. Then, the Ihara formula is described as follows (cf. [31]).

$$Z_X(u)^{-1} = (1-u^2)^{-\chi(X)} \det\big(I - u(D_X - \Delta_X) + u^2(D_X - I)\big).$$

Here, $\chi(X)$ stands for the Euler number of X and D_X stands for the valency operator of X. The above formula was originally established by Y. Ihara in the p-adic setting. Then, it has been generalized in stages by T. Sunada, K. Hashimoto and H. Bass ([2], [14], [15], [16], [17], [18], [21], [27], [28], [29]). If X is a regular graph, this formula describes the relationship between closed geodesics and the spectrum of Δ_X. In this thesis, we call a formula of this type an Ihara type formula.

In 1999, L. Bartholdi introduced the Bartholdi zeta function for finite graphs and established a determinant expression of it ([1]). The Bartholdi zeta function is defined by

$$Z_X(u,t) = \exp\left(\sum_{C \in \mathcal{C}} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)}\right).$$

Here, we denote by $\mathcal{C}$ the set of closed paths in X, by $\ell(C)$ the length of C and by $\mathrm{cbc}(C)$ the cyclic bump count of a closed path C. This is a generalization of the

Ihara zeta function by adding a variable t which plays a role of counting backtrackings of a closed path. Indeed, if t is equal to 0, this zeta function coincides with the Ihara zeta function. The determinant expression of $Z_X(u,t,x_0)$ is as follows ([1]).

$$Z_X(u,t)^{-1} = \big(1-(1-t)^2u^2\big)^{-\chi(X)} \times \det\big(I - u(D_X - \Delta_X) + (1-t)u^2(D_X - (1-t)I)\big).$$

For a finite regular graph, this formula gives an explicit relationship between the number of closed paths and the spectrum of Δ_X. In this thesis, we call a formula of this type a Bartholdi type formula.

Recently, several authors have considered generalizations of the Ihara zeta function from finite graphs to infinite graphs (cf. [4], [5], [6], [8], [11], [12], [13], [26]). In this thesis, we follow [4] essentially. For a vertex-transitive graph X (not necessarily finite) and a fixed vertex x_0, the Ihara zeta function for X was introduced as follows in [4].

$$Z_X(u,x_0) = \Big(\sum_{m=1}^{\infty} \frac{N_m(x_0)}{m} u^m\Big).$$

Here, $N_m(x_0)$ stands for the number of closed geodesics of length m starting at x_0. We remark that this zeta function does not depend on x_0 since X is a vertex-transitive graph. In [4], the definition of the Ihara zeta function for a regular graph X is given (p. 185 in [4]). The definition is a little complicated because we have to introduce another terminology to define the Ihara zeta function for a regular graph besides closed geodesics. Therefore, we do not introduce it (see p. 185 in [4]). G. Chinta, J. Jorgenson and A. Karlsson established the Ihara type formula for the Ihara zeta function for a vertex-transitive graph by giving a new expression of the heat kernel ([4]). This definition works well in the point of studying deeply the relationship between closed geodesics and the spectrum of Δ_X and also as an analogy with heat kernel analysis of rank one symmetric spaces.

This thesis is divided into four parts. In the first part, we give an introduction of our research. In the second part, we survey some basic facts in graph theory which are mainly used in this thesis. In the third part, we introduce an Ihara zeta function for a graph with bounded degree in the third part. This zeta function is a natural generalization of the zeta function which was introduced by G. Chinta, J. Jorgenson and A. Karlsson. Then, we present an Ihara type formula of this zeta function for a graph with bounded degree. Our proof also gives an alternative proof of the proof in [4]. In the final part, we introduce a Bartholdi zeta function for a graph with bounded degree. This zeta function is a generalization of both the Ihara zeta function which is introduced in the third part and the Bartholdi zeta function which was introduced by L. Bartholdi. Then, we present a Bartholdi type formula of this zeta function for a graph with bounded degree. This formula is a generalization of the Bartholdi-Ihara formula from a finite graph to a simple graph with bounded degree. Moreover, we establish a new expression of the heat kernel by using modified Bessel functions. This expression can be regarded as a one-parameter deformation of the expression obtained by G. Chinta, J. Jorgenson and A. Karlsson in [4]. Then, especially for a regular graph (not necessarily finite), we give an alternative proof of the Bartholdi type formula which is presented in this part by using this new expression of the heat kernel. This is an important application of our heat kernel expression. We believe that there should be other applications of our heat kernel expression because it is well-known that there are many applications of the heat kernel at least in the finite graph case such as the distribution of the spectrum of regular graphs.

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2 Preliminaries

In this part, we introduce terminologies which are mainly used throughout in this thesis. We remark that there are other terminologies which are introduced in each part.

2.1 Graphs and Paths

In this section, we give terminologies of graphs and paths used throughout this paper (cf. [1], [27], [30]). A graph X is an ordered pair (VX, EX) of disjoint sets VX and EX with two maps,

$$EX \to VX \times VX, e \mapsto (o(e), t(e)), \quad EX \to EX, e \mapsto \bar{e}$$

such that for each $e \in EX$, $\bar{e} \neq e$, $\bar{\bar{e}} = e$, $o(e) = t(\bar{e})$. For a graph $X = (VX, EX)$, two sets VX and EX are called vertex set and edge set respectively. A graph X is *simple* if X has no loops and multiple edges. For a vertex $x \in VX$, the *degree of* x is the cardinality of the set E_x, where $E_x = \{e \in EX | o(e) = x\}$. We denote the degree of x by $\deg(x)$. A graph X is *countable* if the vertex set is countable. A graph X *has bounded degree* if the supremum of the set of all degrees is not infinite. For a graph X, a *path of length* n is a sequence of edges

$$C = (e_1, \ldots, e_n)$$

such that $t(e_i) = o(e_{i+1})$ for each i. We denote $o(e_1)$ by $o(C)$, $t(e_n)$ by $t(C)$ and the length of C by $\ell(C)$. A path C is *closed* if $o(C) = t(C)$. We regard a vertex as a path of length 0. A path $C = (e_1, \ldots, e_n)$ *has a back-tracking or bump* if there exist i such that $e_{i+1} = \bar{e_i}$. A path $C = (e_1, \ldots, e_n)$ *has a tail* if $e_n = \bar{e_1}$. A path C is a *geodesic* if c has no back-tracking. A closed path $C = (e_1, \ldots, e_n)$ is a *geodesic loop* if C is a geodesic. A closed path $C = (e_1, \ldots, e_n)$ is a *closed geodesic* if C is a geodesic loop and has no tail. For a path $C = (e_1, \ldots, e_n)$, we define the *bump count* of C as follows.

$$\mathrm{bc}(C) = \sharp\{i \in \{1, \ldots, n-1\} \mid e_i = \overline{e_{i+1}}\}.$$

For a closed path $C = (e_1, \ldots, e_n)$, we define the *cyclic bump count* of C as follows.

$$\mathrm{cbc}(C) = \sharp\{i \in \mathbb{Z}/m\mathbb{Z} \mid e_i = \overline{e_{i+1}}\}.$$

For a closed path x_0, we define $\mathrm{bc}(x_0) = \mathrm{cbc}(x_0) = 0$. For a path $C = (e_1, \ldots, e_m)$, we denote e_i by $e_i(C)$.

2.2 The Laplacian of a graph

For the vertex set VX of a graph X, we define *the ℓ^2-space on the vertex set VX* by

$$\ell^2(VX) = \left\{ f\colon VX \to \mathbb{C} \;\middle|\; \sum_{x \in VX} |f(x)|^2 < +\infty \right\}.$$

For a function $f \in \ell^2(VX)$ and a vertex $x \in VX$, we define the *adjacency operator* A_X on X and the *valency operator* D_X on X as follows respectively.

$$(A_X f)(x) = \sum_{e \in E_x} f(t(e)),$$
$$(D_X f)(x) = \deg(x) f(x).$$

Then, we define the *Laplacian* D_X on X by $\Delta_X = D_X - A_X$. The Laplacian is a semipositive and self-adjoint bounded operator under our assumption.

2.3 The heat kernel of a graph

For a graph X with bounded degree and a fixed vertex x_0, the *heat kernel* $K_X(\tau, x_0, x)\colon \mathbb{R}_{\geq 0} \times VX \to \mathbb{R}$ on X is the solution of the heat equation

$$\begin{cases} \left(\Delta_X + \frac{\partial}{\partial \tau}\right) f(\tau, x) = 0, \\ f(0, x) = \delta_{x_0}(x). \end{cases}$$

Here, the function $f(\tau, x)$ is in the class C^1 on $\mathbb{R} \times VX$ for each $x \in VX$ and the function $\delta_{x_0}(x)$ is the Kronecker delta. The heat kernel on X uniquely exists among functions which are bounded on $[0, \tau] \times VX$ for each $\tau \in \mathbb{R}_{\geq 0}$ under our assumptions ([9]). By the uniqueness of the solution of the heat equation, it turns out that the heat kernel $K_X(\tau, x_0, x)$ is an invariant under the automorphism group $\mathrm{Aut}(X)$.

2.4 The modified Bessel function

In this section, we define the modified Bessel function and introduce some well-known properties of them. For $n \in \mathbb{Z}_{\geq 0}$ and $\tau \in \mathbb{R}$, we define the *modified Bessel function of the first kind* by the following power series.

$$\mathrm{I}_n(\tau) = \sum_{m=0}^{\infty} \frac{\left(\tau/2\right)^{n+2m}}{m!(m+n)!}.$$

For $-n \in \mathbb{Z}_{<0}$, we define $\mathrm{I}_{-n}(\tau)$ as follows.

$$\mathrm{I}_{-n}(\tau) = \mathrm{I}_n(\tau).$$

It is well-known that $\mathrm{I}_n(\tau)$ is the power series solution of the following differential equation.

$$\tau^2 \frac{d^2 w}{d\tau^2} + \tau \frac{dw}{d\tau} - (\tau^2 + n^2)w = 0.$$

Moreover, it is also well-known that $\mathrm{I}_n(\tau)$ satisfies the following formula.

$$2\frac{d}{d\tau}\mathrm{I}_n(\tau) = \mathrm{I}_{n-1}(\tau) + \mathrm{I}_{n+1}(\tau). \tag{2.1}$$

In addition, for $n \geq 0$ and $\tau \in \mathbb{R}_{\geq 0}$, $\mathrm{I}_n(\tau)$ has the following trivial bound.

$$\mathrm{I}_n(\tau) \leq \left(\frac{\tau}{2}\right)^n \frac{\mathrm{e}^\tau}{n!}. \tag{2.2}$$

2.5 $G(t)$-transform

For a real valued function $f(\tau)(0 < \tau < \infty)$ which is integrable in every finite interval, we define $G(t)f$ as follows.

$$G(t)f(u) = (u^{-2} - (q+t)(1-t)) \int_0^\infty \mathrm{e}^{-\left((q+t)(1-t)u + \frac{1}{u} - (q+1)\right)\tau} f(\tau) d\tau.$$

We call this transform $G(t)$-*transform*. The following formula holds (cf. [24]). If $0 < u < \frac{1}{\sqrt{(q+t)(1-t)}}$, then, for $k \geq 0$, we have

$$G(t)\Big(\mathrm{e}^{-(q+1)\tau}\left((q+t)(1-t)\right)^{-\frac{k}{2}}\mathrm{I}_k\left(2\sqrt{(q+t)(1-t)}\tau\right)\Big)(u) = u^{k-1}. \tag{2.3}$$

3 An Ihara type formula for graphs with bounded degree

In this part, we establish a generalized Ihara zeta function formula for a simple graph with bounded degree. This is a generalization of the formula obtained by G. Chinta, J. Jorgenson and A. Karlsson from a vertex-transitive graph.

3.1 Introduction

Let X be a connected graph with bounded degree whose vertex set is countable and Δ_X be the combinatorial Laplacian on X. In this paper, a graph with bounded degree means a graph which has the above properties. The relationship between geometric properties of X and the spectrum of Δ_X has been widely studied. Especially, for a finite regular graph, it is well-known that the distribution of the spectrum of Δ_X is deeply related to the number of closed geodesics (cf. [31]). The Ihara zeta function for a finite graph is defined by

$$Z_X(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} u^m\right).$$

Here, we denote by N_m the number of closed geodesics of length m in X. This function is directly related to the number of closed geodesics. The original Ihara zeta function was first defined by Y. Ihara in [20] as a Selberg-type zeta function in the p-adic setting. It can be interpreted in terms of finite regular graphs and has been generalized by T. Sunada, K. Hashimoto and H. Bass ([2], [14], [15], [16], [17], [18], [21], [27], [28], [29]). There are various studies for the Ihara zeta function for a finite graph. The most famous and important formula for the Ihara zeta function for a finite graph is the Ihara determinant formula described as

$$Z_X(u)^{-1} = (1-u^2)^{-\chi(X)} \det\big(I - u(D_X - \Delta_X) + u^2(D_X - I)\big).$$

Here, we denote by $\chi(X)$ the Euler number of X and by D_X the valency operator on X. The above formula for a finite regular graph was originally established by Y. Ihara in the p-adic setting ([20]). Various proofs of this formula are well-known (cf. [2], [21]). This formula can be interpreted as a formula describing a relationship between the number of closed geodesics and the spectrum of the Laplacian on a finite graph.

Recently, several authors have considered generalizations of the Ihara zeta function and the Ihara determinant formula from finite graphs to infinite graphs (cf. [4], [5], [6], [8], [11], [12], [13], [26]). Among them, we follow [4] essentially. In [4], for a regular graph, the Ihara zeta function is defined. The definition is complicated because we have to introduce another terminology besides closed geodesics (cf. p. 185 in [4]). However, if a graph X is a vertex-transitive graph, the Ihara zeta function defined in [4] has a natural form. Namely, for a vertex-transitive graph X, the Ihara zeta function of X defined in [4] is

$$\zeta_X(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m(x_0)}{m} u^m\right).$$

Here, we denote by $N_m(x_0)$ the number of closed geodesics of length m starting at a given vertex x_0. We remark that the above zeta function does not depend on the given vertex since X is a vertex-transitive graph. We also remark that if a regular graph X is not a vertex-transitive graph, the zeta function of X defined in [4] does not always coincide with the above. The idea of giving a generalization of the Ihara zeta function from finite graphs to infinite graphs in [4] is to count not all closed geodesics but only count through a fixed starting vertex. The Ihara's formula for the zeta function defined in [4] is proved by giving a new expression of the heat kernel on a regular graph by using modified Bessel functions. The approach through heat kernel analysis is considered to be successful also from the view point of giving an analogy with quotients of rank one symmetric spaces. Therefore, we define the zeta function for a graph with bounded degree following [4].

The aim of this paper is to continue the study about the relationship between the number of closed geodesics and the spectrum of the Laplacian on a graph. From this standpoint, for a graph X with bounded degree and a vertex $x_0 \in VX$, we define the Ihara zeta function as follows in this paper.

$$Z_X(u, x_0) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m(x_0)}{m} u^m\right).$$

We remark that the above zeta function depends on the vertex x_0. As mentioned in [4], this generalization of the Ihara zeta function can be considered as corresponding to the Hurwitz zeta function which is generalized from the Riemann zeta function. Moreover, as we mentioned above, the Ihara zeta function for a graph X with bounded degree is equal to the Ihara zeta function defined in [4] if X is a vertex-transitive graph. However, our zeta function does not always coincide with the zeta function defined in [4] for a regular graph which is not a vertex-transitive graph.

In this paper, we establish a generalized Ihara zeta function formula for connected simple graphs with bounded degree by using an algebraic method. Here, a simple graph means a graph which have no loops and multiple edges. This is a generalization of the formula for vertex-transitive graphs obtained by G. Chinta, J. Jorgenson and A. Kerlsson in [4]. We remark that we establish the formula for connected simple graphs with bounded degree whereas G. Chinta, J. Jorgenson and A. Karlsson establish the formula for connected vertex-transitive graphs which are not always simple graphs ([4]). Moreover, our proof also gives an alternative proof of the formula obtained by G. Chinta, J. Jorgenson and A. Karlsson for simple vertex-transitive graphs. For finite simple regular graphs, we also derive a generalized Ihara zeta function formula which is regarded as a local version of the original Ihara determinant formula.

3.2 A path counting formula

In this section, we give an explicit formula between the number of geodesic loops and the number of closed geodesics. We remark that a graph which is considered in this section is allowed to have multiple edges and loops. This is a generalization of the path counting formula obtained in [4]. We fix a vertex $x_0 \in VX$. For a vertex $x \in VX$ and a nonnegative integer m, we denote by $c_m(x_0, x)$ the number of geodesic paths of length m from x_0 to x, by $c_m(x) = c_m(x, x)$ the number of geodesic loops of length m starting at x and by $N_m(x_0)$ the number of closed geodesics of length m starting at x_0. For a vertex $x \in VX$ and a nonnegative integer m, we denote $\deg(x)c_m(x) - \sum_{e\in E_x} c_m(t(e))$(resp. $\deg(x)N_m(x) - \sum_{e\in E_x} N_m(t(e))$) by $(\Delta_X c_m)(x)$(resp. $(\Delta_X N_m)(x)$) formally by regarding c_m and N_m as functions on VX. Here, we note that functions c_m and N_m are not in $\ell^2(VX)$ in general. Moreover, we define the following formal power series.

$$C(u : x_0) = \sum_{m=1}^{\infty} c_m(x_0)u^m,$$
$$N(u : x_0) = \sum_{m=1}^{\infty} N_m(x_0)u^m.$$

Our goal in this section is to give the following theorem.

Theorem 3.2.1. *The following formula holds:*

$$N(u : x_0) = (1 - u^2)^{-2}\{1 - (\deg(x_0) - \Delta_X)u^2 + (\deg(x_0) - 1)u^4\}C(u : \cdot)(x_0).$$

First of all, we give the following proposition.

Proposition 3.2.2. *For an integer m greater than 2, the following identity holds:*

$$N_m(x_0) = c_m(x_0) - (\deg(x_0) - 2)\sum_{i=1}^{\lceil \frac{m}{2} \rceil - 1} c_{m-2i}(x_0) + \sum_{i=1}^{\lceil \frac{m}{2} \rceil - 1} i(\Delta_X c_{m-2i})(x_0).$$

Here, the symbol $\lceil\cdot\rceil$ stands for the ceiling function.

Proof. First of all, we define several symbols. For a non-negative integer m, a vertex $x \in VX$ and an edge $e \in EX$ such that $o(e) = x$ or $t(e) = x$, we denote by $c_m(x, e)$(resp. $N_m(x, e)$) the number of geodesic loops (resp. closed geodesics) of length m starting at x through e. Let m be an integer which is greater than 2 and $e \in EX$ be an edge such that $o(e) = x_0$. The number $c_m(x_0, e) - N_m(x_0, e)$ is equal to the number of geodesic loops of length m starting at x_0 through e, which are not closed geodesics. Therefore, we have

$$c_m(x_0, e) - N_m(x_0, e) = c_{m-2}(t(e)) - c_{m-2}(t(e), \bar{e}) - c_{m-2}(t(e), e) + cwt_{m-2}(t(e), \bar{e}).$$

Here, we denote by $cwt_{m-2}(t(e), \bar{e})$ the number of geodesic loops of length $m - 2$ starting at $t(e)$ with tail $\bar{e}$. By this, we have

$$c_m(x_0) - N_m(x_0)$$

$$= \sum_{e \in E_{x_0}} \{c_{m-2}(t(e)) - c_{m-2}(t(e), \bar{e}) - c_{m-2}(t(e), e) + cwt_{m-2}(t(e), \bar{e})\}$$

$$= \deg(x_0) c_{m-2}(x_0) - (\Delta_X c_{m-2})(x_0)$$

$$- \sum_{e \in E_{x_0}} \{c_{m-2}(t(e), \bar{e}) - N_{m-2}(t(e), \bar{e})\} - \sum_{e \in E_{x_0}} \{c_{m-2}(t(e), e) - N_{m-2}(t(e), e)\}$$

$$- \sum_{e \in E_{x_0}} \{N_{m-2}(t(e), \bar{e}) + N_{m-2}(t(e), e)\} + \sum_{e \in E_{x_0}} cwt_{m-2}(t(e), \bar{e}).$$

Further,

$$\sum_{e \in E_{x_0}} \{c_{m-2}(t(e), \bar{e}) - N_{m-2}(t(e), \bar{e})\}$$

$$= \sum_{e \in E_{x_0}} \{c_{m-2}(t(e), e) - N_{m-2}(t(e), e)\}$$

$$= cwt_{m-2}(t(e), \bar{e})$$

and

$$\sum_{e \in E_{x_0}} N_{m-2}(t(e), \bar{e}) = \sum_{e \in E_{x_0}} N_{m-2}(t(e), e) = \sum_{e \in E_{x_0}} N_{m-2}(x_0, e) = N_{m-2}(x_0).$$

Then, we have

$$c_m(x_0) - N_m(x_0)$$

$$= \deg(x_0) c_{m-2}(x_0) - (\Delta_X c_{m-2})(x_0) - 2N_{m-2}(x_0) - \sum_{e \in E_{x_0}} cwt_{m-2}(t(e), \bar{e}). \tag{3.1}$$

Putting $m = 3$ in (3.1), since $cwt_1(t(e), \bar{e}) = 0$, we have

$$c_3(x_0) - N_3(x_0) = \deg(x_0) c_1(x_0) - (\Delta_X c_1)(x_0) - 2N_1(x_0).$$

Therefore,

$$N_3(x_0) = c_3(x_0) - (\deg(x) - 2) c_1(x_0) + (\Delta_X c_1)(x_0). \tag{3.2}$$

By the same argument, we have

$$N_4(x_0) = c_4(x_0) - (\deg(x_0) - 2) c_2(x_0) + (\Delta_X c_2)(x_0).$$

In the case $m \geq 5$, by (3.1), we have

$$c_m(x_0) - N_m(x_0)$$

$$
\begin{aligned}
&= \deg(x_0)c_{m-2}(x_0) - (\Delta_X c_{m-2})(x_0) - 2N_{m-2}(x_0) \\
&\quad - \{N_{m-4}(x_0)\deg(x_0) - 2) + (c_{m-4} - N_{m-4}(x_0))(\deg(x_0) - 1)\} \\
&= \deg(x_0)c_{m-2}(x_0) - (\Delta_X c_{m-2})(x_0) \\
&\quad - 2N_{m-2}(x_0) - (\deg(x_0) - 1)c_{m-4}(x_0) + N_{m-4}(x_0).
\end{aligned}
$$

Therefore, we have the following recursive formula.

$$
\begin{aligned}
&\{N_m(x_0) - N_{m-2}(x_0)\} - \{N_{m-2}(x_0) - N_{m-4}(x_0)\} \\
&= \{c_m(x_0) - c_{m-2}(x_0)\} - (\deg(x_0) - 1)\{c_{m-2}(x_0) - c_{m-4}(x_0)\} + (\Delta_X c_{m-2})(x_0).
\end{aligned}
$$

For $m \geq 5$ which is an odd integer, we have

$$
\begin{aligned}
&N_m(x_0) - N_{m-2}(x_0) \\
&= c_m(x_0) - (\deg(x_0) - 1)c_{m-2}(x_0) + (N_3(x_0) - c_3(x_0)) \\
&\quad + (\deg(x_0) - 1)c_1(x_0) - N_1(x_0) + \sum_{i=1}^{\frac{m-3}{2}} (\Delta_X c_{m-2i})(x_0).
\end{aligned}
$$

Here, we used the above recursive formula in the above equation. By this and (3.2), we have

$$
N_m(x_0) = c_m(x_0) - (\deg(x_0) - 2) \sum_{i=1}^{\frac{m-1}{2}} c_{m-2i}(x_0) + \sum_{i=1}^{\frac{m-1}{2}} i(\Delta_X c_{m-2i})(x_0).
$$

For $m \geq 6$ which is an even integer, by the same argument, we have

$$
N_m(x_0) = c_m(x_0) - (\deg(x_0) - 2) \sum_{i=1}^{\frac{m-2}{2}} c_{m-2i}(x_0) + \sum_{i=1}^{\frac{m-2}{2}} i(\Delta_X c_{m-2i})(x_0).
$$

□

In the rest of this section, we give the proof of Theorem 3.2.1. We define $R_m(x_0)$ and $\tilde{R}_m(x_0)$ as follows.

$$
R_m(x_0) = \sum_{i=1}^{\lceil \frac{m}{2} \rceil - 1} i(\Delta_X c_{m-2i})(x_0),
$$

$$
\tilde{R}_m(x_0) = \begin{cases} R_1(x_0) & \text{if } m = 1, \\ R_2(x_0) & \text{if } m = 2, \\ R_m(x_0) - R_{m-2}(x_0) & \text{if } m \geq 3. \end{cases}
$$

We define the corresponding formal power series as follows.

$$R(u:x_0)=\sum_{m=1}^{\infty}R_m(x_0)u^m,$$
$$\tilde{R}(u:x_0)=\sum_{m=1}^{\infty}\tilde{R}_m(x_0)u^m.$$

By the definition of $R_m(x_0)$ and $R_m(x_0)$, we have

$$R(u:x_0)=u^2(1-u^2)^{-2}\Delta_X C(u:x_0). \tag{3.3}$$

Moreover, for vertices $x_0, x \in VX$, we define $b_m(x_0)$ as follows.

$$b_m(x)=\begin{cases} c_0(x_0,x) & \text{if } m=0,\\ c_1(x_0,x) & \text{if } m=1,\\ c_2(x_0,x) & \text{if } m=2,\\ c_m(x_0,x)-(\deg(x)-2)\sum_{j=1}^{\lceil \frac{m}{2}\rceil-1} c_{m-2j}(x_0,x) & \text{if } m\geq 3. \end{cases}$$

By the definition of this symbol, we have

$$B(u:x_0)=(1-u^2)^{-1}\{1-(\deg(x_0)-1)u^2\}C(u:x_0). \tag{3.4}$$

By Proposition 3.2.2, we have

$$N(u:x_0)=C(u:x_0)-(C(u:x_0)-B(u:x_0))+R(u:x_0)=B(u:x_0)+R(u:x_0).$$

By (3.3) and (3.4), we have

$$\begin{aligned} N(u:x_0)&=(1-u^2)^{-1}\{1-(\deg(x_0)-1)u^2\}C(u:x_0)+u^2(1-u^2)^{-2}\Delta_X C(u:\cdot)(x_0)\\ &=(1-u^2)^{-2}\{1-(\deg(x_0)-\Delta_X)u^2+(\deg(x_0)-1)u^4\}C(u:\cdot)(x_0). \end{aligned}$$

Therefore, we get Theorem 3.2.1.

3.3 An Ihara type formula for simple graphs with bounded degree

In this section, we give a generalized Ihara zeta function formula for a simple graph with bounded degree. Let X be a connected simple graph with bounded degree. We denote the supremum of all degrees of X by M. We remark that M is greater than 1 by our assumption. We denote the set of all bounded operators on $\ell^2(VX)$ equipped with the usual operator norm $\|\cdot\|$ by $\mathcal{B}(\ell^2(VX))$.

First of all, we introduce several bounded operators on $\ell^2(VX)$. For $f \in \ell^2(VX)$ and $m \in \mathbb{Z}_{\geq 0}$, we define C_m as follows.

$$C_m f(x) = \sum_{c \in \mathcal{C}_x, \ell(c)=m} f(t(c)).$$

Here, the symbol $\mathcal{C}_x$ stands for the set of all geodesic paths of length m starting at x. We define Q_X by $D_X - I$ and B_m as follows.

$$B_m = \begin{cases} C_m - (Q-I)\sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} C_{m-2j} & \text{if } m \geq 3, \\ C_m & \text{if } m = 0,1,2. \end{cases}$$

For $f \in \ell^2(VX)$, we define R_m as follows.

$$(R_m f)(x) = \begin{cases} \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} j(\Delta_X c_{m-2j})(x) f(x) & \text{if } m \geq 3, \\ 0 & \text{if } m = 0,1,2. \end{cases}$$

Moreover, we define R_m^+ and $N_{X,m}$ as follows.

$$R_m^+ = \begin{cases} (Q_X - I)\delta_{2\mathbb{Z}}(m) + R_m & \text{if } m \geq 3, \\ 0 & \text{if } m = 0,1,2, \end{cases}$$

$$N_{X,m} = B_m + R_m^+.$$

We remark that the above operators are in $\mathcal{B}(\ell^2(VX))$ since X has a bounded degree. For $B \in \mathcal{B}(\ell^2(VX))$ and for $x_0, x \in VX$, we define $B(x_0, x)$ as follows.

$$B(x_0, x) = B\delta_{x_0}(x).$$

Here, the symbol δ_{x_0} stands for the Kronecker delta. We remark that $B(x_0, x)$ is in $\mathbb{C}$ by the Cauchy-Schwarz inequality since B is in $\mathcal{B}(\ell^2(VX))$.

Then, we have the following proposition.

Proposition 3.3.1. ([12]) *We have the following equation:*

$$C_m = \begin{cases} C_1^2 - Q - I & \text{if } m = 2, \\ C_{m-1}C_1 - C_{m-2}Q & \text{if } m \geq 3. \end{cases}$$

Let $\alpha = \frac{M + \sqrt{M^2 + 4M}}{2}$. *Then, for* $m \in \mathbb{Z}_{\geq 0}$, *we have*

$$\|C_m\| \leq \alpha^m.$$

Moreover, for $|u| < \frac{1}{\alpha}$, *we have the following equations:*

(1) $\left(\sum_{m=0}^{\infty} C_m u^m\right)\big(I - uA_X + u^2 Q_X\big) = (1-u^2)I.$

(2) $\left(\sum_{m=0}^{\infty}\big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor} C_{m-2k}\big)u^m\right)\big(I - uA_X + u^2 Q_X\big) = I.$

By Proposition 3.3.1, we have the following proposition.

Proposition 3.3.2. (1) *For* $|u| < \frac{1}{\alpha}$, *we have*

$$\left(\sum_{m=1}^{\infty} B_m u^m\right)\big(I - uA_X + u^2 Q_X\big) = A_X u - 2Q_X u^2 + \big(Q - I\big)\big(I - uA_X + u^2 Q_X\big)u^2.$$

(2) *For* $|u| < \frac{1}{\alpha}$, *we have*

$$\sum_{m=1}^{\infty} N_{X,m} u^m = u(A_X - 2Q_X u)\big(I - uA_X + u^2 Q_X\big)^{-1} + (Q_X - I)\frac{u^2}{1-u^2} + \sum_{m=3}^{\infty} R_m u^m.$$

It is easy to check by Proposition 3.3.1. Therefore, we omit the proof of Proposition 3.3.2

Let f be a C^1-function on $B_\epsilon = \{u \in \mathbb{C} \,\big|\, |u| < \epsilon\}$ which takes the value to bounded operators on a Hilbert space and satisfies $f(0) = 0$, $\|f(u)\| < 1$ for any $u \in B_\epsilon$. Here, $\|\cdot\|$ stands for the operator norm on this Hilbert space. Then, for $u \in B_\epsilon$, we have

$$-\log(I - f(u)) = \sum_{n=1}^{\infty} \frac{1}{n} f(u)^n.$$

Here, the above series converges in operator norm, uniformly on compact subsets of B_ϵ. By this, we have

$$-\frac{d}{du}\log(I - f(u)) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(u)^j f'(u) f(u)^{n-j-1}.$$

Let $f(u) = A_X u - Q_X u^2$. We remark that $|u| < \frac{1}{\alpha}$ implies $\|f(u)\| < 1$. Then, we have the following proposition.

Proposition 3.3.3. *For $|u| < \frac{1}{\alpha}$, we have*

$$f'(u)(I - f(u))^{-1} = -\frac{d}{du}\log(I - f(u)) + u^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j f(u)^{n-1-j}[A_X, Q_X] f(u)^{j-1}.$$

Here, $[A_X, Q_X] = A_X Q_X - Q_X A_X$.

Proof. By the previous remark, for $|u| < \frac{1}{\alpha}$, we have

$$-\frac{d}{du}\log(I - f(u)) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(u)^j f'(u) f(u)^{n-j-1}.$$

By straightforward calculation, we have

$$[f(u), f'(u)] = (Q_X A_X - A_X Q_X)u^2.$$

Therefore, we get

$$f(u)f'(u) = f'(u)f(u) + [Q_X, A_X]u^2.$$

By this equation, we have

$$\sum_{j=0}^{n-1} f(u)^j f'(u) f(u)^{n-1-j} = n f'(u) f(u)^{n-1} + u^2 \sum_{j=1}^{n-1} j f(u)^{n-1-j}[Q_X, A_X] f(u)^{j-1}.$$

Then, we have

$$\begin{aligned}
-\frac{d}{du}\log(I - f(u)) &= \sum_{n=1}^{\infty} \frac{1}{n}\Big(n f'(u) f(u)^{n-1} + u^2 \sum_{j=1}^{n-1} j f(u)^{n-1-j}[Q_X, A_X] f(u)^{j-1}\Big) \\
&= f'(u) \sum_{n=1}^{\infty} f(u)^{n-1} + u^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j f(u)^{n-1-j}[Q_X, A_X] f(u)^{j-1} \\
&= f'(u)(I - f(u))^{-1} + u^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j f(u)^{n-1-j}[Q_X, A_X] f(u)^{j-1}.
\end{aligned}$$

□

By Proposition 3.3.2 and Proposition 4.3.4, for $|u| < \frac{1}{\alpha}$, we have

$$u\frac{d}{du} \sum_{m=1}^{\infty} \frac{N_{X,m}}{m} u^m = -u\frac{d}{du}\log(I - f(u)) + u^3 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j f(u)^{n-1-j}[A_X, Q_X] f(u)^{j-1}$$

$$+ (Q_X - I)\frac{u^2}{1-u^2} + u\frac{d}{du}\sum_{m=3}^{\infty}\frac{R_m}{m}u^m.$$

Dividing by u and integrating from $u = 0$ to u, we have

$$\sum_{m=1}^{\infty}\frac{N_{X,m}}{m}u^m = -\log(I - f(u)) - \frac{Q_X - I}{2}\log(1-u^2) + \int_0^u z^2 \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1} j f(z)^{n-1-j}[A_X, Q_X] f(z)^{j-1} dz + \sum_{m=3}^{\infty}\frac{R_m}{m}u^m.$$

Therefore, for $x_0, x \in VX$, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{N_{X,m}(x_0,x)}{m}u^m \\
&= -[\log(I - A_X u + Q_X u^2)](x_0,x) - \frac{\deg(x_0)-2}{2}\delta x_0(x)\log(1-u^2) \\
&\quad + \int_0^u z^2 \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1} j\big[f(z)^{n-1-j}[A_X, Q_X] f(z)^{j-1}\big](x_0,x) dz + \sum_{m=3}^{\infty}\frac{R_m(x_0,x)}{m}u^m.
\end{aligned} \tag{3.5}$$

We define $Z_X(u, x_0, x)$ as follows.

$$Z_X(u, x_0, x) = \exp\left(\sum_{m=1}^{\infty}\frac{N_{X,m}(x_0,x)}{m}u^m\right)$$

We remark that $Z_X(u, x_0, x_0) = Z_X(u, x_0)$ by Proposition 3.2.2. Then, we have the following theorem by (3.5).

Theorem 3.3.4. *For $|u| < \frac{1}{\alpha}$, we have*

$$\begin{aligned}
Z_X(u, x_0, x) &= (1-u^2)^{-\frac{\deg(x_0)-2}{2}\delta_{x_0}(x)} \\
&\times \exp\big(-[\log(I - (D_X - \Delta_X)u + (D_X - I)u^2)](x_0,x)\big) \\
&\times \exp\left(\int_0^u z^2 \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1} j\big[f(z)^{n-1-j}[A_X, D_X] f(z)^{j-1}\big](x_0,x) dz\right) \\
&\times \exp\left(\sum_{m=3}^{\infty}\frac{R_m(x_0,x)}{m}u^m\right).
\end{aligned}$$

In particular, if X is a $(q+1)$-regular graph, we have

$$I - (D_X - \Delta_X)u + Q_X u^2 = I - \big((q+1)I - \Delta_X\big)u + qIu^2.$$

Since Δ_X is a self-adjoint bounded operator, there exists a unique spectral measure such that

$$\Delta_X = \int_{\sigma(\Delta_X)} \lambda dE(\lambda).$$

Here, $\sigma(\Delta_X)$ stands for the spectram of Δ_X. For $x_0, x \in VX$, we denote the measure $\langle E(\cdot)\delta_{x_0}, \delta_x \rangle$ by $\mu_{x_0,x}(\cdot)$. Then, by the property of the spectral integral, we have the following corollary by Theorem 4.3.5.

Corollary 3.3.5. *For $|u| < \frac{1}{\alpha}$, we have*

$$\begin{aligned} Z_X(u, x_0, x) = (1-u^2)^{-\frac{q-1}{2}\delta_{x_0}(x)} \\ \times \exp\left(- \int_{\sigma(\Delta_X)} \log(1 - \big((q+1) - \lambda\big)u + qu^2) d\mu_{x_0,x}(\lambda) \right) \\ \times \exp\left(\sum_{m=3}^{\infty} \frac{R_m(x_0, x)}{m} u^m \right). \end{aligned}$$

Before we consider the case that X is a finite $(q+1)$-regular graph, we introduce the notion of the local spectrum ([10]). For a vertex $x \in VX$, we denote x-local multiplicity of λ_i by $\mathrm{m}_x(\lambda_i)$. Here, the *x-local multiplicity* of λ_i is the xx-entry of the primitive idempotent E_{λ_i}. Let $\{\mu_0 = \lambda_0, \mu_1, \ldots, \mu_{d_x}\}$ be the set of eigenvalues whose local multiplicities are positive. For each vertex $x \in VX$, we denote the x-local spectrum by $\sigma_x(X)$. Here, the *x-local spectrum* is $\sigma_x(X) = \{\lambda_0^{\mathrm{m}_x(\lambda_0)}, \mu_1^{\mathrm{m}_x(\mu_1)}, \ldots, \mu_{d_x}^{\mathrm{m}_x(\mu_{d_x})}\}$. Then, we have the following corollary immediately by Corollary 4.3.6.

Corollary 3.3.6. *For $|u| < \frac{1}{\alpha}$, we have*

$$\begin{aligned} Z_X(u, x_0) = (1-u^2)^{-\frac{q-1}{2}} \prod_{\lambda \in \sigma_{x_0}(\Delta_X)} \big(1 - (q+1-\lambda)u + qu^2\big)^{-m_{x_0}(\lambda)} \\ \times \exp\left(\sum_{m=3}^{\infty} \frac{R_m(x_0)}{m} u^m \right). \end{aligned}$$

We remark that this formula holds for a regular graph which is not always a simple graph by Proposition 3.2.2 and the formula obtained in p. 188 in [4]. This formula is regarded as a local version of the original Ihara determinant formula since the above equation gives the explicit relationship between the number of closed geodesics starting at x_0 and the x_0-local spectrum of the Laplacian of X.

4 A Bartholdi type formula for graphs with bounded degree

In this part, we introduce a generalized Bartholdi zeta function for simple graphs with bounded degree. This zeta function is a generalization of both the Bartholdi zeta function which was introduced by L. Bartholdi and the Ihara zeta function which was introduced by G. Chinta, J. Jorgenson and A. Karlsson. Furthermore, we establish a Bartholdi type formula of this Bartholdi zeta function for simple graphs with bounded degree. Moreover, for regular graphs, we give a new expression of the heat kernel which is regarded as a one-parameter deformation of the expression obtained by G. Chinta, J. Jorgenson and A. Karlsson. By applying this formula, we give an alternative proof of the Bartholdi zeta function formula for regular graphs.

4.1 Introduction

All graphs in this paper are assumed to be connected, countable and simple. Let X be a graph with bounded degree and Δ_X be the combinatorial Laplacian of X. It is well-known that the spectrum of Δ_X is closely related to geometric properties and combinatorial properties of X at least from the view point of graph theory, number theory and probability theory. Classically, it is important to study the relationship between closed paths in X and the spectrum of Δ_X. In this paper, we study the relationship from the view point of number theory.

Especially for a finite graph X, it is well-known that closed geodesics of X are deeply related to the spectrum of Δ_X (cf. [31]). The relationship is described as the Ihara formula explicitly. The Ihara zeta function for a finite graph X is defined by

$$Z_X(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} u^m\right).$$

Here, N_m stands for the number of closed geodesics of length m in X. Then, the Ihara formula is described as follows (cf. [31]).

$$Z_X(u)^{-1} = (1-u^2)^{-\chi(X)} \det\big(I - u(D_X - \Delta_X) + u^2(D_X - I)\big).$$

Here, $\chi(X)$ stands for the Euler characteristic of X and D_X stands for the valency operator of X. For finite regular graphs, the above formula was originally established by Y. Ihara in the p-adic setting ([20]). Then, it has been generalized by T. Sunada, K. Hashimoto and H. Bass ([2], [14], [15], [16], [17], [18], [21], [27], [28], [29]). When X is regular, this formula gives an explicit relationship between the number of closed geodesic and the spectrum of Δ_X.

In 1999, L. Bartholdi introduced the Bartholdi zeta function for finite graphs and established a determinant expression of it ([1]). The Bartholdi zeta function is defined by

$$Z_X(u,t) = \exp\left(\sum_{C\in\mathcal{C}} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)}\right).$$

Here, we denote by $\mathcal{C}$ the set of closed paths in X, by $\ell(C)$ the length of C and by $\mathrm{cbc}(C)$ the cyclic bump count of a closed path C. This is a generalization of the Ihara zeta function by adding a variable t which plays a role of counting back-trackings of a closed path. Indeed, if t is equal to 0, this zeta function coincides with the Ihara zeta function $Z_X(u)$. The determinant expression of $Z_X(u,t)$ is described as follows ([1]).

$$\begin{aligned} Z_X(u,t)^{-1} = &\left(1-(1-t)^2u^2\right)^{-\chi(X)} \\ &\times \det\left(I - u(D_X - \Delta_X) + (1-t)u^2(D_X - (1-t)I)\right). \end{aligned}$$

As in the case of the Ihara formula, when X is regular, this formula gives the explicit relationship between the number of closed paths and the spectrum of Δ_X.

Recently, several generalizations of the Ihara zeta function from finite graphs to infinite graphs have been considered (cf. [4], [5], [6], [8], [11], [12], [13], [26]). In this paper, we follow [4] essentially. In 2017, the author introduced the Ihara zeta function for a graph X with bounded degree as follows ([22]).

$$Z_X(u,x_0) = \left(\sum_{m=1}^{\infty} \frac{N_m(x_0)}{m} u^m\right).$$

Here, $N_m(x_0)$ stands for the number of closed geodesics of length m starting at x_0. If X is vertex-transitive, this zeta function coincides with the Ihara zeta function which was introduced in [4]. In [4], the definition of the Ihara zeta function for regular graphs is given (p. 185 in [4]). In general, however, when X is regular, the Ihara zeta function in [4] does not always coincide with our Ihara zeta function. In [4], G. Chinta, J. Jorgenson and A. Karlsson established the Ihara type formula for the Ihara zeta function for vertex-transitive graphs by giving a new expression of the heat kernel ([4]). This definition works well in the point of studying deeply the relationship between closed geodesics and the spectrum of Δ_X and also as an analogy with heat kernel analysis of rank one symmetric spaces. After that, the author established the Ihara type formula for the Ihara zeta function for graphs with bounded degree ([22]). His proof also gives an alternative proof of the formula for vertex-transitive graphs.

In this paper, we study the relationship between closed paths and the spectrum of Δ_X by introducing a Bartholdi zeta function for graphs with bounded degree. For a graph X with bounded degree and a vertex x_0, a Bartholdi zeta function is defined by as follows in this paper.

$$Z_X(u,t,x_0) = \exp\left(\sum_{C \in \mathcal{C}_{x_0}} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)} \right).$$

Here, we denote by $\mathcal{C}_{x_0}$ the set of closed paths starting at x_0. We remark that we introduce a Bartholdi zeta function which is a generalization of the above. However, we do not introduce the definition in Introduction because the definition is a little technical in the sense of being based on a path counting formula. If t is equal to 0, this Bartholdi zeta function coincides with $Z_X(u,x_0)$. If X is a finite graph, by the definition of $Z_X(u,t,x_0)$, the following equality holds.

$$\prod_{x_0 \in VX} Z_X(u,t,x_0) = Z_X(u,t).$$

In this sense, this Bartholdi zeta function is a generalization of the original one. Furthermore, we present a Bartholdi type formula for this Bartholdi zeta function. Especially for regular graphs, this formula describes the relationship between the number of closed paths and the spectrum of Δ_X. We remark that for finite graphs, this formula can be regarded as a refined version of the original Bartholdi zeta function formula.

Moreover, for (possibly infinite) regular graphs, we give a new expression of the heat kernel which is regarded as a one-parameter deformation of the expression obtained in [4]. By applying this formula, we give an alternative proof of the Bartholdi formula for regular graphs. We note that our heat kernel approach to the Bartholdi formula is new even for finite regular graphs. This is an important application of our new heat kernel expression. Many applications of the heat kernel are well-known. Therefore, in addition to the above application, we believe that there should be more applications by using our new expression of the heat kernel.

4.2 A generalized path counting formula

In this section, we give a generalization of the path counting formula obtained by T. Kousaka ([22]). First of all, we introduce several symbols. We take a vertex x_0 and $e \in E_{x_0}$. We denote by $\mathcal{C}_{x_0}$ the set of closed paths starting at x_0 and by $\mathcal{C}_{x_0}^{notail}$ the set of closed paths starting at x_0 which has no tail. For a complex variable t, we define $C_m(t,x_0)$, $N_m(t,x_0,e)$ as follows.

$$C_m(t,x_0) = \sum_{C} t^{\mathrm{cbc}(C)},$$

$$N_m(t, x_0, e) = \sum_C t^{\mathrm{cbc}(C)}.$$

Here, C runs through $\mathcal{C}_{x_0}$ such that $\ell(C) = m$ in the first equality and C runs through $\mathcal{C}_{x_0}^{notail}$ such that $e_1(C) = e, \ell(C) = m$ in the second equality. For $m \in \mathbb{Z}_{\geq 0}$, $f \in \ell^2(VX)$ and $x \in VX$, we define $C_m(t)$ by

$$C_m(t)f(x) = \sum_{C \in \mathcal{B}_x, \ell(C)=m} t^{\mathrm{bc}(C)} f(t(C)).$$

Here, we denote by $\mathcal{B}_x$ the set of paths starting at x. We define $C_m(t)(x_0, e)$ as follows.

$$C_m(t)(x_0, e) = \sum_C t^{\mathrm{bc}(C)}.$$

Here, C runs through $\mathcal{C}_{x_0}$ such that $e_1(C) = e, \ell(C) = m$ in the above equality. Moreover, we define $C_m(t)(x_0, \cdot, \bar{e})$, $N_m(t, x_0, \cdot, \bar{e})$ and $C_m(t)(x_0, e, \bar{e})$ as follows.

$$C_m(t)(x_0, \cdot, \bar{e}) = \sum_C t^{\mathrm{bc}(C)},$$
$$N_m(t, x_0, \cdot, \bar{e}) = \sum_C t^{\mathrm{bc}(C)},$$
$$C_m(t)(x_0, e, \bar{e}) = \sum_C t^{\mathrm{bc}(C)}.$$

Here, C runs through $\mathcal{C}_{x_0}$ such that $e_m(C) = \bar{e}, \ell(C) = m$ in the first equality, C runs through $\mathcal{C}_{x_0}^{notail}$ such that $e_m(C) = \bar{e}, \ell(C) = m$ in the second equality and C runs through $\mathcal{C}_{x_0}$ such that $e_1(C) = e, e_m(C) = \bar{e}, \ell(C) = m$ in the third equality. We denote by $\mathcal{B}(\ell^2(VX))$ the set of bounded operators on $\ell^2(VX)$. We remark that $C_m(t)$ is in $\mathcal{B}(\ell^2(VX))$ for each t. For $B \in \mathcal{B}(\ell^2(VX))$ and $x_1, x_2 \in VX$, we define $B(x_1, x_2)$ as follows.

$$B(x_1, x_2) = B\delta_{x_1}(x_2).$$

Here, the symbol δ_{x_0} stands for the Kronecker delta. We define the following formal power series.

$$C^{\mathrm{cbc}}(t, x_0 : u) = \sum_{m=1}^{\infty} C_m(t, x_0) u^m,$$
$$C(t, x_0 : u) = \sum_{m=1}^{\infty} C_m(t)(x_0, x_0) u^m,$$

$$N(t, x_0 : u) = \sum_{m=1}^{\infty} N_m(t, x_0) u^m.$$

Here, we denote $\sum_{e \in E_{x_0}} N_m(t, x_0, e)$ by $N_m(t, x_0)$. In addition to this, for a vertex $x \in VX$, we denote $\deg(x) C(t, x : u) - \sum_{e \in E_x} C(t, t(e) : u)$ by $\Delta_X C(t, \cdot : u)(x)$ by regarding as an element of $\ell^2(VX)$ formally and for $m \geq 1$, $x \in VX$, we define $R_m(t)(x)$ as follows.

$$R_m(t)(x) = \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} \sum_{i=1}^{j} (1-t)^{2(j-i)} (1-t^2)^{i-1} [\Delta_X C_{m-2j}(t)](x, x).$$

First of all, we prove the following proposition.

Proposition 4.2.1. *For a vertex x_0, we have the following equality:*

$$\begin{aligned}\big(1 - (1-t)^2 u^2\big) N(t, x_0 : u) = \big(1 - (\deg(x_0) - (1-t^2)) u^2\big) C(t, x_0 : u) \\ - \deg(x_0) t u^2 + \frac{u^2}{1 - (1-t^2) u^2} \Delta_X C(t, \cdot : u)(x_0).\end{aligned}$$

Moreover, for $m \geq 3$, we have

$$\begin{aligned}N_m(t, x_0) = C_m(t)(x_0, x_0) - (\deg(x_0) - 2(1-t)) \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} (1-t)^{2(j-1)} C_{m-2j}(t)(x_0, x_0) \\ + R_m(t)(x_0) - \delta_{2\mathbb{Z}}(m)(1-t)^{m-2} t \deg(x_0).\end{aligned}$$

Here, we denote the ceiling function by $\lceil \cdot \rceil$.

Proof. First of all, we prove the first identity. For $m \geq 3$, $x_0 \in VX$ and $e \in E_{x_0}$, we have

$$\begin{aligned}C_m(t)(x_0, e, \bar{e}) = \sum_{C=(e, e_2, \ldots, e_{m-1}, \bar{e}), e_2 = \bar{e}, e_{m-1} \neq e} t^{\mathrm{bc}(C)} + \sum_{C=(e, e_2, \ldots, e_{m-1}, \bar{e}), e_2 = \bar{e}, e_{m-1} = e} t^{\mathrm{bc}(c)} \\ + \sum_{C=(e, e_2, \ldots, e_{m-1}, \bar{e}), e_2 \neq \bar{e}, e_{m-1} \neq e} t^{\mathrm{bc}(C)} + \sum_{C=(e, e_2, \ldots, e_{m-1}, \bar{e}), e_2 \neq \bar{e}, e_{m-1} = e} t^{\mathrm{bc}(C)}.\end{aligned}$$

Then, we have

$$\begin{aligned}&C_m(t)(x_0, e, \bar{e}) \\ &= t\big(C_{m-2}(t)(t(e), \bar{e}) - C_{m-2}(t)(t(e), \bar{e}, e)\big) + t^2 C_{m-2}(t)(t(e), \bar{e}, e) \\ &\quad + \big(C_{m-2}(t)(t(e)) - C_{m-2}(t)(t(e), \bar{e}) - C_{m-2}(t)(t(e), \cdot, e) + C_{m-2}(t)(t(e), \bar{e}, e)\big) \\ &\quad + t\big(C_{m-2}(t)(t(e), \cdot, e) - C_{m-2}(t)(t(e), \bar{e}, e)\big)\end{aligned}$$

$$= C_{m-2}(t)(t(e), t(e)) + (t-1)C_{m-2}(t)(t(e), \bar{e})$$
$$+ (t-1)C_{m-2}(t)(t(e), \cdot, e) + (t-1)^2 C_{m-2}(t)(t(e), \bar{e}, e).$$

By this, we have

$$\begin{aligned}
&C_m(t)(x_0, e, \bar{e}) \\
&= C_{m-2}(t)(t(e), t(e)) + (t-1)\big(C_{m-2}(t)(t(e), \bar{e}) - N_{m-2}(t, t(e), \bar{e})\big) \\
&\quad + (t-1)\big(C_{m-2}(t)(t(e), \cdot, e) - N_{m-2}(t, t(e), \cdot, e)\big) \\
&\quad + (t-1)\big(N_{m-2}(t, t(e), \bar{e}) + N_{m-2}(t, t(e), \cdot, e)\big) + (t-1)^2 C_{m-2}(t)(t(e), \bar{e}, e) \\
&= C_{m-2}(t)(t(e), t(e)) + 2(t-1)N_{m-2}(t, t(e), \bar{e}) + (t^2-1)C_{m-2}(t)(t(e), \bar{e}, e).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
C_m(t)(x_0, e, \bar{e}) = &\, C_{m-2}(t)(t(e), t(e)) + 2(t-1)N_{m-2}(t, t(e), \bar{e}) \\
&+ (t^2-1)C_{m-2}(t)(t(e), \bar{e}, e). \qquad (4.1)
\end{aligned}$$

In the case that $m \geq 5$, by (4.1), we have

$$\begin{aligned}
&C_m(t)(x_0, e, \bar{e}) \\
&= C_{m-2}(t)(t(e), t(e)) + 2(t-1)N_{m-2}(t, t(e), \bar{e}) \\
&\quad + (t^2-1)\Big\{C_{m-4}(t)(x_0, x_0) + 2(t-1)N_{m-4}(t, x_0, e) + (t^2-1)C_{m-4}(t)(x_0, e, \bar{e})\Big\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&C_m(t)(x_0, x_0) - N_m(t, x_0) \\
&= \big(\deg(x_0)C_{m-2}(t)(x_0, x_0) - \Delta_X C_{m-2}(t)(x_0, x_0)\big) \\
&\quad + 2(t-1)N_{m-2}(t, x_0) + (t^2-1)\deg(x_0)C_{m-4}(t)(x_0, x_0) + (t^2-1)2(t-1)N_{m-4}(t, x_0) \\
&\quad + (t^2-1)^2\big(C_{m-4}(t)(x_0, x_0) - N_{m-4}(t, x_0)\big). \qquad (4.2)
\end{aligned}$$

Hence, for $m \geq 1$, we have the following.

$$\begin{aligned}
&N_{m+4}(t, x_0) - 2(1-t)N_{m+2}(t, x_0) + (1-t^2)(1-t)^2 N_m(t, x_0) \\
&= C_{m+4}(t)(x_0, x_0) - \deg(x_0)C_{m+2}(t)(x_0, x_0) - (1-t^2)^2 C_m(t)(x_0, x_0) \\
&\quad + \deg(x_0)(1-t^2)C_m(t)(x_0, x_0) + \Delta_X C_{m+2}(t)(x_0, x_0).
\end{aligned}$$

It is easy to check that this implies the following desired identity:

$$\begin{aligned}
\big(1-(1-t)^2u^2\big)N(t, x_0 : u) = &\,\big(1-(\deg(x_0)-(1-t^2))u^2\big)C(t, x_0 : u) \\
&- \deg(x_0)tu^2 + \frac{u^2}{1-(1-t^2)u^2}\Delta_X C(t, \cdot : u)(x_0).
\end{aligned}$$

Next, we prove the second identity. For $m = 3$, by (4.1), we have

$$\begin{aligned}&N_3(t, x_0) - (1-t)^2 N_1(t, x_0)\\&= C_3(t)(x_0, x_0) - \big(\deg(x_0) - (1-t^2)\big)C_1(t)(x_0, x_0) + \Delta_X C_1(t)(x_0, x_0).\end{aligned}$$

Then, we have

$$N_3(t, x_0) = C_3(t)(x_0, x_0) - \big(\deg(x_0) - 2(1-t)\big)C_1(t)(x_0, x_0) + \Delta_X C_1(t)(x_0, x_0).$$

For $m = 4$, by (4.1), we have

$$\begin{aligned}N_4(t, x_0) &= C_4(t)(x_0, x_0) - (\deg(x_0) - (1-t^2))C_2(t)(x_0, x_0)\\&\quad + \Delta_X C_2(t)(x_0, x_0) + (1-t)^2 N_2(t, x_0)\\&= C_4(t)(x_0, x_0) - (\deg(x_0) - (1-t^2))C_2(t)(x_0, x_0)\\&\quad + \Delta_X C_2(t)(x_0, x_0) + (1-t)^2(C_2(t)(x_0, x_0) - t\deg(x_0))\\&= C_4(t)(x_0, x_0) - (\deg(x_0) - 2(1-t))C_2(t)(x_0, x_0)\\&\quad + \Delta_X C_2(t)(x_0, x_0) - (1-t)^2 t\deg(x_0).\end{aligned}$$

Therefore, the second identity holds for $m = 3, 4$. In the case $m \geq 5$, the identity (4.2) is equivalent to the following identity.

$$\begin{aligned}&N_m(t, x_0) - (1-t)^2 N_{m-2}(t, x_0) - (1-t^2)\big(N_{m-2}(t, x_0) - (1-t)^2 N_{m-4}(t, x_0)\big)\\&= C_m(t)(x_0, x_0) - (1-t^2)C_{m-2}(t)(x_0, x_0)\\&\quad - \big(\deg(x_0 - (1-t^2)\big)\big(C_{m-2}(t)(x_0, x_0) - (1-t^2)C_{m-4}(t)(x_0, x_0)\big)\\&\quad + \Delta_X C_{m-2}(t)(x_0, x_0).\end{aligned} \tag{4.3}$$

By summing the both sides of (4.3), we have the desired identity. □

Next, we prove the following theorem that is our goal in this section.

Theorem 4.2.2. *For a vertex x_0, we have the following identity:*

$$\begin{aligned}&\big(1 - (1-t^2)u^2\big)\big(1 - (1-t)^2u^2\big)C^{\mathrm{cbc}}(t, x_0 : u)\\&= \big\{1 - (1-t)(\deg(x_0) - \Delta_X + 2t)u^2 + (1-t^2)(1-t)\big(\deg(x_0) - (1-t)\big)u^4\big\}C(t, \cdot : u)(x_0)\\&\quad - \big(1 - (1-t^2)\big)t\deg(x_0)(1-t)u^2.\end{aligned}$$

Moreover, for $m \geq 3$, we have the following identity:

$$\begin{aligned}C_m(t, x_0) = C_m(t)(x_0, x_0) &- \frac{\deg(x_0) - 2(1-t)}{1-t}\sum_{j=1}^{\lceil \frac{m}{2}\rceil - 1}(1-t)^{2j}C_{m-2j}(t)(x_0, x_0)\\&+ (1-t)R_m(t)(x_0) - \delta_{2\mathbb{Z}}(m)(1-t)^{m-1}t\deg(x_0).\end{aligned}$$

Proof. For $m \geq 1$, $C_m(t,x_0) - N_m(t,x_0)$ (resp. $C_m(t)(x_0,x_0) - N_m(t,x_0)$) represents the number of closed paths with weight $t^{\mathrm{cbc}(\cdot)}$ (resp. $t^{\mathrm{bc}(\cdot)}$) of length m starting at x_0, which have no tail. Hence, we have

$$C_m(t,x_0) - N_m(t,x_0) = t\big(C_m(t)(x_0,x_0) - N_m(t,x_0)\big).$$

Then, we have

$$C^{\mathrm{cbc}}(t,x_0:u) = tC(t,c_0:u) + (1-t)N(t,x_0:u).$$

By this and Proposition 4.2.1, we have

$$\begin{aligned}
&\big(1-(1-t)^2u^2\big)C^{\mathrm{cbc}}(t,x_0:u)\\
&= \Big((1-t)(1-\big(\deg(x_0)-(1-t^2)\big)u^2) + t(1-(1-t)^2u^2)\Big)C(t,x_0:u)\\
&\quad - (1-t)t\deg(x_0)u^2 + \frac{(1-t)u^2}{1-(1-t^2)u^2}\Delta_X C(t,\cdot:u)(x_0).
\end{aligned}$$

By simple calculation, this implies the first equality. Next, we verify the second equality. By Proposition 4.2.1, for $m \geq 3$, we have

$$\begin{aligned}
C_m(t,x_0) &= t\big(C_m(t)(x_0,x_0) - N_m(t,x_0)\big)\\
&\quad + \big(N_m(t,x_0) - C_m(t)(x_0,x_0)\big) + C_m(t)(x_0,x_0)\\
&= C_m(t)(x_0,x_0) - (1-t)\big(C_m(t)(x_0,x_0) - N_m(t,x_0)\big)\\
&= C_m(t)(x_0,x_0)\\
&\quad - (1-t)(\deg(x_0) - 2(1-t)) \sum_{j=1}^{\lceil \frac{m}{2} \rceil -1} (1-t)^{2(j-1)} C_{m-2j}(t)(x_0,x_0)\\
&\quad + (1-t)R_m(t)(x_0) - \delta_{2\mathbb{Z}}(m)(1-t)^{m-1}t\deg(x_0).
\end{aligned}$$

□

In the end of this section, we note that generating functions which we defined in this section are expressed by $C(t,x_0:u)$. We define the generating function of $R_m(t)(x_0)$ as follows.

$$R(t,x_0:u) = \sum_{m=1}^{\infty} R_m(t)(x_0)u^m.$$

It is straightforward to check that the following holds. If $|t| < 1$, $|t| \neq 0$ and $|u| < \frac{1}{2}$, then, we have

$$R(t,x_0:u) = \frac{u^2}{\big(1-(1-t)^2u^2\big)\big(1-(1-t^2)u^2\big)}\Delta_X C(t,\cdot:u)(x_0).$$

Therefore, all generating functions which we defined in this section are expressed by $C(t,x_0:u)$. We note that the above formula holds for $t = 0$.

4.3 A Bartholdi type formula for simple graphs with bounded degree

In this section, we introduce a Bartholdi zeta function for a graph with bounded degree. This zeta function is a generalization of the Bartholdi zeta function from a finite graph to a graph with bounded degree ([1]).

First of all, we define a Bartholdi zeta function for a graph with bounded degree. For a graph X with bounded degree, a vertex x_0 and complex variables t, u, we define a *Barholdi zeta function* as follows.

$$Z_X(u,t,x_0) = \exp\left(\sum_{C\in\mathcal{C}_{x_0}} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)}\right).$$

This is a natural generalization of the Ihara zeta function for a graph with bounded degree which was introduced in [4] in the spirit of L. Bartholdi although he introduced by using the Euler product expression. Before we give an Ihara type formula for this zeta function, we define several operators and give several properties of $C_m(t)$. We define $I(t)$, Q_X and $Q_X(t)$ as follows.

$$\begin{aligned} I(t) &= (1-t)I, \\ Q_X &= D_X - I, \\ Q_X(t) &= D_X - I(t). \end{aligned}$$

Then, we have the following proposition.

Proposition 4.3.1. *For $m \geq 2$, we have*

$$C_m(t) = \begin{cases} C_1(t)^2 - (1-t)(Q_X + I) & \textit{if } m = 2, \\ C_{m-1}(t)C_1(t) - (1-t)C_{m-2}(t)Q_X(t) & \textit{if } m \geq 3. \end{cases}$$

We give the proof of this proposition although this proposition was proved in [25] because our proof is a little different from [25].

Proof. It is enough to show that for $x_0, x \in VX$,

$$C_m(t)(x_0,x) = \begin{cases} \big(C_1(t)^2 - (1-t)(Q_X + I)\big)(x_0,x) & \text{if } m = 2, \\ \big(C_{m-1}(t)C_1(t) - (1-t)C_{m-2}(t)Q_X(t)\big)(x_0,x) & \text{if } m \geq 3. \end{cases}$$

For $m = 2$, it is obvious. In the case $m \geq 3$, for $x_0, x \in VX$, we have

$$C_{m-1}(t)C_1(t)(x_0,x) = C_{m-1}(t)C_1(t)\delta_{x_0}(x) = \sum_{C\in\mathcal{B}_x, \ell(C)=m-1} \sum_{e\in E^{x_0}_{t(C)}} t^{\mathrm{bc}(C)}.$$

By considering whether a path has backtracking at the last step and comparing $C_{m-1}(t)C_1(t)(x_0,x)$ to $C_m(t)(x_0,x)$, we have

$$\begin{aligned}
&C_{m-1}(t)C_1(t)(x_0,x) - C_{m-2}(t)(x_0,x)t - C_{m-2}(t)(x_0,x)(\deg(x_0)-1)\\
&\quad + C_{m-2}(t)(x_0,x)t^2 + C_{m-2}(t)(x_0,x)(\deg(x_0)-1)t\\
&= C_m(t)(x_0,x).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
C_m(t) &= C_{m-1}(t)C_1(t) - C_{m-2}(t)t - C_{m-2}(t)Q_X + C_{m-2}(t)t^2 + C_{m-2}(t)Q_X t\\
&= C_{m-1}(t)C_1(t) - (1-t)C_{m-2}(t)Q_X(t).
\end{aligned}$$

□

For a complex valuable t, we define $\alpha(t)$ by

$$\alpha(t) = \frac{M + \sqrt{M^2 + 4(|t|+1)M}}{2}.$$

Here, we denote the maximum of all degrees of X by M. Then, we have the following Lemma.

Lemma 4.3.2. *For $|t| < 1$, then for $m \geq 0$, we have*

$$\|C_m(t)\| \leq \alpha(t)^m.$$

Proof. We prove this by induction on m. For $m = 0, 1$, there is nothing to do. We suppose that our assertion holds for $m-1$. Then, we have

$$\begin{aligned}
\|C_m(t)\| &= \|C_{m-1}(t)C_1(t) - (1-t)C_{m-2}(t)Q_X(t)\|\\
&\leq M\alpha(t)^{m-1} + (1+|t|)\alpha(t)^{m-2}(M-1+|t|)\\
&= \alpha(t)^{m-2}\big\{\alpha(t)M + (1+|t|)(M-1+|t|)\big\}\\
&= \alpha(t)^{m-2}(\alpha(t)^2 + |t|^2 - 1) < \alpha(t)^m.
\end{aligned}$$

□

By Proposition 4.3.1 and Lemma 4.3.2, we have the following proposition.

Proposition 4.3.3. *For $|u| < \frac{1}{\alpha(t)}$, $|t| < 1$, we have*

$$\left(\sum_{m=0}^{\infty} C_m(t)u^m\right)\left(I - uA_X + (1-t)Q_X(t)u^2\right) = (1-(1-t)^2u^2)I,$$

$$\left(\sum_{m=0}^{\infty} \Big(\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{m-2j}(t)(1-t)^{2j}\Big)u^m\right)\left(I - uA_X + (1-t)Q_X(t)u^2\right) = I.$$

Next, for $m \leq 0$, t, $x \in VX$ and $f \in \ell^2(VX)$, we define an operator $R_m(t)$ by

$$R_m(t)f(x) = R_m(t)(x)f(x).$$

Then, we define an operator $C_m^{\rm cbc}(t)$ like the operator $N_{X,m}$ introduced in [22] by

$$C_m^{\rm cbc}(t) = \begin{cases} C_m(t) & \text{if } m = 0, 1, \\ tC_2(t) & \text{if } m = 2, \\ C_m(t) - \frac{Q_X(t)-I(t)}{1-t} \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} (1-t)^{2j} C_{m-2j}(t) & \\ +(1-t)R_m(t) - \delta_{2\mathbb{Z}}(m)(1-t)^{m-1} t D_X & \text{if } m \geq 3. \end{cases}$$

We remark that this operator is also a bounded operator by our assumption. We also remark that $C_m^{\rm cbc}(t)(x_0, x_0) = C_m(t, x_0)$ and the following identity holds by Theorem 4.2.2.

$$Z_X(u, t, x_0) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} C_m(t, x_0) u^m \right).$$

Therefore, for $x_0, x \in VX$, we define $Z_X(u, t, x_0, x)$ as follows.

$$Z_X(u, t, x_0, x) = \exp\left(\sum_{m=1}^{\infty} \frac{C_m^{\rm cbc}(t)(x_0, x)}{m} u^m \right).$$

Moreover, we define $f(z)$ as follows.

$$f(z) = zA_X - z^2(1-t)Q_X(t).$$

Then, we have the following Proposition ([22]).

Proposition 4.3.4. *For* $|u| < \frac{1}{\alpha(t)}$, *we have*

$$f'(u)(I - f(u))^{-1} = -\frac{d}{du} \log(I - f(u)) \\ + (1-t)u^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j f(u)^{n-1-j} (A_X Q_X(t) - Q_X(t) A_X) f(u)^{j-1}.$$

Under the above preparation, we give the following theorem.

Theorem 4.3.5. *For* $|u| < \frac{1}{\alpha(t)}, |t| < 1$ *and* $x_0, x \in VX$, *we have*

$$Z_X(u, t, x_0, x) \\ = (1 - (1-t)^2 u^2)^{-\frac{\deg(x_0) - 2}{2} \delta_{x_0}(x)}$$

$$\times \exp\Bigg(- \big[\log(I - u(D_X - \Delta_X) + (1-t)Q_X(t)u^2)\big](x_0, x)\Bigg)$$
$$\times \exp\Bigg(\int_0^u (1-t)z^2 \sum_{n=2}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j \left[f(z)^{n-1-j}(A_X D_X - D_X A_X) f(z)^{j-1}\right](x_0, x) dz \Bigg)$$
$$\times \exp\Bigg(\frac{\big[tD_X - C_2(t)\big](x_0, x)}{2}(1-t)u^2 \Bigg) \exp\Bigg(\sum_{m=3}^{\infty} \frac{(1-t)R_m(t)(x_0, x)}{m} u^m \Bigg).$$

Proof. We consider the following power series that converges in $|u| < \frac{1}{\alpha(t)}, |t| < 1$.

$$\sum_{m=0}^{\infty} C_m^{\mathrm{cbc}}(t) u^m.$$

By the definition of $C_m^{\mathrm{cbc}}(t)$, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} C_m^{\mathrm{cbc}}(t) u^m &= \frac{Q_X(t)}{1-t} \sum_{m=0}^{\infty} C_m(t) u^m - \frac{Q_X(t) - I(t)}{1-t} \sum_{m=0}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{m-2j}(t)(1-t)^{2j} u^m \\
&\quad + \big(Q_X(t) - I(t)\big) \frac{(1-t)u^2}{1-(1-t)^2u^2} + (1-t) \sum_{m=3}^{\infty} R_m(t) u^m \\
&\quad + C_2(t)(t-1)u^2 - \frac{t(1-t)^3u^4}{1-(1-t)^2u^2} D_X.
\end{aligned}$$

Here, we denote the floor function by $\lfloor \cdot \rfloor$. In the right hand side of the above equation, the following equality holds.

$$\begin{aligned}
&\big(Q_X(t) - I(t)\big) \frac{(1-t)u^2}{1-(1-t)^2u^2} - \frac{t(1-t)^3u^4}{1-(1-t)^2u^2} D_X \\
&= t(1-t)u^2 D_X + \frac{(1-t)^2u^2}{1-(1-t)^2u^2}(Q_X - I).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} C_m^{\mathrm{cbc}}(t) u^m &= \frac{Q_X(t)}{1-t} \sum_{m=0}^{\infty} C_m(t) u^m - \frac{Q_X(t) - I(t)}{1-t} \sum_{m=0}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{m-2j}(t)(1-t)^{2j} u^m \\
&\quad + (Q_X - I) \frac{(1-t)^2u^2}{1-(1-t)^2u^2} + (1-t)u^2\big(tD_X - C_2(t)\big) \\
&\quad + (1-t) \sum_{m=3}^{\infty} R_m(t) u^m.
\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{m=1}^{\infty} C_m^{\mathrm{cbc}}(t)u^m &= \frac{Q_X(t)}{1-t}\sum_{m=0}^{\infty} C_m(t)u^m - I - \frac{Q_X(t)-I(t)}{1-t}\sum_{m=0}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} C_{m-2j}(t)(1-t)^{2j}u^m \\ &\quad + (Q_X - I)\frac{(1-t)^2u^2}{1-(1-t)^2u^2} + (1-t)u^2\big(tD_X - C_2(t)\big) \\ &\quad + (1-t)\sum_{m=3}^{\infty} R_m(t)u^m.\end{aligned}$$

By Proposition 4.3.3, we have

$$\begin{aligned}&\frac{Q_X(t)}{1-t}\sum_{m=0}^{\infty} C_m(t)u^m - I - \frac{Q_X(t)-I(t)}{1-t}\sum_{m=0}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} C_{m-2j}(t)(1-t)^{2j}u^m \\ &= \frac{Q_X(t)}{1-t}\big(1-(1-t)^2u^2\big)(I-f(u))^{-1} - I - \frac{Q_X(t)-I(t)}{1-t}(I-f(u))^{-1} \\ &= uf'(u)(I-f(u))^{-1}.\end{aligned}$$

By Proposition 4.3.4, we have

$$\begin{aligned}&\frac{Q_X(t)}{1-t}\sum_{m=0}^{\infty} C_m(t)u^m - I - \frac{Q_X(t)-I(t)}{1-t}\sum_{m=0}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} C_{m-2j}(t)(1-t)^{2j}u^m \\ &= -u\frac{d}{du}\log(I-f(u)) \\ &\quad + (1-t)u^3\sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1} jf(u)^{n-1-j}\big(A_XQ_X(t) - Q_X(t)A_X\big)f(u)^{j-1}.\end{aligned}$$

Therefore, for $x_0, x \in VX$, we have

$$\begin{aligned}&u\frac{d}{du}\sum_{m=1}^{\infty}\frac{C_m^{\mathrm{cbc}}(t)(x_0,x)}{m}u^m \\ &= -\left[u\frac{d}{du}\log(I-f(u))\right](x_0,x) \\ &\quad + (1-t)u^3\sum_{m=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1} j\big[f(u)^{n-1-j}\big(A_XQ_X(t)-Q_X(t)A_X\big)f(u)^{j-1}\big](x_0,x) \\ &\quad - \frac{\deg(x_0)-2}{2}\delta_{x_0}(x)u\frac{d}{du}\big[\log(1-(1-t)^2u^2)\big] + \frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}u\frac{d}{du}\big[(1-t)u^2\big] \\ &\quad + u\frac{d}{du}\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}u^m.\end{aligned}$$

Dividing by u and integrating from 0 to u, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{C_m^{\mathrm{cbc}}(t)(x_0,x)}{m} u^m \\
&= -\Big[\log(I - f(u))\Big](x_0,x_0) \\
&\quad + \int_0^u (1-t)z^2 \sum_{m=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j\big[f(z)^{n-1-j}\big(A_X Q_X(t) - Q_X(t)A_X\big)f(z)^{j-1}\big](x_0,x)dz \\
&\quad - \frac{\deg(x_0)-2}{2}\delta_{x_0}(x)\log(1-(1-t)^2u^2) + \frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}(1-t)u^2 \\
&\quad + \sum_{m=3}^{\infty} \frac{(1-t)R_m(t)(x_0,x)}{m} u^m.
\end{aligned}$$

This implies the following identity

$$\begin{aligned}
&Z_X(u,t,x_0,x) \\
&= (1-(1-t)^2u^2)^{-\frac{\deg(x_0)-2}{2}\delta_{x_0}(x)} \\
&\quad \times \exp\Bigg(- \big[\log(I - u(D_X - \Delta_X) + (1-t)Q_X(t)u^2)\big]\,(x_0,x)\Bigg) \\
&\quad \times \exp\Bigg(\int_0^u (1-t)z^2 \sum_{n=2}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j\,\big[f(z)^{n-1-j}(A_X D_X - D_X A_X)f(z)^{j-1}\big]\,(x_0,x)dz \Bigg) \\
&\quad \times \exp\Bigg(\frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}(1-t)u^2 \Bigg) \exp\Bigg(\sum_{m=3}^{\infty} \frac{(1-t)R_m(t)(x_0,x)}{m} u^m \Bigg).
\end{aligned}$$

□

If X is a $(q+1)$-regular graph, we have

$$I - (D_X - \Delta_X)u + (1-t)Q_X(t)u^2 = I - \big((q+1)I - \Delta_X\big)u + (1-t)(q+t)u^2 I.$$

Since Δ_X is a self-adjoint bounded operator, there exists a unique spectral measure E such that

$$\Delta_X = \int_{\sigma(\Delta_X)} \lambda dE(\lambda).$$

Here, we denote the spectrum of the Laplacian Δ_X by $\sigma(\Delta_X)$. By Theorem 4.3.5 and the property of the spectral integral, we have

$$Z_X(u,t,x_0,x) = (1-(1-t)^2u^2)^{-\frac{q-1}{2}\delta_{x_0}(x)}$$

$$\times \exp\left(\int_{\sigma(\Delta_X)} -\log(1-(q+1-\lambda)u+(1-t)(q+t)u^2)d\mu_{x_0,x}(\lambda)\right)$$
$$\times \exp\left(\frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}(1-t)u^2\right)$$
$$\times \exp\left(\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}u^m\right).$$

Here, we denote $d\langle E(\lambda)\delta_{x_0},\delta_{x_0}\rangle$ by $d\mu_{x_0,x_0}(\lambda)$. Hence, we get the following corollary.

Corollary 4.3.6. *For* $|u| < \frac{1}{\alpha(t)}$, $|t| < 1$, *we have*

$$Z_X(u,t,x_0,x) = (1-(1-t)^2u^2)^{-\frac{q-1}{2}\delta_{x_0}(x)}$$
$$\times \exp\left(\int_{\sigma(\Delta_X)} -\log(1-(q+1-\lambda)u+(1-t)(q+t)u^2)d\mu_{x_0,x}(\lambda)\right)$$
$$\times \exp\left(\frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}(1-t)u^2\right)$$
$$\times \exp\left(\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0)}{m}u^m\right).$$

Moreover, we discuss the case that X is a finite $(q+1)$-regular graph. We introduce the notion of the local spectrum ([10]). For a vertex $x \in VX$, we denote x-local multiplicity of λ_i by $\mathrm{m}_x(\lambda_i)$. Here, the *x-local multiplicity* of λ_i is the xx-entry of the primitive idempotent E_{λ_i}. Let $\{\mu_0 = \lambda_0, \mu_1, \ldots, \mu_{d_x}\}$ be the set of eigenvalues whose local multiplicities are positive. For each vertex $x \in VX$, we denote the x-local spectrum by $\sigma_x(X)$. Here, the *x-local spectrum* is $\sigma_x(X) = \{\lambda_0^{\mathrm{m}_x(\lambda_0)}, \mu_1^{\mathrm{m}_x(\mu_1)}, \ldots, \mu_{d_x}^{\mathrm{m}_x(\mu_{d_x})}\}$.

Then, we have the following corollary immediately by Corollary 4.3.6.

Corollary 4.3.7. *For* $|u| < \frac{1}{\alpha(t)}$, $|t| < 1$, *we have*

$$Z_X(t,u,x_0,x) = (1-(1-t)^2u^2)^{-\frac{q-1}{2}\delta_{x_0}(x)}$$
$$\prod_{\lambda\in\sigma_{x_0}(\Delta_X)} \big(1-(q+1-\lambda)u+(1-t)(q+t)u^2\big)^{-m_{x_0}(\lambda)}$$
$$\times \exp\left(\frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}(1-t)u^2\right)$$
$$\times \exp\left(\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}u^m\right).$$

4.4 The Euler product expression

In this section, we give the Euler product expression of the Bartholdi zeta function which is introduced in Section 4. We have to introduce several terminologies to give the Euler product expression. We take a vertex x_0. A closed path C starting at x_0 is *primitive* if there is no closed paths starting at x_0 whose length is shorter than $\ell(C)$ and of which the multiple is C. We denote by $\mathcal{PK}_{x_0}$ the set of primitive closed paths starting at x_0. Then, the following theorem holds.

Theorem 4.4.1. *For $|u| < \frac{1}{\alpha(t)}$, $|t| < 1$, we have*

$$Z_X(t,u,x_0) = \prod_{C \in \mathcal{PK}_{x_0}} (1 - t^{\mathrm{cbc}(C)} u^{\ell(C)})^{-\frac{1}{\ell(C)}}.$$

Proof. For $|u| < \frac{1}{\alpha(t)}$, $|t| < 1$ and $N \in \mathbb{Z}_{\geq 1}$, we have

$$\begin{aligned}
\log \prod_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} (1 - t^{\mathrm{cbc}(C)} u^{\ell(C)})^{-\frac{1}{\ell(C)}} &= - \sum_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} \frac{1}{\ell(C)} \log(1 - t^{\mathrm{cbc}(C)} u^{\ell(C)}) \\
&= \sum_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} \sum_{m=1}^{\infty} \frac{1}{\ell(C)m} t^{\mathrm{cbc}(C)m} u^{\ell(C)m} \\
&= \sum_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} \sum_{m=1}^{\infty} \frac{1}{\ell(C^m)} t^{\mathrm{cbc}(C^m)} u^{\ell(C^m)} \\
&= \sum_{C} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)}.
\end{aligned}$$

Here, the last sum runs through the set of closed paths starting at x_0 such that the length of primitive paths is less than or equal to N. Therefore, for any N, we have,

$$\exp\left(\sum_{C} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)} \right) = \prod_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} (1 - t^{\mathrm{cbc}(C)} u^{\ell(C)})^{-\frac{1}{\ell(C)}}$$

Here, the sum runs through the same set as the above. Taking the limit of the both sides, we have

$$Z_X(t,u,x_0) = \prod_{C \in \mathcal{PK}_{x_0}} (1 - t^{\mathrm{cbc}(C)} u^{\ell(C)})^{-\frac{1}{\ell(C)}}.$$

□

4.5 The heat kernels on regular graphs

In this section, for a regular graph, we give a new expression of the heat kernels on regular graphs by using the modified Bessel function of the first kind. Let X be a $(q+1)$-regular graph. We denote the heat kernel of X by $K_X(\tau, x_0, x)$. For $j \in \mathbb{Z}_{\geq 0}$ and real variable t which satisfies $|t| < 1$, we define the symbol $d_j(t)$ as follows.

$$d_j(t) = \begin{cases} 1 & \text{if } j = 0, \\ -\frac{q-1+2t}{1-t} & \text{if } j \geq 1. \end{cases}$$

Then, the following theorem holds.

Theorem 4.5.1. *For $\tau \in \mathbb{R}_{\geq 0}$, $x \in VX$ and $|t| < 1$, we have*

$$K_X(\tau, x_0, x) = \sum_{n=0}^{\infty} C_n(t)(x_0, x) \sum_{j=0}^{\infty} d_j(t)\, \mathrm{e}^{-(q+1)\tau} \times (1-t)^{2j} \big((1-t)(q+t)\big)^{-\frac{n+2j}{2}} \mathrm{I}_{n+2j}\left(2\sqrt{(1-t)(q+t)}\tau\right).$$

Proof. We define $g(\tau, x)$ as follows.

$$g(\tau, x) = \mathrm{e}^{(q+1)\tau} f(\tau, x).$$

Then, it turns out that the heat equation is equivalent to the following equation.

$$\begin{cases} \frac{\partial g}{\partial \tau}(\tau, x) - C_1(t) g(\tau, \cdot)(x) = 0, \\ g(0, x) = \delta_{x_0}(x). \end{cases}$$

Here, we remark that $C_1(t)$ is equal to A_X. It is sufficient to show that the following is the solution of the above equation.

$$g(\tau, x) = \sum_{n=0}^{\infty} C_n(t)(x_0, x) \sum_{j=0}^{\infty} d_j(t) \times (1-t)^{2j} \big((1-t)(q+t)\big)^{-\frac{n+2j}{2}} \mathrm{I}_{n+2j}\left(2\sqrt{(1-t)(q+t)}\tau\right).$$

It is obvious that $g(0, x) = \delta_{x_0}(x)$. Therefore, it remains to check that $g(\tau, x)$ is bounded on $[0, T] \times VX$ for each T and $g(\tau, x)$ satisfies the above equation indeed.

First, we check that $g(\tau, x)$ is bounded. By Proposition 4.3.1 and (2.2), we have

$$|g(\tau, x)| \leq \sum_{n=0}^{\infty} \alpha(t)^n \sum_{j=0}^{\infty} |d_j(t)|\, |1-t|^{2j}\, \tau^{n+2j} \frac{\mathrm{e}^{2\sqrt{(q+t)(1-t)}\tau}}{(n+2j)!}.$$

We denote the maximum of $d_j(t)$ by M_t. Then, we have

$$\begin{aligned} |g(\tau,x)| &\le M_t\,\mathrm{e}^{2\sqrt{(q+t)(1-t)}\tau}\sum_{n=0}^{\infty}\alpha(t)^n\sum_{j=0}^{\infty}\frac{\tau^n(2\tau)^{2j}}{(n+2j)!} \\ &\le M_t\,\mathrm{e}^{2\sqrt{(q+t)(1-t)}\tau}\sum_{n=0}^{\infty}\frac{\big(\alpha(t)\tau\big)^n}{n!}\sum_{j=0}^{\infty}\frac{(2\tau)^{2j}}{(2j)!} \\ &= M_t\,\mathrm{e}^{(2\sqrt{(q+t)(1-t)}+\alpha(t))\tau}\cosh(2\tau). \end{aligned}$$

Therefore, $g(\tau,x)$ is bounded on $[0,T]\times VX$ for each T.

Second, we check that $g(\tau,x)$ satisfies the equation. To prove this, we define $g(\tau)$ as follows.

$$\begin{aligned} g(\tau) = {} & \sum_{n=0}^{\infty}C_n(t)\sum_{j=0}^{\infty}d_j(t) \\ & \times(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j}{2}}\,\mathrm{I}_{n+2j}\left(2\sqrt{(1-t)(q+t)}\tau\right). \end{aligned}$$

Then,

$$\begin{aligned} & \frac{\partial g}{\partial\tau}(\tau,x)-C_1(t)g(\tau,\cdot)(x) \\ & = \left\{\frac{\partial g}{\partial\tau}(\tau)-C_1(t)g(\tau)\right\}(x_0,x). \end{aligned}$$

Therefore, it is sufficient to check that

$$\frac{\partial g}{\partial\tau}(\tau)-C_1(t)g(\tau)=0.$$

$$\begin{aligned} & \frac{\partial g}{\partial\tau}(\tau)-C_1(t)g(\tau) \\ & = \sum_{n=0}^{\infty}C_n(t)\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}}\,\mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\ & \quad + \sum_{n=0}^{\infty}C_n(t)\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}}\,\mathrm{I}_{n+2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\ & \quad - \sum_{n=1}^{\infty}C_1(t)C_{n-1}(t) \\ & \qquad \times\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}}\,\mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right). \end{aligned}$$

Here, we used (2.1) and we remark that we are allowed to change the order of the differentiation and power series by (2.2). By Proposition 4.3.1, we have

$$C_n(t) = \begin{cases} C_1(t)^2 - (1-t)(q+1)I & \text{if } n = 2, \\ C_1(t)C_{n-1}(t) - (1-t)(q+t)C_{n-2}(t) & \text{if } n \geq 3. \end{cases}$$

Here, we remark that the operator $C_n(t)$ is a self-adjoint operator. By this relation, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} C_1(t)C_{n-1}(t) \\
&\quad \times \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \, \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&= C_1(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-j} \, \mathrm{I}_{2j}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + C_2(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{1+2j}{2}} \, \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + (1-t)(q+1)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{1+2j}{2}} \, \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + \sum_{n=3}^{\infty} C_n(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \, \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + \sum_{n=1}^{\infty} C_n(t) \\
&\qquad \times \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}} \, \mathrm{I}_{n+2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right).
\end{aligned}$$

To explain our calculation clearly, we put

$$(n=1) = C_1(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-j} \, \mathrm{I}_{2j}\left(2\sqrt{(1-t)(q+t)}\tau\right),$$

$$(n=2-1) = C_2(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{1+2j}{2}} \, \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right),$$

$$(n=2-2) = (1-t)(q+1)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{1+2j}{2}} \, \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right),$$

$$(n=3-1) = \sum_{n=3}^{\infty} C_n(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \, \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right),$$

$$(n=3-2)=\sum_{n=1}^{\infty}C_n(t)\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}}\,\mathrm{I}_{n+2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right).$$

By calculating $(n=1)+(n=2-1)+(n=3-1)$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}C_1(t)C_{n-1}(t)\\
&\quad\times\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}}\,\mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right)\\
&=\sum_{n=1}^{\infty}C_n(t)\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}}\,\mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right)\\
&\quad+(n=2-2)+(n=3-2).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\frac{\partial g}{\partial\tau}(\tau)-C_1(t)g(\tau)\\
&=\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}}\,\mathrm{I}_{2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right)\\
&\quad+\sum_{n=0}^{\infty}C_n(t)\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}}\,\mathrm{I}_{n+2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right)\\
&\quad-(n=2-2)-(n=3-2)\\
&=\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}}\,\mathrm{I}_{2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right)\\
&\quad+\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}}\,\mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right)-(n=2-2)\\
&=\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}}\,\mathrm{I}_{2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right)\\
&\quad+\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\left(1-\frac{q+1}{q+t}\right)\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}}\,\mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right)\\
&=\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}}\,\mathrm{I}_{2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right)\\
&\quad-\sum_{j=0}^{\infty}d_j(t)(1-t)^{2j}\left(\frac{1-t}{q+t}\right)\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}}\,\mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right).
\end{aligned}$$

Here, we note that $C_0(t) = I$. Therefore, we have

$$\frac{\partial g}{\partial \tau}(\tau) - C_1(t)g(\tau) = 0.$$

Here, we used $\mathrm{I}_{-1}(\tau) = \mathrm{I}_1(\tau)$. This completes the proof. $\square$

4.6 An alternative proof of the Bartholdi zeta function formula

In this section, we give an alternative proof of the Bartholdi zeta function formula for a regular graph obtained in Section 4.

Since Δ_X is a self-adjoint bounded operator, there exists a unique spectral measure E such that

$$\Delta_X = \int_{\sigma(\Delta_X)} \lambda dE(\lambda).$$

Here, $\sigma(\Delta_X)$ stands for the spectrum of Δ_X. Therefore, for $x_0, x \in VX$, we have

$$K_X(\tau, x_0, x) = \int_{\sigma(\Delta)} \mathrm{e}^{-\tau\lambda}\, d\mu_{x_0,x}(\lambda).$$

Here, we denote $d\langle E(\lambda)\delta_{x_0}, \delta_x\rangle$ by $d\mu_{x_0,x}(\lambda)$. By applying the $G(t)$-transform and by easy culculation, for $0 < u < \frac{1}{\alpha(t)}$, we have

$$\begin{aligned} G(t)\big(K_X(\tau, x_0, x)\big)(u) &= G(t)\bigg(\int_{\sigma(\Delta)} \mathrm{e}^{-\tau\lambda}\, d\mu_{x_0,x}(\lambda)\bigg)(u) \\ &= \int_{\sigma(\Delta_X)} \frac{u^{-2} - (q+t)(1-t)}{(q+t)(1-t)u + \frac{1}{u} - (q+1-\lambda)} d\mu_{x_0,x}(\lambda). \end{aligned}$$

We remark that we are allowed to change the order of integrations in the above equation by Fubini's theorem. We also remark that if $0 < u < \frac{1}{\alpha(t)}$, then we have

$$(q+t)(1-t)u + \frac{1}{u} - (q+1-\lambda) > 0.$$

On the other hand, by Theorem 4.5.1, we have

$$G(t)\big(K_X(\tau, x_0, x)\big)(u) = G(t)\bigg(\sum_{n=0}^{\infty} C_n(t)(x_0, x) \sum_{j=0}^{\infty} d_j(t)\, \mathrm{e}^{-(q+1)\tau}$$

$$\times (1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j}{2}} \mathrm{I}_{n+2j}\left(2\sqrt{(1-t)(q+t)}\tau\right)\Big)(u)$$
$$= \sum_{n=0}^{\infty} C_n(t)(x_0,x) \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j} u^{n+2j-1}.$$

Here, we used (2.3) in the second equation. Therefore, we have

$$G(t)(K_X(\tau,x_0,x))(u) = \sum_{n=0}^{\infty} C_n(x_0,x)u^{n-1}$$
$$- \frac{q-1+2t}{1-t} \sum_{n=2}^{\infty} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} C_{n-2j}(t)(x_0,x)(1-t)^{2j} u^{n-1}.$$

We note that the following equality holds.

$$\sum_{n=2}^{\infty} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} C_{n-2j}(t)(x_0,x)(1-t)^{2j} u^n = \sum_{n=3}^{\infty} \sum_{j=1}^{\lceil \frac{n}{2} \rceil -1} (1-t)^{2j} C_{n-2j}(t)(x_0,x) u^n$$
$$+ C_0(t)(x_0,x) \frac{(1-t)^4 u^4}{1-(1-t)^2 u^2} + C_0(t)(x_0,x)(1-t)^2 u^2.$$

Hence, we have

$$G(t)(K_X(\tau,x_0,x))(u)$$
$$= \sum_{n=3}^{\infty} C_n(t)(x_0,x)u^{n-1} - \frac{q-1+2t}{1-t} \sum_{n=3}^{\infty} \sum_{j=1}^{\lceil \frac{n}{2} \rceil -1} (1-t)^{2j} C_{n-2j}(t)(x_0,x) u^{n-1}$$
$$+ \frac{1}{u}\big(C_0(t)(x_0,x) + C_1(t)(x_0,x)u + tC_2(t)(x_0,x)u^2\big)$$
$$+ u(1-t)C_2(t)(x_0,x) - (q-1+2t)\frac{(1-t)u}{1-(1-t)^2u^2} C_0(t)(x_0,x).$$

By the definition of $C_m^{\mathrm{cbc}}(t)$, we have

$$G(t)\big(K_X(\tau,x_0,x)\big)(u)$$
$$= \frac{1}{u} C_0(t)(x_0,x) + \frac{1}{u} \sum_{n=1}^{\infty} \big(C_n^{\mathrm{cbc}}(t)(x_0,x) - (1-t)R_m(t)(x_0,x)\big) u^m$$
$$+ (1-t)u\big(C_2(t) - tD_X\big)(x_0,x) - (q-1)\frac{(1-t)^2 u}{1-(1-t)^2u^2} C_0(t)(x_0,x).$$

Therefore, we have

$$\int_{\sigma(\Delta_X)} \frac{u^{-2} - (q+t)(1-t)}{(q+t)(1-t)u + \frac{1}{u} - (q+1-\lambda)} d\mu_{x_0,x}(\lambda)$$

$$
\begin{aligned}
&= \frac{1}{u} C_0(t)(x_0, x) + \frac{1}{u} \sum_{n=1}^{\infty} \big(C_n^{\mathrm{cbc}}(t)(x_0, x) - (1-t) R_m(t)(x_0, x) \big) u^m \\
&\quad + (1-t) u \big(C_2(t) - t D_X \big)(x_0, x) - (q-1) \frac{(1-t)^2 u}{1 - (1-t)^2 u^2} C_0(t)(x_0, x).
\end{aligned}
$$

This is equivalent to the following equation.

$$
\begin{aligned}
&\frac{d}{du} \Bigg\{ \frac{q-1}{2} C_0(t) \log(1 - (1-t)^2 u^2) + \sum_{n=1}^{\infty} \frac{C_n^{\mathrm{cbc}}(t) - (1-t) R_n(t)}{n} u^n \\
&\quad + \frac{C_2(t) - t D_X}{2} (1-t) u^2 \Bigg\} (x_0, x) \\
&= \frac{d}{du} \int_{\sigma(\Delta_X)} - \log \big(1 - (q+1-\lambda) u + (1-t)(q+t) u^2 \big) d\mu_{x_0, x}(\lambda)
\end{aligned}
$$

By integrating both sides from 0 to u and determining the integrating constant, we have

$$
\begin{aligned}
&(1 - (1-t)^2)^{\frac{q-1}{2} \delta_{x_0}(x)} Z_X(u, t, x_0, x) \exp \left(- \sum_{n=3}^{\infty} \frac{(1-t) R_m(t)(x_0, x)}{m} \right) \\
&\quad \times \exp \left(\frac{\big[C_2(t) - t D_X \big](x_0, x)}{2} (1-t) u^2 \right) \\
&= \exp \left(\int_{\sigma(\Delta_X)} - \log \big(1 - (q+1-\lambda) u + (1-t)(q+t) u^2 \big) d\mu_{x_0, x}(\lambda) \right).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
Z_X(u, t, x_0, x) &= (1 - (1-t)^2 u^2)^{-\frac{q-1}{2} \delta_{x_0}(x)} \\
&\quad \times \exp \left(\int_{\sigma(\Delta_X)} - \log(1 - (q+1-\lambda) u + (1-t)(q+t) u^2) d\mu_{x_0, x}(\lambda) \right) \\
&\quad \times \exp \left(\frac{\big[t D_X - C_2(t) \big](x_0, x)}{2} (1-t) u^2 \right) \\
&\quad \times \exp \left(\sum_{m=3}^{\infty} \frac{(1-t) R_m(t)(x_0, x)}{m} u^m \right).
\end{aligned}
$$

This gives an alternative proof of the Bartholdi zeta function formula obtained in Section 4.

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