Stochastic analysis for infinite particle systems related to random matrices

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Stochastic analysis for infinite particle systems related to random matrices
（ランダム行列に関する無限粒子系の確率解析）

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Contents

1 Introduction

1.1 Infinite-dimensional stochastic differential equations

An interacting Brownian motion in infinite dimension is a dynamics for infinitely many Brownian particles moving on $\mathbb{R}^d$ which has free potential Φ and interaction potential Ψ. The dynamics is described by the infinite-dimensional stochastic differential equation (ISDE)

$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla_x \Phi(X_t^i)dt - \frac{\beta}{2}\sum_{j\neq i}\nabla_x \Psi(X_t^i, X_t^j)dt, \quad i \in \mathbb{N}. \tag{1.1}$$

Here $(B_t^i)_{i\in\mathbb{N}}$ is $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion and $\beta > 0$ is an inverse temperature.

Lang began to study (1.1) using Itô's calculus [37, 38]. His work was followed by Fritz [15], Tanemura [70], and others. In these work, interaction potential Ψ is restricted to C_0^3 or exponentially decaying. Thus their results do not work when Ψ is long-range potential, for example, logarithmic potential. However, interacting Brownian motions arising from random matrices has the logarithmic interaction potential.

On the other hand, a Dirichlet form approach also provides a method to solve the ISDE (1.1) [44, 47]. The Dirichlet form approach works under mild assumptions including long-range potential. In fact, Osada constructed an unlabeled diffusion of (1.1) whenever Ψ is logarithmic potential by Dirichlet form techniques [44]. Then, using this unlabeled diffusion, (1.1) was solved by Dirichlet form techniques again [47]. We exposit examples of ISDEs related to random matrices.

Sine$_\beta$ interacting Brownian motion (the Dyson Brownian motion in infinite dimension)

Let $d = 1$. Sine$_\beta$ interacting Brownian motion is given by the following ISDE:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{j\neq i,\, |X_t^i - X_t^j|<r}\frac{1}{X_t^i - X_t^j}\,dt, \quad i \in \mathbb{N}. \tag{1.2}$$

This is also known as the Dyson Brownian motion in infinite dimensions. When $\beta \in \{1, 2, 4\}$, (1.2) was solved by the Dirichlet form approach [47]. For $\beta \geq 1$, Tsai solved (1.2) by another method [76]. Although he constructed only the Dyson Brownian motion, and his method cannot extend to the high-dimensional case $d \geq 2$, this result can be applied to out-of-equilibrium initial conditions, which is stronger than outcomes from the Dirichlet form approach.

Airy$_\beta$ interacting Brownian motion

Let $d = 1$. Airy$_\beta$ interacting Brownian motion is given by the following ISDE:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\Big\{\sum_{|X_t^j|<r, j\neq i}\frac{1}{X_t^i - X_t^j} - \int_{|x|<r}\frac{\varrho(x)}{-x}\,dx\Big\}dt, \quad i \in \mathbb{N}. \tag{1.3}$$

Here $\varrho(x) = 1_{(-\infty,0)}(x)\sqrt{-x}$, which is the shifted and rescaled semicircle function at the right edge. The ISDE (1.3) was solved by the Dirichlet form approach for $\beta \in \{1, 2, 4\}$ [54].

Bessel$_{\alpha,\beta}$ interacting Brownian motion

Let $d = 1$. For $\alpha \in [1, \infty)$, the Bessel$_{\alpha,\beta}$ interacting Brownian motion is defined as the following:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\Big\{\frac{\alpha}{2X_t^i} + \sum_{j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j}\Big\}dt, \quad i \in \mathbb{N}. \tag{1.4}$$

Here particles move on $(0, \infty)$, and each particle especially does not hit the origin. For $\beta = 2$, the ISDE (1.4) was solved by the Dirichlet form approach [17]. When $\beta \in \{1, 2, 4\}$, equilibrium states for sine$_\beta$, Airy$_\beta$ and Bessel$_{\alpha,\beta}$ interacting Brownian motions arise from random matrices with symmetry.

Ginibre interacting Brownian motion

Let $d = 2$ and $\beta = 2$. Ginibre interaction Brownian motion is given by

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt, \quad i \in \mathbb{N}, \tag{1.5}$$

and also

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt, \quad i \in \mathbb{N}. \tag{1.6}$$

Actually, (1.5) and (1.6) have the same solution [47]. A equilibrium state of Ginibre interaction Brownian motion is Ginibre random point field, arising from non-Hermitian random matrices.

1.2 First finite particle approximation theorem and SDE gap for the Dyson Brownian motion

We begin by introducing random matrix models (see [2, 14, 43] for details). Gaussian orthogonal/unitary/symplectic ensembles (GOE/GUE/GSE) are Gaussian ensembles defined on the space of symmetric/Hermitian/self-dual matrices M^N ($N \in \mathbb{N}$) with independent random variables respectively. By definition, the GUE $M^N = [M_{i,j}^N]_{1\leq i,j\leq N}$ is an $N \times N$ Hermite matrix having the form

$$M_{i,j}^N = \begin{cases} \frac{\xi_i}{\sqrt{2}}, & \text{if } i = j, \\ \frac{\tau_{i,j}}{2} + \frac{\sqrt{-1}\tilde{\tau}_{i,j}}{2}, & \text{if } i < j, \end{cases}$$

where $\{\xi_i, \tau_{i,j}, \zeta_{i,j}\}_{i<j}^{\infty}$ are i.i.d. Gaussian random variables with mean zero and unit variance. Similarly the GOE (GSE) is defined as symmetric matrix whose entries are real (quaternion) i.i.d. Gaussian random variables up to symmetry respectively.

The eigenvalues $x_1, \ldots, x_N$ of G(O/U/S)E are real from symmetry and have distribution such that

$$\check{\mu}_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N} |x_i - x_j|^\beta \prod_{k=1}^{N} e^{-\frac{\beta}{2}|x_k|^2}\, d\mathbf{x}_N, \tag{1.7}$$

where $\beta = 1, 2, 4$ for G(O/U/S)E respectively. Here $\mathbf{x}_N = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $Z = Z_{\beta,N}$ is a normalizing constant. Remark that (1.7) shows eigenvalues repel each other by the logarithmic potential, which is long-range potential. Wigner's celebrated semicircle convergence theorem asserts that the empirical measure of eigenvalues converges to the semicircle distribution: for a scaled empirical measure $\mathsf{x}^N = \sum_{1 \le i \le N} \delta_{\frac{x_i}{\sqrt{N}}}$,

$$\lim_{N \to \infty} \mathbb{E}_{\check{\mu}_\beta^N} \left[\frac{1}{N} \mathsf{x}^N(-\infty, s) \right] = \int_{-\infty}^{s} \rho_{\mathrm{sc}}(x)\, dx, \tag{1.8}$$

for $\beta \in \{1.2.4\}$, where ρ_{sc} is the Wigner semicircle law (with radius $\sqrt{2}$)

$$\rho_{\mathrm{sc}}(x) = \frac{1}{\pi} \sqrt{2 - x^2}\, \mathbf{1}_{(-\sqrt{2}, \sqrt{2})}(x).$$

The Wigner semicircle law shows macroscopic statistics (global statistics).

Consider microscopic statistics (local statistics) of Gaussian ensembles. Depending on a centre of a scaling limit on the Wigner semicircle law, two different microscopic statistics appear: the first one is sine random point field, which a bulk scaling limit yields, and the second one is Airy random point field, which a soft-edge scaling limit yields. Hereafter we focus on the case $\beta = 2$ for simplicity: the case in $\beta \in \{1, 4\}$ is almost the same.

Choose a bulk position θ in the Wigner semicircle, that is, fix θ such that

$$\theta \in (-\sqrt{2}, \sqrt{2}). \tag{1.9}$$

Take the bulk scaling $x \mapsto y$ such that

$$x = \frac{y}{\sqrt{N}} + \theta \sqrt{N}. \tag{1.10}$$

Substituting (1.10) to (1.7), we get the scaled eigenvalue distribution $\check{\mu}_{\mathrm{sin},2,\theta}^N$ as follows:

$$\check{\mu}_{\mathrm{sin},2,\theta}^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} \exp\Big(-\Big(\frac{y_k}{\sqrt{N}} + \theta\sqrt{N} \Big)^2 \Big) d\mathbf{x}_N. \tag{1.11}$$

Here the normalize constant Z differs from that in (1.7), but we abuse the notation.

Define $\mu_{\mathrm{sin},2,\theta}^N$ as the random point field whose labeled density is given by $\check{\mu}_{\mathrm{sin},2,\theta}^N$. Let $\rho_{\mathrm{sin},2,\theta}^{N,n}$ be the n-correlation function for $\mu_{\mathrm{sin},2,\theta}^N$. Then for any $n \in \mathbb{N}$ we have

$$\lim_{N \to \infty} \rho_{\mathrm{sin},2,\theta}^{N,n} = \rho_{\mathrm{sin},2,\theta}^{n} \quad \text{compact uniformly}, \tag{1.12}$$

where $\rho_{\mathrm{sin},2,\theta}^{n}$ is the n-correlation function for the sine_2 random point field $\mu_{\mathrm{sin},2,\theta}$. The sine_2 random point field is a determinantal random point field on $\mathbb{R}$ with the sine kernel

$$\mathsf{K}_{\mathrm{sin},\theta}(x, y) = \frac{\sin\{\sqrt{2 - \theta^2}(x - y)\}}{\pi(x - y)}.$$

Then by definition,

$$\rho_{\mathrm{sin},2,\theta}^{n}(x_1, \dots, x_n) = \det[\mathsf{K}_{\mathrm{sin},\theta}(x_i, x_j)]_{i,j=1}^{n}.$$

Compact uniform convergence of the correlation functions (1.12) immediately yields

$$\lim_{N\to\infty} \mu_{\sin,2,\theta}^{N} = \mu_{\sin,2,\theta} \quad \text{weakly.} \tag{1.13}$$

The convergence (1.13) shows that the bulk scaling limit has the universal random point field limit up to the density.

Once universality for random point fields is established, it is natural to ask what is a dynamical counterpart to it: a natural N-particle dynamics associated with $\check{\mu}_{\sin,2,\theta}^{N}$ is the following SDE.

$$dX_t^{N,i} = dB_t^i + \sum_{j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{1}{N} X_t^{N,i}\, dt - \theta\, dt, \quad 1 \le i \le N. \tag{1.14}$$

Actually, the relation between (1.11) and (1.14) is as follows. We first consider the Dirichlet form on $L^2(\mathbb{R}^N, \check{\mu}_{\sin,2,\theta}^{N})$ such that

$$\mathcal{E}^{\check{\mu}_{\sin,2,\theta}^{N}}(f,g) = \int_{\mathbb{R}^N} \frac{1}{2} \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}\, \check{\mu}_{\sin,2,\theta}^{N}(d\mathbf{x}_N).$$

Integration by parts and (1.11) yield a representation of the generator $L^{\check{\mu}_{\sin,2,\theta}^{N}}$ of $\mathcal{E}^{\check{\mu}^N}$ as follows:

$$L^{\check{\mu}_{\sin,2,\theta}^{N}} = \frac{1}{2}\Delta + \sum_{i=1}^{N} \Big\{ \sum_{j;\, j\neq i}^{N} \frac{1}{x_i - x_j} \Big\} \frac{\partial}{\partial x_i} - \sum_{i=1}^{N} \Big\{ \frac{x_i}{N} + \theta \Big\} \frac{\partial}{\partial x_i},$$

which corresponds the SDE (1.14). In other words, a distorted Brownian motion with respect to $\check{\mu}_{\sin,2,\theta}^{N}$ is described as (1.14).

Taking the limit $N \to \infty$ in (1.14), we intuitively expect that a limit ISDE is given by

$$dX_t^i = dB_t^i + \sum_{j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j}\, dt - \theta\, dt, \quad i \in \mathbb{N}. \tag{1.15}$$

However, this intuition fails when $\theta \neq 0$. In fact, a limit ISDE is not (1.15) but the Dyson Brownian motion in infinite dimension for $\beta = 2$, which is described as

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j| < r} \frac{1}{X_t^i - X_t^j}\, dt, \quad i \in \mathbb{N}. \tag{1.16}$$

Therefore the intuitive limit (1.15) and the correct limit (1.16) are different (SDE gap). We expect this SDE gap from the fact that the ISDE (1.16) is the distorted Brownian motion with respect to the sine_2 random point field.

This phenomena is special to long-range correlated systems. Because of the logarithmic interaction, the summation in the drift term in (1.14) does not converge absolutely when N goes to infinity. The limit transition for logarithmic correlated systems is thus a sensitive problem, and we have to consider cancellation for interaction to control the interaction term. A fine estimate shows that the tail part of the interaction term is exactly θ.

Precisely we obtain the above SDE gap as follows. Let $\mathfrak{l}_N$ and $\mathfrak{l}$ be labeling maps. We write $\mathfrak{l}_{N,m}$ and $\mathfrak{l}_m$ as the first m-components of $\mathfrak{l}_N$ and $\mathfrak{l}$ respectively. We assume that for each $m \in \mathbb{N}$

$$\lim_{N\to\infty} \mu_{\sin,2,\theta}^N \circ \mathfrak{l}_{N,m}^{-1} = \mu_{\sin,2,\theta} \circ \mathfrak{l}_m^{-1} \text{ weakly.} \tag{1.17}$$

Let $\mathbf{X}^{\theta,N} = (X^{\theta,N,i})_{i=1}^N$ be a solution of the SDE (1.14) and $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ be a solution of the ISDE (1.16). Assume that $\mathbf{X}_0^{\theta,N} = \mu_{\sin,2,\theta}^N \circ \mathfrak{l}_N^{-1}$ and $\mathbf{X}_0 = \mu_{\sin,2,\theta} \circ \mathfrak{l}^{-1}$ in distribution in addition to (1.9) and (1.17). Then we have for each $m \in \mathbb{N}$

$$\lim_{N\to\infty} (X^{\theta,N,1}, X^{\theta,N,2}, \dots, X^{\theta,N,m}) = (X^1, X^2, \dots, X^m) \tag{1.18}$$

weakly in $C([0,\infty);\mathbb{R}^m)$.

Motivated by (1.18), we established a general theorem of finite particle approximation, which we call the first approximation theorem. An essential assumption for the first theorem is the uniqueness of a solution for an ISDE. The uniqueness was proved for typical ISDEs [53, 76]. Another main assumption is convergence of a drift term in finite-dimensional SDEs. In particular, uniform control of a tail part of an interaction term is crucial.

The first approximation theorem does not depend on the dimension which particles are moving on, inverse temperature, and integrable structure. Thus it is applicable to many other examples related to random matrices, one of which we explain below.

Consider a soft-edge scaling limit rather than the bulk scaling limit, which yields the other microscopic statistics of Gaussian ensembles. The soft-edge scaling $x \mapsto y$ is given by

$$x = \frac{y}{\sqrt{2}N^{\frac{1}{6}}} + \sqrt{2N}. \tag{1.19}$$

The scaling (1.19) means the centre point for the scaling limit is the right edge of the semicircle. Substituting (1.19) to (1.7), we have

$$\check{\mu}_{\text{Airy},2}^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^N |x_i - x_j|^2 \exp\Big\{-\sum_{k=1}^N \Big|\frac{x_k}{\sqrt{2}N^{1/6}} + \sqrt{2N}\Big|^2\Big\} d\mathbf{x}_N.$$

The soft-edge scaling (1.19) means that we focus on the right edge of the semicircle law.

Let $\mu_{\text{Airy},2}^N$ be the random point field with N-particles whose labeled density is $\check{\mu}_{\text{Airy},2}^N$, and $\rho_{\text{Airy},2}^{N,n}$ be the n-correlation function for $\mu_{\text{Airy},2}^N$. Let Airy$_2$ random point field $\mu_{\text{Airy},2}$ be the determinantal random point field with the Airy kernel

$$\mathsf{K}_{\text{Airy}}(x,y) = \frac{\text{Ai}'(x)\text{Ai}(y) - \text{Ai}(x)\text{Ai}'(y)}{x-y}, \tag{1.20}$$

where $\text{Ai}(x)$ is the Airy function such that

$$\text{Ai}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\Big\{i\Big(xk + \frac{k^3}{3}\Big)\Big\}\, dk$$

and $\mathrm{Ai}'(x) = d\mathrm{Ai}(x)/dx$. We set n-correlation function $\rho^n_{\mathrm{Airy},2}$ for $\mu_{\mathrm{Airy},2}$, then by definition

$$\rho^n_{\mathrm{Airy},2} = \det[\mathsf{K}_{\mathrm{Airy}}(x_i, x_j)]^n_{i,j=1}.$$

Then for any n

$$\lim_{N\to\infty} \rho^{N,n}_{\mathrm{Airy},2} = \rho^n_{\mathrm{Airy},2} \quad \text{compact uniformly.} \tag{1.21}$$

From (1.21) we obtain

$$\lim_{N\to\infty} \mu^N_{\mathrm{Airy},2} = \mu_{\mathrm{Airy},2} \quad \text{weakly.}$$

Airy$_2$-random point field has only finite particles on the positive line in $\mathbb{R}$ and the most right particle is Tracy-Widom distributed.

Observe an ISDE associated with Airy$_2$ random point field. Using the same argument which derives (1.14), we obtain an N-particle dynamics associated with $\check{\mu}^N_{\mathrm{Airy},2}$ as follows:

$$dX^{N,i}_t = dB^i_t + \sum_{j=1,\, j\neq i}^{N} \frac{1}{X^{N,i}_t - X^{N,j}_t} dt - \Big\{ N^{1/3} + \frac{1}{2N^{1/3}} X^{N,i}_t \Big\} dt. \tag{1.22}$$

By taking a limit of (1.22) as N to infinity, we may obtain a form of an ISDE associated with Airy$_2$ random point field. However, we cannot easily do the limit transition because the SDE (1.22) has the divergence term.

It is known that an ISDE associated with Airy$_2$ random point field is the following:

$$dX^i_t = dB^i_t + \lim_{r\to\infty} \Big\{ \sum_{|X^j_t|<r, j\neq i} \frac{1}{X^i_t - X^j_t} - \int_{|x|<r} \frac{\varrho(x)}{-x}\, dx \Big\} dt, \quad i \in \mathbb{N}, \tag{1.23}$$

where we recall $\varrho(x) = 1_{(-\infty,0)}(x)\sqrt{-x}$. Once the ISDE related to Airy$_2$ random point field is founded, then it is natural to ask a relation between (1.22) and (1.23). The first approximation theorem also gives a proof of the limit transition from (1.22) to (1.23).

We shall construct the first approximation theorem in Section 2 building upon [28]. Combining this general theorem and concrete calculation using determinantal structure, we prove the SDE gap in Section 3, which is based on [29].

1.3 Second finite particle approximation theorem and dynamical universality for random matrices

The convergence (1.13) is weak universality result in the sense that the limit random point field is sine random point field, which is independent of a bulk position θ in the semicircle. More strongly, it is believed as a universality conjecture for random matrices that sine and Airy random point field are universal and appear as scaling limits for wide class of models more than Gaussian ensembles. Universality for random matrices has been studied intensively in the last two decades.

An $N \times N$ Hermitian matrix M^N is called Wigner (Hermitian) ensemble if M^N is of the form

$$M^N_{i,j} = \begin{cases} \xi_i & \text{if } i = j \\ \tau_{i,j}/\sqrt{2} + \sqrt{-1}\tilde{\tau}_{i,j}/\sqrt{2} & \text{if } i < j, \end{cases}$$

where $\{\xi_i, \tau_{i,j}, \tilde{\tau}_{i,j}\}_{i<j}^{\infty}$, each of which is called atom distribution, are i.i.d. random variables with mean zero and unit variance. When M_N is real symmetric, it is called Wigner symmetric ensemble. In particular, Wigner Hermite ensemble is nothing but the GUE when atom distribution is Gaussian. Consider eigenvalue distribution of M^N. Unlike Gaussian ensembles, eigenvalue distribution does not have explicit formulae generally. However, it is known as a classical result that the Wigner semicircle convergence (1.8) holds for any Wigner ensemble.

Hereafter we focus only on Hermitian ensemble for simplicity. The Wigner semicircle convergence shows that macroscopic statistics of Wigner ensembles is the same as that of Gaussian ensembles. Then it is a natural question to ask what about microscopic statistics of Wigner ensembles. Universality for Wigner Hermite ensembles asserts that microscopic statistics is independent of details of atom distributions, that is, bulk scaling limit and soft-edge scaling limit for Wigner Hermite ensemble gives sine_2 and Airy_2 random point field respectively under some moment condition for atom distributions. More precisely, let $\rho^{N,n}$ be the n-correlation function for eigenvalue distribution of $N \times N$ Wigner ensemble. Take the bulk scaling same as (1.9) and (1.10), and define the bulk scaled correlation function $\rho^{N,n}_{\sin,2,\theta}$ as

$$\rho^{N,n}_{\sin,2,\theta}(x_1,\dots,x_n) = \frac{1}{(\sqrt{N}\rho_{\mathrm{sc}}(\theta))^n}\rho^{N,n}\left(\frac{x_1}{\sqrt{N}\rho_{\mathrm{sc}}(\theta)} + \sqrt{N}\theta, \dots, \frac{x_n}{\sqrt{N}\rho_{\mathrm{sc}}(\theta)} + \sqrt{N}\theta\right).$$

The bulk universality for Wigner ensembles was conjectured in the sense that if $\theta \in (-\sqrt{2}, \sqrt{2})$ the following holds:

$$\lim_{N\to\infty} \rho^{N,n}_{\sin,2,\theta} = \rho^n_{\sin,2} \tag{1.24}$$

for any $n \in \mathbb{N}$, where

$$\rho^n_{\sin,2}(x_1,\dots,x_n) = \det[\mathsf{K}_{\sin}(x_i,x_j)]^n_{i,j=1} \tag{1.25}$$

and $\mathsf{K}_{\sin}$ is the sine kernel

$$\mathsf{K}_{\sin}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}.$$

Here $\rho^n_{\sin,2}$ does not depend on atom distribution and a bulk position θ.

Edge universality for Wigner ensembles is formulated in a similar way. Recalling edge scaling (1.19), define the edge scaled correlation function $\rho^{N,n}_{\mathrm{Airy},2}$ as

$$\rho^{N,n}_{\mathrm{Airy},2}(x_1,\dots,x_n) = \frac{1}{(\sqrt{2}N^{\frac{1}{6}})^n}\rho^{N,n}\Big(\frac{x_1}{\sqrt{2}N^{\frac{1}{6}}} + \sqrt{2N}, \dots, \frac{x_n}{\sqrt{2}N^{\frac{1}{6}}} + \sqrt{2N}\Big). \tag{1.26}$$

Then the edge universality conjecture asserts that for any $n \in \mathbb{N}$

$$\lim_{N\to\infty} \rho^{N,n}_{\mathrm{Airy},2}(x_1,\dots,x_n) = \det[\mathsf{K}_{\mathrm{Airy}}(x_i,x_j)]^n_{i,j=1}, \tag{1.27}$$

where the Airy kernel $\mathsf{K}_{\mathrm{Airy}}$ is defined by (1.20).

In summary the universality conjecture for Wigner ensembles is that microscopic statistics for Wigner ensembles is described as the sine or Airy kernel (equivalently sine_2 or Airy_2 random point field) in the sense of (1.24) and (1.27) according to scaling. As we see (1.12) and (1.21), (1.24) and (1.27) in compact uniform sense hold for Gaussian ensembles. However, (1.24) and (1.27) in compact uniform sense are nonsense generally. The reason is that unlike in the case of Gaussian ensembles, correlation functions may be not functions but distributions as atom distributions are allowed to be discrete random variables. Then unless otherwise noted, universality means that weak convergence of correlation functions holds in this section.

The bulk universality for Wigner ensembles was solved for Gaussian divisible ensembles first. For $t > 0$ an Hermite ensemble M_t which is of the form

$$M_t^N = e^{-\frac{t}{2}} M^N + \sqrt{1 - e^{-t}} \tilde{M}^N$$

is called a Gaussian divisible ensemble, where M^N is the Wigner ensemble and $\tilde{M}^N$ is the GUE independent of M^N. Johansson proved the bulk universality (1.24) for Gaussian divisible ensembles for fixed t [20]. His work was followed by Erdős, Péché, Ramírez, Schlein, and Yau [12]. They extended Johansson's result to Gaussian divisible ensembles for small t depending on N. These works heavily rely on explicit formula of the correlation function for eigenvalue distribution of Gaussian divisible ensembles, which is followed from the Harish-Chandra-Itzykson-Zuber formula.

Simultaneously the soft-edge universality for Wigner ensembles has been studied. The first breakthrough in this area was done by Soshnikov. He proved the soft-edge universality for the Wigner ensembles with symmetric atom distributions [66], and followed by Péché and Soshnikov [57].

One of recent progress has developed by Erdős, Schlein, Yau, Yin, and others. Their idea is reduction from universality problem to analysis of the Dyson Brownian motion (in finite dimensions), and their approach is called a dynamical approach. They proved the bulk and soft-edge universality if atom distributions satisfy subexponential decay [5, 7] (actually their results are stronger and show universality for generalized Wigner ensembles; see for example [5, 7, 13] for the details). An important fact is that eigenvalue distribution of Gaussian divisible ensembles M_t corresponds the distribution of the Dyson Brownian motion with N-particles at time t whose initial distribution is the eigenvalue distribution of M^N. An equilibrium distribution with respect to the Dyson Brownian motion is clearly the eigenvalue distribution for Gaussian ensemble given by (1.7). One of the key steps of the dynamical approach is to estimate how fast the Dyson Brownian motion reaches the equilibrium state. They showed that the dynamics reaches equilibrium for sufficiently short time. After the relaxation time, microscopic statistics of eigenvalue distribution for M_t^N is close to that of the GUE. Furthermore we can see that microscopic statistics of eigenvalue distribution for M_t^N and M^N are the same for large N around the relaxation time, using the relaxation time is sufficiently short. Therefore microscopic statistics for M^N is the same as that of the GUE for large N. See [13] and references therein for more details and history.

Tao and Vu proved the universality for the Wigner ensembles under some moment conditions by a different method. The key result of their approach is four moment theorem, which asserts that for two Wigner matrices, if their atom distributions have same moment up to fourth, then their microscopic statistics correspond. As a result of the theorem, the bulk universality for the Wigner ensembles holds when atom distributions are exponential decay and have at least three points as the support [73], and the result was improved in [74]. Additionally four moment theorem yields the soft-edge universality when the atom distributions are exponential decay and the third moment of the atom distribution is vanish [72]. The main idea to prove four moment theorem is the Lindeberg swapping strategy. Some quantities which we have to estimate for carrying out the swapping strategy are sensitive to eigenvalues being close. Thus it is important to estimate for gap probability between consecutive eigenvalues. Combining gap probability with Hadamard variation formula, eigenvector delocalisation, and so on, they established four moment theorem.

Universality for log-gases has been also studied. Let $V : \mathbb{R} \to \mathbb{R}$ and consider the following log-gas with inverse temperature $\beta > 0$ with N-particles:

$$\check{\mu}_{\beta,V}^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i<j}^N |x_i - x_j|^\beta \prod_{k=1}^N e^{-\frac{\beta}{2} NV(x_k)}\, d\mathbf{x}_N. \tag{1.28}$$

We can recognize V as free potential. When $\beta \in \{1, 2, 4\}$, $\mu_{\beta,V}^N$ corresponds eigenvalue distribution for some invariant random matrix ensemble, and additionally for V is quadratic, $\mu_{\beta,V}^N$ corresponds eigenvalue distribution for Gaussian ensembles. However, we remark that we put N factor in the exponential in (1.28) for convenience, although there is no N factor in (1.7). Then if $\beta \in \{1, 2, 4\}$, we call β classical value, and $\mu_{\beta,V}^N$ classical ensemble. There is no natural matrix model corresponding $\mu_{\beta,V}^N$ for non-classical β except for Gaussian case, that is, V is quadratic.

Assuming a suitable condition for V, there exists a probability density function ρ_V with compact support such that for empirical measure $\mathsf{x}^N = \sum_{i=1}^N \delta_{x_i}$

$$\lim_{N\to\infty} \mathbb{E}_{\mu_{V,\beta}^N}\Big[\frac{1}{N}\mathsf{x}^N((-\infty, s])\Big] = \int_{-\infty}^s \rho_V(x)\, dx.$$

For example, it is enough for analytic V satisfying

$$\lim_{|x|\to\infty} \frac{V(x)}{\log|x|} = \infty, \tag{1.29}$$

and we assume these conditions for simplicity in this section. We call ρ_V an equilibrium measure with respect to $\mu_{\beta,V}^N$. When V is quadratic, ρ_V is nothing but the Wigner semicircle distribution.

Unlike Wigner ensembles, macroscopic statistics ρ_V is not universal and depends on free potential V. However, it is believed that microscopic statistics for log-gases is universal and depends only on β, especially independent of V. Then to consider microscopic statistics take a bulk scaling limit in ρ_V. Fix $\theta \in \mathbb{R}$ satisfying

$$\rho_V(\theta) > 0 \tag{1.30}$$

and take a bulk scaling such that

$$x = \frac{y}{N\rho_{\mathrm{V}}(\theta)} + \theta. \tag{1.31}$$

A bulk scaled measure of (1.28) with respect to (1.31) is the following:

$$\check{\mu}^N_{\sin,\beta,V,\theta}(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^N |x_i - x_j|^\beta \prod_{k=1}^N \exp\Big(-\frac{\beta}{2} NV\Big(\frac{x_k}{N\rho_{\mathrm{V}}(\theta)} + \theta\Big)\Big)\, d\mathbf{x}_N.$$

Define $\mu^N_{\sin,\beta,V,\theta}$ as the random point field whose density is given by $\check{\mu}^N_{\sin,\beta,V,\theta}$. Let $\rho^{N,n}_{\sin,\beta,V,\theta}$ be the n-correlation function with respect to $\mu^N_{\sin,\beta,V,\theta}$, that is,

$$\rho^{N,n}_{\sin,\beta,V,\theta}(x_1,\dots,x_n) = \frac{1}{(N\rho_V(\theta))^n}\rho^{N,n}_{\beta,V}\Big(\frac{x_1}{N\rho_V(\theta)} + \theta,\dots,\frac{x_n}{N\rho_V(\theta)} + \theta\Big),$$

where $\rho^{N,n}_{\beta,V}$ is the n-correlation function with respect to $\mu^N_{\beta,V}$. The bulk universality for log-gases asserts that for any free potential V in a wide class and any θ satisfying (1.30),

$$\lim_{N\to\infty} \rho^{N,n}_{\sin,\beta,V,\theta} = \rho^n_{\sin,\beta}. \tag{1.32}$$

Here $\rho^n_{\sin,\beta}$ is given by the same determinant of the sine kernel as in (1.25) when $\beta = 2$. When $\beta \in \{1,4\}$, $\rho^n_{\sin,\beta}$ is given by some (quaternion) determinant using the sine kernel, but more complicated. For general β, $\rho^n_{\sin,\beta}$ is described in terms of some stochastic process, which was introduced in [77], but there is no explicit formula. In any case, the limit $\{\rho^n_{\sin,\beta}\}_{n\in\mathbb{N}}$ is universal in the sense that it depends only on β and independent of V and θ.

Soft-edge universality is formulated through correlation functions like (1.32):

$$\lim_{N\to\infty} \rho^{N,n}_{\mathrm{Airy},\beta,V} = \rho^n_{\mathrm{Airy},\beta} \tag{1.33}$$

We omit precise definition for the soft-edge scaled correlation function $\rho^{N,n}_{\mathrm{Airy},\beta,V}$ because it is similar to (1.26).

Studies on the universality for log-gases were begun for classical ensemble $\beta \in \{1,2,4\}$. In the case of classical ensembles, correlation functions have explicit expression in terms of orthogonal polynomials. Therefore universality results boil down to asymptotic analysis of orthogonal polynomials, where we can use helpful tools, for example the Christoffel-Darboux formula, Riemann-Hilbert approach, and so on. Pastur and Shcherbina proved bulk universality for $\beta = 2$ [55]. Deift and his collaborators showed bulk universality when $\beta \in \{1,2,4\}$ [10, 11]. Soft-edge universality was also proven for $\beta \in \{1,2,4\}$ in [9]. Other than this, there are a lot of results for classical ensembles, for example see [3, 39, 40, 41, 56, 62, 63]. We should remark that although it is restricted to classical ensembles, these approach gives strong convergence in the sense that (1.32) or (1.33) hold compact uniformly rather than weak convergence.

For non-classical $\beta \notin \{1,2,4\}$, it is difficult to address universality problems because there is no explicit formula of correlation function. Regardless of the lack of explicit formula, the universality for general β log-gases were rapidly established recently.

The dynamical approach, which was used to prove the universality for the Wigner ensembles mentioned above, was improved so that it can apply to log-gases. Because the dynamical approach does not rely on explicit formulae, then we can analyse log-gases for general β. Actually Bourgade, Erdős, and Yau proved the bulk and soft-edge universality for wide class of V for general β [4, 5, 6]. Shcherbina also showed bulk universality for general β log-gases by a different method [64]. Furthermore, Krishnapur, Rider, and Virág proved the soft-edge universality for general β by using random operator [32].

Other than this, universality for non-Hermitian random matrices has been studied. In this case, one of universal microscopic statistics is Ginibre random point field, which is an equilibrium state for Ginibre interacting ISDE. We will skip details for this model, see [1, 75].

Our goal is to establish dynamical universality as the same motivation in Section 1.2. Consider dynamical universality corresponding to (1.32) which is based on bulk universality results in [10, 11]. Let $\beta \in \{1, 2, 4\}$. Assume V is analytic and satisfy (1.29) for $\beta = 2$ and V is even polynomial for $\beta = 1, 4$. Then (1.32) holds compact uniformly.

Doing the same procedure as in Section 1.2, the N-particle dynamics associated with $\mu^N_{\beta,V,\theta}$ is given by the following:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{1 \le j \ne i \le N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{4\rho_V(\theta)} V'\Big(\frac{X_t^{N,i}}{N\rho_V(\theta)} + \theta\Big) dt, \quad 1 \le i \le N. \tag{1.34}$$

For the very same reason as in Section 1.2, we expect that a solution for (1.34) converges as N to infinity to not an informal limit ISDE given by

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{1 \le j \ne i \le \infty} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{4\rho_V(\theta)} V'(\theta) dt, \quad i \in \mathbb{N}, \tag{1.35}$$

but the Dyson Brownian motion given by

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \to \infty} \sum_{j \ne i,\, |X_t^i - X_t^j| < r} \frac{1}{X_t^i - X_t^j} \, dt, \quad i \in \mathbb{N}. \tag{1.2}$$

We see that (1.2) does not depend on V and θ, and depend only on β. Thus we can say that the Dyson Brownian motion is a universal dynamical object. From the relation between random point field and dynamics, we believe the dynamics inherit universal property from the universality for random matrices.

In fact, we can prove it rigorously. Let $\mathbf{X}^{\beta,V,\theta,N} = (X^{\beta,V,\theta,N,1}, X^{\beta,V,\theta,N,2}, \dots, X^{\beta,V,\theta,N,N})$ be a solution for (1.34) satisfying $\mathbf{X}_0^{\beta,V,\theta,N} = \mu^N_{\beta,V,\theta} \circ \mathfrak{l}_N^{-1}$ in distribution and $\mathbf{X}^\beta = (X^{\beta,1}, X^{\beta,2}, \dots,)$ be a solution for (1.2) satisfying $\mathbf{X}_0^\beta = \mu_{\sin,\beta} \circ \mathfrak{l}^{-1}$ in distribution. Suppose that labeling maps $\mathfrak{l}_N$ and $\mathfrak{l}$ satisfy for each $m \in \mathbb{N}$

$$\lim_{N \to \infty} \mu^N_{\beta,V,\theta} \circ \mathfrak{l}_{N,m}^{-1} = \mu_{\sin,\beta} \circ \mathfrak{l}_m^{-1} \quad \text{weakly.}$$

Then we obtain the following dynamical universality for $\beta \in \{1, 2, 4\}$: for each $m \in \mathbb{N}$

$$\lim_{N \to \infty} (X^{\beta,V,\theta,N,1}, X^{\beta,V,\theta,N,2}, \dots, X^{\beta,V,\theta,N,m}) = (X^{\beta,1}, X^{\beta,2}, \dots, X^{\beta,m}) \tag{1.36}$$

weakly in $C([0, \infty); \mathbb{R}^m)$.

One may prove (1.36) by the first approximation theorem, but it is troublesome because we have to estimate the drift term in (1.34). This control of the drift term is sensitive because of logarithmic correlated, and we have to calculate exact cancellation between the interaction term and the second drift term in (1.35) involving V'. Therefore to prove (1.36) in this way requires model dependent argument.

To avoid the problem, we constructed the second approximation theorem, which can prove (1.36) easily. Although it works only for symmetric dynamics whereas the first approximation theorem can be applied to non-symmetric one, the second approximation theorem does not require sensitive estimates. Furthermore once universality of random point field is established, the second approximation theorem accordingly deduces dynamical universality automatically under mild conditions, and model dependent argument is not needed. Therefore, the second approximation theorem yields dynamical universality for the Dyson Brownian motion (1.36). As the second approximation theorem is based on the Dirichlet form approach, then it applicable to Airy_2 interacting ISDE, Ginibre interacting ISDE, and others.

The second approximation theorem essentially needs two conditions. The first essential assumption is the uniqueness of Dirichlet forms associated with μ. There exist two natural Dirichlet forms associated with μ, the upper Dirichlet form and the lower Dirichlet form. Correspondence of such two Dirichlet forms is one of the main assumptions for the second finite particle approximation theorem. We prove that a sufficient condition for the uniqueness of Dirichlet forms is the uniqueness of a solution of the ISDE associated with μ.

The second main assumption is compact uniform convergence of correlation functions for random point field, which is stronger than only weak convergence. As stated, there are a lot of compact uniform convergence results of log-gases for classical values, then we can lift their geometrical universality to dynamical one. Universality results for the Wigner ensembles and general β log-gases is establish in weak convergence, but there are no results of compact uniform convergence yet. Then once their results improve to compact uniform convergence, it immediately derives dynamical universality when the uniqueness of a solution for a corresponding ISDE is established.

Uniqueness of Dirichlet forms is proven in Section 5, based on [31]. Section 6 follows from [30], where we construct the second approximation theorem, and show examples of dynamical universality .

1.4 Density preservation property for interacting Brownian motions

We are interested in the tail preservation property for interacting Brownian motions in infinite dimension. Recall that one of main assumptions for both the first and second approximation theorem is the uniqueness of a solution for ISDEs. Osada and Tanemura established a general framework for the uniqueness of a solution for ISDEs using a property in terms of the tail σ-field, and they revealed that the tail preservation property plays an essential role [53].

Let S be all of configurations on $\mathbb{R}^d$ without accumulation point (configuration space). Let μ be a random point field on $\mathbb{R}^d$ with infinitely many particles, that is, μ is a probability measure on $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$, where the Borel σ-field $\mathcal{B}(\mathsf{S})$ is induced by the vague topology. A sub σ-field of $\mathcal{B}(\mathsf{S})$ which contains only global information about configurations is called the tail σ-field. A random point field μ is called tail trivial if μ is an trivial probability measure with respect to the tail σ-field. Consider a μ-reversible diffusion (X, P) with state space S. Here $\mathsf{X} = \{\mathsf{X}_t\}$ is of the form $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$ and $\mathsf{P} = \{\mathsf{P}_\mathsf{s}\}_{\mathsf{s}\in\mathsf{S}}$ is the diffusion measure.

Suppose (X, P) has an ISDE representation in the sense that labeled dynamics $(X^1, X^2, \ldots)$ solves an ISDE. Then under mild conditions the ISDE has a unique solution if μ is tail trivial. It is also known that determinantal random point fields are tail trivial, which implies that sine_2, Airy_2, and Ginibre random point field are tail trivial [50]. Therefore the uniqueness holds for sine_2, Airy_2, and Ginibre interacting Brownian motion.

In addition, they also discussed the uniqueness of a solution of an ISDE when a random point field is not tail trivial. In this case, the random point field has multiple tails. They proved that a sufficient condition for the uniqueness for an ISDE with respect to μ with multiple tails is the tail preservation property: an unlabeled diffusion which starts on an element of the tail σ-field, stays on the set permanently. However, they could not prove unlabeled diffusion having such property. The tail σ-field is not topologically well behaved: for example, it is not countably determined in general even if the state space is countably determined. Consequently, it is hard to treat the tail σ-field. Is was offered as an open question whether an unlabeled diffusion has the tail preservation property in [53].

We solve this problem in part. Suppose that for μ-a.s. $\mathsf{s} \in \mathsf{S}$, there exists a limit $\lim_{r\to\infty} \mathsf{s}(S_r)/r^d$, where $S_r = \{x \in \mathbb{R}^d \,;\, |x| < r\}$, and let

$$\Phi(\mathsf{s}) = \lim_{r\to\infty} \frac{\mathsf{s}(S_r)}{r^d}.$$

As $\mathsf{s}(S_r)$ is the number of particles on S_r, $\Phi(\mathsf{s})$ describes the density of particles. The limit exists, for example, for a translation invariant random point field like sine or Ginibre random point field. For a fixed positive constant θ, we set $\mathsf{A}_\theta = \{\mathsf{s}\,;\, \Phi(\mathsf{s}) = \theta\}$. Note that the set A_θ is an element of the tail σ-field of S.

From the reversibility of (X, P), we immediately obtain

$$\mathsf{P}_\mu\Big(\lim_{r\to\infty} \frac{\mathsf{X}_t(S_r)}{r^d} = \theta \Big) = \mu(\mathsf{A}_\theta) \ \text{ for any } t. \tag{1.37}$$

We refined (1.37) such that for q.e. $\mathsf{s} \in \mathsf{A}_\theta$,

$$\mathsf{P}_\mathsf{s}\Big(\lim_{r\to\infty} \frac{\mathsf{X}_t(S_r)}{r^d} = \theta \ \text{ for any } t \Big) = 1.$$

In other words, we prove that an unlabeled diffusion starting on a set that is specified in terms of density does not change the density over the course of its time evolution. If the tail σ-field is identified by particle densities, we can discuss the behaviour of an unlabeled diffusion by studying the density instead of the field itself. Then, in some cases the tail preservation property follows from the preservation of density.

This result is intimately related to an ergodic decomposition of unlabeled diffusions. We believe that the ergodic components is given by the tail σ-field. However, because the space of an unlabeled diffusion is huge, it is difficult problem to specify the topological support when infinitely many particles are in motion. Our result is a first step toward addressing this problem.

In Section 4 we prove the density preservation property, based on [27].

2 Finite-particle approximations for interacting Brownian particles with logarithmic potentials

2.1 Introduction

Interacting Brownian motion in infinite dimensions is prototypical of diffusion processes of infinitely many particle systems, initiated by Lang [37, 38], followed by Fritz [15], Tanemura [70], and others. Typically, interacting Brownian motion $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ with Ruelle-class (translation invariant) interaction Ψ and inverse temperature $\beta \geq 0$ is given by

$$dX_t^i = dB_t^i - \frac{\beta}{2}\sum_{j;j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j)dt \quad (i \in \mathbb{N}). \tag{2.1}$$

Here an interaction Ψ is called Ruelle-class if Ψ is super stable in the sense of Ruelle, and integrable at infinity [61].

The system $\mathbf{X}$ is a diffusion process with state space $\mathbf{S}_0 \subset (\mathbb{R}^d)^{\mathbb{N}}$, and has no natural invariant measures. Indeed, such a measure $\check{\mu}$, if exists, is informally given by

$$\check{\mu} = \frac{1}{\mathcal{Z}} e^{-\beta\sum_{(i,j);\,i<j}^{\infty}\Psi(x_i - x_j)} \prod_{k\in\mathbb{N}} dx_k, \tag{2.2}$$

which cannot be justified as it is because of the presence of an infinite product of Lebesgue measures. To rigorize the expression (2.2), the Dobrushin–Lanford–Ruelle (DLR) framework introduces the notion of a Gibbs measure. A point process μ is called a Ψ-canonical Gibbs measure if it satisfies the DLR equation: for each $m \in \mathbb{N}$ and μ-a.s. $\xi = \sum_i \delta_{\xi_i}$

$$\mu_{r,\xi}^m(d\mathsf{s}) = \frac{1}{\mathcal{Z}_{r,\xi}^m} e^{-\beta\{\sum_{i<j,\,s_i,s_j\in S_r}^m \Psi(s_i - s_j) + \sum_{s_i\in S_r,\xi_j\in S_r^c}^m \Psi(s_i - \xi_j)\}} \prod_{k=1}^m ds_k, \tag{2.3}$$

where $\mathsf{s} = \sum_i \delta_{s_i}$, $S_r = \{|x| \leq r\}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$, and ξ is the outer condition. Furthermore, $\mu_{r,\xi}^m$ denotes the regular conditional probability:

$$\mu_{r,\xi}^m(d\mathsf{s}) = \mu(\pi_r(\mathsf{s}) \in d\mathsf{s}|\, \mathsf{s}(S_r) = m,\, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi)).$$

Then μ is a reversible measure of the delabeled dynamics X such that $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$.

If the number of particles is finite, N say, then SDE (2.1) becomes

$$dX_t^{N,i} = dB_t^i - \frac{\beta}{2}\{\nabla\Phi^N(X_t^{N,i}) + \sum_{j;j\neq i}^{N} \nabla\Psi(X_t^{N,i} - X_t^{N,j})\}dt \quad (1 \leq i \leq N), \tag{2.4}$$

where Φ^N is a confining free potential vanishing zero as N goes to infinity. The associated labeled measure is then given by

$$\check{\mu}^N = \frac{1}{\mathcal{Z}} e^{-\beta\{\sum_{i=1}^N \Phi^N(x_i) + \sum_{(i,j);\,i<j}^N \Psi(x_i - x_j)\}} \prod_{k=1}^N dx_k. \tag{2.5}$$

The relation between (2.4) and (2.5) is as follows. We first consider the diffusion process associated with the Dirichlet form with domain $\mathcal{D}^{\check{\mu}^N}$ on $L^2((\mathbb{R}^d)^N, \check{\mu}^N)$, called the distorted Brownian motion, such that

$$\mathcal{E}^{\check{\mu}^N}(f,g) = \int_{(\mathbb{R}^d)^N} \frac{1}{2}\sum_{i=1}^N \nabla_i f \cdot \nabla_i g \, \check{\mu}^N(d\mathbf{x}_N),$$

where $\nabla_i = (\frac{\partial}{\partial x_{ij}})_{j=1}^d$, $\mathbf{x}_N = (x_1,\dots,x_N) \in (\mathbb{R}^d)^N$, and $\cdot$ denotes the inner product in $\mathbb{R}^d$. The generator $L^{\check{\mu}^N}$ of $\mathcal{E}^{\check{\mu}^N}$ is then given by

$$\mathcal{E}^{\check{\mu}^N}(f,g) = (-L^{\check{\mu}^N} f, g)_{L^2((\mathbb{R}^d)^N, \check{\mu}^N)}.$$

Integration by parts yields the representation of the generator of the diffusion process such that

$$L^{\check{\mu}^N} = \frac{1}{2}\Delta - \frac{\beta}{2}\sum_{i=1}^N \{\nabla\Phi^N(x_i) + \sum_{j;\, j\neq i}^N \nabla\Psi(x_i - x_j)\} \cdot \nabla_i,$$

which together with Itô formula yields SDE (2.4).

For a finite or infinite sequence $\mathbf{x} = (x_i)$, we set $\mathfrak{u}(\mathbf{x}) = \sum_i \delta_{x_i}$ and call $\mathfrak{u}$ a delabeling map. For a point process μ, we say a measurable map $\ell = \ell(\mathsf{s})$ defined for μ-a.s. s with value $S^\infty \cup \{\bigcup_{N=1}^\infty S^N\}$ is called a label with respect to μ if $\mathfrak{u} \circ \ell(\mathsf{s}) = \mathsf{s}$. Let ℓ_N be a label with respect to μ^N. We denote by ℓ_m and $\ell_{N,m}$ the first m-components of these labels, respectively. We take Φ^N such that the associated point process $\mu^N = \check{\mu}^N \circ \mathfrak{u}^{-1}$ converges weakly to μ:

$$\lim_{N\to\infty} \mu^N = \mu \quad \text{weakly.} \tag{2.6}$$

The associated delabeling $\mathsf{X}^N = \sum_{i=1}^N \delta_{X^{N,i}}$ is reversible with respect to μ^N. The labeled process $\mathbf{X} = (X^i)$ and $\mathbf{X}^N = (X^{N,i})$ can be recovered from X and X^N by taking suitable initial labels ℓ and ℓ_N, respectively. Choosing the labels in such a way that for each $m \in \mathbb{N}$

$$\lim_{N\to\infty} \mu^N \circ \ell_{N,m}^{-1} = \mu \circ \ell_m^{-1} \quad \text{weakly,} \tag{2.7}$$

we have the convergence of labeled dynamics $\mathbf{X}^N$ to $\mathbf{X}$ such that for each m

$$\lim_{N\to\infty} (X^{N,1},\dots,X^{N,m}) = (X^1,\dots,X^m) \quad \text{in law in } C([0,\infty);(\mathbb{R}^d)^m). \tag{2.8}$$

We expect this convergence because of the absolute convergence of the drift terms in (2.1) and energy in the DLR equation (2.3) for well-behaved initial distributions although it still requires some work to justify this rigorously even if $\Psi \in C_0^3(\mathbb{R}^d)$ [37].

If we take logarithmic functions as interaction potentials, then the situation changes drastically. Consider the soft-edge scaling limit of Gaussian (orthogonal/unitary/symplectic) ensembles. Then the N-labeled density is given by

$$\check{\mu}^N_{\mathrm{Airy},\beta}(d\mathbf{x}_N) = \frac{1}{Z}\{\prod_{i<j}^N |x_i - x_j|^\beta\} \exp\Big\{ -\frac{\beta}{4}\sum_{k=1}^N |2\sqrt{N} + N^{-1/6}x_k|^2 \Big\} d\mathbf{x}_N \tag{2.9}$$

and the associated N-particle dynamics described by SDE

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1,\, j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2}\{N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i}\} dt. \tag{2.10}$$

The correspondence between (2.9) and (2.10) is transparent and same as above. Indeed, we first consider distorted Brownian motion (Dirichlet spaces with $\check{\mu}^N_{\mathrm{Airy},\beta}$ as a common time change and energy measure), then we obtain the generator of the associated diffusion process by integration by parts. SDE (2.10) thus follows from the generator immediately.

It is known that the thermodynamic limit $\mu_{\mathrm{Airy},\beta}$ of the associated point process $\mu^N_{\mathrm{Airy},\beta}$ exists for each $\beta > 0$ [59]. Its m-point correlation function is explicitly given as a determinant of certain kernels if $\beta = 1, 2, 4$ [2, 43]. Indeed, if $\beta = 2$, then the m-point correlation function of the limit point process $\mu_{\mathrm{Airy},2}$ is

$$\rho^m_{\mathrm{Ai},2}(\mathbf{x}_m) = \det[K_{\mathrm{Ai},2}(x_i, x_j)]_{i,j=1}^m,$$

where $K_{\mathrm{Ai},2}$ is the continuous kernel such that, for $x \neq y$,

$$K_{\mathrm{Ai},2}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

We set here $\mathrm{Ai}'(x) = d\mathrm{Ai}(x)/dx$ and denote by $\mathrm{Ai}(\cdot)$ the Airy function given by

$$\mathrm{Ai}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} dk\, e^{i(zk + k^3/3)}, \quad z \in \mathbb{R}.$$

For $\beta = 1, 4$ similar expressions in terms of the quaternion determinant are known [2, 43].

From the convergence of equilibrium states, we may expect the convergence of solutions of SDEs (2.10). The divergence of the coefficients in (2.10) and the very long-range nature of the logarithmic interaction however prove to be problematic. Even an informal representation of the limit coefficients is nontrivial but has been obtained in [54]. Indeed, the limit ISDEs are given by

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ \sum_{|X_t^j|<r, j\neq i} \frac{1}{X_t^i - X_t^j} - \int_{|x|<r} \frac{\varrho(x)}{-x}\, dx\} dt \quad (i \in \mathbb{N}). \tag{2.11}$$

Here $\varrho(x) = 1_{(-\infty,0)}(x)\sqrt{-x}$, which is the shifted and rescaled semicircle function at the right edge.

As an application of our main theorem (Theorem 2.7), we prove the convergence (2.8) of solutions from (2.10) to (2.11) for $\{\mu^N_{\mathrm{Airy},\beta}\}$ with $\beta = 2$. We also prove that the limit points of solutions of (2.10) satisfy ISDE (2.11) with $\beta = 1, 2, 4$.

For general $\beta \neq 1, 2, 4$, the existence and uniqueness of solutions of (2.11) is still an open problem. Indeed, the proof in [54] relies on a general theory developed in [46, 47, 48, 49, 53], which reduces the problem to the quasi-Gibbs property and the existence of the logarithmic derivative of the equilibrium state. These key properties are proved only for $\beta = 1, 2, 4$ at present. We refer to [48, 49] for the definition of the quasi-Gibbs property and Definition 5.4 for the logarithmic derivative.

Another typical example is the Ginibre interacting Brownian motion, which is an infinite-particle system in $\mathbb{R}^2$ (naturally regarded as $\mathbb{C}$), whose equilibrium state is the Ginibre point process μ_{gin}. The m-point correlation function ρ_{gin}^m with respect to Gaussian measure $(1/\pi)e^{-|x|^2}dx$ on $\mathbb{C}$ is then given by

$$\rho_{\mathrm{gin}}^m(\mathbf{x}_m) = \det[e^{x_i\bar{x}_j}]_{i,j=1}^m.$$

The Ginibre point process μ_{gin} is the thermodynamic limit of N-particle point process μ_{gin}^N whose labeled measure is given by

$$\check{\mu}_{\mathrm{gin}}^N(d\mathbf{x}_N) = \frac{1}{\mathcal{Z}}\prod_{i<j}^N |x_i - x_j|^2 e^{-\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N.$$

The associated N-particle SDE is then given by

$$dX_t^{N,i} = dB_t^i - X_t^{N,i}dt + \sum_{j=1, j\neq i}^N \frac{X_t^{N,i} - X_t^{N,j}}{|X_t^{N,i} - X_t^{N,j}|^2}dt \quad (1 \le i \le N). \tag{2.12}$$

We shall prove that the limit ISDEs are

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \quad (i \in \mathbb{N}) \tag{2.13}$$

and

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j| < r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \quad (i \in \mathbb{N}). \tag{2.14}$$

In [47, 53], it is proved that these ISDEs have the same pathwise unique strong solution for $\mu_{\mathrm{gin}} \circ \ell^{-1}$-a.s. $\mathbf{s}$, where ℓ is a label and $\mathbf{s}$ is an initial point. As an example of applications of our second main theorem (Theorem 2.7), we prove the convergence of solutions of (2.12) to those of (2.13) and (2.14). This example indicates again the sensitivity of the representation of the limit ISDE. Such varieties of the limit ISDEs are a result of the long-range nature of the logarithmic potential.

The main purpose of the present paper is to develop a general theory for finite-particle convergence applicable to logarithmic potentials, and in particular, the Airy and Ginibre point processes. Our theory is also applicable to essentially all Gibbs measures with Ruelle-class potentials such as the Lennard-Jones 6-12 potential and Riesz potentials.

In the first main theorem (Theorem 2.2), we present a sufficient condition for a kind of tightness of solutions of stochastic differential equations (SDE) describing finite-particle systems, and prove that the limit points solve the corresponding ISDE. This implies, if in addition the limit ISDE enjoy uniqueness of solutions, then the full sequence converges. We treat non-reversible case in the first main theorem.

In the second main theorem (Theorem 2.7), we restrict to the case of reversible particle systems and simplify the sufficient condition. Because of reversibility, the sufficient condition is reduced to the convergence of logarithmic derivative of μ^N with marginal assumptions. We shall deduce Theorem 2.7 from Theorem 2.2 and apply Theorem 2.7 to all examples in the present paper.

If $\Psi(x) = -\log|x|$, $\beta = 2$ and $d = 1$, there exists an algebraic method to construct the associated stochastic processes [21, 22, 23, 25], and to prove the convergence of finite-particle systems [52, 51]. This method requires that interaction Ψ is the logarithmic function with $\beta = 2$ and depends crucially on an explicit calculation of space-time determinantal kernels. It is thus not applicable to $\beta \neq 2$ even if $d = 1$.

As for Sine_β point processes, Tsai proved the convergence of finite-particle systems for all $\beta \geq 1$ [76]. His method relies on a coupling method based on monotonicity of SDEs, which is very specific to this model.

The organization of the paper is as follows: In Section 2.2, we state the main theorems (Theorem 2.2 and Theorem 2.7). In Section 2.3, we prove Theorem 2.2. In Section 2.4, we prove Theorem 2.7 using Theorem 2.2. In Section 2.5, we present examples.

2.2 Set up and the main theorems

2.2.1 Configuration spaces and Campbell measures

Let S be a closed set in $\mathbb{R}^d$ whose interior S_{int} is a connected open set satisfying $\overline{S_{\mathrm{int}}} = S$ and the boundary ∂S having Lebesgue measure zero. A configuration $\mathsf{s} = \sum_i \delta_{s_i}$ on S is a Radon measure on S consisting of delta masses. We set $S_r = \{s \in S\,;\, |s| \leq r\}$. Let S be the set consisting of all configurations of S. By definition, S is given by

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\; \mathsf{s}(S_r) < \infty \text{ for each } r \in \mathbb{N}\}.$$

By convention, we regard the zero measure as an element of S. We endow S with the vague topology, which makes S a Polish space. S is called the configuration space over S and a probability measure μ on $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$ is called a point process on S.

A symmetric and locally integrable function $\rho^n : S^n \to [0, \infty)$ is called the n-point correlation function of a point process μ on S with respect to the Lebesgue measure if ρ^n satisfies

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_{\mathsf{S}} \prod_{i=1}^m \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} d\mu$$

for any sequence of disjoint bounded measurable sets $A_1, \ldots, A_m \in \mathcal{B}(S)$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. When $\mathsf{s}(A_i) - k_i < 0$, according to our interpretation, $\mathsf{s}(A_i)!/(\mathsf{s}(A_i) - k_i)! = 0$ by convention. Hereafter, we always consider correlation functions with respect to Lebesgue measures.

A point process μ_x is called the reduced Palm measure of μ conditioned at $x \in S$ if μ_x is the regular conditional probability defined as

$$\mu_x = \mu(\cdot - \delta_x | \mathsf{s}(\{x\}) \geq 1).$$

A Radon measure $\mu^{[1]}$ on $S \times \mathsf{S}$ is called the 1-Campbell measure of μ if $\mu^{[1]}$ is given by

$$\mu^{[1]}(dx d\mathsf{s}) = \rho^1(x) \mu_x(d\mathsf{s}) dx. \tag{2.15}$$

2.2.2 Finite-particle approximations (general case)

Let $\{\mu^N\}$ be a sequence of point processes on S such that $\mu^N(\{\mathsf{s}(S) = N\}) = 1$. We assume:

(H1) Each μ^N has a correlation function $\{\rho^{N,n}\}$ satisfying for each $r \in \mathbb{N}$

$$\lim_{N\to\infty} \rho^{N,n}(\mathbf{x}) = \rho^n(\mathbf{x}) \quad \text{uniformly on } S_r^n \text{ for all } n \in \mathbb{N}, \tag{2.16}$$

$$\sup_{N\in\mathbb{N}} \sup_{\mathbf{x}\in S_r^n} \rho^{N,n}(\mathbf{x}) \leq c_1^n n^{c_2 n}, \tag{2.17}$$

where $0 < c_1(r) < \infty$ and $0 < c_2(r) < 1$ are constants independent of $n \in \mathbb{N}$.

It is known that (2.16) and (2.17) imply weak convergence (2.6) [48, Lemma A.1]. As in Section 2.1, let ℓ and ℓ_N be labels of μ and μ^N, respectively. We assume:

(H2) For each $m \in \mathbb{N}$, (2.7) holds. That is,

$$\lim_{N\to\infty} \mu^N \circ \ell_{N,m}^{-1} = \mu \circ \ell_m^{-1} \quad \text{weakly in } S^m. \tag{2.7}$$

We shall later take $\mu^N \circ \ell_N^{-1}$ as an initial distribution of a labeled finite-particle system. Hence **(H2)** means convergence of the initial distribution of the labeled dynamics. There exist infinitely many different labels ℓ, and we choose a label such that the initial distribution of the labeled dynamics converges. **(H2)** will be used in Theorem 2.7 and Theorem 2.2.

For $\mathbf{X} = (X^i)_{i=1}^\infty$ and $\mathbf{X}^N = (X^{N,i})_{i=1}^N$, we set

$$\mathsf{X}_t^{\diamond i} = \sum_{j\neq i}^{\infty} \delta_{X_t^j}, \quad \text{and} \quad \mathsf{X}_t^{N,\diamond i} = \sum_{j\neq i}^{N} \delta_{X_t^{N,j}},$$

where $X_t^{N,\diamond i}$ denotes the zero measure for $N = 1$. Let $\sigma^N, \sigma : S \times \mathsf{S} \to \mathbb{R}^{d^2}$ and $\mathsf{b}^N, \mathsf{b} : S \times \mathsf{S} \to \mathbb{R}^d$ be measurable functions. We introduce the finite-dimensional SDE of $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ with these coefficients such that for $1 \leq i \leq N$

$$dX_t^{N,i} = \sigma^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})dB_t^i + \mathsf{b}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})dt \tag{2.18}$$

$$\mathbf{X}_0^N = \mathbf{s}. \tag{2.19}$$

We assume:

(H3) SDE (2.18) and (2.19) has a unique solution for $\mu^N \circ \ell_N^{-1}$-a.s. $\mathbf{s}$ for each N: this solution does not explode. Furthermore, when ∂S is non-void, particles never hit the boundary.

We set $\mathsf{a}^N = \sigma^{N\,t}\sigma^N$ and assume:

(H4) σ^N are bounded and continuous on $S \times \mathsf{S}$, and converge uniformly to σ on $S_r \times \mathsf{S}$ for each $r \in \mathbb{N}$. Furthermore, a^N are uniformly elliptic on $S_r \times \mathsf{S}$ for each $r \in \mathbb{N}$ and $\nabla_x \mathsf{a}^N$ are uniformly bounded on $S \times \mathsf{S}$.

From **(H4)** we see that a^N converge uniformly to $\mathsf{a} := \sigma^t\sigma$ on each compact set $S_r \times \mathsf{S}$, and that a^N and a are bounded and continuous on $S \times \mathsf{S}$. There thus exists a positive constant c_3 such that

$$||\mathsf{a}||_{S\times\mathsf{S}},\ ||\nabla_x\mathsf{a}||_{S\times\mathsf{S}},\ \sup_{N\in\mathbb{N}}||\mathsf{a}^N||_{S\times\mathsf{S}},\ \sup_{N\in\mathbb{N}}||\nabla_x\mathsf{a}^N||_{S\times\mathsf{S}} \le c_3. \tag{2.20}$$

Here $\|\cdot\|_{S\times\mathsf{S}}$ denotes the uniform norm on $S\times\mathsf{S}$. Furthermore, we see that a is uniformly elliptic on each $S_r \times \mathsf{S}$. From these, we expect that SDEs (2.18) have a sub-sequential limit.

$$\begin{aligned}\lim_{N\to\infty}\{X_t^{N,i} - X_0^{N,i}\} &= \lim_{N\to\infty}\int_0^t \sigma^N(X_t^{N,i},\mathsf{X}_t^{N,\diamond i})dB_u^i + \lim_{N\to\infty}\int_0^t \mathsf{b}^N(X_t^{N,i},\mathsf{X}_t^{N,\diamond i})du \\ &= \int_0^t \sigma(X_t^{N,i},\mathsf{X}_t^{N,\diamond i})dB_u^i + \lim_{N\to\infty}\int_0^t \mathsf{b}^N(X_t^{N,i},\mathsf{X}_t^{N,\diamond i})du.\end{aligned}$$

To identify the second term on the right-hand side and to justify the convergence, we make further assumptions. As the examples in Section 2.1 suggest, the identification of the limit is a sensitive problem, which is at the heart of the present paper.

We set the maximal module variable $\overline{\mathbf{X}}^{N,m}$ of the first m-particles by

$$\overline{\mathbf{X}}^{N,m} = \max_{i=1}^{m}\sup_{t\in[0,T]}|X_t^{N,i}|.$$

and by $\mathcal{L}_r^N$ the maximal label with which the particle intersects S_r; that is,

$$\mathcal{L}_r^N = \max\{i\in\mathbb{N}\cup\{\infty\}\,;\,|X_t^{N,i}|\le r \text{ for some } 0\le t\le T\}.$$

We assume the following.
(I1) For each $m\in\mathbb{N}$

$$\lim_{a\to\infty}\liminf_{N\to\infty}P^{\mu^N\circ\ell_N^{-1}}(\overline{\mathbf{X}}^{N,m}\le a) = 1 \tag{2.21}$$

and there exists a constant $c_4 = c_4(m,a)$ such that for $0\le t,u\le T$

$$\sup_{N\in\mathbb{N}}\sum_{i=1}^m \mathrm{E}^{\mu^N\circ\ell_N^{-1}}[|X_t^{N,i} - X_u^{N,i}|^4;\overline{\mathbf{X}}^{N,m}\le a] \le c_4|t-u|^2. \tag{2.22}$$

Furthermore, for each $r\in\mathbb{N}$

$$\lim_{L\to\infty}\liminf_{N\to\infty}P^{\mu^N\circ\ell_N^{-1}}(\mathcal{L}_r^N\le L) = 1. \tag{2.23}$$

Let $\mu^{N,[1]}$ be the one-Campbell measure of μ^N defined as (2.15). Set $c_5(r,N) = \mu^{N,[1]}(S_r\times\mathsf{S})$. Then by (2.17) $\sup_N c_5(r,N) < \infty$ for each $r\in\mathbb{N}$. Without loss of generality, we can assume that $c_5 > 0$ for all r, N. Let $\mu_r^{N,[1]} = \mu^{N,[1]}(\cdot\cap\{S_r\times\mathsf{S}\})$. Let $\bar{\mu}_r^{N,[1]}$ be the probability measure defined as $\bar{\mu}_r^{N,[1]}(\cdot) = \mu^{N,[1]}(\cdot\cap\{S_r\times\mathsf{S}\})/c_5$. Let $\varpi_{r,s}$ be a map from $S_r\times\mathsf{S}$ to itself such that $\varpi_{r,s}(x,\mathsf{s}) = (x,\sum_{|x-s_i|<s}\delta_{s_i})$, where $\mathsf{s}=\sum_i\delta_{s_i}$. Let

$\mathcal{F}_{r,s} = \sigma[\varpi_{r,s}]$ be the sub-σ-field of $\mathcal{B}(S_r \times \mathsf{S})$ generated by $\varpi_{r,s}$. Because S_r is a subset of S, we can and do regard $\mathcal{F}_{r,s}$ as a σ-field on $S \times \mathsf{S}$, which is trivial outside $S_r \times \mathsf{S}$.

We set a tail-truncated coefficient $\mathsf{b}_{r,s}^N$ of b^N and their tail parts $\mathsf{b}_{r,s}^{N,\mathrm{tail}}$ by

$$\mathsf{b}_{r,s}^N = \mathrm{E}^{\bar{\mu}_r^{N,[1]}}[\mathsf{b}^N|\mathcal{F}_{r,s}], \quad \mathsf{b}^N = \mathsf{b}_{r,s}^N + \mathsf{b}_{r,s}^{N,\mathrm{tail}}. \tag{2.24}$$

We can and do take a version of $\mathsf{b}_{r,s}^N$ such that

$$\mathsf{b}_{r,s}^N(x,\mathsf{y}) = 0 \quad \text{for } x \notin S_r, \tag{2.25}$$

$$\mathsf{b}_{r,s}^N(x,\mathsf{y}) = \mathsf{b}_{r+1,s}^N(x,\mathsf{y}) \quad \text{for } x \in S_r. \tag{2.26}$$

We next introduce a cut-off coefficient $\mathsf{b}_{r,s,\mathsf{p}}^N$ of $\mathsf{b}_{r,s}^N$. Let $\mathsf{b}_{r,s,\mathsf{p}}^N$ be a continuous and $\mathcal{F}_{r,s}$-measurable function on $S \times \mathsf{S}$ such that

$$\mathsf{b}_{r,s,\mathsf{p}}^N(x,\mathsf{y}) = 0 \quad \text{for } x \notin S_r$$

$$\mathsf{b}_{r,s,\mathsf{p}}^N(x,\mathsf{y}) = \mathsf{b}_{r+1,s,\mathsf{p}}^N(x,\mathsf{y}), \qquad \text{for } x \in S_{r-1}$$

and that, for $(S \times \mathsf{S})_{r,\mathsf{p}} = \{(x,\mathsf{y}) \in S_r \times \mathsf{S};\ |x-y_i| \le 1/2^{\mathsf{p}} \text{ for some } y_i\}$, where $\mathsf{y} = \sum_i \delta_{y_i}$,

$$\mathsf{b}_{r,s,\mathsf{p}}^N(x,\mathsf{y}) = 0 \quad \text{for } (x,\mathsf{y}) \in (S \times \mathsf{S})_{r,\mathsf{p}+1}, \tag{2.27}$$

$$\mathsf{b}_{r,s,\mathsf{p}}^N(x,\mathsf{y}) = \mathsf{b}_{r,s}^N(x,\mathsf{y}) \quad \text{for } (x,\mathsf{y}) \notin (S \times \mathsf{S})_{r,\mathsf{p}}. \tag{2.28}$$

The main requirements for b^N and $\mathsf{b}_{r,s,\mathsf{p}}^N$ are the following:

(I2) There exists a $\hat{p}$ such that $1 < \hat{p}$ and that for each $r \in \mathbb{N}$

$$\limsup_{N\to\infty} \int_{S_r \times \mathsf{S}} |\mathsf{b}^N|^{\hat{p}} d\mu^{N,[1]} < \infty. \tag{2.29}$$

Furthermore, for each $r, i \in \mathbb{N}$, there exists a constant c_6 such that

$$\sup_{\mathsf{p}\in\mathbb{N}} \sup_{N\in\mathbb{N}} E^{\mu^N\circ\ell_N^{-1}}[\int_0^T |\mathsf{b}_{r,s,\mathsf{p}}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})|^{\hat{p}} dt] \le c_6. \tag{2.30}$$

We set $\mathsf{S}_r^m = \{\mathsf{s}\,;\mathsf{s}(S_r) = m\}$. Let $\|\cdot\|_{S\times\mathsf{S}_r^m}$ denote the uniform norm on $S\times\mathsf{S}_r^m$ and set $L^{\hat{p}}(\mu_r^{N,[1]}) = L^{\hat{p}}(S_r\times\mathsf{S}, \mu^{N,[1]})$. For a function f on $S\times\mathsf{S}_r^m$ we denote by $\nabla f = (\nabla_x \check{f}, \nabla_{y_i}\check{f})$, where $\check{f}$ is a function on $S_r \times S_r^m$ such that $\check{f}(x,(y_i)_{i=1}^m)$ is symmetric in $(y_i)_{i=1}^m$ for each x and $f(x,\sum_i \delta_{y_i}) = \check{f}(x,(y_i)_{i=1}^m)$. We decompose $\mathsf{b}_{r,s}^N$ as

$$\mathsf{b}_{r,s}^N = \mathsf{b}_{r,s,\mathsf{p}}^N + \mathsf{b}_{r,s}^N - \mathsf{b}_{r,s,\mathsf{p}}^N \tag{2.31}$$

and we assume:

(I3) For each $m, \mathsf{p}, r, s \in \mathbb{N}$ such that $r < s$, there exists $\mathsf{b}_{r,s,\mathsf{p}}$ such that

$$\lim_{N\to\infty} \|\mathsf{b}_{r,s,\mathsf{p}}^N - \mathsf{b}_{r,s,\mathsf{p}}\|_{S\times\mathsf{S}_r^m} = 0. \tag{2.32}$$

Moreover, $\mathsf{b}^N_{r,s,\mathsf{p}}$ are differentiable in x and satisfying the bounds:

$$\sup_{N\in\mathbb{N}} \|\nabla \mathsf{b}^N_{r,s,\mathsf{p}}\|_{S\times\mathsf{S}^m_r} < \infty, \tag{2.33}$$

$$\lim_{\mathsf{p}\to\infty}\sup_{N\in\mathbb{N}} \|\mathsf{b}^N_{r,s,\mathsf{p}} - \mathsf{b}^N_{r,s}\|_{L^{\hat{p}}(\mu_r^{N,[1]})} = 0. \tag{2.34}$$

Furthermore, we assume for each $i, r < s \in \mathbb{N}$

$$\lim_{\mathsf{p}\to\infty}\limsup_{N\to\infty} E^{\mu^N\circ\ell_N^{-1}}[\int_0^T |\{\mathsf{b}^N_{r,s,\mathsf{p}} - \mathsf{b}^N_{r,s}\}(X^{N,i}_t, \mathsf{X}^{N,\diamond i}_t)|^{\hat{p}}dt] = 0, \tag{2.35}$$

$$\lim_{\mathsf{p}\to\infty} E^{\mu\circ\ell^{-1}}[\int_0^T |\{\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b}_{r,s}\}(X^i_t, \mathsf{X}^{\diamond i}_t)|^{\hat{p}}dt] = 0, \tag{2.36}$$

where $\mathsf{b}_{r,s}$ is such that

$$\mathsf{b}_{r,s}(x,\mathsf{y}) = \lim_{N\to\infty} \mathsf{b}^N_{r,s}(x,\mathsf{y}) \quad \text{for each } (x,\mathsf{y}) \in \bigcup_{\mathsf{p}\in\mathbb{N}} (S\times\mathsf{S})^c_{r,\mathsf{p}}. \tag{2.37}$$

Remark 2.1. We see that $\bigcup_{\mathsf{p}\in\mathbb{N}}(S\times\mathsf{S})^c_{r,\mathsf{p}} = \{S^c_r\times\mathsf{S}\}\cup\{(x,\mathsf{y}); x\neq y_i \text{ for all } i\}$ by definition and $\mathsf{b}_{r,s}(x,\mathsf{y}) = 0$ for $x\notin S_r$ by (2.25). The limit in (2.37) exists because of (2.27), (2.28), and (2.32).

(I4) There exists a $\mathsf{b}^{\mathrm{tail}} \in C(S;\mathbb{R}^d)$ independent of $r\in\mathbb{N}$ and $\mathsf{s}\in\mathsf{S}$ such that

$$\lim_{s\to\infty}\limsup_{N\to\infty} \|\mathsf{b}^{N,\mathrm{tail}}_{r,s} - \mathsf{b}^{\mathrm{tail}}\|_{L^{\hat{p}}(\mu_r^{N,[1]})} = 0. \tag{2.38}$$

Furthermore, for each $r, i\in\mathbb{N}$:

$$\lim_{s\to\infty}\limsup_{N\to\infty} E^{\mu^N\circ\ell_N^{-1}}[\int_0^T |(\mathsf{b}^{N,\mathrm{tail}}_{r,s} - \mathsf{b}^{\mathrm{tail}})(X^{N,i}_t, \mathsf{X}^{N,\diamond i}_t)|^{\hat{p}}dt] = 0. \tag{2.39}$$

We remark that $\mathsf{b}^{\mathrm{tail}}$ is automatically independent of r for consistency (2.28). By assumption, $\mathsf{b}^{\mathrm{tail}} = \mathsf{b}^{\mathrm{tail}}(x)$ is a function of x. From (2.24) and (2.31) we have

$$\mathsf{b}^N = \mathsf{b}^N_{r,s,\mathsf{p}} + \mathsf{b}^{\mathrm{tail}} + \{\mathsf{b}^N_{r,s} - \mathsf{b}^N_{r,s,\mathsf{p}}\} + \{\mathsf{b}^{N,\mathrm{tail}}_{r,s} - \mathsf{b}^{\mathrm{tail}}\}. \tag{2.40}$$

In **(I3)** and **(I4)**, we have assumed that the last two terms $\{\mathsf{b}^N_{r,s} - \mathsf{b}^N_{r,s,\mathsf{p}}\}$ and $\{\mathsf{b}^{N,\mathrm{tail}}_{r,s} - \mathsf{b}^{\mathrm{tail}}\}$ in (2.40) are asymptotically negligible.

Under these assumptions, we prove in Lemma 2.11 that there exists b such that for each $r\in\mathbb{N}$

$$\lim_{s\to\infty} \|\mathsf{b}_{r,s} - \mathsf{b}\|_{L^{\hat{p}}(\mu_r^{N,[1]})} = 0. \tag{2.41}$$

We assume:
(I5) For each $i, r\in\mathbb{N}$

$$\lim_{s\to\infty} E^{\mu\circ\ell^{-1}}[\int_0^T |(\mathsf{b}_{r,s} - \mathsf{b})(X^i_t, \mathsf{X}^{\diamond i}_t)|^{\hat{p}}dt] = 0. \tag{2.42}$$

We say a sequence $\{\mathbf{X}^N\}$ of $C([0,T];S^N)$-valued random variables is tight if for any subsequence we can choose a subsequence denoted by the same symbol such that $\{\mathbf{X}^{N,m}\}_{N\geq m}$ is convergent in law in $C([0,T];S^m)$ for each $m\in\mathbb{N}$. With these preparations, we state the main theorem in this section.

Theorem 2.2. Assume **(H1)**–**(H4)** and **(I1)**–**(I5)**. Then, $\{\mathbf{X}^N\}_{N\in\mathbb{N}}$ is tight in $C([0,T];S^{\mathbb{N}})$ and, any limit point $\mathbf{X}=(X^i)_{i\in\mathbb{N}}$ of $\{\mathbf{X}^N\}_{N\in\mathbb{N}}$ is a solution of the ISDE

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{\diamond i})dB_t^i + \{\mathsf{b}(X_t^i, \mathsf{X}_t^{\diamond i}) + \mathsf{b}^{\mathrm{tail}}(X_t^i)\}dt. \tag{2.43}$$

Remark 2.3. If diffusion processes are symmetric, we can dispense with (2.22), (2.30), (2.35), (2.36), (2.39), and (2.42) as we see in Subsection 2.2.3. Indeed, using the Lyons-Zheng decomposition we can derive these from *static* conditions **(H4)**, (2.29), (2.32), (2.34), (2.38), and (2.41). We remark that we can apply Theorem 2.2 to non-symmetric diffusion processes by assuming these *dynamical* conditions.

2.2.3 Finite-particle approximations (reversible case)

For a subset A, we set $\pi_A : \mathsf{S}\to\mathsf{S}$ by $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot\cap A)$. We say a function f on S is local if f is $\sigma[\pi_K]$-measurable for some compact set K in S. For a local function f on S, we say f is smooth if $\check{f}$ is smooth, where $\check{f}(x_1,\ldots)$ is a symmetric function such that $\check{f}(x_1,\ldots) = f(\mathsf{x})$ for $\mathsf{x}=\sum_i \delta_{x_i}$. Let $\mathcal{D}_\circ$ be the set of all bounded, local smooth functions on S. We write $f\in L^p_{\mathrm{loc}}(\mu^{[1]})$ if $f\in L^p(S_r\times\mathsf{S},\mu^{[1]})$ for all $r\in\mathbb{N}$. Let $C_0^\infty(S)\otimes\mathcal{D}_\circ = \{\sum_{i=1}^N f_i(x)g_i(\mathsf{y})\,;\, f_i\in C_0^\infty(S),\, g_i\in\mathcal{D}_\circ,\, N\in\mathbb{N}\}$ denote the algebraic tensor product of $C_0^\infty(S)$ and $\mathcal{D}_\circ$.

Definition 2.4. A $\mathbb{R}^d$-valued function $\mathsf{d}^\mu\in L^1_{\mathrm{loc}}(\mu^{[1]})$ is called *the logarithmic derivative* of μ if, for all $\varphi\in C_0^\infty(S)\otimes\mathcal{D}_\circ$,

$$\int_{S\times\mathsf{S}} \mathsf{d}^\mu(x,\mathsf{y})\varphi(x,\mathsf{y})\mu^{[1]}(dxd\mathsf{y}) = -\int_{S\times\mathsf{S}} \nabla_x\varphi(x,\mathsf{y})\mu^{[1]}(dxd\mathsf{y}).$$

Remark 2.5. (1) The logarithmic derivative d^μ is determined uniquely (if exists).
(2) If the boundary ∂S is nonempty and particles hit the boundary, then d^μ would contain a term arising from the boundary condition. For example, if the Neumann boundary condition is imposed on the boundary, then there would be local time-type drifts. We shall later assume that particles never hit the boundary, and the above formulation is thus sufficient in the present situation.
(3) A sufficient condition for the explicit expression of the logarithmic derivative of point processes is given in [47, Theorem 45]. Using this, one can obtain the logarithmic derivative of point processes appearing in random matrix theory such as sine_β, Airy_β, $(\beta=1,2,4)$, $\mathrm{Bessel}_{2,\alpha}$ $(1\le\alpha)$, and the Ginibre point process (see Examples in Section 2.5). For canonical Gibbs measures with Ruelle-class interaction potentials, one can easily calculate the logarithmic derivative employing DLR equation [53, Lemma 10.10].

We assume:

(J1) Each μ^N has a logarithmic derivative d^N, and the coefficient b^N is given as

$$\mathsf{b}^N = \frac{1}{2}\{\nabla_x\mathsf{a}^N + \mathsf{a}^N\mathsf{d}^N\}. \tag{2.44}$$

Furthermore, the vector-valued functions $\{\nabla_x\mathsf{a}^N\}_N$ are continuous and converge to $\nabla_x\mathsf{a}$ uniformly on each $S_r\times\mathsf{S}$, where $\nabla_x\mathsf{a}^N$ is the d-dimensional column vector such that

$$\nabla_x\mathsf{a}^N(x,\mathsf{y}) = {}^t\Big(\sum_{i=1}^d \frac{\partial}{\partial x_i}\mathsf{a}^N_{1i}(x,\mathsf{y}),\ldots,\sum_{i=1}^d \frac{\partial}{\partial x_i}\mathsf{a}^N_{di}(x,\mathsf{y})\Big). \tag{2.45}$$

Remark 2.6. From **(J1)** we see that the delabeled dynamics $\mathsf{X}^N = \sum_{i=1}^N \delta_{X^i}$ of $\mathbf{X}^N$ is reversible with respect to μ^N. Thus **(J1)** relates the measure μ^N with the labeled dynamics $\mathbf{X}^N$. For each $N < \infty$, $\mathbf{X}^N$ has a reversible measure. Indeed, the symmetrization $(\mu^N \circ \ell_N^{-1})_{\mathrm{sym}}$ of $\mu^N \circ \ell_N^{-1}$ is a reversible measure of $\mathbf{X}^N$ as we see for $\check{\mu}^N$ in Introduction, where $(\mu^N \circ \ell_N^{-1})_{\mathrm{sym}} = \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N}(\mu^N \circ \ell_N^{-1})\circ\sigma^{-1}$ and $\mathfrak{S}_N$ is the symmetric group of order N. When $N = \infty$, $\mathbf{X}$ does not have any reversible measure in general. For example, infinite-dimensional Brownian motion $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ on $(\mathbb{R}^d)^{\mathbb{N}}$ has no reversible measures. We also remark that the Airy$_\beta$ ($\beta = 1, 2, 4$) interacting Brownian motion defined by (2.11) has a reversible measure given by $\mu_{\mathrm{Airy},\beta} \circ \ell^{-1}$ with label $\ell(\mathsf{s}) = (s_1, s_2, \ldots)$ such that $s_i > s_{i+1}$ for all $i \in \mathbb{N}$ because ℓ gives a bijection from (a subset of) S to $\mathbb{R}^{\mathbb{N}}$ defined for $\mu_{\mathrm{Airy},\beta}$-a.s. s, and thus the relation $\mathbf{X}_t = \ell(\mathsf{X}_t)$ holds for all t.

We prove that convergence of the logarithmic derivative implies weak convergence of the solutions of the associated SDEs. Each logarithmic derivative d^N belongs to a different L^p-space $L^p(\mu^{N,[1]})$, and $\mu^{N,[1]}$ are mutually singular. Hence we decompose d^N to define a kind of L^p-convergence.

Let u, u^N, $w : S \to \mathbb{R}^d$ and g, g^N, v, $v^N : S^2 \to \mathbb{R}^d$ be measurable functions. We set

$$\mathsf{g}_s(x, \mathsf{y}) = \int_S \chi_s(x-y)v(x,y)dy + \sum_i \chi_s(x-y_i)g(x,y_i), \tag{2.46}$$

$$\mathsf{g}_s^N(x, \mathsf{y}) = \int_S \chi_s(x-y)v^N(x,y)dy + \sum_i \chi_s(x-y_i)g^N(x,y_i),$$

$$\mathsf{w}_s^N(x, \mathsf{y}) = \int_S \{1-\chi_s(x-y)\}v^N(x,y)dy + \sum_i (1-\chi_s(x-y_i))g^N(x,y_i),$$

where $\mathsf{y} = \sum_i \delta_{y_i}$ and $\chi_s \in C_0^\infty(S)$ is a cut-off function such that $0 \le \chi_s \le 1$, $\chi_s(x) = 0$ for $|x| \ge s+1$, and $\chi_s(x) = 1$ for $|x| \le s$. We assume the following.

(J2) Each μ^N has a logarithmic derivative d^N such that

$$\mathsf{d}^N(x, \mathsf{y}) = u^N(x) + \mathsf{g}_s^N(x, \mathsf{y}) + \mathsf{w}_s^N(x, \mathsf{y}). \tag{2.47}$$

Furthermore, we assume that

(1) u^N are in $C^1(S)$. Furthermore, u^N and ∇u^N converge uniformly to u and ∇u, respectively, on each compact set in S.

(2) For each $s \in \mathbb{N}$, $\int_S \chi_s(x-y)v^N(x,y)dy$ are in $C^1(S)$. Furthermore, functions $\int_S \chi_s(x-y)v^N(x,y)dy$ and $\nabla_x \int_S \chi_s(x-y)v^N(x,y)dy$ converge uniformly to $\int_S \chi_s(x-y)v(x,y)dy$ and $\nabla_x \int_S \chi_s(x-y)v(x,y)dy$, respectively, on each compact set in S.

(3) g^N are in $C^1(S^2 \cap \{x \neq y\})$. Furthermore, g^N and $\nabla_x g^N$ converge uniformly to g and $\nabla_x g$, respectively, on $S^2 \cap \{|x-y| \ge 2^{-\mathsf{p}}\}$ for each $\mathsf{p} > 0$. In addition, for each $r \in \mathbb{N}$,

$$\lim_{\mathsf{p}\to\infty} \limsup_{N\to\infty} \int_{x\in S_r, |x-y|\le 2^{-\mathsf{p}}} \chi_s(x-y)|g^N(x,y)|^{\hat{p}}\, \rho_x^{N,1}(y)dxdy = 0, \tag{2.48}$$

where $\rho_x^{N,1}$ is a one-correlation function of the reduced Palm measure μ_x^N.

(4) There exists a continuous function $w : S \to \mathbb{R}$ such that

$$\lim_{s\to\infty}\limsup_{N\to\infty}\int_{S_r\times\mathsf{S}} |\mathsf{w}_s^N(x,\mathsf{y}) - w(x)|^{\hat{p}} d\mu^{N,[1]} = 0, \quad w \in L^{\hat{p}}_{\mathrm{loc}}(S,dx). \tag{2.49}$$

Let p be such that $1 < p < \hat{p}$. Assume **(H1)** and **(J2)**. Then from [47, Theorem 45] we see that the logarithmic derivative d^{μ} of μ exists in $L^p_{\mathrm{loc}}(\mu^{[1]})$ and is given by

$$\mathsf{d}^{\mu}(x,\mathsf{y}) = u(x) + \mathsf{g}(x,\mathsf{y}) + w(x). \tag{2.50}$$

Here $\mathsf{g}(x,\mathsf{y}) = \lim_{s\to\infty}\mathsf{g}_s(x,\mathsf{y})$ and the convergence of $\lim \mathsf{g}_s$ takes place in $L^p_{\mathrm{loc}}(\mu^{[1]})$. We now introduce the ISDE of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$:

$$dX_t^i = \sigma(X_t^i,\mathsf{X}_t^{\diamond i})dB_t^i + \frac{1}{2}\{\nabla_x\mathsf{a}(X_t^i,\mathsf{X}_t^{\diamond i}) + \mathsf{a}(X_t^i,\mathsf{X}_t^{\diamond i})\mathsf{d}^{\mu}(X_t^i,\mathsf{X}_t^{\diamond i})\}dt \tag{2.51}$$

$$\mathbf{X}_0 = \mathsf{s}. \tag{2.52}$$

Here $\nabla_x\mathsf{a}$ is defined similarly as (2.45). If σ is the unit matrix and **(J2)** is satisfied, we have

$$dX_t^i = dB_t^i + \frac{1}{2}\{u(X_t^i) + w(X_t^i) + \mathsf{g}(X_t^i,\mathsf{X}_t^{\diamond i})\}dt. \tag{2.53}$$

In the sequel, we give a sufficient condition for solving ISDE (2.51) (and (2.53)).

Let $\mathbb{D}$ be the standard square field on S such that for any $f,g \in \mathcal{D}_\circ$ and $\mathsf{s} = \sum_i \delta_{s_i}$

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\{\sum_i \nabla_i\check{f}\cdot\nabla_i\check{g}\}(\mathsf{s}),$$

where $\cdot$ is the inner product in $\mathbb{R}^d$. Since the function $\sum_i \nabla_i\check{f}(\mathbf{s})\cdot\nabla_i\check{g}(\mathbf{s})$, where $\mathbf{s} = (s_i)_i$ and $\mathsf{s} = \sum_i \delta_{s_i}$, is symmetric in $(s_i)_i$, we regard it as a function of s. We set $L^2(\mu) = L^2(\mathsf{S},\mu)$ and let

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g](\mathsf{s})\mu(d\mathsf{s}),$$
$$\mathcal{D}_\circ^{\mu} = \{f \in \mathcal{D}_\circ \cap L^2(\mu)\,;\, \mathcal{E}^{\mu}(f,f) < \infty\}.$$

We assume:

(J3) $(\mathcal{E}^{\mu},\mathcal{D}_\circ^{\mu})$ is closable on $L^2(\mu)$.

From **(J3)** and the local boundedness of correlation functions given by **(H1)**, we deduce that the closure $(\mathcal{E}^{\mu},\mathcal{D}^{\mu})$ of $(\mathcal{E}^{\mu},\mathcal{D}_\circ^{\mu})$ becomes a quasi-regular Dirichlet form [44, Theorem 1]. Hence, using a general theory of quasi-regular Dirichlet forms, we deduce the existence of the associated S-valued diffusion (P,X) [42]. By construction, (P,X) is μ-reversible.

If one takes μ as Poisson point process with Lebesgue intensity, then the diffusion (P,X) thus obtained is the standard S-valued Brownian motion B such that $\mathsf{B}_t = \sum_{i\in\mathbb{N}}\delta_{B_t^i}$, where $\{B^i\}_{i\in\mathbb{N}}$ are independent copies of the standard Brownian motions on $\mathbb{R}^d$. This is the reason why we call $\mathbb{D}$ the standard square field.

Let Cap^{μ} denote the capacity given by the Dirichlet space $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}, L^2(\mu))$ [16]. Let

$$\mathsf{S}_{\text{s.i.}} = \{\mathsf{s} \in \mathsf{S}\,;\ \mathsf{s}(x) \le 1 \text{ for all } x \in S,\ \mathsf{s}(S) = \infty\}$$

and assume:

(J4) $\mathrm{Cap}^{\mu}(\{\mathsf{S}_{\text{s.i.}}\}^c) = 0$.

Let $\mathrm{Erf}(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-|x|^2/2}\, dx$ be the error function. Let $S_r = \{|x| < r\}$ as before. We assume:

(J5) There exists a $Q > 0$ such that for each $R > 0$

$$\liminf_{r\to\infty} \sup_{N\in\mathbb{N}} \Big\{ \int_{S_{r+R}} \rho^{N,1}(x)\, dx \Big\} \mathrm{Erf}\Big(\frac{r}{\sqrt{(r+R)Q}} \Big) = 0. \tag{2.54}$$

We write $s_i = \ell_N(\mathsf{s})_i$ and assume for each $r \in \mathbb{N}$

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sum_{i>L} \int_{\mathsf{S}} \mathrm{Erf}(\frac{|s_i| - r}{\sqrt{c_3 T}}) \mu^N(d\mathsf{s}) = 0. \tag{2.55}$$

We remark that (2.55) is easy to check. Indeed, we prove in Lemma 2.22 that, if $s_i = \ell_N(\mathsf{s})_i$ is taken such that

$$|s_1| \le |s_2| \le \cdots, \tag{2.56}$$

then (2.55) follows from **(H1)** and (2.57) below.

$$\lim_{q\to\infty} \limsup_{N\to\infty} \int_{S\backslash S_q} \mathrm{Erf}(\frac{|x| - r}{\sqrt{c_3 T}}) \rho^{N,1}(x) dx = 0. \tag{2.57}$$

Let ℓ be the label as before. Let $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ be a family of solution of (2.51) satisfying $\mathbf{X}_0 = \mathbf{s}$ for $\mu \circ \ell^{-1}$-a.s. $\mathbf{s}$. We call $\mathbf{X}$ satisfies μ-absolute continuity condition if

$$\mu_t \prec \mu \quad \text{for all } t \ge 0, \tag{2.58}$$

where μ_t is the distribution of X_t and $\mu_t \prec \mu$ means μ_t is absolutely continuous with respect to μ. Here $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$, for $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}}$. By definition $\mathsf{X} = \{\mathsf{X}_t\}$ is the delabeled dynamics of $\mathbf{X}$ and by construction $\mathsf{X}_0 = \mu$ in distribution.

We say ISDE (2.51) has μ-uniqueness of solutions in law if $\mathbf{X}$ and $\mathbf{X}'$ are solutions with the same initial distributions satisfying the μ-absolute continuity condition, then they are equivalent in law. We assume:

(J6) ISDE (2.51) has μ-uniqueness of solutions in law.

Let $\mathbf{X}^N$ be a solution of (2.18). From (2.44) we can rewrite (2.18) as

$$dX_t^{N,i} = \sigma^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i}) dB_t^i + \frac{1}{2}\{\nabla_x \mathsf{a}^N + \mathsf{a}^N \mathsf{d}^N\}(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i}) dt. \tag{2.59}$$

We set $\mathbf{X}^{N,m} = (X^{N,1}, X^{N,2}, \dots, X^{N,m})$ $1 \le m \le N$ and $\mathbf{X}^m = (X^1, X^2, \dots, X^m)$. We say $\{\mathbf{X}^N\}$ is tight in $C([0,\infty); S^{\mathbb{N}})$ if each subsequence $\{\mathbf{X}^{N'}\}$ contains a subsequence $\{\mathbf{X}^{N''}\}$ such that $\{\mathbf{X}^{N'',m}\}$ is convergent weakly in $C([0,\infty); S^m)$ for each $m \in \mathbb{N}$.

Theorem 2.7. Assume **(H1)**–**(H4)** and **(J1)**–**(J5)**. Assume that $\mathbf{X}_0^N = \mu^N \circ \ell_N^{-1}$ in distribution. Then $\{\mathbf{X}^N\}$ is tight in $C([0,\infty); S^{\mathbb{N}})$ and each limit point $\mathbf{X}$ of $\{\mathbf{X}^N\}$ is a solution of (2.51) with initial distribution $\mu \circ \ell^{-1}$. Furthermore, if we assume **(J6)** in addition, then for any $m \in \mathbb{N}$

$$\lim_{N\to\infty} \mathbf{X}^{N,m} = \mathbf{X}^m \quad \text{weakly in } C([0,\infty), S^m). \tag{2.60}$$

Remark 2.8. To prove (2.60) it is sufficient to prove the convergence in $C([0,T]; S^m)$ for each $T \in \mathbb{N}$. We do this in the following sections.

Remark 2.9. (1) A sufficient condition for **(J3)** is obtained in [48, 49]. Indeed, if μ is a (Φ, Ψ)-quasi-Gibbs measure with upper semi-continuous potential (Φ, Ψ), then **(J3)** is satisfied. This condition is mild and is satisfied by all examples in the present paper. We refer to [48, 49] for the definition of quasi-Gibbs property.
(2) From the general theory of Dirichlet forms, we see that **(J4)** is equivalent to the non-collision of particles [16]. We refer to [19] for a necessary and sufficient condition of this non-collision property of interacting Brownian motions in finite-dimensions, which gives a sufficient condition of non-collision in infinite dimensions. We also refer to [45] for a sufficient condition for non-collision property of interacting Brownian motions in infinite-dimensions applicable to, in particular, determinantal point processes.
(3) From (2.54) of **(J5)**, we deduce that each tagged particle X^i does not explode [16, 46]. We remark that the delabeled dynamics $\mathsf{X} = \sum_i \delta_{X^i}$ are μ-reversible, and they thus never explode. Indeed, as for configuration-valued diffusions, explosion occurs if and only if infinitely many particles gather in a compact domain, so the explosion of tagged particle does not imply that of the configuration-valued process.
(4) It is known that, if we suppose **(H1)**, **(J1)**–**(J5)**, then ISDE (2.51) has a solution for $\mu \circ \ell^{-1}$-a.s. $\mathbf{s}$ satisfying the non-collision and non-explosion property [47]. Indeed, let $\mathbf{X} = (X^i)$ be the $S^{\mathbb{N}}$-valued continuous process consisting of tagged particles X^i of the delabeled diffusion process $\mathsf{X} = \sum_{i\in\mathbb{N}} \delta_{X^i}$ given by the Dirichlet form of **(J3)**. Then from **(J4)** and **(J5)** (2.54) we see $\mathbf{X}$ is uniquely determined by its initial starting point. It was proved that $\mathbf{X}$ is a solution of (2.51) in [47].

Remark 2.10. Assumption **(J6)** follows from tail triviality of μ [53], where tail triviality of μ means that the tail σ-field $\mathcal{T} = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}]$ is μ-trivial. Indeed, from tail triviality of μ and marginal assumptions (**(E1)**, **(F1)**, and **(F2)** in [53]), we obtain **(J6)**. Tail triviality holds for all determinantal point processes [50] and grand canonical Gibbs measures with sufficiently small inverse temperature $\beta > 0$.

2.3 Proof of Theorem 2.2

The purpose of this section is to prove Theorem 2.2. We assume the same assumptions as Theorem 2.2 throughout this section. We begin by proving (2.41).

Lemma 2.11. (2.41) holds.

Proof. From **(H1)** and (2.32), we obtain

$$\lim_{N\to\infty} \mathsf{b}_{r,s,\mathsf{p}}^N = \mathsf{b}_{r,s,\mathsf{p}} \quad \text{for } \bar{\mu}_r^{[1]}\text{-a.s. and in } L^{\hat{p}}(\bar{\mu}_r^{[1]}). \tag{2.61}$$

We next prove the convergence of $\{\mathsf{b}_{r,s,\mathsf{p}}\}$ as $\mathsf{p}\to\infty$. Note that

$$\begin{aligned}&\|\mathsf{b}_{r,s,\mathsf{p}}-\mathsf{b}_{r,s,\mathsf{q}}\|_{L^{\hat{p}}(\bar{\mu}_r^{[1]})} \\ &\le \|\mathsf{b}_{r,s,\mathsf{p}}-\mathsf{b}_{r,s,\mathsf{p}}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{[1]})}+\|\mathsf{b}_{r,s,\mathsf{p}}^N-\mathsf{b}_{r,s,\mathsf{q}}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{[1]})}+\|\mathsf{b}_{r,s,\mathsf{q}}^N-\mathsf{b}_{r,s,\mathsf{q}}\|_{L^{\hat{p}}(\bar{\mu}_r^{[1]})}.\end{aligned}\tag{2.62}$$

From (2.34) for each ϵ there exists a p_0 such that for all $\mathsf{p},\mathsf{q}\ge\mathsf{p}_0$

$$\sup_{N\in\mathbb{N}}\|\mathsf{b}_{r,s,\mathsf{p}}^N-\mathsf{b}_{r,s,\mathsf{q}}^N\|_{L^{\hat{p}}(\mu_r^{N,[1]})}<\epsilon \tag{2.63}$$

By (2.61) there exists an $N=N_{\mathsf{p},\mathsf{q}}$ such that

$$\|\mathsf{b}_{r,s,\mathsf{p}}-\mathsf{b}_{r,s,\mathsf{p}}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{[1]})}<\epsilon,\quad \|\mathsf{b}_{r,s,\mathsf{q}}-\mathsf{b}_{r,s,\mathsf{q}}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{[1]})}<\epsilon. \tag{2.64}$$

Putting (2.63) and (2.64) into (2.62), we deduce that $\{\mathsf{b}_{r,s,\mathsf{p}}\}_{\mathsf{p}\in\mathbb{N}}$ is a Cauchy sequence in $L^{\hat{p}}(\bar{\mu}_r^{[1]})$. Hence from (2.28), (2.34), and (2.37) we see

$$\lim_{\mathsf{p}\to\infty}\mathsf{b}_{r,s,\mathsf{p}}=\mathsf{b}_{r,s}\quad\text{ in } L^{\hat{p}}(\bar{\mu}_r^{[1]}). \tag{2.65}$$

Recall that $\mathsf{b}_{r,s}^N=\mathrm{E}^{\bar{\mu}_r^{N,[1]}}[\mathsf{b}^N|\mathcal{F}_{r,s}]$ by (2.24). Then, because $\mathcal{F}_{r,s}\subset\mathcal{F}_{r,s+1}$, we have

$$\mathsf{b}_{r,s}^N=\mathrm{E}^{\bar{\mu}_r^{N,[1]}}[\mathsf{b}_{r,s+1}^N|\mathcal{F}_{r,s}]. \tag{2.66}$$

From $\mathsf{b}_{r,s}^N=\mathrm{E}^{\bar{\mu}_r^{N,[1]}}[\mathsf{b}^N|\mathcal{F}_{r,s}]$ we have

$$\|\mathsf{b}_{r,s}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{N,[1]})}\le\|\mathsf{b}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{N,[1]})}$$

From this and (2.29) we obtain

$$\sup_{r<s}\limsup_{N\to\infty}\|\mathsf{b}_{r,s}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{N,[1]})}\le\limsup_{N\to\infty}\|\mathsf{b}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{N,[1]})}<\infty. \tag{2.67}$$

Combining (2.37), (2.66) and (2.67), we have

$$\mathsf{b}_{r,s}=\lim_{N\to\infty}\mathsf{b}_{r,s}^N=\lim_{N\to\infty}\mathrm{E}^{\bar{\mu}_r^{N,[1]}}[\mathsf{b}_{r,s+1}^N|\mathcal{F}_{r,s}]=\mathrm{E}^{\bar{\mu}_r^{[1]}}[\mathsf{b}_{r,s+1}|\mathcal{F}_{r,s}]. \tag{2.68}$$

From **(H1)**, (2.37), (2.67), and Fatou's lemma, we see that

$$\sup_{r<s}\|\mathsf{b}_{r,s}\|_{L^{\hat{p}}(\bar{\mu}_r^{[1]})}\le\sup_{r<s}\liminf_{N\to\infty}\|\mathsf{b}_{r,s}^N\|_{L^{\hat{p}}(\bar{\mu}_r^{N,[1]})}<\infty. \tag{2.69}$$

From (2.68) we deduce that $\{\mathsf{b}_{r,s}\}_{s=r+1}^{\infty}$ is martingale in s. Applying the martingale convergence theorem to $\{\mathsf{b}_{r,s}\}_{s=r+1}^{\infty}$ and using (2.69), we deduce that there exists a b_r such that

$$\mathsf{b}_{r,s}=\mathrm{E}^{\bar{\mu}_r^{[1]}}[\mathsf{b}_r|\mathcal{F}_{r,s}] \tag{2.70}$$

and that

$$\lim_{s\to\infty}\mathsf{b}_{r,s}=\mathsf{b}_r\quad\text{for } \bar{\mu}_r^{[1]}\text{-a.s. and in } L^{\hat{p}}(\bar{\mu}_r^{[1]}).$$

By the consistency of $\{\bar{\mu}_r^{[1]}\}_{r\in\mathbb{N}}$ in r, the function b_r in (2.70) can be taken to be independent of r. This together with (2.65) completes the proof of (2.41). $\square$

We proceed with the proof of the latter half of Theorem 2.2. Recall SDE (2.18). Then

$$X_t^{N,i} - X_0^{N,i} = \int_0^t \sigma^N(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})dB_u^i + \int_0^t \mathsf{b}^N(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})du. \tag{2.71}$$

Using the decomposition in (2.40), we see from (2.71) that

$$\begin{aligned} X_t^{N,i} - X_0^{N,i} = &\int_0^t \sigma^N(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})dB_u^i + \int_0^t \{\mathsf{b}_{r,s,\mathsf{p}}^N + \mathsf{b}^{\mathrm{tail}}\}(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})du \\ &+ \int_0^t \Big[\{\mathsf{b}_{r,s}^N - \mathsf{b}_{r,s,\mathsf{p}}^N\} + \{\mathsf{b}_{r,s}^{N,\mathrm{tail}} - \mathsf{b}^{\mathrm{tail}}\}\Big](X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})du. \end{aligned} \tag{2.72}$$

Let $\partial_{i,j} = \frac{\partial}{\partial x_{i,j}}$, $x_i = (x_{i,j})_{j=1}^d \in \mathbb{R}^d$, and $\mathbf{x}_m = (x_i)_{i=1}^m \in (\mathbb{R}^d)^m$. Set $\nabla_i = (\partial_{i,j})_{j=1}^d$. Let $\psi \in C_0^\infty(S^m)$ and $\mathsf{a}_i^N \nabla_i \nabla_i \psi(\mathbf{x}_m) = \sum_{k,l=1}^d \mathsf{a}_{kl}^N(x_i)\partial_{i,k}\partial_{i,l}\psi(\mathbf{x}_m)$. Applying the Itô formula to ψ and (2.72), and putting $\mathbf{X}_t^{N,m} = (X_t^{N,1}, \dots, X_t^{N,m})$, we deduce that

$$\begin{aligned} \psi(\mathbf{X}_t^{N,m}) - \psi(\mathbf{X}_0^{N,m}) = &\sum_{i=1}^m \Big(\int_0^t \nabla_i \psi(\mathbf{X}_u^{N,m}) \cdot \sigma^N(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})dB_u^i \\ &+ \int_0^t \frac{1}{2}\mathsf{a}_i^N \nabla_i \nabla_i \psi(\mathbf{X}_u^{N,m}) + \{\mathsf{b}_{r,s,\mathsf{p}}^N + \mathsf{b}^{\mathrm{tail}}\}(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i}) \cdot \nabla_i \psi(\mathbf{X}_u^{N,m})du \Big) \\ &+ \sum_{i=1}^m \int_0^t \nabla_i \psi(\mathbf{X}_u^{N,m}) \cdot \{\mathsf{b}_{r,s}^N - \mathsf{b}_{r,s,\mathsf{p}}^N\}(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})du \\ &+ \sum_{i=1}^m \int_0^t \nabla_i \psi(\mathbf{X}_u^{N,m}) \cdot \{\mathsf{b}_{r,s}^{N,\mathrm{tail}} - \mathsf{b}^{\mathrm{tail}}\}(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})du. \end{aligned} \tag{2.73}$$

We set

$$\mathsf{Q}_{r,s,\mathsf{p}}^N = \sum_{i=1}^m \int_0^T \Big|\{\mathsf{b}_{r,s}^N - \mathsf{b}_{r,s,\mathsf{p}}^N\}(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})\Big|du,$$

$$\mathsf{R}_{r,s}^N = \sum_{i=1}^m \int_0^T \Big|\{\mathsf{b}_{r,s}^{N,\mathrm{tail}} - \mathsf{b}^{\mathrm{tail}}\}(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i})\Big|du.$$

Lemma 2.12. For each $m, r < s \in \mathbb{N}$

$$\lim_{\mathsf{p}\to\infty} \limsup_{N\to\infty} \mathrm{E}^{\mu^N \circ \ell_N^{-1}} \big[(\mathsf{Q}_{r,s,\mathsf{p}}^N)^{\hat{p}}\big] = 0,$$

$$\lim_{s\to\infty} \limsup_{N\to\infty} \mathrm{E}^{\mu^N \circ \ell_N^{-1}} \big[(\mathsf{R}_{r,s}^N)^{\hat{p}}\big] = 0.$$

Proof. Lemma 2.12 follows from (2.35) and (2.39) immediately. □

Let $\Xi = S^m \times (\mathbb{R}^{d^2})^m \times (\mathbb{R}^d)^m$ and $\psi \in C_0^\infty(S^m)$. Let $F : C([0,T];\Xi) \to C([0,T];\mathbb{R})$ such that

$$\begin{aligned} F(\xi,\eta,\zeta)(t) = &\psi(\xi(t)) - \psi(\xi(0)) - \int_0^t \sum_{i=1}^m \zeta_i(u) \cdot \nabla_i \psi(\xi(u))du \\ &- \int_0^t \sum_{i=1}^m \Big(\frac{1}{2}\eta_i(u)\Delta_i \psi(\xi(u)) + \mathsf{b}^{\mathrm{tail}}(\xi_i(u)) \cdot \nabla_i \psi(\xi(u))\Big)du, \end{aligned} \tag{2.74}$$

where $\xi = (\xi_i)_{i=1}^m$, $\eta = (\eta_i)_{i=1}^m$, $\eta_i = (\eta_{i,kl})_{k,l=1}^d$, $\zeta = (\zeta_i)_{i=1}^m$, and $\Delta_i = \sum_{j=1}^d \partial_{i,j}^2$.

As $\psi \in C_0^\infty(S^m)$ and $\mathsf{b}^{\mathrm{tail}} \in C(S^m)$ by definition, we see that F satisfies the following.
(1) F is continuous.
(2) $F(\xi, \eta, \zeta)$ is bounded in (ξ, η) for each ζ, and linear in ζ for each (ξ, η).

Let $\mathbf{A}^{N,m} = (\mathsf{A}^{N,i})_{i=1}^m$ and $\mathbf{B}_{r,s,\mathsf{p}}^{N,m} = (\mathsf{B}_{r,s,\mathsf{p}}^{N,i})_{i=1}^m$ such that

$$\mathsf{A}^{N,i}(t) = \mathsf{a}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i}), \quad \mathsf{B}_{r,s,\mathsf{p}}^{N,i}(t) = \mathsf{b}_{r,s,\mathsf{p}}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i}). \tag{2.75}$$

Then we see from (2.73)–(2.75) that for each $m \in \mathbb{N}$

$$\begin{aligned} &\Big| F(\mathbf{X}^{N,m}, \mathbf{A}^{N,m}, \mathbf{B}_{r,s,\mathsf{p}}^{N,m}) - \sum_{i=1}^m \int_0^{\cdot} \nabla_i \psi(\mathbf{X}_u^{N,m}) \cdot \sigma^N(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i}) dB_u^i \Big| \\ &\le c_7 \{\mathsf{Q}_{r,s,\mathsf{p}}^N + \mathsf{R}_{r,s}^N\}, \end{aligned} \tag{2.76}$$

where $c_7 = c_7(\psi)$ is the constant such that $c_7 = \max_{i=1}^m \|\nabla_i \psi\|_{S^m}$ ($\|\cdot\|_A$ is the uniform norm over A as before). We take the limit of each term in (2.76) in the sequel.

Lemma 2.13. $\{X^{N,i}\}_{N \in \mathbb{N}}$, $\{\mathsf{A}^{N,i}\}_{N \in \mathbb{N}}$ and $\{\mathsf{B}_{r,s,\mathsf{p}}^{N,i}\}_{N \in \mathbb{N}}$ are tight for each $i, r, s, \mathsf{p} \in \mathbb{N}$.

Proof. The tightness of $\{X^{N,i}\}_{N \in \mathbb{N}}$ is clear from **(I1)**.

We note that $\{\nabla_x \mathsf{a}^N\}_N$ is uniformly bounded on $S_r \times \mathsf{S}$ for each $r \in \mathbb{N}$ by **(H4)**. Hence from this and **(I1)** there exists a constant c_8 independent of N such that for all $0 \le u, v \le T$

$$\mathrm{E}^{\mu^N \circ \ell_N^{-1}}[|\mathsf{A}^{N,i}(u) - \mathsf{A}^{N,i}(v)|^4; \sup_{t \in [0,T]} |X_t^{N,i}| \le a] \le c_8 |u - v|^2.$$

By **(I1)** we see that $\{\mathsf{A}^{N,i}(0)\}_{N \in \mathbb{N}}$ is tight. Combining these deduces the tightness of $\{\mathsf{A}^{N,i}\}_{N \in \mathbb{N}}$.

Recall that $\mathsf{B}_{r,s,\mathsf{p}}^{N,i}(t) = \mathsf{b}_{r,s,\mathsf{p}}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})$ and that $\mathsf{b}_{r,s,\mathsf{p}}^N$ is $\mathcal{F}_{r,s}$-measurable by assumption. By construction

$$P^{\mu^N \circ \ell_N^{-1}}(X_t^{N,j} \in S_r \text{ for all } 1 \le j \le m,\, 0 \le t \le T |\ \mathcal{L}_{r+s}^N \le m) = 1. \tag{2.77}$$

Let $c_9 = \sup_{N \in \mathbb{N}} \|\nabla \mathsf{b}_{r,s,\mathsf{p}}^N\|_{S \times \mathsf{S}_s^{m-1}}$. From (2.75), (2.33), (2.77), and (2.22) we see

$$\begin{aligned} &E^{\mu^N \circ \ell_N^{-1}}[|\mathsf{B}_{r,s,\mathsf{p}}^{N,i}(u) - \mathsf{B}_{r,s,\mathsf{p}}^{N,i}(v)|^4; \sup_{t \in [0,T]} |X_t^{N,i}| \le a,\ \mathcal{L}_{r+s}^N \le m] \\ &= E^{\mu^N \circ \ell_N^{-1}}[|\mathsf{b}_{r,s,\mathsf{p}}^N(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i}) - \mathsf{b}_{r,s,\mathsf{p}}^N(X_v^{N,i}, \mathsf{X}_v^{N,\diamond i})|^4; \sup_{t \in [0,T]} |X_t^{N,i}| \le a,\ \mathcal{L}_{r+s}^N \le m] \\ &\le E^{\mu^N \circ \ell_N^{-1}}[\sum_{j=1}^m c_9^4 |X_u^{N,j} - X_v^{N,j}|^4; \sup_{t \in [0,T]} |X_t^{N,i}| \le a,\ \mathcal{L}_{r+s}^N \le m] \\ &\le c_9^4 c_6 |u - v|^2 \quad \text{for all } 0 \le u, v \le T. \end{aligned}$$

From this, (2.21), and (2.23), we deduce the tightness of $\{\mathsf{B}_{r,s,\mathsf{p}}^{N,i}\}_{N \in \mathbb{N}}$. $\square$

Lemma 2.14. $\{((X^{N,i}, \mathsf{A}^{N,i}, \mathsf{B}_{r,s,\mathsf{p}}^{N,i}))_{i=1}^m\}_{N \in \mathbb{N}}$ is tight in $C([0,T], \Xi^m)$ for each $m, r, s, \mathsf{p} \in \mathbb{N}$.

Proof. Lemma 2.14 is obvious from Lemma 2.13. Indeed, the tightness of the probability measures on a countable product space follows from that of the distribution of each component. $\square$

Assumption **(I1)** and Lemma 2.14 combined with the diagonal argument imply that for any subsequence of $\{((X^{N,i}, \mathsf{A}^{N,i}, \mathsf{B}^{N,i}_{r,s,\mathsf{p}}))_{i=1}^m\}_{N,\,\mathsf{p}\in\mathbb{N},\,r<s<\infty}$, there exists a convergent-in-law subsequence, denoted by the same symbol. That is, for each $\mathsf{p}, s, r, m \in \mathbb{N}$,

$$\lim_{N\to\infty} (X^{N,i}, \mathsf{A}^{N,i}, \mathsf{B}^{N,i}_{r,s,\mathsf{p}})_{i=1}^m = (X^i, \mathsf{A}^i, \mathsf{B}^i_{r,s,\mathsf{p}})_{i=1}^m \quad \text{in law.} \tag{2.78}$$

We thus assume (2.78) in the rest of this section.

Let $\mathbf{A}^m = (\mathsf{A}^i)_{i=1}^m$, $\mathbf{B}^{N,m}_{r,s,\mathsf{p}} = (\mathsf{B}^{N,i}_{r,s,\mathsf{p}})_{i=1}^m$, and $\mathbf{X}^m = (X^i)_{i=1}^m$ for $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ in Theorem 2.2.

Lemma 2.15. For each $m \in \mathbb{N}$

$$\lim_{N\to\infty} F(\mathbf{X}^{N,m}, \mathbf{A}^{N,m}, \mathbf{B}^{N,m}_{r,s,\mathsf{p}}) = F(\mathbf{X}^m, \mathbf{A}^m, \mathbf{B}^m_{r,s,\mathsf{p}}) \quad \text{in law.} \tag{2.79}$$

Moreover, A^i and $\mathsf{B}^i_{r,s,\mathsf{p}}$ are given by

$$\mathsf{A}^i(t) = \mathsf{a}(X^i_t, \mathsf{X}^{\diamond i}_t), \quad \mathsf{B}^i_{r,s,\mathsf{p}}(t) = \mathsf{b}_{r,s,\mathsf{p}}(X^i_t, \mathsf{X}^{\diamond i}_t). \tag{2.80}$$

Proof. Recall that $F(\xi, \eta, \zeta)$ is continuous. Hence (2.79) follows from (2.78). By **(H4)** we see $\{\mathsf{a}^N\}$ converges to a uniformly on each $S_r \times \mathsf{S}$. Then, from this, (2.32), and (2.75) we obtain (2.80). $\square$

Lemma 2.16. For each $m \in \mathbb{N}$

$$\lim_{N\to\infty} \sum_{i=1}^m \int_0^{\cdot} \nabla_i \psi(\mathbf{X}^{N,m}_u) \cdot \sigma^N(X^{N,i}_u, \mathsf{X}^{N,\diamond i}_u) dB^i_u = \sum_{i=1}^m \int_0^{\cdot} \nabla_i \psi(\mathbf{X}^m_u) \cdot \sigma(X^i_u, \mathsf{X}^{\diamond i}_u) d\hat{B}^i_u \quad \text{in law,}$$

where $(\hat{B}^i)_{i=1}^m$ is the first m-components of a $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion $(\hat{B}^i)_{i\in\mathbb{N}}$.

Proof. By the calculation of quadratic variation, we see

$$\begin{aligned} &\langle \int_0^{\cdot} \partial_{i,k}\psi(\mathbf{X}^{N,m}_u) \sum_{n=1}^d \sigma^N_{kn}(X^{N,i}_u, \mathsf{X}^{N,\diamond i}_u) dB^{i,n}_u, \int_0^{\cdot} \partial_{j,l}\psi(\mathbf{X}^{N,m}_u) \sum_{n=1}^d \sigma^N_{ln}(X^{N,j}_u, \mathsf{X}^{N,\diamond j}_u) dB^{j,n}_u \rangle_u \\ &= \delta_{ij} \int_0^{\cdot} \mathsf{a}^N_{kl}(X^{N,i}_u, \mathsf{X}^{N,\diamond i}_u) \partial_{i,k}\psi(\mathbf{X}^{N,m}_u) \partial_{i,l}\psi(\mathbf{X}^{N,m}_u) du. \end{aligned}$$

From **(H4)**, we see that a^N converges to a uniformly on S_r for each $r \in \mathbb{N}$. Hence we deduce from **(I1)** and $\psi \in C_0^\infty(S^m)$ the convergence in law such that

$$\begin{aligned} \lim_{N\to\infty} &\sum_{i=1}^m \int_0^{\cdot} \mathsf{a}^N_{kl}(X^{N,i}_u, \mathsf{X}^{N,\diamond i}_u) \partial_{i,k}\psi(\mathbf{X}^{N,m}_u) \partial_{i,l}\psi(\mathbf{X}^{N,m}_u) du \\ &= \sum_{i=1}^m \int_0^{\cdot} \mathsf{a}_{kl}(X^i_u, \mathsf{X}^{\diamond i}_u) \partial_{i,k}\psi(\mathbf{X}^m_u) \partial_{i,l}\psi(\mathbf{X}^m_u) du. \end{aligned}$$

Then the right-hand side gives the quadratic variation of $\sum_{i=1}^m \int_0^{\cdot} \nabla_i \psi(\mathbf{X}^m_u) \cdot \sigma(X^i_u, \mathsf{X}^{\diamond i}_u) d\hat{B}^i_u$. This completes the proof. $\square$

We are now ready for the proof of Theorem 2.2.

Proof of Theorem 2.2. From Lemma 2.12 and (2.76) we deduce that

$$\begin{aligned}\limsup_{N\to\infty} \mathrm{E}^{\mu^N\circ\ell_N^{-1}}\Big[\sup_{0\le t\le T}\Big|&F(\mathbf{X}^{N,m},\mathbf{A}^{N,m},\mathbf{B}^{N,m}_{r,s,\mathsf{p}})(t)\\ &-\sum_{i=1}^m\int_0^t \nabla_i\psi(\mathbf{X}^{N,m}_u)\cdot\sigma^N(X^{N,i}_u,\mathsf{X}^{N,\diamond i}_u)dB^i_u\Big|^{\hat{p}}\Big]\\ &\le \limsup_{N\to\infty}\mathrm{E}^{\mu^N\circ\ell_N^{-1}}\big[(\mathsf{Q}^N_{r,s,\mathsf{p}})^{\hat{p}}+(\mathsf{R}^N_{r,s})^{\hat{p}}\big]=:c_{10}(s,\mathsf{p}),\end{aligned}$$

where $0\le c_{10}(s,\mathsf{p})=c_{10}(s,\mathsf{p},\psi)\le\infty$ is a constant depending on s,p,ψ. Applying Lemma 2.15 and Lemma 2.16 to (2.76), we then deduce that

$$\mathrm{E}^{\mu\circ\ell^{-1}}\big[\sup_{0\le t\le T}\big|F(\mathbf{X}^m,\mathbf{A}^m,\mathbf{B}^m_{r,s,\mathsf{p}})(t)-\sum_{i=1}^m\int_0^t\nabla_i\psi(\mathbf{X}^m_u)\cdot\sigma(X^i_u,\mathsf{X}^{\diamond i}_u)d\hat{B}^i_u\big|^{\hat{p}}\big]\le c_{10}(s,\mathsf{p}).$$

From this and (2.74), we obtain that

$$\begin{aligned}\mathrm{E}^{\mu\circ\ell^{-1}}\big[\sup_{0\le t\le T}\big|&\psi(\mathbf{X}^m_t)-\psi(\mathbf{X}^m_0)-\sum_{i=1}^m\int_0^t\nabla_i\psi(\mathbf{X}^m_u)\cdot\sigma(X^i_u,\mathsf{X}^{\diamond i}_u)d\hat{B}^i_u\\ &-\sum_{i=1}^m\int_0^t\frac{1}{2}\mathsf{a}(X^i_u,\mathsf{X}^{\diamond i}_u)\nabla_i\nabla_i\psi(\mathbf{X}^m_u)+\mathsf{b}^{\mathrm{tail}}(X^i_u)\cdot\nabla_i\psi(\mathbf{X}^m_u)du\\ &-\sum_{i=1}^m\int_0^t\mathsf{b}_{r,s,\mathsf{p}}(X^i_u,\mathsf{X}^{\diamond i}_u)\cdot\nabla_i\psi(\mathbf{X}^m_u)du\big|^{\hat{p}}\big]\\ &\le c_{10}(s,\mathsf{p}).\end{aligned}\tag{2.81}$$

Take $\psi=\psi_R\in C_0(S^m)$ such that $\psi(x_1,\dots,x_m)=x_i$ for $\{|x_j|\le R; j=1,\dots,m\}$ while keeping $|\nabla_i\psi|$ bounded in such a way that

$$c_{10}(\mathsf{p},s)=\sup_R c_{10}(\mathsf{p},s,R)=o(\mathsf{p},s).$$

Then we deduce from (2.81) that

$$\begin{aligned}\mathrm{E}^{\mu\circ\ell^{-1}}\big[\sup_{0\le t\le T}\big|&X^i_{t\wedge\tau_R}-X^i_0-\int_0^{t\wedge\tau_R}\sigma(X^i_u,\mathsf{X}^{\diamond i}_u)d\hat{B}^i_u\\ &-\int_0^{t\wedge\tau_R}\{\mathsf{b}_{r,s,\mathsf{p}}(X^i_u,\mathsf{X}^{\diamond i}_u)+\mathsf{b}^{\mathrm{tail}}(X^i_u)\}du\big|^{\hat{p}}\big]\le c_{10}(s,\mathsf{p}),\end{aligned}\tag{2.82}$$

where τ_R is a stopping time such that, for $\mathbf{X}^m=(X^i,\mathsf{X}^{\diamond i})_{i=1}^m\in C([0,T];(S\times\mathsf{S})^m)$,

$$\tau_R=\inf\{t>0;|X^i_t|\ge R \text{ for some } i=1,\dots,m\}.$$

As $R > 0$ is arbitrary, (2.82) holds for all $R > 0$. Taking $R \to \infty$, we thus obtain

$$\begin{aligned}
&\mathrm{E}^{\mu\circ\ell^{-1}}\big[\sup_{0\le t\le T}\big|X_t^i - X_0^i - \int_0^t \sigma(X_u^i, \mathsf{X}_u^{\diamond i})d\hat{B}_u^i \\
&\qquad\qquad - \int_0^t \{\mathsf{b}_{r,s,\mathsf{p}}(X_u^i, \mathsf{X}_u^{\diamond i}) + \mathsf{b}^{\mathrm{tail}}(X_u^i)\}du\big|\big] \\
&\le \liminf_{R\to\infty}\mathrm{E}^{\mu\circ\ell^{-1}}\big[\sup_{0\le t\le T}\big|X_{t\wedge\tau_R}^i - X_0^i - \int_0^{t\wedge\tau_R} \sigma(X_u^i, \mathsf{X}_u^{\diamond i})d\hat{B}_u^i \\
&\qquad\qquad - \int_0^{t\wedge\tau_R} \{\mathsf{b}_{r,s,\mathsf{p}}(X_u^i, \mathsf{X}_u^{\diamond i}) + \mathsf{b}^{\mathrm{tail}}(X_u^i)\}du\big|\big] \\
&\le c_{10}(s,\mathsf{p})^{1/\hat{p}}.
\end{aligned} \tag{2.83}$$

We note here that the integrands in the first and second lines of (2.83) are uniformly integrable because of (2.82). Taking $\mathsf{p} \to \infty$, then $s \to \infty$ in (2.83), and using assumptions (2.36) and (2.42) we thus obtain

$$\mathrm{E}^{\mu\circ\ell^{-1}}\big[\sup_{0\le t\le T}\big|X_t^i - X_0^i - \int_0^t \sigma(X_u^i, \mathsf{X}_u^{\diamond i})d\hat{B}_u^i - \int_0^t \{\mathsf{b}(X_u^i, \mathsf{X}_u^{\diamond i}) + \mathsf{b}^{\mathrm{tail}}(X_u^i)\}du\big|\big] = 0.$$

This implies for all $0 \le t \le T$

$$X_t^i - X_0^i - \int_0^t \sigma(X_u^i, \mathsf{X}_u^{\diamond i})d\hat{B}_u^i - \int_0^t \{\mathsf{b}(X_u^i, \mathsf{X}_u^{\diamond i}) + \mathsf{b}^{\mathrm{tail}}(X_u^i)\}du = 0. \tag{2.84}$$

We deduce (2.43) from (2.84), which completes the proof of Theorem 2.2. $\square$

2.4 Proof of Theorem 2.7

Is this section we prove Theorem 2.7 using Theorem 2.2. **(H1)**–**(H4)** are commonly assumed in Theorem 2.7 and Theorem 2.2. Hence our task is to derive condition **(I1)**–**(I5)** from conditions stated in Theorem 2.7. From **(J2)** we easily deduce that

$$\lim_{N\to\infty} u^N = u \quad \text{in } L^{\hat{p}}_{\mathrm{loc}}(S, dx), \tag{2.85}$$

$$\lim_{N\to\infty} \mathsf{g}_s^N = \mathsf{g}_s \quad \text{in } L^{\hat{p}}_{\mathrm{loc}}(\mu^{[1]}) \quad \text{for all } s. \tag{2.86}$$

Lemma 2.17. μ has a logarithmic derivative d^{μ} in $L^p_{\mathrm{loc}}(\mu^{[1]})$, where $1 \le p < \hat{p}$.

Proof. We use a general theory developed in [47]. **(H1)** corresponds to (4.1) and (4.2) in [47]. (2.85), (2.86), (2.47), and (2.49) correspond to (4.15), (4.30), (4.29), and (4.31) in [47]. Then all the assumptions of [47, Theorem 45] are satisfied. We thus deduce Lemma 2.17 from [47, Theorem 45]. $\square$

Let $\{\mathbf{X}^N\}_{N\in\mathbb{N}}$ be a sequence of solutions in (2.18) and (2.19). We set the m-labeling

$$\mathbf{X}^{N,[m]} = (X^{N,1}, \dots, X^{N,m}, \sum_{j=1+m}^{N} \delta_{X^{N,j}}). \tag{2.87}$$

It is known [46, 47] that $\mathbf{X}^{N,[m]}$ is a diffusion process associated with the Dirichlet form $\mathcal{E}^{\mu^{N,[m]}}$ on $L^2(S^m \times \mathsf{S}, \mu^{N,[m]})$ such that

$$\mathcal{E}^{\mu^{N,[m]}}(f,g) = \int_{S^m \times \mathsf{S}} \frac{1}{2}\{\sum_{i=1}^{m} \nabla_i f \cdot \nabla_i g\} + \mathbb{D}[f,g] d\mu^{N,[m]}, \tag{2.88}$$

where the domain $\mathcal{D}^{[m]}$ is taken as the closure of $\mathcal{D}_0^{[m]} = C_0^\infty(S^m) \otimes \mathcal{D}_\circ$. Note that the coordinate function $x_i = x_i \otimes 1$ is locally in $\mathcal{D}^{[m]}$. From this we can regard $\{X_t^{N,i}\}$ as a Dirichlet process of the m-labeled diffusion $\mathbf{X}^N$ associated with the Dirichlet space as above. In other words, we can write

$$X_t^{N,i} - X_0^{N,i} = f_i(\mathbf{X}_t^N) - f_i(\mathbf{X}_0^N) =: A_t^{[f_i]},$$

where $f_i(\mathsf{x}, \mathsf{s}) = x_i \otimes 1$, $x_i \in \mathbb{R}^d$, and $\mathbf{x} = (x_j)_{j=1}^m \in (\mathbb{R}^d)^m$. By the Fukushima decomposition of $X_t^{N,i}$, there exist a unique continuous local martingale additive functional $\mathsf{M}^{N,i} = \{\mathsf{M}_t^{N,i}\}$ and an additive functional of zero energy $\mathsf{N}^{N,i} = \{\mathsf{N}_t^{N,i}\}$ such that

$$X_t^{N,i} - X_0^{N,i} = \mathsf{M}_t^{N,i} + \mathsf{N}_t^{N,i}.$$

We refer to [16, Chapter 5] for the Fukushima decomposition. Because of (2.18), we then have

$$\mathsf{M}_t^{N,i} = \int_0^t \sigma^N(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i}) dB_u^i, \quad \mathsf{N}_t^{N,i} = \int_0^t \mathsf{b}^N(X_u^{N,i}, \mathsf{X}_u^{N,\diamond i}) du.$$

Lemma 2.18. Let $r_T : C([0,T]; S) \to C([0,T]; S)$ be such that $r_T(X)_t = X_{T-t}$. Suppose that $\mathbf{X}_0^{N,[m]} = \mu^{N,[m]}$ in law. Then

$$X_t^{N,i} - X_0^{N,i} = \frac{1}{2}\mathsf{M}_t^{N,i} + \frac{1}{2}(\mathsf{M}_{T-t}^{N,i}(r_T) - \mathsf{M}_T^{N,i}(r_T)) \quad \text{a.s..} \tag{2.89}$$

Proof. Applying the Lyons-Zheng decomposition [16, Theorem 5.7.1] to additive functionals $A^{[f_i]}$ for $1 \le i \le m$, we obtain (2.89). □

Lemma 2.19. (I1) holds.

Proof. Although $\mathsf{M}^{N,i}$ is a d-dimensional martingale by definition, we assume $d = 1$ here and prove only this case for simplicity. The general case $d \ge 1$ can be proved in a similar fashion. Let c_3 be the constant in (2.20) (under the assumption $d = 1$). Then we note that for $u \ge v$

$$0 \le \langle \mathsf{M}^{N,i} \rangle_u - \langle \mathsf{M}^{N,i} \rangle_v = \int_v^u \mathsf{A}^{N,i}(t) dt \le c_3(u - v) \tag{2.90}$$

We begin by proving (2.22). From a standard calculation of martingales and (2.90), we obtain

$$\begin{aligned} \mathrm{E}^{\mu^N \circ \ell_N^{-1}}[|\mathsf{M}_u^{N,i} - \mathsf{M}_v^{N,i}|^4] &= \mathrm{E}^{\mu^N \circ \ell_N^{-1}}[|B_{\langle \mathsf{M}^{N,i} \rangle_u} - B_{\langle \mathsf{M}^{N,i} \rangle_v}|^4] \\ &= 3\mathrm{E}^{\mu^N \circ \ell_N^{-1}}\big[|\langle \mathsf{M}^{N,i} \rangle_u - \langle \mathsf{M}^{N,i} \rangle_v|^2\big] \\ &\le c_{11}|u - v|^2, \end{aligned}$$

where $c_{11} = 3c_3^2$ and $\{B_t\}$ is a one-dimensional Brownian motion. Applying the same calculation to $\mathsf{M}^{N,i}_{T-t}(r_T) - \mathsf{M}^{N,i}_T(r_T)$, we have

$$\mathrm{E}^{\mu^N \circ \ell_N^{-1}}[|\mathsf{M}^{N,i}_{T-t}(r_T) - \mathsf{M}^{N,i}_{T-u}(r_T)|^4] \le c_{11}|t-u|^2 \quad \text{for each } 0 \le t, u \le T. \tag{2.91}$$

Combining (2.89) and (2.91) with the Lyons-Zheng decomposition (2.89), we thus obtain

$$\mathrm{E}^{\mu^N \circ \ell_N^{-1}}[|X^{N,i}_t - X^{N,i}_u|^4] \le 2c_{11}|t-u|^2 \quad \text{for each } 0 \le t, u \le T. \tag{2.92}$$

Taking a sum over $i = 1, \dots, m$ in (2.92), we deduce (2.22).

We next prove (2.21). From (2.89) we have

$$2|X^{N,i}_t - X^{N,i}_0| \le |\mathsf{M}^{N,i}_t| + |\mathsf{M}^{N,i}_{T-t}(r_T) - \mathsf{M}^{N,i}_T(r_T)| \quad \text{a.s..}$$

From this and a representation theorem of martingales, we obtain

$$\begin{aligned} & P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,T]} |X^{N,i}_t - X^{N,i}_0| \ge a) \\ \le & P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,T]} |\mathsf{M}^{N,i}_t| \ge a) + P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,T]} |\mathsf{M}^{N,i}_{T-t}(r_T) - \mathsf{M}^{N,i}_T(r_T)| \ge a) \\ = & 2P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,T]} |\mathsf{M}^{N,i}_t| \ge a) \\ = & 2P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,T]} |B_{\langle \mathsf{M}^{N,i}\rangle_t}| \ge a). \end{aligned} \tag{2.93}$$

A direct calculation shows

$$P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,T]} |B_{\langle \mathsf{M}^{N,i}\rangle_t}| \ge a) \le P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,\sqrt{c_3}T]} |B_t| \ge a) \le \mathrm{Erf}(\frac{a}{\sqrt{c_3}T}) \tag{2.94}$$

From (2.93), (2.94), and **(H2)**, we obtain (2.21).

We proceed with the proof of (2.23). Similarly as (2.93) and (2.94), we deduce

$$\begin{aligned} P^{\mu^N \circ \ell_N^{-1}}(\inf_{t\in[0,T]} |X^{N,i}_t| \le r) \le & P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,T]} |X^{N,i}_t - X^{N,i}_0| \ge |X^{N,i}_0| - r) \\ \le & 2P^{\mu^N \circ \ell_N^{-1}}(\sup_{t\in[0,T]} |\mathsf{M}^{N,i}_t| \ge |X^{N,i}_0| - r) \\ \le & 2\int_{\mathsf{S}} \mathrm{Erf}(\frac{|s_i| - r}{\sqrt{c_3}T})\mu^N(d\mathsf{s}), \end{aligned} \tag{2.95}$$

where $s_i = \ell_N(\mathsf{s})_i$. We note that $X^{N,i}_0 = s_i$ by construction. From (2.95) and (2.55), we deduce

$$\begin{aligned} \limsup_{N\to\infty} P^{\mu^N \circ \ell_N^{-1}}(\mathcal{L}^N_r > L) \le & \limsup_{N\to\infty} \sum_{i>L} P^{\mu^N \circ \ell_N^{-1}}(\inf_{t\in[0,T]} |X^{N,i}_t| \le r) \\ \le & 2\limsup_{N\to\infty} \sum_{i>L} \int_{\mathsf{S}} \mathrm{Erf}(\frac{|s_i| - r}{\sqrt{c_3}T})\mu^N(d\mathsf{s}) \\ \to & 0 \quad (L \to \infty). \end{aligned}$$

This completes the proof. $\square$

Lemma 2.20. **(I2)** holds.

Proof. (2.29) follows from (2.85), (2.86), and (2.49). For each $i \in \mathbb{N}$ we deduce that

$$E^{\mu^N \circ \ell_N^{-1}}[\int_0^T |\mathsf{b}_{r,s,\mathsf{p}}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})|^{\hat{p}} dt] \le \sum_{i=1}^N E^{\mu^N \circ \ell_N^{-1}}[\int_0^T |\mathsf{b}_{r,s,\mathsf{p}}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})|^{\hat{p}} dt] \tag{2.96}$$
$$= E^{\mu^N \circ \ell_N^{-1}}[\sum_{i=1}^N \int_0^T |\mathsf{b}_{r,s,\mathsf{p}}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})|^{\hat{p}} dt]$$
$$= E^{\mu^{N,[1]}}[\int_0^T |\mathsf{b}_{r,s,\mathsf{p}}^N(\mathbf{X}_t^{N,[1]})|^{\hat{p}} dt].$$

Diffusion process $\mathbf{X}^{N,[1]}$ in (2.87) with $m = 1$ given by the Dirichlet form $\mathcal{E}^{\mu^{N,[1]}}$ in (2.88) is $\mu^{N,[1]}$-symmetric. Hence we see that for all $0 \le t \le T$

$$E^{\mu^{N,[1]}}[|\mathsf{b}_{r,s,\mathsf{p}}^N(\mathbf{X}_t^{N,[1]})|^{\hat{p}}] \le \int_{S \times \mathsf{S}} |\mathsf{b}_{r,s,\mathsf{p}}^N|^{\hat{p}} d\mu^{N,[1]}.$$

This yields

$$\int_0^T dt\, E^{\mu^{N,[1]}}[|\mathsf{b}_{r,s,\mathsf{p}}^N(\mathbf{X}_t^{N,[1]})|^{\hat{p}}] \le T \int_{S \times \mathsf{S}} |\mathsf{b}_{r,s,\mathsf{p}}^N|^{\hat{p}} d\mu^{N,[1]}. \tag{2.97}$$

From (2.96) and (2.97) we obtain (2.30). □

Lemma 2.21. **(I3)**–**(I5)** hold.

Proof. Conditions (2.32), (2.33), and (2.34) follow from **(J1)**, **(J2)**, **(I1)**, **(I2)**, and (2.46). Similarly, as Lemma 2.20, we obtain for each $i \in \mathbb{N}$

$$E^{\mu^N \circ \ell_N^{-1}}[\int_0^T |(\mathsf{b}_{r,s}^N - \mathsf{b}_{r,s,\mathsf{p}}^N)(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})|^{\hat{p}} dt] \le T \int_{S \times \mathsf{S}} |\mathsf{b}_{r,s}^N - \mathsf{b}_{r,s,\mathsf{p}}^N|^{\hat{p}} d\mu^{N,[1]}. \tag{2.98}$$

Hence (2.35) follows from (2.98) and (2.34). (2.36) follows from (2.65) and an inequality similar to (2.98). We have thus obtained **(I3)**. Condition (2.38) follows from **(J1)** and **(J2)**. Similarly, as Lemma 2.20, we obtain for each $i \in \mathbb{N}$

$$E^{\mu^N \circ \ell_N^{-1}}[\int_0^T |(\mathsf{b}_{r,s}^{N,\mathrm{tail}} - \mathsf{b}^{\mathrm{tail}})(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})|^{\hat{p}} dt] \le T \int_{S \times \mathsf{S}} |\mathsf{b}_{r,s}^{N,\mathrm{tail}} - \mathsf{b}^{\mathrm{tail}}|^{\hat{p}} d\mu^{N,[1]}.$$

This together with (2.38) implies (2.39). Hence we have **(I4)**. Similarly as Lemma 2.20, we obtain (2.42) from (2.41). We have thus obtained **(I5)**. □

Proof of Theorem 2.7. **(I1)**–**(I5)** follows from Lemma 2.19–Lemma 2.21. Hence we deduce Theorem 2.7 from Theorem 2.2. □

We finally present a sufficient condition of (2.55).

Lemma 2.22. Assume **(H1)** and (2.57) for each $r \in \mathbb{N}$ as Section 2.2. We take the label ℓ_N as (2.56). Then (2.55) holds.

Proof. Let $c_{12} = c_{12}(N)$ be such that

$$c_{12} = \int_S \mathrm{Erf}(\frac{|x| - r}{\sqrt{c_3 T}})\rho^{N,1}(x)dx.$$

Let $c_{13} = \limsup_{N\to\infty} c_{12}(N)$. Then from **(H1)** and (2.57), we see that for each large r

$$\begin{aligned} c_{13} \le & \lim_{N\to\infty} \int_{S_r} \mathrm{Erf}(\frac{|x| - r}{\sqrt{c_3 T}})\rho^{N,1}(x)dx + \limsup_{N\to\infty} \int_{S\backslash S_r} \mathrm{Erf}(\frac{|x| - r}{\sqrt{c_3 T}})\rho^{N,1}(x)dx \\ & < \infty. \end{aligned} \tag{2.99}$$

From **(H1)** we see that $\{\mu^N\}_{N\in\mathbb{N}}$ converges to μ weakly. Hence $\{\mu^N\}_{N\in\mathbb{N}}$ is tight. This implies that there exists a sequence of increasing sequences of natural numbers $\mathbf{a}_n = \{a_n(m)\}_{m=1}^{\infty}$ such that $\mathbf{a}_n < \mathbf{a}_{n+1}$ and that for each m

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu^N(\mathsf{s}(S_m) \ge a_n(m)) = 0.$$

Without loss of generality, we can take $a_n(m) > m$ for all $m, n \in \mathbb{N}$. Then from this, we see that there exists a sequence $\{\mathsf{p}(L)\}_{L\in\mathbb{N}}$ converging to ∞ such that $\mathsf{p}(L) < L$ for all $L \in \mathbb{N}$ and that

$$\lim_{L\to\infty} \limsup_{N\to\infty} \mu^N(\mathsf{s}(S_{\mathsf{p}(L)}) \ge L) = 0. \tag{2.100}$$

Recall that the label $\ell_N(\mathsf{s}) = (s_i)_{i\in\mathbb{N}}$ satisfies $|s_1| \le |s_2| \le \cdots$. Using this, we divide the set S as in such a way that

$$\{s_L \in S_{\mathsf{p}(L)}\} \text{ and } \{s_L \notin S_{\mathsf{p}(L)}\}.$$

Then $\mathsf{s} \in \{s_L \in S_{\mathsf{p}(L)}\}$ if and only if $\mathsf{s}(S_{\mathsf{p}(L)}) \ge L$. Hence we easily see that

$$\sum_{i>L} \int_{\mathsf{S}} \mathrm{Erf}(\frac{|s_i| - r}{\sqrt{c_3 T}})\mu^N(d\mathsf{s}) \le c_{12}(N)\mu^N(\{\mathsf{s}(S_{\mathsf{p}(L)}) \ge L\}) + \int_{S\backslash S_{\mathsf{p}(L)}} \mathrm{Erf}(\frac{|x| - r}{\sqrt{c_3 T}})\rho^{N,1}(x)dx.$$

Taking the limits on both sides, we obtain

$$\begin{aligned} & \lim_{L\to\infty} \limsup_{N\to\infty} \sum_{i>L} \int_{\mathsf{S}} \mathrm{Erf}(\frac{|s_i| - r}{\sqrt{c_3 T}})\mu^N(d\mathsf{s}) \le \\ & c_{13} \lim_{L\to\infty} \limsup_{N\to\infty} \mu^N(\{\mathsf{s}(S_{\mathsf{p}(L)}) \ge L\}) + \lim_{L\to\infty} \limsup_{N\to\infty} \int_{S\backslash S_{\mathsf{p}(L)}} \mathrm{Erf}(\frac{|x| - r}{\sqrt{c_3 T}})\rho^{N,1}(x)dx. \end{aligned}$$

Applying (2.99) and (2.100) to the second term, and (2.57) to the third, we deduce (2.55). □

2.5 Examples

The finite-particle approximation in Theorem 2.7 contains many examples such as Airy_β point processes ($\beta = 1, 2, 4$), $\mathrm{Bessel}_{2,\alpha}$ point process, the Ginibre point process, the Lennard–Jones 6-12 potential, and Riesz potentials.The first three examples are related to random matrix theory and the interaction $\Psi(x) = -\log|x|$, the logarithmic function. We present these in this section. For this we shall confirm the assumptions in Theorem 2.7, that is, assumptions **(H1)**–**(H4)** and **(J1)**–**(J6)**.

Assumption **(H1)** is satisfied for the first three examples [43, 67]. As for the last two examples, we assume **(H1)**. We also assume **(H2)**. **(H3)** can be proved in the same way as given in [53]. In all examples, a is always a unit matrix. Hence it holds that **(H4)** is satisfied and that (2.44) in **(J1)** becomes $\mathsf{b}^N = \frac{1}{2}\mathsf{d}^N$. From this we see that SDEs (2.59) and (2.51) become

$$dX_t^{N,i} = dB_t^{N,i} + \frac{1}{2}\mathsf{d}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})\,dt \quad (1 \le i \le N), \tag{2.101}$$

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, \mathsf{X}_t^{\diamond i})\,dt \quad (i \in \mathbb{N}), \tag{2.102}$$

where d^μ is the logarithmic derivative of μ given by (2.50). Assumption **(J6)** for the first three examples with $\beta = 2$ can be proved in the same way as [53] as we explained in Remark 2.10. Thus, in the rest of this section, our task is to check assumptions **(J2)**–**(J5)**.

2.5.1 The Airy$_\beta$ interacting Brownian motion ($\beta = 1, 2, 4$)

Let $\mu_{\mathrm{Airy},\beta}^N$ and $\mu_{\mathrm{Airy},\beta}$ be as in Section 2.1. Recall SDEs (2.10) and (2.11) in Section 2.1. Let $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ and $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ be solutions of

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j=1,\,j\ne i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}}dt - \frac{\beta}{2}\{N^{1/3} + \frac{1}{2N^{1/3}}X_t^{N,i}\}dt, \tag{2.10}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\{\sum_{|X_t^j|<r,j\ne i} \frac{1}{X_t^i - X_t^j} - \int_{|x|<r}\frac{\varrho(x)}{-x}\,dx\}dt \quad (i \in \mathbb{N}). \tag{2.11}$$

Proposition 2.23. If $\beta = 1, 4$, then each sub-sequential limit of solutions $\mathbf{X}^N$ of (2.10) satisfies (2.11). If $\beta = 2$, then the full sequence converges to (2.11).

Proof. Conditions **(J2)**–**(J5)** other than (2.48) can be proved in the same way as given in [54]. In [54], we take $\chi_s(x) = 1_{S_s}(x)$; its adaptation to the present case is easy.

We consider estimates of correlation functions such that

$$\inf_{N\in\mathbb{N}} \rho_{\mathrm{Airy},\beta}^{N,1}(x) \ge c_{14} \quad \text{for all } x \in S_r \tag{2.103}$$

$$\sup_{N\in\mathbb{N}} \rho_{\mathrm{Airy},\beta}^{N,2}(x,y) \le c_{15}|x-y| \quad \text{for all } x, y \in S_r, \tag{2.104}$$

where $c_{14}(r)$ and $c_{15}(r)$ are positive constants. The first estimate is trivial because $\rho_{\mathrm{Airy},\beta}^{N,1}$ converges to $\rho_{\mathrm{Airy},\beta}^1$ uniformly on S_r and, all these correlation functions are continuous and positive. The second estimate follows from the determinantal expression of the correlation functions and bounds on derivative of determinantal kernels. Estimates needed for the proof can be found in [54] and the detail of the proof of (2.104) is left to the reader.

Equation (2.48) follows from (2.103) and (2.104). Indeed, the integral in (2.48) is taken on the bounded domain and the singularity of integral of $g^N(x,y) = \beta/(x-y)$ near

$\{x = y\}$ is logarithmic. Furthermore, the one-point correlation function $\rho^{N,1}_{\mathrm{Airy},\beta,x}$ of the reduced Palm measure conditioned at x is controlled by the upper bound of the two-point correlation function and the lower bound of one-point correlation function because

$$\rho^{N,1}_{\mathrm{Airy},\beta,x}(y) = \frac{\rho^{N,2}_{\mathrm{Airy},\beta}(x,y)}{\rho^{N,1}_{\mathrm{Airy},\beta}(x)}.$$

Using these facts, we see that (2.103) and (2.104) imply (2.48). □

2.5.2 The Bessel$_{2,\alpha}$ interacting Brownian motion

Let $S = [0,\infty)$ and $\alpha \in [1,\infty)$. We consider the Bessel$_{2,\alpha}$ point process $\mu_{\mathrm{bes},2,\alpha}$ and their N-particle version. The Bessel$_{2,\alpha}$ point process $\mu_{\mathrm{bes},2,\alpha}$ is a determinantal point process with kernel

$$\begin{aligned}\mathsf{K}_{\mathrm{bes},2,\alpha}(x,y) &= \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)} \\ &= \frac{\sqrt{x}J_{\alpha+1}(\sqrt{x})J_\alpha(\sqrt{y}) - J_\alpha(\sqrt{x})\sqrt{y}J_{\alpha+1}(\sqrt{y})}{2(x-y)},\end{aligned} \tag{2.105}$$

where J_α is the Bessel function of order α [67, 17]. The density $\mathsf{m}^N_\alpha(\mathbf{x})d\mathbf{x}$ of the associated N-particle systems $\mu^N_{\mathrm{bes},2,\alpha}$ is given by

$$\mathsf{m}^N_\alpha(\mathbf{x}) = \frac{1}{\mathcal{Z}^N_\alpha} e^{-\sum_{i=1}^N x_i/4N} \prod_{j=1}^N x_j^\alpha \prod_{k<l}^N |x_k - x_l|^2. \tag{2.106}$$

It is known that $\mu^N_{\mathrm{bes},2,\alpha}$ is also determinantal [67, 945p] and [14, 91p] The Bessel$_{2,\alpha}$ interacting Brownian motion is given by the following [17].

$$dX^{N,i}_t = dB^i_t + \{-\frac{1}{8N} + \frac{\alpha}{2X^{N,i}_t} + \sum_{j=1,j\neq i}^N \frac{1}{X^{N,i}_t - X^{N,j}_t}\}dt \quad (1 \le i \le N), \tag{2.107}$$

$$dX^i_t = dB^i_t + \{\frac{\alpha}{2X^i_t} + \sum_{j\neq i}^\infty \frac{1}{X^i_t - X^j_t}\}dt \quad (i \in \mathbb{N}). \tag{2.108}$$

This appears at the hard edge of one-dimensional systems.

Proposition 2.24. Assume $\alpha > 1$. Then (2.60) holds for (2.107) and (2.108).

Proof. **(J2)**–**(J5)** except (2.55) are proved in [17]. We easily see that the assumptions of Lemma 2.22 hold and yield (2.55). We thus obtain **(J5)**. □

Remark 2.25. There exist other natural ISDEs and N-particle systems related to the Bessel point processes. They are the non-colliding square Bessel processes and their square root. The non-colliding square Bessel processes are reversible to the Bessel$_{2,\alpha}$ point processes, but the associated Dirichlet forms are different from the Bessel$_{2,\alpha}$ interacting Brownian motion. Indeed, the coefficients a^N and a in Section 2.2 are taken to be $\mathsf{a}^N(x.\mathsf{y}) = \mathsf{a}(x.\mathsf{y}) = 2x$. On the other hand, each square root of the non-colliding Bessel processes is not reversible to the $\mathrm{Bessel}_{2,\alpha}$ point processes, but has the same type of Dirichlet forms as the $\mathrm{Bessel}_{2,\alpha}$ interacting Brownian motion. In particular, the coefficients a^N and a in Section 2.2 are taken to be $\mathsf{a}^N(x.\mathsf{y}) = \mathsf{a}(x.\mathsf{y}) = 1/4$. That is, they are constant time change of distorted Brownian motion with the standard square field.

We refer to [25, 26, 52] for these processes. For reader's convenience we provide an ISDE describing the non-colliding square Bessel processes and their square root. We note that SDE (2.110) is a constant time change of that in [26, 52]. Let $\mathbf{Y}^N = (Y^{N,i})_{i=1}^N$ and $\mathbf{Y} = (Y^i)_{i\in\mathbb{N}}$ be the non-colliding square Bessel processes. Then

$$dY_t^{N,i} =\sqrt{Y_t^{N,i}}dB_t^i + \{-\frac{Y_t^{N,i}}{8N} + \frac{\alpha+1}{2} + \sum_{j=1,j\neq i}^{N} \frac{Y_t^{N,i}}{Y_t^{N,i} - Y_t^{N,j}}\}dt \quad (1 \le i \le N), \tag{2.109}$$

$$dY_t^i =\sqrt{Y_t^i}dB_t^i + \{\frac{\alpha+1}{2} + \sum_{j\neq i}^{\infty} \frac{Y_t^i}{Y_t^i - Y_t^j}\}dt \quad (i \in \mathbb{N}). \tag{2.110}$$

Let $\mathbf{Z}^N = (Z^{N,i})_{i=1}^N$ and $\mathbf{Z} = (Z^i)_{i\in\mathbb{N}}$ be square root of the non-colliding square Bessel processes. Then applying Itô formula we obtain from (2.109) and (2.110)

$$dZ_t^{N,i} =\frac{1}{2}dB_t^i + \frac{1}{4}\{-\frac{Z_t^{N,i}}{4N} + \frac{\alpha+\frac{1}{2}}{Z_t^{N,i}} + \sum_{j=1,j\neq i}^{N} \frac{2Z_t^{N,i}}{(Z_t^{N,i})^2 - (Z_t^{N,j})^2}\}dt \ (1 \le i \le N), \tag{2.111}$$

$$dZ_t^i =\frac{1}{2}dB_t^i + \frac{1}{4}\{\frac{\alpha+\frac{1}{2}}{Z_t^i} + \sum_{j\neq i}^{\infty} \frac{2Z_t^{N,i}}{(Z_t^i)^2 - (Z_t^j)^2}\}dt \quad (i \in \mathbb{N}). \tag{2.112}$$

We remark that Theorem 2.7 can be applied to the non-colliding square Bessel processes because the equilibrium states are the same as the Bessel interacting Brownian motion and coefficients are well-behaved as $\mathsf{a}^N(x.\mathsf{y}) = \mathsf{a}(x.\mathsf{y}) = 2x$.

2.5.3 The Ginibre interacting Brownian motion

Let $S = \mathbb{R}^2$. Let μ_{gin}^N and μ_{gin} be as in Section 2.1. Let $\Phi^N = |x|^2$ and $\Psi(x) = -\log|x|$. Then the N-particle systems are given by

$$dX_t^{N,i} =dB_t^i - X_t^{N,i}dt + \sum_{j=1,j\neq i}^{N} \frac{X_t^{N,i} - X_t^{N,j}}{|X_t^{N,i} - X_t^{N,j}|^2}dt \quad (1 \le i \le N). \tag{2.12}$$

The limit ISDEs are

$$dX_t^i =dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \quad (i \in \mathbb{N}) \tag{2.13}$$

and

$$dX_t^i =dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \quad (i \in \mathbb{N}). \tag{2.14}$$

Proposition 2.26. (2.60) holds for (2.12) and both (2.13) and (2.14).

Proof. **(J2)**–**(J5)** except (2.55) are proved in [48, 47]. (2.55) is obvious for a Ginibre point process because their one-correlation functions with respect to the Lebesgue measure have a uniform bound such that $\rho_{\mathrm{gin}}^{N,1} \le 1/\pi$. This estimate follows from (6.4) in [47] immediately. Let d_1 and d_2 be the logarithmic derivative associated with ISDEs (2.13) and (2.14). Then $\mathsf{d}_1 = \mathsf{d}_2$ a.s. [47]. Hence we conclude Proposition 2.26 $\square$

2.5.4 Gibbs measures with Ruelle-class potentials

Let μ^{Ψ} be Gibbs measures with Ruelle-class potential $\Psi(x,y) = \Psi(x-y)$ that are smooth outside the origin. Let $\Phi^N \in C^\infty(S)$ be a confining potential for the N- particle system. We assume that the correlation functions of $\mu^{\Phi^N,\Psi}$ satisfy bounds $\sup_N \rho^{N,m} \le c_{16}^m$ for some constants c_{16}; see the construction of [61]. Then one can see in the same fashion as [53] that μ^{Ψ} satisfy **(J2)**–**(J5)** except (2.55). Under the condition $\sup_N \rho^{N,m} \le c_{16}^m$, (2.55) is obvious. Moreover, if μ^{Ψ} is a grand canonical Gibbs measure with sufficiently small inverse temperature β, then μ^{Ψ} is tail trivial. Hence we can obtain **(J6)** in the same way as [53] in this case. We present two concrete examples below.

2.5.5 Lennard–Jones 6-12 potentials

Let $S = \mathbb{R}^3$ and $\beta > 0$. Let $\Psi_{6-12}(x) = |x|^{-12} - |x|^{-6}$ be the Lennard-Jones potential. The corresponding ISDEs are given by the following.

$$
\begin{aligned}
dX_t^{N,i} =& dB_t^i + \frac{\beta}{2}\{\nabla\Phi^N(X_t^{N,i}) + \sum_{\substack{j=1,\\ j\neq i}}^{N} \frac{12(X_t^{N,i} - X_t^{N,j})}{|X_t^{N,i} - X_t^{N,j}|^{14}} - \frac{6(X_t^{N,i} - X_t^{N,j})}{|X_t^{N,i} - X_t^{N,j}|^{8}}\}dt \ (1 \le i \le N),\\
dX_t^i =& dB_t^i + \frac{\beta}{2}\sum_{j=1,j\neq i}^{\infty}\{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N}).
\end{aligned}
$$

2.5.6 Riesz potentials

Let $d < a \in \mathbb{N}$ and $\beta > 0$. Let $\Psi_a(x) = \frac{\beta}{a}|x|^{-a}$ the Riesz potential. The corresponding SDEs are given by

$$
\begin{aligned}
dX_t^{N,i} =& dB_t^i + \frac{\beta}{2}\{\nabla\Phi^N(X_t^{N,i}) + \sum_{j=1,j\neq i}^{N} \frac{X_t^{N,i} - X_t^{N,j}}{|X_t^{N,i} - X_t^{N,j}|^{2+a}}\}dt \quad (1 \le i \le N),\\
dX_t^i =& dB_t^i + \frac{\beta}{2}\sum_{j=1,j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{2+a}}dt \quad (i \in \mathbb{N}).
\end{aligned}
$$

3 Dynamical bulk scaling limit of Gaussian unitary ensembles and stochastic-differential-equation gaps

3.1 Introduction

Gaussian unitary ensembles (GUE) are Gaussian ensembles defined on the space of random matrices M^N $(N \in \mathbb{N})$ with independent random variables, the matrices of which are Hermitian. By definition, $M^N = [M^N_{i,j}]_{i,j=1}^N$ is then an $N \times N$ matrix having the form

$$M^N_{i,j} = \begin{cases} \xi_i & \text{if } i = j \\ \tau_{i,j}/\sqrt{2} + \sqrt{-1}\zeta_{i,j}/\sqrt{2} & \text{if } i < j, \end{cases}$$

where $\{\xi_i, \tau_{i,j}, \zeta_{i,j}\}_{i<j}^{\infty}$ are i.i.d. Gaussian random variables with mean zero and a half variance. Then the eigenvalues $\lambda_1, \ldots, \lambda_N$ of M^N are real and have distribution $\check{\mu}^N$ such that

$$\check{\mu}^N(d\mathbf{x}_N) = \frac{1}{Z^N} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} e^{-|x_k|^2}\, d\mathbf{x}_N, \tag{3.1}$$

where $\mathbf{x}_N = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and Z^N is a normalizing constant [2]. Wigner's celebrated semicircle law asserts that their empirical distributions converge in distribution to a semicircle distribution:

$$\lim_{N\to\infty} \frac{1}{N}\{\delta_{\lambda_1/\sqrt{N}} + \cdots + \delta_{\lambda_N/\sqrt{N}}\} = \frac{1}{\pi} 1_{(-\sqrt{2},\sqrt{2})}(x)\sqrt{2 - x^2}dx.$$

One may regard this convergence as a law of large numbers because the limit distribution is a *non-random* probability measure.

We consider the scaling of the next order in such a way that the distribution is supported on the set of configurations. That is, let θ be the position of the macro scale given by

$$-\sqrt{2} < \theta < \sqrt{2} \tag{3.2}$$

and take the scaling $x \mapsto y$ such that

$$x = \frac{y}{\sqrt{N}} + \theta\sqrt{N}. \tag{3.3}$$

Let μ_θ^N be the point process for which the labeled density $\mathbf{m}_\theta^N d\mathbf{x}_N$ is given by

$$\mathbf{m}_\theta^N(\mathbf{x}_N) = \frac{1}{Z^N} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} e^{-|x_k + \theta N|^2/N}. \tag{3.4}$$

The position θ in (3.2) is called the bulk and the scaling in (3.3) the bulk scaling (of the point processes). It is well known that the rescaled point processes μ_θ^N satisfy

$$\lim_{N\to\infty} \mu_\theta^N = \mu_\theta \quad \text{in distribution,} \tag{3.5}$$

where μ_θ is the determinantal point process with sine kernel K_θ:

$$\mathsf{K}_\theta(x,y)=\frac{\sin\{\sqrt{2-\theta^2}(x-y)\}}{\pi(x-y)}.$$

By definition μ_θ is the point process on $\mathbb{R}$ for which the m-point correlation function ρ_θ^m with respect to the Lebesgue measure is given by

$$\rho_\theta^m(x_1,\dots,x_m)=\det[\mathsf{K}_\theta(x_i,x_j)]_{i,j=1}^m.$$

We hence see that the limit is universal in the sense that it is the Sine_2 point process and independent of the macro-position θ up to the dilation of determinantal kernels K_θ. This may be regarded as a first step of the universality of the Sine_2 point process, which has been extensively studied for general inverse temperature β and a wide class of free potentials (see [5] and references therein).

Once a static universality is established, then it is natural to enquire of its dynamical counter part. Indeed, we shall prove the dynamical version of (3.5) and present a phenomenon called stochastic-differential-equation (SDE) gaps for $\theta\neq 0$.

Two natural N-particle dynamics are known for GUE. One is Dyson's Brownian motion corresponding to time-inhomogeneous N-particle dynamics given by the time evolution of eigenvalues of time-dependent Hermitian random matrices $\mathcal{M}^N(t)$ for which the coefficients are Brownian motions $B_t^{i,j}$ [43].

The other is a diffusion process $\mathbf{X}^{\theta,N}=(X^{\theta,N,i})_{i=1}^N=\{(X_t^{\theta,N,i})_{i=1}^N\}_t$ given by the SDE such that for $1\le i\le N$

$$dX_t^{\theta,N,i}=dB_t^i+\sum_{j\neq i}^N\frac{1}{X_t^{\theta,N,i}-X_t^{\theta,N,j}}dt-\frac{1}{N}X_t^{\theta,N,i}\,dt-\theta\,dt, \tag{3.6}$$

which has a unique strong solution for $\mathbf{X}_0^{\theta,N}\in\mathbb{R}^N\backslash\mathcal{N}$ and $\mathbf{X}^{\theta,N}$ never hits $\mathcal{N}$, where $\mathcal{N}=\{\mathbf{x}=(x_k)_{k=1}^N;\,x_i=x_j\text{ for some }i\neq j\}$ [19].

The derivation of (3.6) is as follows: Let $\check{\mu}_\theta^N(d\mathbf{x}_N)=\mathbf{m}_\theta^N(\mathbf{x}_N)d\mathbf{x}_N$ be the labeled symmetric distribution of μ_θ^N. Consider a Dirichlet form on $L^2(\mathbb{R}^N,\check{\mu}_\theta^N)$ such that

$$\mathcal{E}^{\check{\mu}_\theta^N}(f,g)=\int_{\mathbb{R}^N}\frac{1}{2}\sum_{i=1}^N\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}\check{\mu}_\theta^N(d\mathbf{x}_N).$$

Then using (3.4) and integration by parts, we specify the generator A^N of $\mathcal{E}^{\check{\mu}_\theta^N}$ on $L^2(\mathbb{R}^N,\check{\mu}_\theta^N)$ such that

$$A^N=\frac{1}{2}\Delta+\sum_{i=1}^N\{\sum_{j\neq i}^N\frac{1}{x_i-x_j}\}\frac{\partial}{\partial x_i}-\sum_{i=1}^N\{\frac{x_i}{N}+\theta\}\frac{\partial}{\partial x_i}.$$

From this we deduce that the associated diffusion $\mathbf{X}^{\theta,N}$ is given by (3.6).

Taking the limit $N\to\infty$ in (3.6), we *intuitively* obtain the infinite-dimensional SDE (ISDE) of $\mathbf{X}^\theta=(X^{\theta,i})_{i\in\mathbb{N}}$ such that

$$dX_t^{\theta,i}=dB_t^i+\sum_{j\neq i}^\infty\frac{1}{X_t^{\theta,i}-X_t^{\theta,j}}\,dt-\theta\,dt, \tag{3.7}$$

which was introduced in [68] with $\theta = 0$. For each θ, we have a unique, strong solution $\mathbf{X}^\theta$ of (3.7) such that $\mathbf{X}^\theta_0 = \mathbf{s}$ for $\mu_\theta \circ \mathfrak{l}^{-1}$-a.s. $\mathbf{s}$, where $\mathfrak{l}$ is a labeling map. Although only the $\theta = 0$ ISDE of $\mathbf{X}^0 =: \mathbf{X} = (X^i)_{i\in\mathbb{N}}$ is studied in [53, 76], the general $\theta \neq 0$ ISDE is nevertheless follows easily using the transformation

$$X^{\theta,i}_t = X^i_t - \theta t.$$

Let $\mathsf{X}^\theta_t = \sum_i \delta_{X^{\theta,i}_t}$ be the associated delabeled process. Then $\mathsf{X}^\theta = \{\mathsf{X}^\theta_t\}$ takes μ_θ as an invariant probability measure, and is *not* μ_θ-symmetric for $\theta \neq 0$.

The precise meaning of the drift term in (3.7) is the substitution of $\mathbf{X}^\theta_t = (X^{\theta,i}_t)_{i\in\mathbb{N}}$ for the function $b(x, \mathsf{y})$ given by the conditional sum

$$b(x, \mathsf{y}) = \lim_{r\to\infty} \{ \sum_{|x-y_i|<r} \frac{1}{x - y_i} \} - \theta \quad \text{in } L^1_{\text{loc}}(\mu^{[1]}_\theta), \tag{3.8}$$

where $\mathsf{y} = \sum_i \delta_{y_i}$ and $\mu^{[1]}_\theta$ is the one-Campbell measure of μ_θ (see (3.17)). We do this in such a way that $b(X^{\theta,i}_t, \sum_{j\neq i} \delta_{X^{\theta,j}_t})$. Because μ_θ is translation invariant, it can be easily checked that (3.8) is equivalent to (3.9):

$$b(x, \mathsf{y}) = \lim_{r\to\infty} \{ \sum_{|y_i|<r} \frac{1}{x - y_i} \} - \theta \quad \text{in } L^1_{\text{loc}}(\mu^{[1]}_\theta). \tag{3.9}$$

Let $\mathfrak{l}_N$ and $\mathfrak{l}$ be labeling maps. We denote by $\mathfrak{l}_{N,m}$ and $\mathfrak{l}_m$ the first m-components of $\mathfrak{l}_N$ and $\mathfrak{l}$, respectively. We assume that, for each $m \in \mathbb{N}$,

$$\lim_{N\to\infty} \mu^N_\theta \circ \mathfrak{l}^{-1}_{N,m} = \mu_\theta \circ \mathfrak{l}^{-1}_m \text{ weakly }. \tag{3.10}$$

Let $\mathbf{X}^{\theta,N} = (X^{\theta,N,i})_{i=1}^N$ and $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ be solutions of SDEs (3.6) and (3.11), respectively, such that

$$dX^{\theta,N,i}_t = dB^i_t + \sum_{j\neq i}^N \frac{1}{X^{\theta,N,i}_t - X^{\theta,N,j}_t} dt - \frac{1}{N} X^{\theta,N,i}_t \, dt - \theta \, dt, \tag{3.6}$$

$$dX^i_t = dB^i_t + \lim_{r\to\infty} \sum_{j\neq i,\, |X^i_t - X^j_t|<r}^\infty \frac{1}{X^i_t - X^j_t} \, dt. \tag{3.11}$$

We now state the first main result of the present paper.

Theorem 3.1. Assume (3.2) and (3.10). Assume that $\mathbf{X}^{\theta,N}_0 = \mu^N_\theta \circ \mathfrak{l}^{-1}_N$ in distribution and $\mathbf{X}_0 = \mu_\theta \circ \mathfrak{l}^{-1}$ in distribution. Then, for each $m \in \mathbb{N}$,

$$\lim_{N\to\infty} (X^{\theta,N,1}, X^{\theta,N,2}, \dots, X^{\theta,N,m}) = (X^1, X^2, \dots, X^m) \tag{3.12}$$

weakly in $C([0,\infty), \mathbb{R}^m)$. In particular, the limit $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ does not satisfy (3.7) for any θ other than $\theta = 0$.

We next consider non-reversible initial distributions. Let $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ and $\mathbf{Y}^\theta = (Y^{\theta,i})_{i\in\mathbb{N}}$ be solutions of (3.13) and (3.14), respectively, such that

$$dX_t^{N,i} = dB_t^i + \sum_{j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{1}{N} X_t^{N,i}\, dt, \tag{3.13}$$

$$dY_t^{\theta,i} = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i,\, |Y_t^{\theta,i} - Y_t^{\theta,j}|<r}^{\infty} \frac{1}{Y_t^{\theta,i} - Y_t^{\theta,j}}\, dt + \theta\, dt. \tag{3.14}$$

Note that $\mathbf{X}^N = \mathbf{X}^{0,N}$ and that $\mathbf{X}^N$ is not reversible with respect to $\mu_\theta^N \circ \mathfrak{l}_N^{-1}$ for any $\theta \neq 0$. We remark that the delabeld process $\mathsf{Y}^\theta = \{\sum_{i\in\mathbb{N}} \delta_{Y_t^{\theta,i}}\}$ of $\mathbf{Y}^\theta$ has invariant probability measure μ_θ and is *not* symmetric with respect to μ_θ for $\theta \neq 0$. We state the second main theorem.

Theorem 3.2. Assume (3.2) and (3.10). Assume that $\mathbf{X}_0^N = \mu_\theta^N \circ \mathfrak{l}_N^{-1}$ in distribution and $\mathbf{Y}_0^\theta = \mu_\theta \circ \mathfrak{l}^{-1}$ in distribution. Then for each $m \in \mathbb{N}$

$$\lim_{N\to\infty} (X^{N,1}, X^{N,2}, \ldots, X^{N,m}) = (Y^{\theta,1}, Y^{\theta,2}, \ldots, Y^{\theta,m}) \tag{3.15}$$

weakly in $C([0,\infty), \mathbb{R}^m)$.

- We refer to the second claim in Theorem 3.1, and (3.15) as the SDE gaps. The convergence in (3.15) of Theorem 3.2 resembles the "Propagation of Chaos" in the sense that the limit equation (3.14) depends on the initial distribution, although it is a linear equation. Because the logarithmic potential is by its nature long-ranged, the effect of initial distributions μ_θ^N still remains in the limit ISDE, and the rigidity of the Sine_2 point process makes the residual effect a non-random drift term θdt.

  Our result is the dynamical universality of Dyson's Brownian motion in infinite-dimension. There are similar result of dynamical universality of Dyson's Brownian motion in [36], but finite N result, then it is somehow different from ours.

- Let S_θ be a Borel set such that $\mu_\theta(\mathsf{S}_\theta) = 1$, where $-\sqrt{2} < \theta < \sqrt{2}$. In [27], the first author proves that one can choose S_θ such that $\mathsf{S}_\theta \cap \mathsf{S}_{\theta'} = \emptyset$ if $\theta \neq \theta'$ and that for each $\mathsf{s} \in \mathsf{S}_\theta$ (3.11) has a strong solution $\mathbf{X}$ such that $\mathbf{X} = \mathfrak{l}(\mathsf{s})$ and that

  $$\mathsf{X}_t := \sum_{i=1}^{\infty} \delta_{X_t^i} \in \mathsf{S}_\theta \quad \text{ for all } t \in [0,\infty).$$

  This implies that the state space of solutions of (3.11) can be decomposed into uncountable disjoint components. We conjecture that the component S_θ is ergodic for each $\theta \in (-\sqrt{2}, \sqrt{2})$.

- For $\theta = 0$, the convergence (3.12) is also proved in [52]. The proof in [52] is algebraic and valid only for dimension $d = 1$ and inverse temperature $\beta = 2$ with the logarithmic potential. It relies on an explicit calculation of the space-time correlation functions, the strong Markov property of the stochastic dynamics given by the algebraic construction, the identity of the associated Dirichlet forms constructed by two completely different methods, and the uniqueness of solutions of ISDE (3.7). Although one may prove (3.10) for $\theta \neq 0$ using the algebraic method in [52], this requires a lot of work as mentioned above. We remark that, as a corollary and an application, Theorem 3.1 proves the weak convergence of finite-dimensional distributions explicitly given by the space-time correlation functions. We refer to [24, 52] for the representation of these correlation functions.

- Tsai proves the pathwise uniqueness and the existence of strong solutions of

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j|<r}^{\infty} \frac{1}{X_t^i - X_t^j}\, dt \quad (i \in \mathbb{N}) \tag{3.16}$$

for general $\beta \in [1, \infty)$ in [76]. The proof uses the classical stochastic analysis and crucially depends on a specific monotonicity of SDEs (3.16). For $\beta = 1, 4$, we have a good control of the correlation functions as for $\beta = 2$. Hence our method can be applied to $\beta = 1, 4$ and the same result as for $\beta = 2$ in Theorem 3.1 holds. We shall return to this point in future.

The key point of the proof of Theorem 3.1 is to prove the convergence of the drift coefficient $b^N(x, \mathsf{y})$ of the N-particle system to the drift coefficient $b(x, \mathsf{y})$ of the limit ISDE even if $\theta \neq 0$. That is, as $N \to \infty$,

$$b^N(x, \mathsf{y}) = \{\sum_{i=1}^{N} \frac{1}{x - y_i}\} - \theta \quad \Longrightarrow \quad b(x, \mathsf{y}) = \lim_{r\to\infty} \{ \sum_{|y_i|<r} \frac{1}{x - y_i} \}.$$

Note that support of the coefficients $b^N(x, \mathsf{y})$ and $b(x, \mathsf{y})$ are mutually disjoint, and that the sum in b^N is not neutral for any $\theta \neq 0$. We shall prove uniform bounds of the tail of the coefficients using fine estimates of the correlation functions, and cancel out the deviation of the sum in b^N with θ. Because of rigidity of the Sine_2 point process, we justify this cancellation not only for static but also dynamical instances.

The organization of the paper is as follows: In Section 3.2, we prepare general theories for interacting Brownian motion in infinite dimensions. In Section 3.3, we quote estimates for the oscillator wave functions and determinantal kernels. In Section 3.4, we prove key estimates (3.37)–(3.40). In Section 3.5, we complete the proof of Theorem 3.1. In Section 3.6, we prove Theorem 3.2.

3.2 Preliminaries from general theory

In this section we present the general theory described in [47, 48, 53, 28] in a reduced form sufficient for the current purpose. In particular, we take the space where particles move in $\mathbb{R}$ rather than $\mathbb{R}^d$ as in the cited articles.

3.2.1 μ-reversible diffusions

Let $S_r = \{s \in \mathbb{R}\,; |s| < r\}$. The configuration space S over $\mathbb{R}$ is a Polish space equipped with the vague topology such that

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\, \mathsf{s}(S_r) < \infty \text{ for all } r \in \mathbb{N}\}.$$

Each element $\mathsf{s} \in \mathsf{S}$ is called a configuration regarded as countable delabeled particles. A probability measure μ on $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$ is called a point process (a random point field).

A locally integrable symmetric function $\rho^n : \mathbb{R}^n \to [0, \infty)$ is called the n-point correlation function of μ with respect to the Lebesgue measure if ρ^n satisfies

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(s_1, \ldots, s_n)\, d\mathbf{s}_n = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} \mu(d\mathsf{s})$$

for any sequence of disjoint bounded measurable subsets $A_1, \ldots, A_m \subset \mathbb{R}$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. Here we assume that $\mathsf{s}(A_i)!/(\mathsf{s}(A_i) - k_i)! = 0$ for $\mathsf{s}(A_i) - k_i < 0$.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ be measurable functions called free and interaction potentials, respectively. Let $\mathcal{H}_r$ be the Hamiltonian on S_r given by

$$\mathcal{H}_r(\mathsf{x}) = \sum_{x_i \in S_r} \Phi(x_i) + \sum_{j \neq k, x_j, x_k \in S_r} \Psi(x_j, x_k) \quad \text{ for } \mathsf{x} = \sum_i \delta_{x_i}.$$

For each $m, r \in \mathbb{N}$ and μ-a.s. $\xi \in \mathsf{S}$, let $\mu_{r,\xi}^m$ denote the regular conditional probability such that

$$\mu_{r,\xi}^m = \mu(\pi_{S_r}(\mathsf{x}) \in \cdot \,|\, \pi_{S_r^c}(\mathsf{x}) = \pi_{S_r^c}(\xi),\, \mathsf{x}(S_r) = m).$$

Here for a subset A, we set $\pi_A : \mathsf{S} \to \mathsf{S}$ by $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot \cap A)$.

Let Λ_r denote the Poisson point process with intensity being a Lebesgue measure on S_r. We set $\Lambda_r^m(\cdot) = \Lambda_r(\cdot \cap \mathsf{S}_r^m)$, where $\mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S_r) = m\}$.

Definition 3.3 ([48], [49])**.** A point process μ is said to be a (Φ, Ψ)-quasi Gibbs measure if its regular conditional probabilities $\mu_{r,\xi}^m$ satisfy, for any $r, m \in \mathbb{N}$ and μ-a.s. ξ,

$$c_{17}^{-1} e^{-\mathcal{H}_r(\mathsf{x})} \Lambda_r^m(d\mathsf{x}) \leq \mu_{r,\xi}^m(d\mathsf{x}) \leq c_{17} e^{-\mathcal{H}_r(\mathsf{x})} \Lambda_r^m(d\mathsf{x}).$$

Here c_{17} is a positive constant depending on r, m, ξ.

The significance of the quasi-Gibbs property is to guarantee the existence of μ-reversible diffusion process $\{P_{\mathsf{s}}\}$ on S given by the natural Dirichlet form associated with μ, in analogy with distorted Brownian motion in finite-dimensions.

To introduce the Dirichlet form, we provide some notations. We say a function f on S is local if f is $\sigma[\pi_K]$-measurable for some compact set K in $\mathbb{R}$. For a local function f on S, we say f is smooth if $\check{f}$ is smooth, where $\check{f}(x_1, \ldots)$ is the symmetric function such that $\check{f}(x_1, \ldots) = f(\mathsf{x})$ for $\mathsf{x} = \sum_i \delta_{x_i}$. Let $\mathcal{D}_\circ$ be the set of all bounded, locally smooth functions on S.

Let $\mathbb{D}$ be the standard square field on S such that for $f, g \in \mathcal{D}_\circ$ and $\mathsf{s} = \sum_i \delta_{s_i}$

$$\mathbb{D}[f, g](\mathsf{s}) = \frac{1}{2} \{\sum_i (\nabla_i \check{f})(\nabla_i \check{g})\}\,(\mathbf{s}).$$

We write $\mathbf{s} = (s_i)_i$. Because the function $\sum_i (\nabla_i \check{f})(\mathbf{s})(\nabla_i \check{g})(\mathbf{s})$ is symmetric in $\mathbf{s} = (s_i)_i$, we regard it as a function of s. We set $L^2(\mu) = L^2(\mathsf{S}, \mu)$ and let

$$\mathcal{E}^\mu(f, g) = \int_{\mathsf{S}} \mathbb{D}[f, g](\mathsf{s}) \mu(d\mathsf{s}), \quad \mathcal{D}_\circ^\mu = \{f \in \mathcal{D}_\circ \cap L^2(\mu)\,;\, \mathcal{E}^\mu(f, f) < \infty\}.$$

We quote:

Lemma 3.4 ([48])**.** Assume that μ is a (Φ, Ψ)-quasi Gibbs measure with upper semi-continuous (Φ, Ψ). Assume that the correlation functions $\{\rho^n\}$ are locally bounded for all $n \in \mathbb{N}$. Then $(\mathcal{E}^\mu, \mathcal{D}^\mu_\circ)$ is closable on $L^2(\mu)$. Furthermore, there exists a μ-reversible diffusion process $\{P_{\mathsf{s}}\}$ associate with the Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ on $L^2(\mu)$. Here $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is the closure of $(\mathcal{E}^\mu, \mathcal{D}^\mu_\circ)$ on $L^2(\mu)$.

3.2.2 Infinite-dimensional SDEs

Suppose that diffusion $\{P_{\mathsf{s}}\}$ in Lemma 3.4 is collision-free and that each tagged particle does not explode. Then we can construct labeled dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{Z}}$ by introducing the initial labeling $\mathfrak{l} = (\mathfrak{l}_i)_{i\in\mathbb{Z}}$ such that

$$\mathbf{X}_0 = \mathfrak{l}(\mathsf{X}_0).$$

Indeed, once the label $\mathfrak{l}$ is given at time zero, then each particle retains the tag for all time because of the collision-free and explosion-free property.

To specify the ISDEs satisfied by $\mathbf{X}$ above, we introduce the notion of the logarithmic derivative of μ, which was introduced in [47].

A point process μ_x is called the reduced Palm measure of μ conditioned at $x \in \mathbb{R}$ if μ_x is the regular conditional probability defined as

$$\mu_x = \mu(\cdot - \delta_x | \mathsf{s}(\{x\}) \geq 1).$$

A Radon measure $\mu^{[1]}$ on $\mathbb{R} \times \mathsf{S}$ is called the 1-Campbell measure of μ if

$$\mu^{[1]}(dxd\mathsf{s}) = \rho^1(x)\mu_x(d\mathsf{s})dx. \tag{3.17}$$

We write $f \in L^p_{\mathrm{loc}}(\mu^{[1]})$ if $f \in L^p(S_r \times \mathsf{S}, \mu^{[1]})$ for all $r \in \mathbb{N}$.

Definition 3.5. A $\mathbb{R}$-valued function $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mu^{[1]})$ is called the *logarithmic derivative* of μ if, for all $\varphi \in C_0^\infty(\mathbb{R}) \otimes \mathcal{D}_\circ$,

$$\int_{\mathbb{R}\times\mathsf{S}} \mathsf{d}^\mu(x,\mathsf{y})\varphi(x,\mathsf{y})\mu^{[1]}(dxd\mathsf{y}) = -\int_{\mathbb{R}\times\mathsf{S}} \nabla_x\varphi(x,\mathsf{y})\mu^{[1]}(dxd\mathsf{y}).$$

Under these assumptions, we obtain the following:

Lemma 3.6 ([47])**.** Assume that $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ is the collision-free and explosion-free. Then $\mathbf{X}$ is a solution of the following ISDE:

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, \mathsf{X}_t^{\diamond i})dt \quad (i \in \mathbb{N}) \tag{3.18}$$

with initial condition $\mathbf{X}_0 = \mathbf{s}$ for $\mu \circ \mathfrak{l}^{-1}$-a.s. $\mathbf{s}$, where $\mathsf{X}_t^{\diamond i} = \sum_{j\neq i}^\infty \delta_{X_t^j}$.

3.2.3 Finite-particle approximations

Let μ be a point process with correlaton functions $\{\rho^n\}_{n\in\mathbb{N}}$. Let $\{\mu^N\}_{N\in\mathbb{N}}$ be a sequence of point processes on $\mathbb{R}$ such that $\mu^N(\{\mathsf{s}(\mathbb{R}) = N\}) = 1$. We assume:

(A1) Each μ^N has correlation functions $\{\rho^{N,n}\}_{n\in\mathbb{N}}$ satisfying, for each $r \in \mathbb{N}$,

$$\lim_{N\to\infty} \rho^{N,n}(\mathbf{x}) = \rho^n(\mathbf{x}) \quad \text{uniformly on } S_r^n \text{ for each } n \in \mathbb{N}, \tag{3.19}$$

$$\sup_{N\in\mathbb{N}} \sup_{\mathbf{x}\in S_r^n} \rho^{N,n}(\mathbf{x}) \leq c_{18}^n n^{c_{19}n}, \tag{3.20}$$

where $0 < c_{18}(r) < \infty$ and $0 < c_{19}(r) < 1$ are constants independent of $n \in \mathbb{N}$.

It is known that (3.19) and (3.20) imply the weak convergence of $\{\mu^N\}$ to μ [48, Lemma A.1]. As in Section 3.1, let $\mathfrak{l}$ and $\mathfrak{l}_N$ be labels of μ and μ^N, respectively. We assume:

(A2) For each $m \in \mathbb{N}$,

$$\lim_{N\to\infty} \mu^N \circ \mathfrak{l}_{N,m}^{-1} = \mu \circ \mathfrak{l}_m^{-1} \quad \text{weakly in } \mathbb{R}^m.$$

We shall later take $\mu^N \circ \mathfrak{l}_N^{-1}$ as an initial distribution of labeled finite particle system. Therefore, **(A2)** means the convergence of the initial distribution of the labeled dynamics.

For a labeled process $\mathbf{X}^N = (X^{N,i})_{i=1}^N$, where $N \in \mathbb{N}$, we set

$$\mathsf{X}_t^{N,\diamond i} = \sum_{j\neq i}^N \delta_{X_t^{N,j}},$$

where $\mathsf{X}_t^{N,\diamond i}$ denotes the zero measure for $N = 1$. Let $\mathsf{b}^N, \mathsf{b} : \mathbb{R} \times \mathsf{S} \to \mathbb{R}$ be measurable functions. We introduce the finite-dimensional SDE of $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ with these coefficients such that for $1 \le i \le N$

$$dX_t^{N,i} = dB_t^i + \mathsf{b}^N(X_t^{N,i}, \mathsf{X}_t^{N,\diamond i})dt. \tag{3.21}$$

We assume:

(A3) SDE (3.21) with initial condition $\mathbf{X}_0^N = \mathbf{s}$ has a unique solution for $\mu^N \circ \mathfrak{l}_N^{-1}$-a.s. $\mathbf{s}$ for each N. This solution does not explode.

Let $u,\ u^N,\ w : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be measurable functions. We set

$$\mathsf{g}_r(x, \mathsf{y}) = \sum_i \chi_r(x - y_i) g(x, y_i), \tag{3.22}$$

$$w_r(x, \mathsf{y}) = \sum_i (1 - \chi_r(x - y_i)) g(x, y_i), \tag{3.23}$$

where $\mathsf{y} = \sum_i \delta_{y_i}$ and $\chi_r \in C_0^\infty(\mathbb{R})$ is a cut-off function such that $0 \le \chi_r \le 1$, $\chi_r(x) = 0$ for $|x| \ge r + 1$, and $\chi_r(x) = 1$ for $|x| \le r$. We assume the following.

(A4) Each μ^N has a logarithmic derivative d^N such that

$$\mathsf{d}^N(x, \mathsf{y}) = u^N(x) + \mathsf{g}_r(x, \mathsf{y}) + w_r(x, \mathsf{y}). \tag{3.24}$$

Furthermore, we assume that

(1) u^N are in $C^1(\mathbb{R})$. Furthermore, u^N and ∇u^N converge uniformly to u and ∇u, respectively, on each compact set in $\mathbb{R}$.

(2) $g \in C^1(\mathbb{R}^2 \cap \{x \neq y\})$. There exists a $\hat{p} > 1$ such that, for each $R \in \mathbb{N}$,

$$\lim_{\mathsf{p}\to\infty} \limsup_{N\to\infty} \int_{x\in S_R, |x-y|\le 2^{-\mathsf{p}}} \chi_r(x - y)|g(x, y)|^{\hat{p}} \rho_x^{N,1}(y)dxdy = 0, \tag{3.25}$$

where $\rho_x^{N,1}$ is a one-correlation function of the reduced Palm measure μ_x^N.

(3) There exists a continuous function $w : \mathbb{R} \to \mathbb{R}$ such that for each $R \in \mathbb{N}$

$$\lim_{r\to\infty} \limsup_{N\to\infty} \int_{S_R\times\mathsf{S}} |w_r(x,\mathsf{y}) - w(x)|^{\hat{p}} d\mu^{N,[1]} = 0. \tag{3.26}$$

Let p be such that $1 < p < \hat{p}$. Assume **(A1)** and **(A4)**. Then [47, Theorem 45] deduces that the logarithmic derivative d^{μ} of μ exists in $L^p_{\mathrm{loc}}(\mu^{[1]})$ and is given by

$$\mathsf{d}^{\mu}(x,\mathsf{y}) = u(x) + \mathsf{g}(x,\mathsf{y}) + w(x). \tag{3.27}$$

Here $\mathsf{g}(x,\mathsf{y}) = \lim_{r\to\infty} \mathsf{g}_r(x,\mathsf{y})$ and the convergence of $\lim \mathsf{g}_r$ takes place in $L^p_{\mathrm{loc}}(\mu^{[1]})$. Taking (3.27) into account, we introduce the ISDE of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$:

$$dX^i_t = dB^i_t + \frac{1}{2}\{u(X^i_t) + \mathsf{g}(X^i_t, \mathsf{X}^{\diamond i}_t) + w(X^i_t)\}dt. \tag{3.28}$$

Under the assumptions of Lemma 3.6, ISDE (3.28) with $\mathbf{X}_0 = \mathbf{s}$ has a solution for $\mu \circ \mathfrak{l}^{-1}$-a.s. $\mathbf{s}$. Moreover, the associated delabeled diffusion $\mathsf{X} = \{\mathsf{X}_t\}$ is μ-reversible, where $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X^i_t}$ for $\mathbf{X}_t = (X^i_t)_{i\in\mathbb{N}}$. As for uniqueness, we recall the notion of μ-absolute continuity solution introduced in [53].

Let $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ be a family of solution of (3.28) satisfying $\mathbf{X}_0 = \mathbf{s}$ for $\mu \circ \mathfrak{l}^{-1}$-a.s. $\mathbf{s}$. Let μ_t be the distribution of the delabeled process $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X^i_t}$ at time t with initial distribution μ. That is, μ_t is given by

$$\mu_t = \int_{\mathsf{S}} P_{\mathsf{s}}(\mathsf{X}_t \in \cdot) d\mu$$

We say that $\mathbf{X}$ satisfies the μ-absolute continuity condition if

$$\mu_t \prec \mu \quad \text{for all } t \geq 0, \tag{3.29}$$

where $\mu_t \prec \mu$ means that μ_t is absolutely continuous with respect to μ. If X is μ-reversible, then (3.29) is satisfied.

We say ISDE (3.28) has μ-uniqueness in law of solutions if $\mathbf{X}$ and $\mathbf{X}'$ are solutions with the same initial distributions satisfying the μ-absolute continuity condition, then they are equivalent in law. We assume:

(A5) ISDE (3.28) has μ-uniqueness in law of solutions.

It is proved in [53] that ISDE (3.18) has a μ-pathwise unique strong solution if μ is tail trivial, the logarithmic derivative d^{μ} has a sort of off-diagonal smoothness, and the one-correlation function has sub-exponential growth at infinity. This results implies μ-uniqueness in law. We refer to Theorems 2.1 and 9.3 in [53] for details. The next result is a special case of [28, Theorem 2.1].

Lemma 3.7 ([28, Theorem 2.1])**.** Make the same assumptions in Lemma 3.4 and Lemma 3.6. Assume **(A1)**–**(A4)**. Assume that $\mathbf{X}^N_0 = \mu^N \circ \mathfrak{l}^{-1}_N$ in distribution. Then $\{\mathbf{X}^N\}_{N\in\mathbb{N}}$ is tight in $C([0,\infty);\mathbb{R}^{\mathbb{N}})$ and each limit point $\mathbf{X}$ of $\{\mathbf{X}^N\}_{N\in\mathbb{N}}$ is a solution of (3.28) with initial distribution $\mu \circ \mathfrak{l}^{-1}$. If, in addition, we assume **(A5)**, then for any $m \in \mathbb{N}$

$$\lim_{N\to\infty} (X^{N,1}, \ldots, X^{N,m}) = (X^1, \ldots, X^m).$$

weakly in $C([0,\infty),\mathbb{R}^m)$. Here $\mathbf{X}^N = (X^{N,i})^N_{i=1}$ and $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ as before.

3.2.4 Reduction of Theorem 3.1 to (3.26)

In this subsection, we deduce Theorem 3.1 from Lemma 3.7 by assuming (3.26). We take μ_θ^N and μ_θ as in Section 3.1. Then the logarithmic derivative $\mathsf{d}^{\mu_\theta^N}$ of μ_θ^N is given by

$$\mathsf{d}^{\mu_\theta^N}(x, \mathsf{y}) = \sum_{i=1}^{N} \frac{2}{x - y_i} - \frac{2x}{N} - 2\theta, \tag{3.30}$$

where $\mathsf{y} = \sum_i \delta_{y_i}$. From (3.30), we take coefficients in **(A4)** as follows:

$$u^N(x) = -\frac{2x}{N} - 2\theta, \quad u(x) = -2\theta, \quad w(x) = 2\theta, \tag{3.31}$$

$$g(x, y) = \frac{2}{x - y}. \tag{3.32}$$

Other functions are given by (3.22) and (3.23).

Lemma 3.8. Assume (3.26) holds with $\hat{p} = 2$ for the coefficients as above. Then (3.12) holds.

Proof. To prove Lemma 3.8, we check the assumptions in Lemma 3.7, that is, the assumptions in Lemma 3.4, Lemma 3.6, and **(A1)**–**(A5)**.

The assumptions in Lemma 3.4 are proved in [48]. The assumptions in Lemma 3.6 are checked in [47]. **(A1)** is well known. **(A2)** is assumed by (3.10). **(A3)** is obvious as the interaction is smooth outside the origin, and the capacity of the colliding set $\{x_i = x_j \text{ for some } i \neq j\}$ is zero (see [45, 19]). Furthermore, the one-correlation functions are bounded, which guarantees explosion-free of tagged particles. We take functions in **(A4)** as (3.31) and (3.32). These satisfy (3.24), (3.25), and (1) of **(A4)**. (3.26) is satisfied by assumption. It is known that μ_θ is tail trivial [50]. Then **(A5)** follows from tail triviality of μ_θ and [53, Theorem 3.1]. All the assumptions in Lemma 3.7 are thus satisfied, and hence yields (3.12). □

3.2.5 A sufficient condition for (3.26)

The most crucial step to apply Lemma 3.7 is to check (3.26). Indeed, it only remains to prove (3.26) for Theorem 3.1. We quote then a sufficient condition for (3.26) in terms of correlation functions from [47]. Lemma 3.10 below is a special case of [47, Lemma 53].

Let $\mu_{\theta,x}^N$ be the reduced Palm measure of μ_θ^N conditioned at x. We denote the supremum norm in x over S_R by $\|\cdot\|_R$. Let $\mathrm{E}^{\cdot}$ and $\mathrm{Var}^{\cdot}$ denote the expectation and variance with resoect to $\cdot$, respectively.

Lemma 3.9. Assume $|\theta| < \sqrt{2}$. Let w_r be as in (3.23) with $g(x, y)$ given by (3.32). Let $w(x) = 2\theta$ as in (3.31). Then (3.26) follows from (3.33)–(3.36).

$$\lim_{r\to\infty} \limsup_{N\to\infty} \left\| \mathrm{E}^{\mu_\theta^N}[w_r(x, \mathsf{y})] - 2\theta \right\|_R = 0, \tag{3.33}$$

$$\lim_{r\to\infty} \limsup_{N\to\infty} \left\| \mathrm{E}^{\mu_\theta^N}[w_r(x, \mathsf{y})] - \mathrm{E}^{\mu_{\theta,x}^N}[w_r(x, \mathsf{y})] \right\|_R = 0, \tag{3.34}$$

$$\lim_{r\to\infty} \limsup_{N\to\infty} \left\| \mathrm{Var}^{\mu_\theta^N}[w_r(x, \mathsf{y})] \right\|_R = 0, \tag{3.35}$$

$$\lim_{r\to\infty} \limsup_{N\to\infty} \left\| \mathrm{Var}^{\mu_\theta^N}[w_r(x, \mathsf{y})] - \mathrm{Var}^{\mu_{\theta,x}^N}[w_r(x, \mathsf{y})] \right\|_R = 0. \tag{3.36}$$

Proof. Lemma 3.9 follows from [47, Lemma 52]. Indeed, (3.33), (3.34), (3.35), and (3.36) in the present paper correspond to (5.4), (5.2), (5.5), and (5.3) in [47], respectively. We note that in [47] we use $1_{S_r}(x)$ instead of $\chi_r(x)$. This slight modification yields no difficulty. $\square$

Multiplying $w_r(x, \mathsf{y})$ by a half, we give a sufficient condition of (3.33)–(3.36) in terms of correlation functions. Let $\rho_{\theta,x}^{N,m}$ and $\rho_{\theta}^{N,m}$ be the m-point correlation functions of $\mu_{\theta,x}^{N}$ and μ_{θ}^{N}, respectively. Let

$$S_{r,\infty}(x) = \{y \in \mathbb{R}\, ; r < |x-y| < \infty\}.$$

Lemma 3.10. Assume $|\theta| < \sqrt{2}$. Then (3.33)–(3.36) follow from (3.37)–(3.40).

$$\lim_{r\to\infty} \limsup_{N\to\infty} \Big\| \int_{S_{r,\infty}(x)} \frac{\rho_{\theta}^{N,1}(y)}{x-y} dy - \theta \Big\|_R = 0, \tag{3.37}$$

$$\lim_{r\to\infty} \limsup_{N\to\infty} \Big\| \int_{S_{r,\infty}(x)} \frac{\rho_{\theta,x}^{N,1}(y) - \rho_{\theta}^{N,1}(y)}{x-y} dy \Big\|_R = 0, \tag{3.38}$$

$$\lim_{r\to\infty} \limsup_{N\to\infty} \Big\| \int_{S_{r,\infty}(x)} \frac{\rho_{\theta}^{N,1}(y)}{(x-y)^2} dy + \int_{S_{r,\infty}(x)^2} \frac{\rho_{\theta}^{N,2}(y,z) - \rho_{\theta}^{N,1}(y)\rho_{\theta}^{N,1}(z)}{(x-y)(x-z)} dydz \Big\|_R = 0, \tag{3.39}$$

$$\begin{aligned} \lim_{r\to\infty} \limsup_{N\to\infty} \Big\| & \int_{S_{r,\infty}(x)} \frac{\rho_{\theta,x}^{N,1}(y) - \rho_{\theta}^{N,1}(y)}{(x-y)^2} dy \\ & + \int_{S_{r,\infty}(x)^2} \frac{\rho_{\theta,x}^{N,2}(y,z) - \rho_{\theta,x}^{N,1}(y)\rho_{\theta,x}^{N,1}(z) - \{\rho_{\theta}^{N,2}(y,z) - \rho_{\theta}^{N,1}(y)\rho_{\theta}^{N,1}(z)\}}{(x-y)(x-z)} dydz \Big\|_R = 0. \end{aligned} \tag{3.40}$$

Proof. Lemma 3.10 follows immediately from a standard calculation of correlation functions and the definitions of w_r and χ_r. $\square$

3.3 Subsidiary estimates

Keeping Lemma 3.10 in mind, our task is to prove (3.37)–(3.40). To control the correlation functions in Lemma 3.10 we prepare in this section estimates of the oscillator wave functions and determinantal kernels. We shall use these estimates in Section 3.4.

3.3.1 Oscillator wave functions

Let $H_n(x) = (-1)^n e^{x^2} (\frac{d}{dx})^n e^{-x^2}$ be Hermite polynomials. Let $\psi_n(x)$ denote the oscillator wave functions defined by

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} e^{-\frac{x^2}{2}} H_n(x).$$

Note that $\{\psi_n\}_{n=0}^{\infty}$ is an orthonormal system; $\int_{\mathbb{R}} \psi_n(x)\psi_m(x)\, dx = \delta_{nm}$.

The following estimates for these oscillator wave functions are essentially due to Plancherel-Rotach [58]. We quote here a version from Katori-Tanemura [25].

Lemma 3.11 ([25])**.** Let C^1_{nm}, C^2_{nm}, and D^1_{nm} be the constants introduced in [25] (see **(A.1)** in [25, 572 p]). Let $l = -1, 0, 1$ and $N, L \in \mathbb{N}$. Then we have the following.
(1) Let $0 < \tau \le \frac{\pi}{2}$. Assume that $N \sin^3 \tau \ge CN^{\varepsilon}$ for some $C > 0$ and $\varepsilon > 0$. Then

$$\psi_{N+l}(\sqrt{2N}\cos\tau) = \frac{1+\mathcal{O}(N^{-1})}{\sqrt{\pi\sin\tau}}\Big(\frac{2}{N}\Big)^{\frac{1}{4}}$$
$$\times\Big[\sum_{n=0}^{L-1}\sum_{m=0}^{n} C^1_{nm}(N+l,\tau)\sin\Big\{\frac{N}{2}(2\tau-\sin 2\tau)+D^1_{nm}(\tau)-(1+l)\tau\Big\}+\mathcal{O}(\frac{1}{N\sin\tau})\Big].$$

(2) Let $\tau > 0$. Assume that $N\sinh^3\tau \ge CN^{\varepsilon}$ for some $C > 0$ and $\varepsilon > 0$. Then

$$\psi_{N+l}(\sqrt{2N}\cosh\tau) = \frac{1+\mathcal{O}(N^{-1})}{\sqrt{2\pi\sinh\tau}}\Big(\frac{1}{2N}\Big)^{\frac{1}{4}}$$
$$\times\exp\Big[\Big(\frac{N+1+l}{2}\Big)(2\tau-\sinh 2\tau)+(1+l)\tau\Big]\Big[\sum_{n=0}^{L-1}\sum_{m=0}^{n} C^2_{nm}(\tau, N+l)+\mathcal{O}\Big(\frac{\cosh^3\tau}{N\sinh\tau}\Big)\Big].$$

Proof. (1) and (2) follow from (5.5) and (5.10) in [25], respectively. □

We next quote estimates from [25, 54].

Lemma 3.12 ([25], [54])**.** (1) Let $y = \sqrt{2N}\cos\tau$ with $N \in \mathbb{N}$ and $0 < \tau \le \frac{\pi}{2}$. Assume that $N\sin^3\tau \ge CN^{\varepsilon}$ for some $C > 0$ and $\varepsilon > 0$. Then,

$$\sum_{k=0}^{N-1}\psi_k(y)^2 = \frac{1}{\pi}\sqrt{2N-y^2}+\mathcal{O}\Big(\frac{\sqrt{N}}{2N-y^2}\Big).$$

(2) Let $y = \sqrt{2N}\cosh\tau$ with $N \in \mathbb{N}$ and $\tau > 0$. Assume that $N\sinh^3\tau \ge CN^{\varepsilon}$ for some $C > 0$ and $\varepsilon > 0$. Then

$$\sum_{k=0}^{N-1}\psi_k(y)^2 = \mathcal{O}\Big(\frac{\sqrt{N}}{y^2-2N}\Big). \tag{3.41}$$

(3) There is a positive constant c_{20} such that for all $N \in \mathbb{N}$

$$\sup_{y\in\mathbb{R}}|\psi_N(y)| \le c_{20}N^{-\frac{1}{12}}. \tag{3.42}$$

Proof. (1) follows from Lemma 5.2 (i) in [25]. (2) follows from Lemma 5.2 (ii) in [25]. From Lemma 6.9 in [54] there exists a constant c_{20} such that

$$|N^{\frac{1}{12}}\psi_N(2\sqrt{N}+yN^{-\frac{1}{6}})| \le \frac{c_{20}}{(1\vee|y|)^{\frac{1}{4}}},\quad y\in[-2N^{\frac{2}{3}},\infty).$$

Hence we have

$$|\psi_N(y)| \le \frac{c_{20}}{N^{\frac{1}{12}}(1\vee\{N^{\frac{1}{6}}|y-2\sqrt{N}|\})^{\frac{1}{4}}},\quad y\in[0,\infty). \tag{3.43}$$

Claim (3.42) is immediate from (3.43) and the well-known property such that $\psi_N(y) = \psi_N(-y)$ if N is even and that $\psi_N(y) = -\psi_N(-y)$ if N is odd. □

3.3.2 Determinantal kernels of N-particle systems

We recall the definition of determinantal point processes. Let $K : \mathbb{R}^2 \to \mathbb{C}$ be a measurable kernel. A probability measure μ on S is called a determinantal point process with kernel K if, for each n, its n-point correlation function is given by

$$\rho^n(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n. \tag{3.44}$$

If K is an Hermitian symmetric and of locally trace class such that $0 \le \mathrm{Spec}(K) \le 1$, then there exists a unique determinantal point process with kernel K [65, 67].

The distribution of the delabeled eigenvalues of GUE associated with (3.1) is a determinantal point process with kernel K^N such that

$$\mathsf{K}^N(x, y) = \sum_{k=0}^{N-1} \psi_k(x)\psi_k(y). \tag{3.45}$$

The Christoffel-Darboux formula and a simple calculation yield the following.

$$\mathsf{K}^N(x, y) = \sqrt{\frac{N}{2}} \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x - y}. \tag{3.46}$$

From the scaling (3.3), μ_θ^N is a determinantal point process with kernel

$$\mathsf{K}_\theta^N(x, y) = \frac{1}{\sqrt{N}} \mathsf{K}^N\Big(\frac{x + N\theta}{\sqrt{N}}, \frac{y + N\theta}{\sqrt{N}}\Big). \tag{3.47}$$

Let $x_N = \sqrt{N}x$ and $y_N = \sqrt{N}y$. We set

$$\mathsf{L}^N(x, y) = \frac{1}{\sqrt{N}} \mathsf{K}^N(x_N, y_N) = \frac{1}{\sqrt{N}} \mathsf{K}^N(\sqrt{N}x, \sqrt{N}y). \tag{3.48}$$

From (3.47) and (3.48) we then clearly see that

$$\begin{aligned} \mathsf{K}_\theta^N(x, y) &= \mathsf{L}^N\Big(\frac{x}{N} + \theta, \frac{y}{N} + \theta\Big), \\ \mathsf{L}^N(x, y) &= \mathsf{K}_\theta^N(N(x - \theta), N(y - \theta)). \end{aligned} \tag{3.49}$$

From (3.46) we deduce

$$\mathsf{L}^N(x, x) = (1/\sqrt{2})\{\psi_{N-1}(x_N)\psi'_N(x_N) - \psi_N(x_N)\psi'_{N-1}(x_N)\}. \tag{3.50}$$

Using the Schwartz inequality to (3.45) we see from (3.46) and (3.48) that

$$\mathsf{L}^N(y, z)^2 \le \mathsf{L}^N(y, y)\mathsf{L}^N(z, z). \tag{3.51}$$

From here on, we assume

$$-\frac{2}{3} < \alpha < -\frac{1}{2}. \tag{3.52}$$

We set

$$\mathsf{B}^N = (-\sqrt{2} - N^\alpha, -\sqrt{2} + N^\alpha) \cup (\sqrt{2} - N^\alpha, \sqrt{2} + N^\alpha). \tag{3.53}$$

The next lemma will be used in Section 3.4.

Lemma 3.13. We set $\mathsf{U}^N = \mathbb{R}\backslash\mathsf{B}^N$. Then the following holds.
(1) There exists a constant c_{21} such that for all $N \in \mathbb{N}$

$$\sup_{x,y\in\mathbb{R}} |\mathsf{L}^N(x,y)| \le c_{21}N^{\frac{1}{3}}, \tag{3.54}$$

$$\sup_{x,y\in\mathsf{U}^N} |\mathsf{L}^N(x,y)| \le c_{21}. \tag{3.55}$$

(2) Assume (3.52). Then there exists a constant c_{22} such that

$$|\mathsf{L}^N(x,y)| \le \frac{c_{22}}{N|x-y|} \quad \text{for each } x,y \in \mathsf{U}^N,\ N \in \mathbb{N}. \tag{3.56}$$

Proof. It is well known that

$$\sqrt{2}\psi_n'(x) = \sqrt{n}\psi_{n-1}(x) - \sqrt{n+1}\psi_{n+1}(x).$$

From this and (3.50), we see that with a simple calculation

$$\begin{aligned}\mathsf{L}^N(x,x) &= \frac{1}{\sqrt{2}}\{\psi_{N-1}\psi_N' - \psi_N\psi_{N-1}'\}(x_N) \qquad (3.57)\\ &= \frac{N^{\frac{1}{2}}}{2}\{\psi_{N-1}^2 + \psi_N^2 - \sqrt{1-N^{-1}}\psi_{N-2}\psi_N - \sqrt{1+N^{-1}}\psi_{N-1}\psi_{N+1}\}(x_N).\end{aligned}$$

Combining this with (3.42) we obtain

$$\mathsf{L}^N(x,x) \le \frac{N^{\frac{1}{2}}}{2}5c_{20}^2N^{-\frac{1}{6}} = \frac{5c_{20}^2}{2}N^{\frac{1}{3}}.$$

From this and (3.51) we deduce (3.54). From Lemma 3.11 and (3.57), we see that

$$\sup_{N\in\mathbb{N}}\sup_{y\in\mathsf{U}^N} \mathsf{L}^N(y,y) < \infty.$$

We deduce (3.55) from this and (3.51). Taking a constant c_{21} in (3.54) and (3.55) in common completes the proof of (1).

Claim (3.56) follows from Lemma 3.11, (3.46), and (3.48). □

3.4 Proof of (3.37)–(3.40)

As we see in Section 3.2, the point of the proof of Theorem 3.1 is to check conditions (3.37)–(3.40) in Lemma 3.10. The purpose of this section is to prove these equations. We recall a property of the reduced Palm measures of determinantal point processes.

Lemma 3.14 ([65])**.** Let μ be a determinantal point process with kernel K. Assume that $K(x,y) = \overline{K(y,x)}$ and $0 \le \mathrm{Spec}(K) \le 1$. Then the reduced Palm measure μ_x is a determinantal point process with kernel K_x given by

$$K_x(y,z) = K(y,z) - \frac{K(y,x)K(x,z)}{K(x,x)} \tag{3.58}$$

for x such that $K(x,x) > 0$.

Let K_θ^N be the determinantal kernel of μ_θ^N given by (3.47). Let $\mu_{\theta,x}^N$ be as in Lemma 3.10. Recall that $\mathsf{K}_\theta^N(y,z) = \mathsf{K}_\theta^N(z,y)$ by definition. Then from this, (3.47), and (3.58), $\mu_{\theta,x}^N$ is a determinantal point process with kernel

$$\mathsf{K}_{\theta,x}^N(y,z) = \mathsf{K}_\theta^N(y,z) - \frac{\mathsf{K}_\theta^N(x,y)\mathsf{K}_\theta^N(x,z)}{\mathsf{K}_\theta^N(x,x)}. \tag{3.59}$$

From (3.44) and (3.59), we calculate correlation functions in (3.37)–(3.40) as follows.

$$\rho_\theta^{N,1}(y) = \mathsf{K}_\theta^N(y,y), \tag{3.60}$$

$$\rho_{\theta,x}^{N,1}(y) - \rho_\theta^{N,1}(y) = -\frac{\mathsf{K}_\theta^N(x,y)^2}{\mathsf{K}_\theta^N(x,x)}, \tag{3.61}$$

$$\rho_\theta^{N,2}(y,z) - \rho_\theta^{N,1}(y)\rho_\theta^{N,1}(z) = -\mathsf{K}_\theta^N(y,z)^2, \tag{3.62}$$

$$\begin{aligned} &\rho_{\theta,x}^{N,2}(y,z) - \rho_{\theta,x}^{N,1}(y)\rho_{\theta,x}^{N,1}(z) - \{\rho_\theta^{N,2}(y,z) - \rho_\theta^{N,1}(y)\rho_\theta^{N,1}(z)\} \\ &\quad = -\mathsf{K}_{\theta,x}^N(y,z)^2 + \mathsf{K}_\theta^N(y,z)^2 \\ &\quad = 2\frac{\mathsf{K}_\theta^N(y,z)\mathsf{K}_\theta^N(x,y)\mathsf{K}_\theta^N(x,z)}{\mathsf{K}_\theta^N(x,x)} - \frac{\mathsf{K}_\theta^N(x,y)^2\mathsf{K}_\theta^N(x,z)^2}{\mathsf{K}_\theta^N(x,x)^2}. \end{aligned} \tag{3.63}$$

Using these and (3.49) we rewrite (3.37)–(3.40) as follows.

Lemma 3.15. To simplify the notation, let

$$\mathsf{x}_\theta^N = \frac{x}{N} + \theta, \quad T_{r,\infty}^N(x) = \{y \in \mathbb{R}\,;\, \frac{r}{N} \le |\mathsf{x}_\theta^N - y| < \infty\}. \tag{3.64}$$

Then (3.37)–(3.40) are equivalent to (3.65)–(3.68) below, respectively.

$$\lim_{r\to\infty} \limsup_{N\to\infty} \Big\| \int_{T_{r,\infty}^N(x)} \frac{\mathsf{L}^N(y,y)}{\mathsf{x}_\theta^N - y} dy - \theta \Big\|_R = 0, \tag{3.65}$$

$$\lim_{r\to\infty} \limsup_{N\to\infty} \Big\| \int_{T_{r,\infty}^N(x)} \frac{1}{\mathsf{x}_\theta^N - y} \frac{\mathsf{L}^N(\mathsf{x}_\theta^N, y)^2}{\mathsf{L}^N(\mathsf{x}_\theta^N, \mathsf{x}_\theta^N)} dy \Big\|_R = 0. \tag{3.66}$$

Furthermore,

$$\lim_{r\to\infty} \limsup_{N\to\infty} \Big\| \int_{T_{r,\infty}^N(x)} \frac{1}{N} \frac{\mathsf{L}^N(y,y)}{|\mathsf{x}_\theta^N - y|^2} dy - \int_{T_{r,\infty}^N(x)^2} \frac{\mathsf{L}^N(y,z)^2}{(\mathsf{x}_\theta^N - y)(\mathsf{x}_\theta^N - z)} dydz \Big\|_R = 0, \tag{3.67}$$

$$\begin{aligned} &\lim_{r\to\infty} \limsup_{N\to\infty} \Big\| \int_{T_{r,\infty}^N(x)} \frac{1}{N} \frac{1}{|\mathsf{x}_\theta^N - y|^2} \frac{\mathsf{L}^N(\mathsf{x}_\theta^N, y)^2}{\mathsf{L}^N(\mathsf{x}_\theta^N, \mathsf{x}_\theta^N)} dy \\ &+ \int_{T_{r,\infty}^N(x)^2} \frac{1}{(\mathsf{x}_\theta^N - y)(\mathsf{x}_\theta^N - z)} \\ &\qquad \Big\{ 2\frac{\mathsf{L}^N(y,z)\mathsf{L}^N(\mathsf{x}_\theta^N, y)\mathsf{L}^N(\mathsf{x}_\theta^N, z)}{\mathsf{L}^N(\mathsf{x}_\theta^N, \mathsf{x}_\theta^N)} - \frac{\mathsf{L}^N(\mathsf{x}_\theta^N, y)\mathsf{L}^N(\mathsf{x}_\theta^N, z)}{\mathsf{L}^N(\mathsf{x}_\theta^N, \mathsf{x}_\theta^N)^2} \Big\} dydz \Big\|_R = 0. \end{aligned} \tag{3.68}$$

Proof. Recall that $\mathsf{L}^N(x,y) = \mathsf{K}_\theta^N(N(x-\theta), N(y-\theta))$ by (3.49). Then Lemma 3.15 follows easily from (3.60)–(3.63). $\square$

Let B^N and U^N be as in Lemma 3.13. Decompose U^N into U_1^N and U_2^N such that

$$\mathsf{U}_1^N = [-\sqrt{2}+N^\alpha, \sqrt{2}-N^\alpha], \quad \mathsf{U}_2^N = \mathbb{R}\backslash(-\sqrt{2}-N^\alpha, \sqrt{2}+N^\alpha).$$

Then clearly $\mathsf{U}^N = \mathsf{U}_1^N \cup \mathsf{U}_2^N$ and $\mathbb{R} = \mathsf{U}_1^N \cup \mathsf{U}_2^N \cup \mathsf{B}^N$. We begin by the integral outside U_1^N.

Lemma 3.16. Let $0 < q < 3/2$. Then

$$\limsup_{N\to\infty} \Big\| \int_{\mathbb{R}\backslash \mathsf{U}_1^N} \frac{\mathsf{L}^N(y,y)^q}{|\mathsf{x}_\theta^N - y|} dy \Big\|_R = 0. \tag{3.69}$$

Proof. From (3.54), (3.64), and the definition of B^N, we obtain that

$$\begin{aligned}
&\limsup_{N\to\infty} \Big\| \int_{\mathsf{B}^N} \frac{\mathsf{L}^N(y,y)^q}{|\mathsf{x}_\theta^N - y|} dy \Big\|_R \qquad (3.70)\\
\le&\limsup_{N\to\infty} \Big\| \int_{\mathsf{B}^N} \frac{c_{21}^q N^{\frac{q}{3}}}{|\mathsf{x}_\theta^N - y|} dy \Big\|_R \\
\le&\limsup_{N\to\infty} \Big\| c_{21}^q N^{\frac{q}{3}} \Big\{ \log\Big|\frac{x}{N}+\theta-(\sqrt{2}-N^\alpha)\Big| - \log\Big|\frac{x}{N}+\theta-(\sqrt{2}+N^\alpha)\Big| \Big\} \\
&+c_{21}^q N^{\frac{q}{3}} \Big\{ \log\Big|\frac{x}{N}+\theta-(-\sqrt{2}-N^\alpha)\Big| - \log\Big|\frac{x}{N}+\theta-(-\sqrt{2}+N^\alpha)\Big| \Big\} \Big\|_R \\
=&\mathcal{O}(N^{\frac{q}{3}+\alpha}) = 0 \quad \text{ as } N\to\infty.
\end{aligned}$$

Here we used $q < 3/2$ and $\alpha < -1/2$ in the last line.

Note that $|y| \ge \sqrt{2}+N^\alpha$ for $y \in \mathsf{U}_2^N$. Let $y = \sqrt{2}\cosh\tau$. Then we see that

$$\begin{aligned}
N\sinh^3\tau &= N(\cosh^2\tau - 1)^{\frac{3}{2}} \\
&= N2^{-\frac{3}{2}}(y^2-2)^{\frac{3}{2}} \ge N2^{-\frac{3}{2}}(2\sqrt{2}N^\alpha + N^{2\alpha})^{\frac{3}{2}}.
\end{aligned}$$

From this, $q > 0$, and $\alpha > -2/3$, we apply (3.41) to obtain $c_{23} > 0$ such that,

$$\limsup_{N\to\infty} \Big\| \int_{\mathsf{U}_2^N} \frac{\mathsf{L}^N(y,y)^q}{|\mathsf{x}_\theta^N - y|} dy \Big\|_R \le \limsup_{N\to\infty} \Big\| \int_{\mathsf{U}_2^N} \frac{c_{23}}{|\mathsf{x}_\theta^N - y| N^q (y^2-2)^q} dy \Big\|_R = 0,$$

which combined with (3.70) yields (3.69). □

Lemma 3.17. (3.65) holds.

Proof. Let $y = \sqrt{2}\cos\tau$. Then $N\sin^3\tau \ge N2^{-\frac{3}{2}}(2\sqrt{2}N^\alpha - N^{2\alpha})$ for $y \in \mathsf{U}_1^N$. Then applying Lemma 3.12 (1) we deduce that for each $r > 0$

$$\begin{aligned}
&\limsup_{N\to\infty} \Big\| \int_{T_{r,\infty}^N(x)\cap \mathsf{U}_1^N} \frac{\mathsf{L}^N(y,y)}{\mathsf{x}_\theta^N - y} dy - \theta \Big\|_R \\
=&\limsup_{N\to\infty} \Big\| \Big\{ \int_{-\sqrt{2}+N^\alpha}^{\mathsf{x}_\theta^N - \frac{r}{N}} + \int_{\mathsf{x}_\theta^N + \frac{r}{N}}^{\sqrt{2}-N^\alpha} \Big\} \frac{1}{\mathsf{x}_\theta^N - y} \frac{1}{\pi}\sqrt{2-y^2}\, dy - \theta \Big\|_R \\
=&\Big| \mathrm{P.V.} \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{\theta - y}\frac{1}{\pi}\sqrt{2-y^2} dy - \theta \Big| = 0.
\end{aligned}$$

Combining this with (3.69), we obtain (3.65). □

It is well known that $\mathsf{K}_\theta^N(x,x)$ are positive and continuous in x, and $\{\mathsf{K}_\theta^N(x,x)\}_{N\in\mathbb{N}}$ converges to $\mathsf{K}_\theta(x,x)=\sqrt{2-\theta^2}/\pi$ uniformly on each compact set. Then we see

$$\sup_{N\in\mathbb{N}}\sup_{x\in S_R}\frac{1}{\mathsf{K}_\theta^N(x,x)}<\infty.$$

From this, (3.49), and (3.64), we see that the following constant c_{24} is finite.

$$c_{24}:=\sup_{N\in\mathbb{N}}\sup_{x\in S_R}\frac{1}{\mathsf{L}^N(\mathsf{x}_\theta^N,\mathsf{x}_\theta^N)}<\infty. \tag{3.71}$$

Lemma 3.18. (3.72) and (3.73) below hold. In particular, (3.66) holds.

$$\lim_{r\to\infty}\limsup_{N\to\infty}\Big\|\int_{T_{r,\infty}^N(x)}\frac{\mathsf{L}^N(\mathsf{x}_\theta^N,y)^2}{|\mathsf{x}_\theta^N-y|\mathsf{L}^N(\mathsf{x}_\theta^N,\mathsf{x}_\theta^N)}dy\Big\|_R=0, \tag{3.72}$$

$$\lim_{r\to\infty}\limsup_{N\to\infty}\Big\|\int_{T_{r,\infty}^N(x)}\frac{\mathsf{L}^N(\mathsf{x}_\theta^N,y)}{|\mathsf{x}_\theta^N-y|\mathsf{L}^N(\mathsf{x}_\theta^N,\mathsf{x}_\theta^N)}dy\Big\|_R=0. \tag{3.73}$$

Proof. From (3.51) and (3.69) we deduce that as $N\to\infty$

$$\Big\|\int_{\mathbb{R}\backslash\mathsf{U}_1^N}\frac{\mathsf{L}^N(\mathsf{x}_\theta^N,y)^2}{|\mathsf{x}_\theta^N-y|\mathsf{L}^N(\mathsf{x}_\theta^N,\mathsf{x}_\theta^N)}dy\Big\|_R\le\Big\|\int_{\mathbb{R}\backslash\mathsf{U}_1^N}\frac{\mathsf{L}^N(y,y)}{|\mathsf{x}_\theta^N-y|}dy\Big\|_R\to 0. \tag{3.74}$$

From (3.56) and (3.71) for each $N\in\mathbb{N}$ and $r>0$

$$\begin{aligned}\Big\|\int_{T_{r,\infty}^N(x)\cap\mathsf{U}_1^N}\frac{\mathsf{L}^N(\mathsf{x}_\theta^N,y)^2\,dy}{|\mathsf{x}_\theta^N-y|\mathsf{L}^N(\mathsf{x}_\theta^N,\mathsf{x}_\theta^N)}\Big\|_R&\le\Big\|\int_{T_{r,\infty}^N(x)\cap\mathsf{U}_1^N}\frac{c_{22}^2c_{24}\,dy}{N^2|\mathsf{x}_\theta^N-y|^3}\Big\|_R\\&\le\frac{c_{22}^2c_{24}}{r^2}.\end{aligned} \tag{3.75}$$

Hence (3.72) follows from (3.74) and (3.75). This completes the proof of (3.72).

We next prove (3.73). From (3.51), (3.69), and (3.71) we see for each $r>0$

$$\limsup_{N\to\infty}\Big\|\int_{T_{r,\infty}^N(x)\backslash\mathsf{U}_1^N}\frac{\mathsf{L}^N(\mathsf{x}_\theta^N,y)}{|\mathsf{x}_\theta^N-y|\mathsf{L}^N(\mathsf{x}_\theta^N,\mathsf{x}_\theta^N)}dy\Big\|_R=0. \tag{3.76}$$

From (3.56) and (3.71) we see that for each $N\in\mathbb{N}$ and $r>0$

$$\begin{aligned}\Big\|\int_{T_{r,\infty}^N(x)\cap\mathsf{U}_1^N}\frac{\mathsf{L}^N(\mathsf{x}_\theta^N,y)\,dy}{|\mathsf{x}_\theta^N-y|\mathsf{L}^N(\mathsf{x}_\theta^N,\mathsf{x}_\theta^N)}\Big\|_R&\le\Big\|\int_{T_{r,\infty}^N(x)\cap\mathsf{U}_1^N}\frac{c_{22}c_{24}\,dy}{N|\mathsf{x}_\theta^N-y|^2}\Big\|_R\\&\le\frac{2c_{22}c_{24}}{r}.\end{aligned} \tag{3.77}$$

Combining (3.76) and (3.77) we obtain (3.73). □

Lemma 3.19. (3.78) and (3.79) below hold. In particular, (3.67) holds.

$$\lim_{r\to\infty}\limsup_{N\to\infty}\Big\|\int_{T_{r,\infty}^N(x)}\frac{\mathsf{L}^N(y,y)}{N|\mathsf{x}_\theta^N-y|^2}\,dy\Big\|_R=0, \tag{3.78}$$

$$\lim_{r\to\infty}\limsup_{N\to\infty}\Big\|\int_{T_{r,\infty}^N(x)^2}\frac{\mathsf{L}^N(y,z)^2}{|\mathsf{x}_\theta^N-y||\mathsf{x}_\theta^N-z|}dydz\Big\|_R=0. \tag{3.79}$$

Proof. Note that $\mathsf{L}^N(y,y) \le c_{21}$ on U^N by (3.55). Then by the definition of $T^N_{r,\infty}(x)$,

$$\int_{T^N_{r,\infty}(x)\cap \mathsf{U}^N} \frac{\mathsf{L}^N(y,y)}{N|\mathsf{x}^N_\theta - y|^2}\, dy \le \frac{c_{21}}{N}\frac{2N}{r} = \frac{2c_{21}}{r}. \tag{3.80}$$

By (3.54) we see $\mathsf{L}^N(y,y) \le c_{21}N^{\frac{1}{3}}$ on $\mathbb{R}$. Recall that $|\mathsf{B}^N| = 4N^\alpha$ by construction. Furthermore, $c_{25} := \limsup_{N\to\infty} \sup_{y\in\mathsf{B}^N} \||\mathsf{x}^N_\theta - y|^{-2}\|_R < \infty$. Hence for each $r>0$

$$\limsup_{N\to\infty} \int_{T^N_{r,\infty}(x)\cap \mathsf{B}^N} \frac{\mathsf{L}^N(y,y)}{N|\mathsf{x}^N_\theta - y|^2}\, dy \le \limsup_{N\to\infty} \frac{c_{21}N^{\frac{1}{3}}4N^\alpha c_{25}}{N} = 0. \tag{3.81}$$

Here we used $\alpha < -1/2$. We thus obtain (3.78) from (3.80) and (3.81).

We proceed with the proof of (3.79). We first consider the integral away from the diagonal line. By (3.56) and the Schwartz inequality, we see that

$$\begin{aligned}
&\Big\| \int_{(T^N_{r,\infty}(x)\cap\mathsf{U}^N)^2\cap\{|y-z|\ge\frac{1}{N}\}} \frac{\mathsf{L}^N(y,z)^2}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz \Big\|_R \\
\le& \Big\| \int_{(T^N_{r,\infty}(x)\cap\mathsf{U}^N)^2\cap\{|y-z|\ge\frac{1}{N}\}} \frac{c_{22}^2}{N^2|y-z|^2|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz \Big\|_R \\
\le& \Big\| \Big\{ \int_{T^N_{r,\infty}(x)^2\cap\{|y-z|\ge\frac{1}{N}\}} \frac{c_{22}^2}{N^2|y-z|^2|\mathsf{x}^N_\theta - y|^2} dydz \Big\}^{\frac{1}{2}} \\
&\qquad \Big\{ \int_{T^N_{r,\infty}(x)^2\cap\{|y-z|\ge\frac{1}{N}\}} \frac{c_{22}^2}{N^2|y-z|^2|\mathsf{x}^N_\theta - z|^2} dydz \Big\}^{\frac{1}{2}} \Big\|_R \\
=& \Big\| \int_{T^N_{r,\infty}(x)^2\cap\{|y-z|\ge\frac{1}{N}\}} \frac{c_{22}^2}{N^2|y-z|^2|\mathsf{x}^N_\theta - y|^2} dydz \Big\|_R \\
\le& c_{22}^2 \frac{2N}{N^2}\Big\{\frac{2N}{r}\Big\} = \frac{4c_{22}^2}{r}.
\end{aligned}$$

The last line follows from a straightforward calculation. Indeed, first integrating z over $\{|y-z| \ge \frac{1}{N}\}$, and then integrating y over $T^N_{r,\infty}(x)$, we obtain the inequality in the last line. We therefore see that

$$\lim_{r\to\infty}\lim_{N\to\infty} \Big\| \int_{(T^N_{r,\infty}(x)\cap\mathsf{U}^N)^2\cap\{|y-z|\ge\frac{1}{N}\}} \frac{\mathsf{L}^N(y,z)^2}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz \Big\|_R = 0. \tag{3.82}$$

We next consider the integral near the diagonal. From (3.55), we see that

$$\begin{aligned}
&\Big\| \int_{(T^N_{r,\infty}(x)\cap\mathsf{U}^N)^2\cap\{|y-z|\le\frac{1}{N}\}} \frac{\mathsf{L}^N(y,z)^2}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz \Big\|_R \\
\le& \Big\| \int_{(T^N_{r,\infty}(x)\cap\mathsf{U}^N)^2\cap\{|y-z|\le\frac{1}{N}\}} \frac{c_{21}^2}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz \Big\|_R \\
\le& \Big\| \int_{T^N_{r,\infty}(x)^2\cap\{|y-z|\le\frac{1}{N}\}} \frac{c_{21}^2}{2}\Big\{\frac{1}{|\mathsf{x}^N_\theta - y|^2} + \frac{1}{|\mathsf{x}^N_\theta - z|^2}\Big\} dydz \Big\|_R \\
=& \frac{2c_{21}^2}{N} \Big\| \int_{T^N_{r,\infty}(x)} \frac{1}{|\mathsf{x}^N_\theta - y|^2} dy \Big\|_R = \frac{2c_{21}^2}{N}\frac{2N}{r} = \frac{4c_{21}^2}{r}.
\end{aligned} \tag{3.83}$$

From (3.82) and (3.83), we have

$$\lim_{r\to\infty}\lim_{N\to\infty}\Big\| \int_{(T^N_{r,\infty}(x)\cap \mathsf{U}^N)^2} \frac{\mathsf{L}^N(y,z)^2}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz\Big\|_R = 0. \tag{3.84}$$

We next consider the integral on $\mathsf{B}^N \times \mathsf{B}^N$. Let

$$c_{26} = \limsup_{N\to\infty} \sup_{x\in S_R, y\in \mathsf{B}^N} |\mathsf{x}^N_\theta - y|^{-1}.$$

Then, we deduce from (3.54) and the definition of B^N given by (3.53) that

$$\begin{aligned}&\limsup_{N\to\infty}\Big\| \int_{(T^N_{r,\infty}(x)\cap \mathsf{B}^N)^2} \frac{\mathsf{L}^N(y,z)^2}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz\Big\|_R \\ &\le \lim_{N\to\infty} c_{21}^2 c_{26}^2 N^{\frac{2}{3}} (4N^\alpha)^2 = 0.\end{aligned} \tag{3.85}$$

Here we used $|\mathsf{B}^N| = 4N^\alpha$ for the inequality and $\alpha < -1/2$ for the last equality.

We finally consider the case $\mathsf{U}^N \times \mathsf{B}^N$. Then a similar argument yields

$$\begin{aligned}&\Big\| \int_{(T^N_{r,\infty}(x)\cap \mathsf{U}^N)\times(T^N_{r,\infty}(x)\cap \mathsf{B}^N)} \frac{\mathsf{L}^N(y,z)^2}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz\Big\|_R \\ \le&\Big\| \int_{(T^N_{r,\infty}(x)\cap \mathsf{U}^N)\times(T^N_{r,\infty}(x)\cap \mathsf{B}^N)} \frac{\mathsf{L}^N(y,y)\mathsf{L}^N(z,z)}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|} dydz\Big\|_R \\ =&\Big\| \int_{T^N_{r,\infty}(x)\cap \mathsf{U}^N} \frac{\mathsf{L}^N(y,y)}{|\mathsf{x}^N_\theta - y|} dy \int_{T^N_{r,\infty}(x)\cap \mathsf{B}^N} \frac{\mathsf{L}^N(z,z)}{|\mathsf{x}^N_\theta - z|} dz\Big\|_R \\ =&\mathcal{O}(\log N)\mathcal{O}(N^{\frac{1}{3}+\alpha}) \to 0 \quad \text{as } N\to\infty.\end{aligned} \tag{3.86}$$

Collecting (3.84), (3.85), and (3.86), we conclude (3.79). □

Lemma 3.20. (3.68) holds.

Proof. We shall estimate the three terms in (3.68) beginning with the first. From (3.51) and (3.78) we have

$$\begin{aligned}&\lim_{r\to\infty}\limsup_{N\to\infty}\Big\| \int_{T^N_{r,\infty}(x)} \frac{\mathsf{L}^N(\mathsf{x}^N_\theta, y)^2 dy}{N|\mathsf{x}^N_\theta - y|^2 \mathsf{L}^N(\mathsf{x}^N_\theta, \mathsf{x}^N_\theta)}\Big\|_R \\ &\le \lim_{r\to\infty}\limsup_{N\to\infty}\Big\| \int_{T^N_{r,\infty}(x)} \frac{\mathsf{L}^N(y,y) dy}{N|\mathsf{x}^N_\theta - y|^2}\Big\|_R = 0.\end{aligned} \tag{3.87}$$

Next, using the Schwartz inequality, we have for the second term

$$\begin{aligned}&\Big\| \int_{T^N_{r,\infty}(x)^2} \frac{\mathsf{L}^N(y,z)\mathsf{L}^N(\mathsf{x}^N_\theta, y)\mathsf{L}^N(\mathsf{x}^N_\theta, z)\, dydz}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|\mathsf{L}^N(\mathsf{x}^N_\theta, \mathsf{x}^N_\theta)}\Big\|_R \\ \le&\Big\| \int_{T^N_{r,\infty}(x)^2} \frac{\mathsf{L}^N(y,z)^2 dydz}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|}\Big\|_R^{\frac{1}{2}} \Big\| \int_{T^N_{r,\infty}(x)^2} \frac{\mathsf{L}^N(\mathsf{x}^N_\theta, y)^2\mathsf{L}^N(\mathsf{x}^N_\theta, z)^2 dydz}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|\mathsf{L}^N(\mathsf{x}^N_\theta, \mathsf{x}^N_\theta)^2}\Big\|_R^{\frac{1}{2}} \\ =&\Big\| \int_{T^N_{r,\infty}(x)^2} \frac{\mathsf{L}^N(y,z)^2 dydz}{|\mathsf{x}^N_\theta - y||\mathsf{x}^N_\theta - z|}\Big\|_R^{\frac{1}{2}} \Big\| \int_{T^N_{r,\infty}(x)} \frac{\mathsf{L}^N(\mathsf{x}^N_\theta, y)^2}{|\mathsf{x}^N_\theta - y|\mathsf{L}^N(\mathsf{x}^N_\theta, \mathsf{x}^N_\theta)} dy\Big\|_R.\end{aligned}$$

Applying (3.79) and (3.72) to the last line, we obtain

$$\lim_{r\to\infty}\limsup_{N\to\infty}\Big\| \int_{T^N_{r,\infty}(x)^2} \frac{\mathsf{L}^N(y,z)\mathsf{L}^N(\mathsf{x}^N_\theta,y)\mathsf{L}^N(\mathsf{x}^N_\theta,z)\,dydz}{|\mathsf{x}^N_\theta-y||\mathsf{x}^N_\theta-z|\mathsf{L}^N(\mathsf{x}^N_\theta,\mathsf{x}^N_\theta)}\Big\|_R=0. \tag{3.88}$$

We finally estimate the third term. From (3.73), as $N\to\infty$, we have

$$\begin{aligned}&\Big\| \int_{T^N_{r,\infty}(x)^2} \frac{\mathsf{L}^N(\mathsf{x}^N_\theta,y)\mathsf{L}^N(\mathsf{x}^N_\theta,z)\,dydz}{|\mathsf{x}^N_\theta-y||\mathsf{x}^N_\theta-z|\mathsf{L}^N(\mathsf{x}^N_\theta,\mathsf{x}^N_\theta)^2}\Big\|_R \\ =&\Big\|\Big\{ \int_{T^N_{r,\infty}(x)} \frac{\mathsf{L}^N(\mathsf{x}^N_\theta,y)\,dy}{|\mathsf{x}^N_\theta-y|\mathsf{L}^N(\mathsf{x}^N_\theta,\mathsf{x}^N_\theta)}\Big\}^2\Big\|_R \\ =&\Big\| \int_{T^N_{r,\infty}(x)} \frac{\mathsf{L}^N(\mathsf{x}^N_\theta,y)\,dy}{|\mathsf{x}^N_\theta-y|\mathsf{L}^N(\mathsf{x}^N_\theta,\mathsf{x}^N_\theta)}\Big\|_R^2\to 0 \quad \text{by (3.73).}\end{aligned} \tag{3.89}$$

From (3.87), (3.88), and (3.89) we obtain (3.68). This completes the proof. □

3.5 Proof of Theorem 3.1

From Lemma 3.17–Lemma 3.20 we deduce that all the assumptions (3.37)–(3.40) in Lemma 3.10 are satisfied. Hence (3.26) is proved by Lemma 3.10. Then Theorem 3.1 follows from Lemma 3.8, Lemma 3.9, and Lemma 3.10.

3.6 Proof of Theorem 3.2

In this section we prove Theorem 3.2 using Theorem 3.1. It is sufficient for the proof of Theorem 3.2 to prove (3.15) in $C([0,T];\mathbb{R}^m)$ for each $T\in\mathbb{N}$. Hence we fix $T\in\mathbb{N}$. Let $\mathbf{X}^N=(X^{N,i})_{i=1}^N$ be as in (3.13). Let $Y^{\theta,N,i}=\{Y^{\theta,N,i}_t\}$ such that

$$Y^{\theta,N,i}_t=X^{N,i}_t+\theta t. \tag{3.90}$$

Then from (3.13) we see that $\mathbf{Y}^{\theta,N}=(Y^{\theta,N,i})_{i=1}^N$ is a solution of

$$dY^{\theta,N,i}_t=dB^i_t+\sum_{j\neq i}^N\frac{1}{Y^{\theta,N,i}_t-Y^{\theta,N,j}_t}dt-\frac{1}{N}Y^{\theta,N,i}_t\,dt+\frac{\theta}{N}\,dt \tag{3.91}$$

with the same initial condition as $\mathbf{X}^N$. Let $P^{\theta,N}$ and $Q^{\theta,N}$ be the distributions of $\mathbf{X}^N$ and $\mathbf{Y}^{\theta,N}$ on $C([0,T];\mathbb{R}^N)$, respectively. Then applying the Girsanov theorem [18, pp.190-195] to (3.91), we see that

$$\begin{aligned}\frac{dQ^{\theta,N}}{dP^{\theta,N}}(\mathbf{W})&=\exp\{\int_0^T\sum_{i=1}^N\frac{\theta}{N}dB^i_t-\frac{1}{2}\int_0^T\sum_{i=1}^N\Big|\frac{\theta}{N}\Big|^2dt\}\\ &=\exp\{\frac{\theta}{N}\sum_{i=1}^N B^i_T-\frac{\theta^2T}{2N}\},\end{aligned} \tag{3.92}$$

where we write $\mathbf{W}=(W^i)\in C([0,T];\mathbb{R}^N)$ and $\{B^i\}_{i=1}^N$ under $P^{\theta,N}$ are independent copies of Brownian motions starting at the origin.

Lemma 3.21. For each $\epsilon > 0$,

$$\lim_{N\to\infty} Q^{\theta,N}\Big(\Big|\frac{dP^{\theta,N}}{dQ^{\theta,N}}(\mathbf{W}) - 1\Big| \geq \epsilon\Big) = 0. \tag{3.93}$$

Proof. It is sufficient for (3.93) to prove, for each $\epsilon > 0$,

$$\lim_{N\to\infty} P^{\theta,N}\Big(\Big|\frac{dQ^{\theta,N}}{dP^{\theta,N}}(\mathbf{W}) - 1\Big| \geq \epsilon\Big) = 0.$$

This follows from (3.92) immediately. □

Proof of Theorem 3.2. We write $\mathbf{W}^m = (W^1, \dots, W^m) \in C([0,T];\mathbb{R}^m)$ for $\mathbf{W} = (W^i)_{i=1}^N$, where $m \leq N \leq \infty$. Let Q^θ be the distribution of the solution $\mathbf{Y}^\theta$ with initial distribution $\mu_\theta \circ \mathfrak{l}^{-1}$. From Theorem 3.1 and (3.90) we deduce that for each $m \in \mathbb{N}$

$$\lim_{N\to\infty} Q^{\theta,N}(\mathbf{W}^m \in \cdot) = Q^\theta(\mathbf{W}^m \in \cdot)$$

weakly in $C([0,T];\mathbb{R}^m)$. Then from this, for each $F \in C_b(C([0,T];\mathbb{R}^m))$,

$$\lim_{N\to\infty} \int_{C([0,T];\mathbb{R}^N)} F(\mathbf{W}^m) dQ^{\theta,N} = \int_{C([0,T];\mathbb{R}^{\mathbb{N}})} F(\mathbf{W}^m) dQ^\theta. \tag{3.94}$$

We obtain from (3.93) and (3.94) that

$$\begin{aligned}
\lim_{N\to\infty} \int_{C([0,T];\mathbb{R}^N)} F(\mathbf{W}^m) dP^{N,\theta} &= \lim_{N\to\infty} \int_{C([0,T];\mathbb{R}^N)} F(\mathbf{W}^m) \frac{dP^{\theta,N}}{dQ^{\theta,N}}(\mathbf{W}) dQ^{\theta,N} \\
&= \lim_{N\to\infty} \int_{C([0,T];\mathbb{R}^N)} F(\mathbf{W}^m) dQ^{\theta,N} \\
&= \int_{C([0,T];\mathbb{R}^{\mathbb{N}})} F(\mathbf{W}^m) dQ^\theta.
\end{aligned}$$

This implies (3.15). We have thus completed the proof of Theorem 3.2. □

4 Density preservation of unlabeled diffusion in systems with infinitely many particles

4.1 Introduction

Let S be a configuration space over $\mathbb{R}^d$ for $d \in \mathbb{N}$. We endow S with the vague topology. Let μ be a random point field on $\mathbb{R}^d$ with infinitely many particles, and consider a μ-reversible diffusion (X, P) with state space S. Here $\mathsf{X} = \{\mathsf{X}_t\}$ is of the form $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$ and $\mathsf{P} = \{\mathsf{P}_\mathsf{s}\}_{\mathsf{s}\in\mathsf{S}}$ is the diffusion measure.

Suppose that for μ-a.s. s, there exists a limit $\lim_{r\to\infty} \mathsf{s}(S_r)/r^d$, where $S_r = \{x \in \mathbb{R}^d\,;\, |x| < r\}$, and let

$$\Phi(\mathsf{s}) = \lim_{r\to\infty} \frac{\mathsf{s}(S_r)}{r^d}.$$

This assumption holds, for example, if μ is translation invariant. Note that Φ is tail σ-field measurable random variable by definition [see (4.4) below]. For a fixed positive constant θ, we set $\mathsf{A}_\theta = \{\mathsf{s}\,;\, \Phi(\mathsf{s}) = \theta\}$. Then, from the reversibility of (X, P),

$$\mathsf{P}_\mu\Big(\lim_{r\to\infty} \frac{\mathsf{X}_t(S_r)}{r^d} = \theta\Big) = \mu(\mathsf{A}_\theta) \ \text{ for any } t. \tag{4.1}$$

The purpose of this paper is to refine (4.1) such that for q.e. $\mathsf{s} \in \mathsf{A}_\theta$,

$$\mathsf{P}_\mathsf{s}\Big(\lim_{r\to\infty} \frac{\mathsf{X}_t(S_r)}{r^d} = \theta \ \text{ for any } t\Big) = 1.$$

We prove that an unlabeled diffusion starting on a set that is specified in terms of density does not change the density over the course of its time evolution. This property is useful for the study of the dynamics of infinite particle systems.

Note that the set A_θ is an element of the tail σ-field of S. The tail σ-field plays an important role in the study of the properties of unlabeled diffusions. Indeed, the tail σ-field contains global information about infinite particle systems. A typical example is the particle density, as mentioned above. We are particularly interested in the tail-preserving property of unlabeled diffusions, that is, whether an unlabeled diffusion starts on an element of the tail σ-field, then it stays on the set permanently. However, the tail σ-field is not topologically well behaved; for example, it is not countably determined in general even if the state space is countably determined. Consequently, it is hard to treat the tail σ-field directly. Conversely, if the tail σ-field is identified by particle densities, we can discuss the behavior of an unlabeled diffusion on the field by studying the density instead of the field itself. Then, in some cases the tail-preserving property follows from the preservation of density.

Our result is closely related to the ergodic decomposition of unlabeled diffusions. Because the space of an unlabeled diffusion is huge, it is an important and difficult problem to specify the topological support when infinitely many particles are in motion. Our result is a first step toward addressing this problem.

Density preservation is also important from the point of view of infinite-dimensional stochastic differential equations (ISDEs), because the tail preserving property implies the strong uniqueness of a solution of an ISDE. We consider interacting Brownian motions with infinitely many particles having an interaction potential Ψ. The dynamics is described by the ISDE

$$dX_t^i = dB_t^i - \frac{1}{2}\sum_{i\neq j} \nabla_x \Psi(X_t^i, X_t^j)dt,\ 1 \le i < \infty. \tag{4.2}$$

Lang began to study (4.2) using Itô's calculus [37, 38]. In this work, he assumed that Ψ is C_0^3 or exponentially decaying. Lang's result therefore does not work if Ψ is a long-range potential, for example, logarithmic. This work was followed by Fritz [15], Tanemura [70], and others. Recently, Tsai [76] solved (4.2) for the case in which Ψ is logarithmic and $d = 1$, that is, Dyson's Brownian motion in infinite dimensions. This result can be applied to out-of-equilibrium initial conditions, then this is a strong way to study ISDEs. On the other hand, the Dirichlet form approach can also solve (4.2) under assumptions including long-range potentials. In fact, Osada [44] constructed an unlabeled diffusion of (4.2) whenever Ψ is logarithmic potential using this approach. Then, using this unlabeled diffusion, (4.2) was again solved using Dirichlet forms [47]. Furthermore, the sufficiency condition that an ISDE of the form given by (4.2) has a unique strong solution has been shown by Osada and Tanemura [53]. They identified the sufficient conditions in the context of a random point field. Their results guarantee that an ISDE in the form of (4.2) has a unique strong solution when a random point field is tail trivial.

In addition, they also discussed the strong uniqueness of a solution of an ISDE when a random point field is *not* tail trivial. In this case, the random point field has multiple tails. They proved that if a solution of an ISDE satisfies the absolute continuity condition with respect to the random point field conditioned by the tail σ- field, then strong uniqueness holds. That is, so long as a solution has the tail-preserving property, strong uniqueness holds. However, they could not exclude existence of a solution that does not satisfy this condition. Proving that there is no solution such that the tail-preserving condition is not satisfied remain an open question in [53].

Our result addresses this problem in part. We can demonstrate the strong uniqueness of an ISDE in a more general situation than considered in [53]. In particular, this general theory can be applied to an ISDE related to random matrices. One of the most important examples of this is Dyson's Brownian motion with infinitely many particles, which has a logarithmic interaction potential. Then we can show that the strong uniqueness of Dyson's Brownian motion with multiple tails holds as a corollary of our result, but we do not pursue this topic here.

Density preservation is also important from the point of view of finite particle approximations of ISDEs. We will demonstrate that a solution of a finite dimensional stochastic differential equation converges to that of the corresponding ISDE as the particle number goes to infinity. One of the key points of the proof in the finite particle approximation is the uniqueness of a solution of an ISDE in the limit. Therefore, we can employ the finite particle approximation of an ISDE associated with many random point fields if we can prove that the tail-preserving property holds for an unlabeled diffusion associated with the random point fields.

This paper is organized as follows. In Section 4.2, we describe our framework and the main results. In Section 4.3, we prove the main result.

4.2 Set up and main results

We begin by defining a random point field and introducing an unlabeled diffusion. Set $S = \mathbb{R}^d$ $(d \geq 1)$ and let S be a configuration space over S defined by

$$\mathsf{S} = \Big\{ \mathsf{s} = \sum_i \delta_{s_i} \,;\, s_i \in S \text{ with } \mathsf{s} \text{ is a Radon measure} \Big\}.$$

A probability measure μ on S is called a *random point field.*

A symmetric function $\rho^n : S^n \to \mathbb{C}$ is called the *n-correlation function* of μ with respect to the Lebesgue measure if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(x_1,\ldots,x_n) \prod_{i=1}^n dx_i = \int_{\mathsf{S}} \prod_{i=1}^m \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!}\, d\mu(\mathsf{s})$$

holds for any disjoint compact sets $A_i \in \mathcal{B}(S)$ and $k_i \in \mathbb{N}$ such that $\sum_{i=1}^m k_i = n$.

Next, we define the Dirichlet form associated with μ. We set $S_r = \{x \in S\,;\, |x| \leq r\}$ and let $\pi_r : \mathsf{S} \to \mathsf{S}$ be a mapping such that $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$. A function f on S is called *local* if there exists an $r \in \mathbb{N}$ such that f is $\sigma[\pi_r]$-measurable and called *smooth* if $\check{f}$ is smooth, where $\check{f}((s_i)_i)$ is a permutation invariant function in $(s_i)_i \in \bigcup_{n\in\mathbb{N}} S^n \cup S^{\mathbb{N}}$ such that $f(\mathsf{s}) = \check{f}((s_i)_i)$. Let $\mathcal{D}_\circ$ be the set of all of local smooth functions on S.

For $f, g \in \mathcal{D}_\circ$, we define a bilinear form as

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_i \nabla_{s_i}\check{f}(\mathbf{s}) \cdot \nabla_{s_i}\check{g}(\mathbf{s}),$$

where $\mathsf{s} = \sum_i \delta_{s_i}$ and $\mathbf{s} = (s_i)_i$. We use the notation $\mathbb{D}[f]$ for $\mathbb{D}[f,f]$. Define a bilinear form $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ on $L^2(\mathsf{S},\mu)$ as

$$\mathcal{E}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g](\mathsf{s})\, d\mu(\mathsf{s}) \text{ for } f,g \in \mathcal{D}_\circ^\mu,$$
$$\mathcal{D}_\circ^\mu = \{f \in \mathcal{D}_\circ \cap L^2(\mathsf{S},\mu)\,;\, \mathcal{E}(f,f) < \infty\}.$$

We further assume that

(A1) ρ^n is locally bounded for each $n \in \mathbb{N}$; and
(A2) $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mathsf{S},\mu)$.

Let $(\mathcal{E}, \mathcal{D})$ be the closure of $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ on $L^2(\mathsf{S},\mu)$. It is known that, given **(A1)** and **(A2)**, $(\mathcal{E}, \mathcal{D})$ is a local, quasi-regular Dirichlet form [44]. In particular, there exists an associated S-valued diffusion $(\mathsf{X}, \{\mathsf{P}_\mathsf{s}\}_{\mathsf{s}\in\mathsf{H}})$ with state space $\mathsf{H} \subset \mathsf{S}$ such that $\mu(\mathsf{H}) = 1$. This S-valued diffusion is called the *unlabeled diffusion.*

Throughout this paper, we assume the random point field μ has infinitely many particles with probability 1, that is,

$$\mu(\mathsf{S}_\infty) = 1, \quad \text{where } \mathsf{S}_\infty = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S) = \infty\}. \tag{4.3}$$

In addition to (4.3), we assume the following:

(A3) $\mathrm{Cap}^\mu(\mathsf{S}_\infty^c) = 0$.

Recall that for a subset $\mathsf{A} \subset \mathsf{S}$, $\mathrm{Cap}^\mu(\mathsf{A})$ denotes the capacity of A with respect to the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(\mathsf{S},\mu))$.

Remark 4.1. It is known that if each tagged particle of $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ does not explode, then **(A3)** holds [46]. We do not explain about non-explosion property in this paper. Refer to [46] for this.

For **(A3)**, we can regard $(\mathcal{E}, \mathcal{D}, L^2(\mathsf{S}, \mu))$ as $(\mathcal{E}, \mathcal{D}, L^2(\mathsf{S}_\infty, \mu))$ and the associated S-valued diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ as being S_∞-valued.

Hereafter, we fix the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(\mathsf{S}_\infty, \mu))$ and consider the unlabeled diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ associated with it. We use the general concepts of Dirichlet form theory (see [16]).

For a non-decreasing function $f : [0,\infty) \to (0,\infty)$ that satisfy $\lim_{r\to\infty} f(r) = \infty$, we define random variables $\Phi_\pm(\mathsf{s}) : \mathsf{S}_\infty \to [0,\infty]$ as

$$\Phi_+(\mathsf{s}) = \limsup_{r\to\infty} \frac{\mathsf{s}(S_r)}{f(r)},$$
$$\Phi_-(\mathsf{s}) = \liminf_{r\to\infty} \frac{\mathsf{s}(S_r)}{f(r)}.$$

Let $\mathcal{T}(\mathsf{S}_\infty)$ be the tail σ-field given by

$$\mathcal{T}(\mathsf{S}_\infty) = \bigcap_{r\in\mathbb{N}} \sigma[\pi_r^c], \tag{4.4}$$

where $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$. For each $i \in \{+,-\}$, we define A_i as

$$\mathsf{A}_i = \{\mathsf{s}\,;\, \Phi_i(\mathsf{s}) = 1\}. \tag{4.5}$$

Note that $\mathsf{A}_i \in \mathcal{T}(\mathsf{S}_\infty)$, because $\Phi_\pm$ is $\mathcal{T}(\mathsf{S}_\infty)$-measurable.

Theorem 4.2. With assumptions **(A1)**–**(A3)**, if f satisfies

$$\lim_{r\to\infty} \frac{f(r+1)}{f(r)} = 1, \tag{4.6}$$

then the associated unlabeled diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ satisfies, for $i \in \{+,-\}$,

$$\mathsf{P}_{\mathsf{s}}(\tau_{\mathsf{A}_i} = \infty) = 1 \text{ for q.e. } \mathsf{s} \in \mathsf{A}_i, \tag{4.7}$$

Here, τ_{A} is the first exit time from A defined as

$$\tau_{\mathsf{A}} = \inf\{t > 0\,;\, \mathsf{X}_t \notin \mathsf{A}\}.$$

Recall that "q.e." in (4.7) is the abbreviation of "quasi-everywhere," which means that the equations holds with the exception of a set of zero capacity ([16, p.68]). Then (4.7) means that for fixed f and i, there exists $\mathsf{N}_{f,i}$ such that $\mathrm{Cap}^\mu(\mathsf{N}_{f,i}) = 0$ and $\mathsf{P}_{\mathsf{s}}(\tau_{\mathsf{A}_i} = \infty) = 1$ for any $\mathsf{s} \in \mathsf{A}_i \setminus \mathsf{N}_{f,i}$.

Remark 4.3. If f is a polynomial growth function, then it satisfies (4.6). Exponential growth functions do not.

Remark 4.4. Note that $(\mathsf{X}, \{\mathsf{P_s}\}_{\mathsf{s}\in\mathsf{H}})$ is μ-reversible by construction, and thus the following trivially holds:

$$\mathsf{P}_\mu(\mathsf{X}_t \in \mathsf{A}_i) = \mu(\mathsf{A}_i) \text{ for any } t \in [0, \infty). \tag{4.8}$$

Equation (4.8) implies that the probability of X_t being in A_i is invariant for each $t \in [0, \infty)$. It does not, however, provide information about the trajectory of the diffusion. In contrast, what we prove in Theorem 4.2 is that for q.e. $\mathsf{s} \in \mathsf{A}_i$,

$$\mathsf{P_s}(\mathsf{X}_t \in \mathsf{A}_i \text{ for any } t \in [0, \infty)) = 1, \tag{4.9}$$

that is, A_i is an invariant set of the diffusion.

We next provide an application of Theorem 4.2. Fix a positive constant $\theta \in (0, \infty)$. Let A_θ represent all of the configurations with density θ given by

$$\mathsf{A}_\theta = \Big\{\mathsf{s}\,;\, \lim_{r\to\infty} \frac{\mathsf{s}(S_r)}{\mathrm{vol}(S_r)} = \theta\Big\}.$$

From Theorem 4.2 by choosing $f(r) = \theta\, \mathrm{vol}(S_r)$, we obtain the corollary that the associated unlabeled diffusion does not change its density over the time evolution:

Corollary 4.5. Given assumptions **(A1)**–**(A3)**, for each $\theta \in (0, \infty)$ the associated unlabeled diffusion satisfies

$$\mathsf{P_s}(\tau_{\mathsf{A}_\theta} = \infty) = 1 \text{ for q.e. } \mathsf{s} \in \mathsf{A}_\theta.$$

4.3 Proof of Theorem 4.2

In this section, we give a proof of Theorem 4.2. We begin by introducing cut off functions. Let $\mathfrak{u} : \mathrm{S}^{\mathbb{N}} \to \mathsf{S}_\infty$ be an unlabeled map defined as

$$\mathfrak{u}(\mathbf{s}) = \sum_{i\in\mathbb{N}} \delta_{s_i} \text{ for } \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathrm{S}^{\mathbb{N}}.$$

A mapping $\mathfrak{l} : \mathsf{S}_\infty \to \mathrm{S}^{\mathbb{N}}$ is called a labeled map if $\mathfrak{l}$ is measurable and $\mathfrak{u} \circ \mathfrak{l}$ is the identity.

We fix a non-decreasing sequence $\mathbf{a} = \{a_r\}_{r\in\mathbb{N}} \subset \mathbb{N}$ and a label $\mathfrak{l} = (\mathfrak{l}_1, \mathfrak{l}_2, \ldots)$ satisfying $|\mathfrak{l}_j(\mathsf{s})| \le |\mathfrak{l}_{j+1}(\mathsf{s})|$ for any $j \in \mathbb{N}$. Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth function such that

$$\rho(t) = \begin{cases} 1, & t \in (-\infty, 0], \\ 0, & t \in [1, \infty), \end{cases}$$

and let c_{27} be a positive constant given by

$$c_{27} := \sup_{x\in\mathbb{R}} |\rho\prime(x)| (< \infty).$$

We set

$$J_{r,\mathsf{s},+} = \{j\,;\, j > a_r, \mathfrak{l}_j(\mathsf{s}) \in S_r\}.$$

Then, for each $m \in \mathbb{N}$, we define $\chi_+^m[\mathbf{a}] : \mathsf{S}_\infty \to [0,1]$ as

$$\chi_+^m[\mathbf{a}](\mathsf{s}) = \rho \circ h_{\mathbf{a},+}^m(\mathsf{s}),$$

where

$$h_{\mathbf{a},+}^m(\mathsf{s}) = \frac{\log(d_{\mathbf{a},+}^m(\mathsf{s}) + 1)}{\log 2}, \quad d_{\mathbf{a},+}^m(\mathsf{s}) = \sum_{r=m}^{\infty} \sum_{j \in J_{r,\mathsf{s},+}} (r - |\mathfrak{l}_j(\mathsf{s})|)^2.$$

Similarly, we set

$$J_{r,\mathsf{s},-} = \{j\, ;\, j < a_r, \mathfrak{l}_j(\mathsf{s}) \in S_r^c\},$$

and for each $m \in \mathbb{N}$, define $\chi_-^m[\mathbf{a}] : \mathsf{S}_\infty \to [0,1]$ as

$$\chi_-^m[\mathbf{a}](\mathsf{s}) = \rho \circ h_{\mathbf{a},-}^m(\mathsf{s}),$$

where

$$h_{\mathbf{a},-}^m(\mathsf{s}) = \frac{\log(d_{\mathbf{a},-}^m(\mathsf{s}) + 1)}{\log 2}, \quad d_{\mathbf{a},-}^m(\mathsf{s}) = \sum_{r=m}^{\infty} \sum_{j \in J_{r,\mathsf{s},-}} (r - |\mathfrak{l}_j(\mathsf{s})|)^2.$$

In addition, we prepare maps approximating $\chi_i^m[\mathbf{a}]$ for $i \in \{+,-\}$. Let

$$\chi_i^{m,s}[\mathbf{a}](\mathsf{s}) = \rho \circ h_{\mathbf{a},i}^{m,s}(\mathsf{s}),$$

where

$$h_{\mathbf{a},i}^{m,s}(\mathsf{s}) = \frac{\log(d_{\mathbf{a},i}^{m,s}(\mathsf{s}) + 1)}{\log 2}, \quad d_{\mathbf{a},i}^{m,s}(\mathsf{s}) = \sum_{r=m}^{s} \sum_{j \in J_{r,\mathsf{s},i}} (r - |\mathfrak{l}_j(\mathsf{s})|)^2.$$

Clearly, $\chi_+^{m,s}[\mathbf{a}]$ is thus $\sigma[\pi_s]$-measurable. By the definition of S_∞, $\chi_-^{m,s}[\mathbf{a}]$ is $\sigma[\pi_{s+1}]$-measurable. Therefore, for each $i \in \{+,-\}$,

$$\chi_i^{m,s}[\mathbf{a}] \in \mathcal{D}_\circ.$$

Furthermore, it is easily deduced that $\lim_{s\to\infty} \chi_i^{m,s}[\mathbf{a}] = \chi_i^m[\mathbf{a}]$ in $L^2(\mathsf{S}_\infty, \mu)$.

Lemma 4.6. Recall that $c_{27} = \sup_{x\in\mathbb{R}} |\rho\prime(x)| < \infty$. For each $i \in \{+,-\}$ and each $m, s \in \mathbb{N}$,

$$\mathbb{D}[\chi_i^{m,s}[\mathbf{a}]](\mathsf{s}) \le \frac{2c_{27}^2}{(\log 2)^2} \frac{d_{\mathbf{a},i}^{m,s}(\mathsf{s})}{(d_{\mathbf{a},i}^{m,s}(\mathsf{s}) + 1)^2}. \tag{4.10}$$

In particular, there exists a positive constant c_{28} independent of m, $\mathbf{a}$, i, and s such that

$$\mathbb{D}[\chi_i^{m,s}[\mathbf{a}]](\mathsf{s}) \le c_{28}. \tag{4.11}$$

Proof. Easy calculation yields (4.10). In fact, we have

$$\begin{aligned}
\mathbb{D}[\chi_i^{m,s}[\mathbf{a}]](\mathsf{s}) &= \frac{1}{2} \sum_{r=m}^{s} \sum_{j \in J_{r,\mathsf{s},i}} \left\{ \frac{\rho\prime(h_{\mathbf{a},i}^{m,s}(\mathsf{s}))}{\log 2} \frac{2(r - |\mathfrak{l}_j(\mathsf{s})|)}{d_{\mathbf{a},i}^{m,s}(\mathsf{s}) + 1} \right\}^2 \\
&\le \frac{2c_{27}^2}{(\log 2)^2} \cdot \frac{d_{\mathbf{a},i}^{m,s}(\mathsf{s})}{(d_{\mathbf{a},i}^{m,s}(\mathsf{s}) + 1)^2}.
\end{aligned}$$

Equation (4.11) then follows from (4.10) immediately. $\square$

For a given non-decreasing sequence $\mathbf{a} = \{a_r\}_{r\in\mathbb{N}}$, we set

$$\mathsf{S}_+^m[\mathbf{a}] = \{\mathsf{s} \in \mathsf{S}_\infty \,;\, \mathsf{s}(S_r) \le a_r \text{ for any } r \ge m\},$$
$$\mathsf{S}_-^m[\mathbf{a}] = \{\mathsf{s} \in \mathsf{S}_\infty \,;\, \mathsf{s}(S_r) \ge a_r \text{ for any } r \ge m\}.$$

Clearly, the $\mathsf{S}_\pm^m[\mathbf{a}]$ are non-decreasing sets with respect to m. For given $\mathbf{a}$, we define new sequences $\mathbf{a}^\pm = \{a_{r\pm1}\}_{r\in\mathbb{N}}$. We use the bilinear form $\mathcal{E}_1$ given by $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2(\mathsf{S}_\infty,\mu)}$ for $u, v \in \mathcal{D}$. Below, $||u||_{\mathcal{E}_1}$ denotes the norm with respect to $\mathcal{E}_1(u,u)$. Note that $(\mathcal{E}_1, \mathcal{D})$ is a Hilbert space.

Lemma 4.7. For each $i \in \{+,-\}$ and each $m \in \mathbb{N}$, the following hold:
(i) $\chi_i^m[\mathbf{a}] = 1$ on $\mathsf{S}_i^m[\mathbf{a}]$ and $\chi_i^m[\mathbf{a}] = 0$ on $(\mathsf{S}_i^m[\mathbf{a}^i])^c$.
(ii)

$$\lim_{s\to\infty} \chi_i^{m,s}[\mathbf{a}] = \chi_i^m[\mathbf{a}] \text{ weakly in } (\mathcal{E}_1, \mathcal{D}). \tag{4.12}$$

(iii) $\chi_i^m[\mathbf{a}] \in \mathcal{D}$.

Proof. From the definition of $\chi_+^m[\mathbf{a}]$ and $\chi_-^m[\mathbf{a}]$, we obtain (i) immediately.

Equation (4.11) implies that $\sup_{s\in\mathbb{N}} ||\chi_i^{m,s}[\mathbf{a}]||_{\mathcal{E}_1} \le \sqrt{1 + c_{28}}$. Using this together with the $L^2(\mu)$-convergence of $\chi_i^{m,s}[\mathbf{a}]$, we obtain (ii)

Clearly, (iii) follows from (ii). □

Lemma 4.8. For each $i \in \{+,-\}$, $\{\chi_i^m[\mathbf{a}]\}_{m\in\mathbb{N}}$ is a Cauchy sequences in $\mathcal{E}_1$.

Proof. We prove only the case in which $i = +$; the $(-)$–case can be demonstrated similarly.

Let δ be a constant satisfying $0 < \delta < 1$. We define subsets $\mathsf{S}_1^{M,\delta}$ and $\mathsf{S}_2^{M,\delta}$ for each $M \in \mathbb{N}$ as

$$\mathsf{S}_1^{M,\delta} = \{\mathsf{s} \in \mathsf{S}_\infty \,;\, d_{\mathbf{a},+}^M(\mathsf{s}) < \delta\},$$
$$\mathsf{S}_2^{M,\delta} = \{\mathsf{s} \in \mathsf{S}_\infty \,;\, \delta \le d_{\mathbf{a},+}^M(\mathsf{s}) < \infty\}.$$

We can and do take M sufficiently large that

$$\mu(\mathsf{S}_2^{M,\delta}) \le \delta. \tag{4.13}$$

From (4.12), we have

$$\begin{aligned} &||\chi_+^l[\mathbf{a}] - \chi_+^m[\mathbf{a}]||_{\mathcal{E}_1}^2 \qquad\qquad (4.14)\\ &\le \liminf_{s\to\infty} ||\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]||_{\mathcal{E}_1}^2 \\ &= \liminf_{s\to\infty} \Big\{ \int_{\mathsf{S}_\infty} |\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]|^2 \, d\mu(\mathsf{s}) + \int_{\mathsf{S}_\infty} \mathbb{D}[\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]] \, d\mu(\mathsf{s}) \Big\}. \end{aligned}$$

We set $\mathsf{S}^{M,\delta} = \mathsf{S}_1^{M,\delta} + \mathsf{S}_2^{M,\delta}$ and

$$\mathsf{S}^{l,m,s} = \{\mathsf{s} \,;\, d_{\mathbf{a},+}^{l,s}(\mathsf{s}) < 1 \text{ or } d_{\mathbf{a},+}^{m,s}(\mathsf{s}) < 1\}.$$

Clearly,

$$\lim_{s\to\infty}\mu((\mathsf{S}^{M,\delta})^c\cap\mathsf{S}^{l,m,s})=0. \tag{4.15}$$

By the definition of $\chi_+^{m,s}[\mathbf{a}]$,

$$\chi_+^{l,s}[\mathbf{a}]=\chi_+^{m,s}[\mathbf{a}] \text{ on } (\mathsf{S}^{l,m,s})^c \text{ for } l,m\in\mathbb{N}.$$

From this and (4.11), we have

$$\begin{aligned}
&\int_{(\mathsf{S}^{M,\delta})^c}|\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]|^2\,d\mu(\mathsf{s})+\int_{(\mathsf{S}^{M,\delta})^c}\mathbb{D}[\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]]\,d\mu(\mathsf{s})\\
&=\int_{(\mathsf{S}^{M,\delta})^c\cap\mathsf{S}^{l,m,s}}|\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]|^2\,d\mu(\mathsf{s})+\int_{(\mathsf{S}^{M,\delta})^c\cap\mathsf{S}^{l,m,s}}\mathbb{D}[\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]]\,d\mu(\mathsf{s})\\
&\le(1+4c_{28})\mu((\mathsf{S}^{M,\delta})^c\cap\mathsf{S}^{l,m,s}).
\end{aligned}$$

Combining this and (4.15), we conclude

$$\lim_{s\to\infty}\{\int_{(\mathsf{S}^{M,\delta})^c}|\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]|^2\,d\mu(\mathsf{s})+\int_{(\mathsf{S}^{M,\delta})^c}\mathbb{D}[\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]]\,d\mu(\mathsf{s})\}=0. \tag{4.16}$$

By virtue of Lipschitz continuity, there exists a positive constant c_{29} such that

$$|\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]|\le c_{29}|d_{\mathbf{a},+}^{l,s}(\mathsf{s})-d_{\mathbf{a},+}^{m,s}(\mathsf{s})|. \tag{4.17}$$

Note that $d_{\mathbf{a},+}^{m}(\mathsf{s})\le d_{\mathbf{a},+}^{M}(\mathsf{s})$ for $m\ge M$ and $d_{\mathbf{a},+}^{m,s}(\mathsf{s})\le d_{\mathbf{a},+}^{m}(\mathsf{s})$. Then, for $s\ge m\ge M$,

$$d_{\mathbf{a},+}^{m,s}(\mathsf{s})<\delta \text{ on } \mathsf{S}_1^{M,\delta}. \tag{4.18}$$

Therefore, for each $l,m\ge M$, we have from (4.10), (4.17), and (4.18),

$$\begin{aligned}
&\int_{\mathsf{S}_1^{M,\delta}}|\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]|^2\,d\mu(\mathsf{s})+\int_{\mathsf{S}_1^{M,\delta}}\mathbb{D}[\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]]\,d\mu(\mathsf{s})\\
&<c_{29}^2\delta^2+\frac{8c_{27}^2}{(\log 2)^2}\delta.
\end{aligned}\tag{4.19}$$

From (4.11) and (4.13), we deduce that

$$\begin{aligned}
&\int_{\mathsf{S}_2^{M,\delta}}|\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]|^2\,d\mu(\mathsf{s})+\int_{\mathsf{S}_2^{M,\delta}}\mathbb{D}[\chi_+^{l,s}[\mathbf{a}]-\chi_+^{m,s}[\mathbf{a}]]\,d\mu(\mathsf{s})\\
&\le\mu(\mathsf{S}_2^{M,\delta})(1+4c_{28})\\
&\le\delta(1+4c_{28}).
\end{aligned}\tag{4.20}$$

Combining (4.14), (4.16), (4.19), and (4.20), we conclude that for any δ satisfying $0<\delta<1$, there exists $M\in\mathbb{N}$ such that for any $l,m\ge M$,

$$||\chi_+^{l}[\mathbf{a}]-\chi_+^{m}[\mathbf{a}]||_{\mathcal{E}_1}<\{c_{29}^2\delta^2+\frac{8c_{27}^2}{(\log 2)^2}\delta+\delta(1+4c_{28})\}^{1/2}.$$

Hence, $\{\chi_+^{m}[\mathbf{a}]\}_{m\in\mathbb{N}}$ is a Cauchy sequences in $\mathcal{E}_1$. □

For a subset $\mathsf{B} \subset \mathsf{S}_\infty$, an element $e_{\mathsf{B}} \in \mathcal{D}$ is called the 1-equilibrium potential of B if $\tilde{e_{\mathsf{B}}} = 1$ q.e. on B and $\mathcal{E}_1(e_{\mathsf{B}}, v) \geq 0$ for any $v \in \mathcal{D}$ satisfying $\tilde{v} \geq 0$ q.e. on B. Here, $\tilde{u}$ is a quasi-continuous μ-version of $u \in \mathcal{D}$.

Lemma 4.9. Take $i \in \{+, -\}$ and set

$$\mathsf{S}_i^{\mathbf{a}} = \bigcup_{m \in \mathbb{N}} \mathsf{S}_i^m[\mathbf{a}], \quad \mathsf{S}_i^{\mathbf{a}^i} = \bigcup_{m \in \mathbb{N}} \mathsf{S}_i^m[\mathbf{a}^i].$$

Assume that

$$\mu((\mathsf{S}_i^{\mathbf{a}})^c \cap \mathsf{S}_i^{\mathbf{a}^i}) = 0. \tag{4.21}$$

Then

$$\lim_{m \to \infty} \chi_i^m[\mathbf{a}] = e_{\mathsf{S}_i^{\mathbf{a}}} \text{ in } \mathcal{E}_1, \tag{4.22}$$

$$1 - \lim_{m \to \infty} \chi_i^m[\mathbf{a}] = e_{(\mathsf{S}_i^{\mathbf{a}})^c} \text{ in } \mathcal{E}_1. \tag{4.23}$$

Furthermore, we have

$$\tilde{e}_{\mathsf{S}_i^{\mathbf{a}}} = 0 \text{ for q.e. } \mathsf{s} \in (\mathsf{S}_i^{\mathbf{a}})^c, \tag{4.24}$$

$$\tilde{e}_{(\mathsf{S}_i^{\mathbf{a}})^c} = 0 \text{ for q.e. } \mathsf{s} \in \mathsf{S}_i^{\mathbf{a}}. \tag{4.25}$$

Proof. We give a proof only for $i = +$; The $(-)$–case can be proved similarly.

First, there exists a $u \in \mathcal{D}$ such that $\lim_{m \to \infty} \chi_+^m[\mathbf{a}] = u$ in $\mathcal{E}_1$ from Lemma 4.8. To show (4.22), it is enough to prove that $\tilde{u} = 1$ q.e. on $\mathsf{S}_+^{\mathbf{a}}$ and $\mathcal{E}_1(u, v) \geq 0$ for any $v \in \mathcal{D}$ with $\tilde{v} \geq 0$ q.e. on $\mathsf{S}_+^{\mathbf{a}}$.

From Lemma 4.7 (i), we have $u = 1$ μ-a.e. on $\mathsf{S}_+^{\mathbf{a}}$. Here, we use the monotonicity of $\mathsf{S}_+^m[\mathbf{a}]$. Therefore, we can take $\tilde{u}$ as a version of u such that $\tilde{u} = 1$ q.e. on $\mathsf{S}_+^{\mathbf{a}}$.

Next, we take $v \in \mathcal{D}$ such that $\tilde{v} \geq 0$ q.e. on $\mathsf{S}_+^{\mathbf{a}}$. We use the result that $u = 1$ μ-a.e. on $\mathsf{S}_+^{\mathbf{a}}$ to obtain

$$\int_{\mathsf{S}_+^{\mathbf{a}} \cap \mathsf{S}_+^{\mathbf{a}^+}} u(\mathsf{s}) v(\mathsf{s}) \, d\mu(\mathsf{s}) \geq 0. \tag{4.26}$$

We have $u = 0$ μ-a.e. on $(\mathsf{S}_+^{\mathbf{a}^+})^c$ from Lemma 4.7 (i) and the monotonicity. From this and (4.26), we deduce

$$\begin{aligned} \int_{\mathsf{S}_\infty} u(\mathsf{s}) v(\mathsf{s}) \, d\mu(\mathsf{s}) &= \int_{\mathsf{S}_+^{\mathbf{a}^+}} u(\mathsf{s}) v(\mathsf{s}) \, d\mu(\mathsf{s}) \\ &= \Big\{ \int_{\mathsf{S}_+^{\mathbf{a}} \cap \mathsf{S}_+^{\mathbf{a}^+}} + \int_{(\mathsf{S}_+^{\mathbf{a}})^c \cap \mathsf{S}_+^{\mathbf{a}^+}} \Big\} u(\mathsf{s}) v(\mathsf{s}) \, d\mu(\mathsf{s}) \\ &\geq 0. \end{aligned} \tag{4.27}$$

Here we have used the fact that the second term in the second line in (4.27) vanishes because of (4.21).

Next we consider $\mathcal{E}(u,v)$. Let $\{v_m\}_{m=1}^{\infty} \subset \mathcal{D}_\circ$ such that $\lim_{m\to\infty} v_m = v$ in $\mathcal{E}_1$. Recall that $\lim_{m\to\infty} \chi_+^m[\mathbf{a}] = u$ in $\mathcal{E}_1$. Then

$$\begin{aligned} \mathcal{E}(u,v)^2 &\le \mathcal{E}(u,u)\mathcal{E}(v,v) \\ &= \lim_{m\to\infty} \mathcal{E}(\chi_+^m[\mathbf{a}], \chi_+^m[\mathbf{a}])\mathcal{E}(v,v) \\ &\le \lim_{m\to\infty} \liminf_{s\to\infty} \mathcal{E}(\chi_+^{m,s}[\mathbf{a}], \chi_+^{m,s}[\mathbf{a}])\mathcal{E}(v,v). \end{aligned} \tag{4.28}$$

We set

$$\mathsf{S}_+^{m,s}[\mathbf{a}] = \{\mathsf{s} \in \mathsf{S}_\infty \,;\, \mathsf{s}(S_r) \le a_r \text{ for any } r \text{ satisfying } s \ge r \ge m\}.$$

Note that $\mathsf{S}_+^{m,s}[\mathbf{a}]$ is a non-increasing set with respect to s. Because $\chi_+^{m,s}[\mathbf{a}]$ is constant on $\mathsf{S}_+^{m,s}[\mathbf{a}] \cup (\mathsf{S}_+^{m,s}[\mathbf{a}^+])^c$ by definition,

$$\mathbb{D}[\chi_+^{m,s}[\mathbf{a}]](\mathsf{s}) = 0 \text{ on } \mathsf{S}_+^{m,s}[\mathbf{a}] \cup (\mathsf{S}_+^{m,s}[\mathbf{a}^+])^c.$$

From this, we have

$$\begin{aligned} \mathcal{E}(\chi_+^{m,s}[\mathbf{a}], \chi_+^{m,s}[\mathbf{a}]) &= \int_{\mathsf{S}} \mathbb{D}[\chi_+^{m,s}[\mathbf{a}]](\mathsf{s})\, d\mu(\mathsf{s}) \\ &= \int_{(\mathsf{S}_+^{m,s}[\mathbf{a}])^c \cap \mathsf{S}_+^{m,s}[\mathbf{a}^+]} \mathbb{D}[\chi_+^{m,s}[\mathbf{a}]](\mathsf{s})\, d\mu(\mathsf{s}). \end{aligned} \tag{4.29}$$

From (4.11) and $(\mathsf{S}_+^{m,s}[\mathbf{a}])^c \subset (\mathsf{S}_+^m[\mathbf{a}])^c$,

$$\begin{aligned} \int_{(\mathsf{S}_+^{m,s}[\mathbf{a}])^c \cap \mathsf{S}_+^{m,s}[\mathbf{a}^+]} \mathbb{D}[\chi_+^{m,s}[\mathbf{a}]](\mathsf{s})\, d\mu(\mathsf{s}) &\le c_{28}\mu((\mathsf{S}_+^{m,s}[\mathbf{a}])^c \cap \mathsf{S}_+^{m,s}[\mathbf{a}^+]) \\ &\le c_{28}\mu((\mathsf{S}_+^{m}[\mathbf{a}])^c \cap \mathsf{S}_+^{m,s}[\mathbf{a}^+]). \end{aligned}$$

Combining this and (4.29), we have

$$\begin{aligned} \liminf_{s\to\infty} \mathcal{E}(\chi_+^{m,s}[\mathbf{a}], \chi_+^{m,s}[\mathbf{a}]) &\le c_{28} \liminf_{s\to\infty} \mu((\mathsf{S}_+^m[\mathbf{a}])^c \cap \mathsf{S}_+^{m,s}[\mathbf{a}^+]) \\ &= c_{28}\mu((\mathsf{S}_+^m[\mathbf{a}])^c \cap \mathsf{S}_+^m[\mathbf{a}^+]). \end{aligned} \tag{4.30}$$

We use the monotonicity of $\mathsf{S}_+^{m,s}[\mathbf{a}^+]$ in the last line. From (4.28), (4.30), and the monotonicity of $\mathsf{S}_+^m[\mathbf{a}]$ with respect to m, we have

$$\begin{aligned} \mathcal{E}(u,v)^2 &\le \lim_{m\to\infty} c_{28}\mu((\mathsf{S}_+^m[\mathbf{a}])^c \cap \mathsf{S}_+^m[\mathbf{a}^+])\mathcal{E}(v,v) \\ &\le \lim_{m\to\infty} c_{28}\mu((\mathsf{S}_+^m[\mathbf{a}])^c \cap \mathsf{S}_+^{\mathbf{a}^+})\mathcal{E}(v,v) \\ &= c_{28}\mu((\mathsf{S}_+^{\mathbf{a}})^c \cap \mathsf{S}_+^{\mathbf{a}^+})\mathcal{E}(v,v). \end{aligned}$$

Consequently, we find that $\mathcal{E}(u,v) = 0$ by virtue of (4.21). Combining this and (4.27), we have $\mathcal{E}_1(u,v) \ge 0$. Then we conclude $u = e_{\mathsf{S}_+^{\mathbf{a}}}$. Equation (4.24) is clear because we have $u = 0$ μ-a.e. on $(\mathsf{S}_i^{\mathbf{a}})^c$ from the discussion above.

Finally, (4.23) and (4.25) are deduced easily from (4.22) and (4.24). $\square$

Theorem 4.2 follows from Lemma 4.9 with an appropriate choice of $\mathbf{a}$. For small $\varepsilon > 0$, we set

$$\mathbf{a}_\varepsilon = \{(f(r)(1-\varepsilon))\}_{r\in\mathbb{N}}, \quad \mathbf{b}_\varepsilon = \{(f(r)(1+\varepsilon))\}_{r\in\mathbb{N}}, \tag{4.31}$$

as a non-decreasing sequence. We further set

$$\begin{aligned}\mathsf{A}_\varepsilon &= \{\mathsf{s}\,;\, \Phi_+(\mathsf{s}) < 1-\varepsilon\},\ \mathsf{B}_\varepsilon = \{\mathsf{s}\,;\, \Phi_+(\mathsf{s}) > 1+\varepsilon\},\\ \mathsf{C}_\varepsilon &= \{\mathsf{s}\,;\, \Phi_-(\mathsf{s}) < 1-\varepsilon\},\ \mathsf{D}_\varepsilon = \{\mathsf{s}\,;\, \Phi_-(\mathsf{s}) > 1+\varepsilon\}.\end{aligned}$$

Lemma 4.10. With $\mathsf{A}_\pm$ as in (4.5), the following hold:

$$\mathsf{A}_\varepsilon \subset \bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{a}_\varepsilon],\ \mathsf{B}_\varepsilon \subset \Big(\bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{b}_\varepsilon]\Big)^c, \tag{4.32}$$

$$\mathsf{C}_\varepsilon \subset \Big(\bigcup_{m\in\mathbb{N}} \mathsf{S}_-^m[\mathbf{a}_\varepsilon]\Big)^c,\ \mathsf{D}_\varepsilon \subset \bigcup_{m\in\mathbb{N}} \mathsf{S}_-^m[\mathbf{b}_\varepsilon], \tag{4.33}$$

and

$$\mathsf{A}_+ \subset \Big(\bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{a}_\varepsilon]\Big)^c \cap \bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{b}_\varepsilon], \tag{4.34}$$

$$\mathsf{A}_- \subset \bigcup_{m\in\mathbb{N}} \mathsf{S}_-^m[\mathbf{a}_\varepsilon] \cap \Big(\bigcup_{m\in\mathbb{N}} \mathsf{S}_-^m[\mathbf{b}_\varepsilon]\Big)^c. \tag{4.35}$$

Proof. The first inclusion relation in (4.32) is obvious by

$$\bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{a}_\varepsilon] = \bigcup_{m\in\mathbb{N}} \{\mathsf{s}\,;\, \mathsf{s}(S_r) \le f(r)(1-\varepsilon), {}^\forall r \ge m\}.$$

The second inclusion relation in (4.32) follows from

$$\begin{aligned}\big(\bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{b}_\varepsilon]\big)^c &= \big(\bigcup_{m\in\mathbb{N}} \{\mathsf{s}\,;\, \mathsf{s}(S_r) \le f(r)(1+\varepsilon), {}^\forall r \ge m\}\big)^c\\ &= \bigcap_{m\in\mathbb{N}} \{\mathsf{s}\,;\, \mathsf{s}(S_r) > f(r)(1+\varepsilon), {}^\exists r \ge m\}.\end{aligned}$$

Equations (4.33), (4.34), and (4.35) can be checked in a similar way. $\square$

Proof of Theorem 4.2. We use Lemma 4.9 for the non-decreasing sequence (4.31). For (4.6), we can take arbitrary small $\varepsilon > 0$ in (4.31), which yields (4.21). In fact, let $\mathsf{S}_+^{\mathbf{a}_\varepsilon}$ and $\mathsf{S}_+^{\mathbf{a}_\varepsilon^+}$ be as in Lemma 4.9, then we have

$$(\mathsf{S}_+^{\mathbf{a}_\varepsilon})^c = \bigcap_{m\in\mathbb{N}} \{\mathsf{s}(S_r) > f(r)(1-\varepsilon), {}^\exists r \ge m\} \subset \{\mathsf{s}\,;\, \Phi_+(\mathsf{s}) \ge 1-\varepsilon\}, \tag{4.36}$$

and

$$\mathsf{S}_+^{\mathbf{a}_\varepsilon^+} = \bigcup_{m\in\mathbb{N}} \{\mathsf{s}(S_r) \le f(r+1)(1-\varepsilon), {}^\forall r \ge m\} \subset \{\mathsf{s}\,;\, \Phi_+(\mathsf{s}) \le 1-\varepsilon\}. \tag{4.37}$$

Combining (4.36) and (4.37), we obtain

$$(\mathsf{S}_+^{\mathbf{a}_\varepsilon})^c \cap \mathsf{S}_+^{\mathbf{a}_\varepsilon^+} \subset \{\mathsf{s}\,;\,\Phi_+(\mathsf{s}) = 1-\varepsilon\}.$$

From this, (4.21) is satisfied for $\mathbf{a} = \mathbf{a}_\varepsilon$ with an arbitrary small $\varepsilon > 0$.

Therefore, we combine Lemma 4.9 with Lemma 4.10 to obtain

$$\mathsf{P}_\mathsf{s}(\tau_{\mathsf{A}_\varepsilon^c} = \infty) = \mathsf{P}_\mathsf{s}(\tau_{\mathsf{B}_\varepsilon^c} = \infty) = 1 \text{ for q.e. } \mathsf{s} \in \mathsf{A}_+, \tag{4.38}$$

$$\mathsf{P}_\mathsf{s}(\tau_{\mathsf{C}_\varepsilon^c} = \infty) = \mathsf{P}_\mathsf{s}(\tau_{\mathsf{D}_\varepsilon^c} = \infty) = 1 \text{ for q.e. } \mathsf{s} \in \mathsf{A}_-. \tag{4.39}$$

Here we have used the fact that for a nearly Borel set A, $p^1_{\mathsf{S}_\infty \setminus \mathsf{A}}(\cdot) = \mathsf{E}_\cdot[e^{-\tau_\mathsf{A}}]$ is a quasi-continuous μ-version of $e_{\mathsf{S}_\infty \setminus \mathsf{A}}$. Because (4.38) and (4.39) hold for arbitrarily small ε, we arrive at (4.7). $\square$

5 Uniqueness of Dirichlet forms related to infinite systems of interacting Brownian motions

5.1 Introduction

An infinite system of interacting Brownian motions in $\mathbb{R}^d$ can be represented by $(\mathbb{R}^d)^{\mathbb{N}}$-valued stochastic process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ [37, 38, 44, 48]. This process is realized using several probabilistic constructs such as stochastic differential equation, Dirichlet form theory, and martingale problems. Among them, the second author constructed in a general setting processes using the technique of Dirichlet forms [44, 48].

Specifically, the Dirichlet form introduced, $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ is obtained by the smallest extension of the bilinear form $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\circ})$ on $L^2(\mathsf{S}, \mu)$ with domain $\mathcal{D}^{\mu}_{\circ}$ defined by

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g](\mathsf{s})\,\mu(d\mathsf{s}),$$

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_{i=1}^{\infty} \nabla_{s_i}\check{f}\cdot\nabla_{s_i}\check{g},$$

$$\mathcal{D}^{\mu}_{\circ} = \{f \in \mathcal{D}_{\circ}\cap L^2(\mathsf{S},\mu)\,;\, \mathcal{E}^{\mu}(f,f) < \infty\},$$

where $\mathcal{D}_{\circ}$ is the set of all local smooth functions on the (unlabeled) configuration space S introduced in (5.6), $\check{f}$ is a symmetric function such that $\check{f}(s_1, s_2, \dots) = f(\mathsf{s})$, $\cdot$ is the inner product in $\mathbb{R}^d$, and $\mathsf{s} = \sum_i \delta_{s_i}$ denotes a configuration. If we take μ to be the Poisson random point field, the intensity of which is the Lebesgue measure, then the diffusion given by the Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ is the unlabeled Brownian motion B such that $\mathsf{B}_t = \sum_{i=1}^{\infty}\delta_{B^i_t}$, where $\{B^i\}_{i=1}^{\infty}$ is a system of independent copies of the standard Brownian motion.

This Dirichlet form is a decreasing limit of Dirichlet forms associated with finite systems of interacting Brownian motions in bounded domains $S_R = \{x \in \mathbb{R}^d; |x| \le R\}$ with a boundary condition. Because of the boundary condition, Brownian particles that touch the boundary disappear. Also, particles enter the domain from the boundary according to the reversible measure μ.

In contrast, Lang constructed the infinite system of Brownian motions as a limit of stochastic dynamics in bounded domains S_R by considering finite systems with another boundary condition [37, 38]. In his finite systems, a particle hitting the boundary is reflected and hence the number of particles in the domain is invariant. His process is associated with the Dirichlet form $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ that is the increasing limit of the Dirichlet forms associated with finite systems with the reflecting boundary condition.

In this paper, we discuss the relation between these Dirichlet forms, $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ and $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$. The main purpose of this paper is to give a sufficient condition for

$$(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}}) = (\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}}). \tag{5.1}$$

By construction the inequality

$$(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}}) \le (\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}}) \tag{5.2}$$

always holds whereas (5.1) does not necessarily hold in general. Although the problem is quite natural and general, little is known about the equality (5.1). To the best of our knowledge, the unique example for which the equality (5.1) holds is the system of hard-core Brownian balls proved by the third author [71].

The study of infinite systems of interacting Brownian motions was initiated by Lang [37, 38] and continued by Fritz [15], the third author [70], and others. In their respective work, the free potential Φ is assumed to be zero and the interaction potentials Ψ are of class $C_0^3(\mathbb{R}^d)$ or exponentially decay at infinity and satisfying the super-stability in the sense of Ruelle. The infinite-dimensional stochastic differential equation (ISDE) is given by

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j=1,\, j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j)dt. \tag{5.3}$$

Here $\beta > 0$ is an inverse temperature. Lang constructed a solution for the μ-a.s. x unlabeled initial point, where μ is a grand canonical Gibbs measure with interaction potential Ψ.

Indeed, Lang and others solved the ISDE as a limit of solutions of finite-volume stochastic differential equations (SDE), describing particles in S_R with reflecting boundary condition on ∂S_R. That is, the SDE is given by

$$\begin{aligned} dX_t^i =& dB_t^i - \frac{\beta}{2} \sum_{j\neq i}^{\mathsf{x}(S_R)} \nabla\Psi(X_t^i - X_t^j)dt - \frac{\beta}{2} \sum_{j>\mathsf{x}(S_R)}^{\infty} \nabla\Psi(X_t^i - x_j)dt \\ &+ \frac{1}{2}\mathbf{n}^R(X_t^i)dL_t^{R,i} \qquad \text{for } 1 \le i \le \mathsf{x}(S_R), \\ X_t^i =& x_i \qquad\qquad\qquad\qquad\quad \text{for } i > \mathsf{x}(S_R) \end{aligned} \tag{5.4}$$

with the initial condition $\mathbf{X}_0 = (x_i)_{i=1}^{\infty}$ such that $|x_i| < |x_{i+1}|$ for all $i \in \mathbb{N}$, and $\mathsf{x}(S_R)$ coincides with the number of particles in S_R. The process $L^{R,i} = \{L_t^{R,i}\}$ denotes the local time-type drift arising from the reflecting boundary condition on ∂S_R (see (5.29) for $L^{R,i}$) and $\mathbf{n}^R(x)$ is the inward normal vector at $x \in \partial S_R$.

In contrast, the labeled diffusion in S_R introduced in [44] is given by the SDE

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j\neq i,\ X_t^j \in S_R} \nabla\Psi(X_t^i - X_t^j)dt \tag{5.5}$$

with the foregoing boundary condition. These SDEs have thus different boundary conditions. The solutions of (5.4) are non-ergodic, whereas the solutions of (5.5) are ergodic. Indeed, the system in (5.4) keeps the initial number of particles in S_R. In the second dynamics, the number of particles in S_R varies. The state space of solutions in (5.5) therefore consists of a unique ergodic component (regarded as $\{\bigcup_{m=0}^{\infty}(S_R^{\mathrm{int}})^m\}$-valued process, where $(S_R^{\mathrm{int}})^0 = \{\emptyset\}$ and S_R^{int} is the interior of S_R).

Let $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})$ be the Dirichlet form on $L^2(\mathsf{S}, \mu)$ associated with (5.4), that is, the Dirichlet form $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})$ describes the motion of unlabeled dynamics related to (5.4). Let $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_R^{\mathsf{upr}})$ be the Dirichlet form on $L^2(\mathsf{S}, \mu)$ associated with (5.5). Here we use the notation $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_R^{\mathsf{upr}})$ rather than $(\mathcal{E}_R^{\mathsf{upr}}, \mathcal{D}_R^{\mathsf{upr}})$ because $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_R^{\mathsf{upr}})$ is the closure of $(\mathcal{E}, \mathcal{D}_{\circ}^{\mu} \cap \mathcal{B}_R(\mathsf{S}))$ (see Lemma 5.1 see for notational details), whereas $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})$ is the closure with respect to the energy form $\mathcal{E}_R$ different from $\mathcal{E}$. As we shall see later, these two Dirichlet forms satisfy the relation

$$(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}}) \le (\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_R^{\mathsf{upr}}).$$

Furthermore, as $R \to \infty$, $\{(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})\}$ is an increasing scheme of Dirichlet forms, whereas $\{(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_R^{\mathsf{upr}})\}$ is decreasing. This fact implies the obvious relation (5.2).

The difference in these schemes lies in the boundary condition. Therefore, our task is to control the effect of the boundary condition to prove it becomes negligible as $R \to \infty$.

The main examples of our models have a logarithmic interaction potential, which is a very long rang potential that has quite strong long-range effect. We emphasize that the ISDEs arising from random matrix theory usually have logarithmic interaction potentials, and hence this class of interacting Brownian motions is significant.

The typical ISDE for logarithmic potentials is the Ginibre interacting Brownian motion in $\mathbb{R}^2$ with the ISDE

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j|<r,\, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}),$$

and

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}).$$

Surprisingly, these two ISDEs have the same solution that defines the Ginibre interacting Brownian motion. This is a consequence of the long-rang effect of the logarithmic interaction potential whereby the motion of the particles is suppressed very strongly.

Our result proves nevertheless the uniqueness of Dirichlet forms for which (5.1) holds, a phenomenon similar to short-range interaction potentials.

In the first main theorem (Theorem 5.14), we shall prove that any limit point of the solutions of (5.4) is a solution of the ISDE (5.3) satisfying well-behaved properties (see Theorem 5.14). The limit points of the solutions of (5.5) were proved to satisfy the ISDE (5.3) with the same well-behaved properties [44, 47]. Hence, assuming the uniqueness of solutions of (5.3) under the foregoing well-behaved properties, these two limits of the solutions are the same. This establishes the coincidence of the two Dirichlet forms $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ and $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$.

The motivation of our work lies in the recent rapid and outstanding progress of random matrix theory in proving that the random point fields arising from Gaussian random matrices (invariant random matrices) such as sine_β, Airy_β, and Ginibre random point fields, are universal. Indeed, these random point fields are obtained as scaling limits of eigenvalue distributions of a quite general class of random matrices and also log gases with general free potentials. Once this static universality is established, it is natural to pursue its dynamical counter part. In a forthcoming paper, the first and second authors will prove that the natural reversible stochastic dynamics associated with these random point fields are also universal objects. Examples of universal stochastic dynamics are the sine, Airy, and Ginibre interacting Brownian motions (see Section 5.8). They are limits of the stochastic dynamics related to N-particle systems with reversible random point fields that converge to those universal random point fields mentioned above. This result is established by assuming the limits of the lower and upper Dirichlet forms in (5.1) are equal in addition to a certain strong convergence of the random point fields. Hence our main theorem (Theorem 3.2) plays a crucial role in the dynamical universality of random matrices in the sense given above.

The organization of the paper is as follows: In Section 5.2, we prepare the two schemes of the Dirichlet forms describing interacting Brownian motions, and quote related results. In Section 5.3, we state the main theorems (Theorem 5.14 and Theorem 5.15). In Section 5.4, we prove Theorem 5.14. In Section 5.5, we prove Theorem 5.15. In Section 5.6, we comment on a generalization to the uniformly elliptic case. In Section 5.7, we construct cut-off coefficients $\mathsf{b}_{r,s,\mathsf{p}}$ appearing in **(A6)**. In Section 5.8, we present examples. In Appendix (Section 5.9) we present a set $\mathcal{D}_\bullet$ used in Section 5.2 and prove Lemma 5.7. In Section 5.10, we give concluding remarks with some open questions.

5.2 Preliminaries

5.2.1 Two schemes of Dirichlet forms

Let S be a closed set in $\mathbb{R}^d$ with interior S^{int} which is a connected open set satisfying $\overline{S^{\mathrm{int}}} = S$ and the boundary ∂S having Lebesgue measure zero.

A configuration $\mathsf{s} = \sum_i \delta_{s_i}$ on S is a Radon measure on S consisting of delta masses δ_{s_i}. Let S be the configuration space over S. Then, by definition, S is the set given by

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\, ;\ \mathsf{s}(K) < \infty \text{ for all compact set } K\}. \tag{5.6}$$

By convention, we regard the zero measure as an element of S. We endow S with the vague topology, which makes S to be a Polish space.

A probability measure μ on $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$ is called a random point field on S. We assume μ is supported on the set consisting of infinitely-many particles:

$$\mu(\{\mathsf{s} \in \mathsf{S}; \mathsf{s}(S) = \infty\}) = 1.$$

Let $S_r = \{s \in S\, ;\, |s| \le r\}$ and $S_r^m = S_r \times \cdots \times S_r$ be the m-product of S_r. Let $\mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}\, ;\, \mathsf{s}(S_r) = m\}$ for $r, m \in \mathbb{N}$. We set maps $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$ such that

$$\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r) \quad \text{and} \quad \pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c).$$

For $\mathsf{s} \in \mathsf{S}_r^m$, $\mathbf{x}_r^m(\mathsf{s}) = (x_r^i(\mathsf{s}))_{i=1}^m \in S_r^m$ is called a S_r^m-coordinate of s, if $\pi_r(\mathsf{s}) = \sum_{i=1}^m \delta_{x_r^i(\mathsf{s})}$.

For a function $f : \mathsf{S} \to \mathbb{R}$ we set $f_r^m(\mathsf{s}, \mathbf{x}) = f_{r,\mathsf{s}}^m(\mathbf{x})$ such that $f_r^m : \mathsf{S} \times S_r^m \to \mathbb{R}$ and that $f_{r,\mathsf{s}}^m$, with $r, m \in \mathbb{N}$, satisfies
(1) $f_{r,\mathsf{s}}^m$ is a permutation invariant function on S_r^m for each $\mathsf{s} \in \mathsf{S}$.
(2) $f_{r,\mathsf{s}(1)}^m = f_{r,\mathsf{s}(2)}^m$ if $\pi_r^c(\mathsf{s}(1)) = \pi_r^c(\mathsf{s}(2))$ for $\mathsf{s}(1), \mathsf{s}(2) \in \mathsf{S}_r^m$.
(3) $f_{r,\mathsf{s}}^m(\mathbf{x}_r^m(\mathsf{s})) = f(\mathsf{s})$ for $\mathsf{s} \in \mathsf{S}_r^m$, where $\mathbf{x}_r^m(\mathsf{s})$ is a S_r^m-coordinate of s.
(4) $f_{r,\mathsf{s}}^m(s) = 0$ for $\mathsf{s} \notin \mathsf{S}_r^m$.

The function $f_{r,\mathsf{s}}^m$ is called the S_r^m-representation of f. Note that $f_{r,\mathsf{s}}^m$ is unique and $f(\mathsf{s}) = \sum_{m=0}^\infty f_{r,\mathsf{s}}^m(\mathbf{x}_r^m(\mathsf{s}))$ for each $r \in \mathbb{N}$. When f is $\sigma[\pi_r]$-measurable, the S_r^m-representations are independent of s. In this case we often write f_r^m instead of $f_{r,\mathsf{s}}^m$. We set

$$\mathcal{B}_r(\mathsf{S}) = \{f : \mathsf{S} \to \mathbb{R}\,;\ f \text{ is } \sigma[\pi_r]\text{-measurable}\}, \quad \mathcal{B}_\infty(\mathsf{S}) = \bigcup_{r=1}^{\infty} \mathcal{B}_r(\mathsf{S}),$$
$$\mathcal{D}_\circ = \{f \in \mathcal{B}_\infty(\mathsf{S})\,;\ f_{r,\mathsf{s}}^m \text{ is smooth on } S_r^m \text{ for all } m, r \in \mathbb{N},\, \mathsf{s} \in \mathsf{S}\}.$$

Note that $\mathcal{D}_\circ \cap L^2(\mathsf{S}, \mu)$ is dense in $L^2(\mathsf{S}, \mu)$ and $\mathcal{D}_\circ \subset C(\mathsf{S})$, where $C(\mathsf{S})$ is the set of all continuous functions on S. We remark that, if $f \in \mathcal{D}_\circ$ and f is $\sigma[\pi_r]$-measurable, then $f_{r,\mathsf{s}}^m(x_1, \dots, x_m)$ is constant in x_m on the boundary ∂S_r for each $(x_1, \dots, x_{m-1}) \in S_r^{m-1}$, and its derivatives vanishes on ∂S_r^m.

For $\mathsf{s} = \sum_i \delta_{s_i}$ we set $\nabla_{s_i} = (\frac{\partial}{\partial s_{i1}}, \dots, \frac{\partial}{\partial s_{id}})$. For $f, g \in \mathcal{D}_\circ$ let

$$\mathbb{D}_r^m[f,g](\mathsf{s}) = \begin{cases} \frac{1}{2} \sum_{i\,;\, s_i \in S_r} \nabla_{s_i} f_{r,\mathsf{s}}^m(\mathbf{x}_r^m(\mathsf{s})) \cdot \nabla_{s_i} g_{r,\mathsf{s}}^m(\mathbf{x}_r^m(\mathsf{s})) & \text{for } \mathsf{s} \in \mathsf{S}_r^m, \\ 0 \quad \text{for } \mathsf{s} \notin \mathsf{S}_r^m. \end{cases} \tag{5.7}$$

Moreover, we set

$$\mathbb{D}_r = \sum_{m=1}^{\infty} \mathbb{D}_r^m. \tag{5.8}$$

Note that $\mathbb{D}_r^m[f,g]$ is independent of the choice of the S_r^m-coordinate $\mathbf{x}_r^m(\mathsf{s})$ and is well-defined. We now define bilinear forms on $\mathcal{D}_\circ$:

$$\mathcal{E}_r^{\mu,m}(f,g) = \int_{\mathsf{S}} \mathbb{D}_r^m[f,g](\mathsf{s})\mu(d\mathsf{s}) \quad \text{and} \quad \mathcal{E}_r^{\mu} = \int_{\mathsf{S}} \mathbb{D}_r[f,g](\mathsf{s})\mu(d\mathsf{s}). \tag{5.9}$$

Then clearly $\mathcal{E}_r^{\mu} = \sum_{m=1}^{\infty} \mathcal{E}_r^{\mu,m}$.

Let $(\mathcal{E}^{\mu}, \mathcal{D}_\circ^{\mu})$ be a bilinear form on $L^2(\mathsf{S}, \mu)$ with domain $\mathcal{D}_\circ^{\mu}$ defined by

$$\mathcal{E}^{\mu}(f,f) = \lim_{r \to \infty} \mathcal{E}_r^{\mu}(f,f), \tag{5.10}$$
$$\mathcal{D}_\circ^{\mu} = \{f \in \mathcal{D}_\circ \cap L^2(\mathsf{S}, \mu)\,;\, \mathcal{E}^{\mu}(f,f) < \infty\}.$$

We note that $\mathcal{E}_r^{\mu}(f,f)$ is nondecreasing in r, and hence the limit in (5.10) exists. We assume

$$(\mathcal{E}_r^{\mu,m}, \mathcal{D}_\circ^{\mu}) \text{ is closable on } L^2(\mathsf{S}, \mu) \text{ for each } m, r \in \mathbb{N}. \tag{5.11}$$

We present later a sufficient condition regarding (5.11); see **(A1)** in Section 5.2.

Lemma 5.1. ([44, Lemma 2.2, Theorem 2]) Assume (5.11). Then the following hold.
(1) $(\mathcal{E}^{\mu}, \mathcal{D}_\circ^{\mu} \cap \mathcal{B}_r(\mathsf{S}))$ and $(\mathcal{E}_r^{\mu}, \mathcal{D}_\circ^{\mu})$ are closable on $L^2(\mathsf{S}, \mu)$ for each r.
(2) $(\mathcal{E}^{\mu}, \mathcal{D}_\circ^{\mu})$ is closable on $L^2(\mathsf{S}, \mu)$.

For symmetric bilinear forms $(\mathcal{E}, \mathcal{D})$ and $(\mathcal{E}', \mathcal{D}')$ we write $(\mathcal{E}, \mathcal{D}) \le (\mathcal{E}', \mathcal{D}')$ if $\mathcal{D} \supset \mathcal{D}'$ and $\mathcal{E}(f,f) \le \mathcal{E}'(f,f)$ for all $f \in \mathcal{D}'$. We say a sequence of symmetric bilinear forms $\{(\mathcal{E}_n, \mathcal{D}_n)\}_{n \in \mathbb{N}}$ is increasing if $(\mathcal{E}_n, \mathcal{D}_n) \le (\mathcal{E}_{n+1}, \mathcal{D}_{n+1})$ for all n. Replacing $\le$ by $\ge$, we call $\{(\mathcal{E}_n, \mathcal{D}_n)\}_{n \in \mathbb{N}}$ decreasing. Let $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_r^{\mathsf{upr}})$ and $(\mathcal{E}_r^{\mathsf{lwr}}, \mathcal{D}_r^{\mathsf{lwr}})$ denote the closures of $(\mathcal{E}^{\mu}, \mathcal{D}_\circ^{\mu} \cap \mathcal{B}_r(\mathsf{S}))$ and $(\mathcal{E}_r^{\mu}, \mathcal{D}_\circ^{\mu})$ on $L^2(\mathsf{S}, \mu)$, respectively. Then we quote:

Lemma 5.2 ([44, Lemma 2.1]). Assume (5.11). Then
(1) $\{(\mathcal{E}_r^{\mathsf{lwr}}, \mathcal{D}_r^{\mathsf{lwr}})\}_{r\in\mathbb{N}}$ is increasing.
(2) $\{(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_r^{\mathsf{upr}})\}_{r\in\mathbb{N}}$ is decreasing.

Let $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ be the increasing limit of $\{(\mathcal{E}_r^{\mathsf{lwr}}, \mathcal{D}_r^{\mathsf{lwr}})\}_{r\in\mathbb{N}}$, that is,

$$\mathcal{E}^{\mathsf{lwr}}(f,f) = \lim_{r\to\infty} \mathcal{E}_r^{\mathsf{lwr}}(f,f), \quad \text{and} \quad \mathcal{D}^{\mathsf{lwr}} = \{f \in \bigcap_{r\in\mathbb{N}} \mathcal{D}_r^{\mathsf{lwr}}\,;\, \lim_{r\to\infty} \mathcal{E}_r^{\mathsf{lwr}}(f,f) < \infty\}.$$

Then $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ is closed on $L^2(\mathsf{S},\mu)$ by construction. Recall that $(\mathcal{E}^{\mu}, \mathcal{D}_{\circ}^{\mu})$ is closable on $L^2(\mathsf{S},\mu)$ by Lemma 5.1. We then denote by $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ the closure of $(\mathcal{E}^{\mu}, \mathcal{D}_{\circ}^{\mu})$ on $L^2(\mathsf{S},\mu)$. Let $G_{r,\alpha}^{\mathsf{lwr}}$, $G_{\alpha}^{\mathsf{lwr}}$, $G_{r,\alpha}^{\mathsf{upr}}$, and $G_{\alpha}^{\mathsf{upr}}$ be resolvent of $(\mathcal{E}_r^{\mathsf{lwr}}, \mathcal{D}_r^{\mathsf{lwr}})$, $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$, $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_r^{\mathsf{upr}})$, and $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ on $L^2(\mathsf{S},\mu)$, respectively.

Lemma 5.3. ([44, Lemma 2.1, Theorem 2]) Assume (5.11). Then
(1) $\{G_{r,\alpha}^{\mathsf{lwr}}\}_{r\in\mathbb{N}}$ converges to $G_{\alpha}^{\mathsf{lwr}}$ strongly in $L^2(\mathsf{S},\mu)$ as $r\to\infty$ for each $\alpha>0$.
(2) $\{G_{r,\alpha}^{\mathsf{upr}}\}_{r\in\mathbb{N}}$ converges to $G_{\alpha}^{\mathsf{upr}}$ strongly in $L^2(\mathsf{S},\mu)$ as $r\to\infty$ for each $\alpha>0$.

By construction we have for each r

$$(\mathcal{E}_r^{\mathsf{lwr}}, \mathcal{D}_r^{\mathsf{lwr}}) \le (\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_r^{\mathsf{upr}}). \tag{5.12}$$

Hence taking $r\to\infty$ we see that

$$(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}}) \le (\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}}). \tag{5.13}$$

We call $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ the lower Dirichlet form and $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ the upper Dirichlet form. We also call $\{(\mathcal{E}_r^{\mathsf{lwr}}, \mathcal{D}_r^{\mathsf{lwr}})\}_{r\in\mathbb{N}}$ a lower scheme and $\{(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}_r^{\mathsf{upr}})\}_{r\in\mathbb{N}}$ an upper scheme. The relations (5.12) and (5.13) justify the names of these schemes.

5.2.2 Quasi-Gibbs measures, unlabeled diffusions, and labeled dynamics

Let Λ_r be the Poisson random point field whose intensity is the Lebesgue measure on S_r and set $\Lambda_r^m = \Lambda_r(\cdot\cap\mathsf{S}_r^m)$. Let $\Phi : S\to\mathbb{R}\cup\{\infty\}$ and $\Psi : S^2\to\mathbb{R}\cup\{\infty\}$ be measurable functions such that $\Psi(x,y)=\Psi(y,x)$. Following [48, 49] we quote:

Definition 5.4. A random point field μ is called a (Φ,Ψ)-quasi Gibbs measure if its regular conditional probabilities

$$\mu_{r,\mathsf{s}}^m = \mu(\,\pi_r(\mathsf{x})\in\cdot\,|\,\pi_r^c(\mathsf{x}) = \pi_r^c(\mathsf{s}),\, \mathsf{x}(S_r)=m)$$

satisfy, for all $r,m\in\mathbb{N}$ and μ-a.s. s,

$$c_{30}^{-1}e^{-\mathcal{H}_r^m(\mathsf{x})}\Lambda_r^m(d\mathsf{x}) \le \mu_{r,\mathsf{s}}^m(d\mathsf{x}) \le c_{30}e^{-\mathcal{H}_r^m(\mathsf{x})}\Lambda_r^m(d\mathsf{x}). \tag{5.14}$$

Here $c_{30}=c_{30}(r,m,\mathsf{s})$ is a positive constant depending on r, m, s. For two measures μ,ν on a σ-field $\mathcal{F}$, we write $\mu\le\nu$ if $\mu(A)\le\nu(A)$ for all $A\in\mathcal{F}$. Moreover, $\mathcal{H}_r^m$ is the Hamiltonian on S_r defined by

$$\mathcal{H}_r^m(\mathsf{x}) = \sum_{\substack{x_i\in S_r\\ 1\le i\le m}} \Phi(x_i) + \sum_{\substack{x_j,x_k\in S_r\\ 1\le j<k\le m}} \Psi(x_j,x_k) \quad \text{for } \mathsf{x}=\sum_i \delta_{x_i}.$$

Remark 5.5. (1) From (5.14), we see that for all $r, m \in \mathbb{N}$ and μ-a.s. s, $\mu^m_{r,\mathsf{s}}(d\mathsf{x})$ have (unlabeled) Radon–Nikodym densities $m^m_{r,\mathsf{s}}(\mathsf{x})$ with respect to Λ^m_r. Clearly, the canonical Gibbs measures μ with potentials (Φ, Ψ) are quasi-Gibbs measures, and their densities $m^m_{r,\mathsf{s}}(\mathsf{x})$ with respect to Λ^m_r are given by the Dobrushin–Lanford–Ruelle (DLR) equation. That is, for μ-a.s. $\mathsf{s} = \sum_j \delta_{s_j}$

$$m^m_{r,\mathsf{s}}(\mathsf{x}) = \frac{1}{\mathcal{Z}^m_{r,\mathsf{s}}} \exp\{-\mathcal{H}^m_r(\mathsf{x}) - \sum_{\substack{x_i \in S_r,\, s_j \in S_r^c \\ 1 \le i \le m}} \Psi(x_i, s_j)\}.$$

Here $\mathcal{Z}^m_{r,\mathsf{s}}$ is the normalizing constant. For random point fields appearing in random matrix theory, interaction potentials are logarithmic functions, where the DLR equations do not make sense as stated because the term $\sum_{x_i \in S_r,\, s_j \in S_r^c} \Psi(x_i, s_j)$ diverges. The notion of a quasi-Gibbs measure still makes sense for logarithmic potentials.
(2) We refer to [48, 49] for sufficient conditions of quasi-Gibbs measures. These conditions give us the quasi-Gibbs property of random point fields appearing in random matrix theory, such as sine_β, Airy_β $(\beta = 1, 2, 4)$, and $\mathrm{Bessel}_{2,\alpha}$ $(1 \le \alpha)$, and Ginibre random point fields [48, 49, 54, 17].

We make the following assumption.

(A1) μ is a (Φ, Ψ)-quasi Gibbs measure. Furthermore, there exists upper semi-continuous functions $(\hat{\Phi}, \hat{\Psi})$ and positive constants c_{31} and c_{32} satisfying

$$c_{31}^{-1} \hat{\Phi}(x) \le \Phi(x) \le c_{31} \hat{\Phi}(x), \quad c_{32}^{-1} \hat{\Psi}(x, y) \le \Psi(x, y) \le c_{32} \hat{\Psi}(x, y),$$

where Ψ and $\hat{\Psi}$ satisfy $\Psi(x, y) = \Psi(y, x)$ and $\hat{\Psi}(x, y) = \hat{\Psi}(y, x)$.

If these interaction potentials are translation invariant, we often write $\Psi(x, y) = \Psi(x - y)$ and $\hat{\Psi}(x, y) = \hat{\Psi}(x - y)$. The importance of **(A1)** lies in the fact that it gives a sufficient condition of the basic assumption (5.11). We quote:

Lemma 5.6 ([48, 45-46pp])**.** Assume **(A1)**. Then $(\mathcal{E}^{\mu,m}_r, \mathcal{D}^\mu_\circ)$ is closable on $L^2(\mathsf{S}, \mu)$ for each $m, r \in \mathbb{N}$. In particular, $(\mathcal{E}^\mu_r, \mathcal{D}^\mu_\circ)$ is closable on $L^2(\mathsf{S}, \mu)$.

We now recall two basic notions on random point fields: correlation functions and density functions.

A symmetric and locally integrable function $\rho^n : S^n \to [0, \infty)$ is called the n-point correlation function of a random point field μ on S with respect to the Lebesgue measure if ρ^n satisfies

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_{\mathsf{S}} \prod_{i=1}^m \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} d\mu$$

for any sequence of disjoint bounded measurable sets $A_1, \ldots, A_m \in \mathcal{B}(S)$ and a sequence of positive integers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. When $\mathsf{s}(A_i) - k_i < 0$, according to our interpretation, $\mathsf{s}(A_i)!/(\mathsf{s}(A_i) - k_i)! = 0$ by convention. We assume that μ has n-point correlation function ρ^n for each $n \in \mathbb{N}$.

A symmetric function $\sigma_r^k : S_r^k \to [0,\infty)$ is called the k-point density function of a random point field μ on S_r with respect to the Lebesgue measure if for all non-negative, bounded $\sigma[\pi_r]$-measurable function f with S_r^k-representation f_r^k

$$\frac{1}{k!}\int_{S_r^k} f_r^k \sigma_r^k d\mathbf{x}^k = \int_{\mathsf{S}_r^k} f d\mu < \infty.$$

Let $\mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S_r) = m\}$ as before. We make the following assumption.

(A2) $\sum_{m=1}^{\infty} m\mu(\mathsf{S}_r^m) < \infty$ for all $r \in \mathbb{N}$.

A family of probability measures $\{\mathsf{P}_{\mathsf{x}}\}_{\mathsf{x}\in\mathsf{S}}$ on $C([0,\infty);\mathsf{S})$ is called a diffusion if the canonical process $\mathsf{X} = \{\mathsf{X}_t\}$ under P_{x} is a continuous process having a strong Markov property starting at x. Furthermore, $\{\mathsf{P}_{\mathsf{x}}\}_{\mathsf{x}\in\mathsf{S}}$ is called conservative if it has an invariant probability measure.

Assume **(A1)**. Then we deduce from Lemma 5.1 and Lemma 5.6 that the non-negative form $(\mathcal{E}^{\mu}, \mathcal{D}_{\circ}^{\mu})$ is closable on $L^2(\mathsf{S},\mu)$. Therefore, let $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ be its closure on $L^2(\mathsf{S},\mu)$. The next result is a refinement of [44, 119p. Corollary 1] and can be proved in a similar fashion. We postpone the proof to Appendix (see Section 5.9).

Lemma 5.7. Assume **(A1)** and **(A2)**. Then there exists a μ-reversible diffusion $\{\mathsf{P}_{\mathsf{x}}^{\mathsf{upr}}\}_{\mathsf{x}\in\mathsf{S}}$ associated with the Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ on $L^2(\mathsf{S},\mu)$.

We note that **(A2)** is used to guarantee the existence of the diffusion. The μ-reversibility of the diffusion follows from $1 \in \mathcal{D}^{\mathsf{upr}}$ and symmetry of $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$.

By construction, such a family of diffusion measures $\mathsf{P}^{\mathsf{upr}} = \{\mathsf{P}_{\mathsf{x}}^{\mathsf{upr}}\}$ with quasi-continuity in x is unique for quasi-everywhere starting point x. Equivalently, there exists a set S_0 such that the complement of S_0 has capacity zero and the family of diffusion measures $\mathsf{P}^{\mathsf{upr}} = \{\mathsf{P}_{\mathsf{x}}^{\mathsf{upr}}\}$ associated with the Dirichlet space above with quasi-continuity in x is unique for all $\mathsf{x} \in \mathsf{S}_0$ and $\mathsf{P}_{\mathsf{x}}^{\mathsf{upr}}(\mathsf{X}_t \in \mathsf{S}_0 \text{ for all } t) = 1$ for all $\mathsf{x} \in \mathsf{S}_0$.

We next lift the unlabeled dynamics X to a labeled dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$. Under $\mathsf{P}^{\mathsf{upr}} = \{\mathsf{P}_{\mathsf{x}}^{\mathsf{upr}}\}$, we can write $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$, where each $X^i = \{X_t^i\}$ is a continuous process with time parameter of the form $[0,b)$ or (a,b). We call X^i tagged particles and $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ labeled dynamics. Note that for a given unlabeled process X, there exist plural labeled dynamics in general. We next give a condition such that $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ is determined uniquely. For this purpose, we impose the following condition:

(A3) Under $\mathsf{P}^{\mathsf{upr}} = \{\mathsf{P}_{\mathsf{x}}^{\mathsf{upr}}\}$, each tagged particle $\{X^i\}_{i\in\mathbb{N}}$ does not collide with another. Furthermore, $\{X^i\}_{i\in\mathbb{N}}$ does not hit the boundary ∂S of S.

This condition is equivalent to both the capacity of multiple points and that of configurations with particles on the boundary ∂S being zero:

$$\mathrm{Cap}^{\mu}(\{\mathsf{s} \in \mathsf{S}; \mathsf{s}(\{x\}) \geq 2 \text{ for some } x \in S\}) = 0,$$
$$\mathrm{Cap}^{\mu}(\{\mathsf{s} \in \mathsf{S}; \mathsf{s}(\partial S) \geq 1\}) = 0.$$

Here Cap^{μ} denotes the one-capacity with respect to the Dirichlet space $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$ on $L^2(\mathsf{S},\mu)$ (see [16] for the definition of capacity).

Let $\mathrm{Erf}(t) = \int_t^{\infty} (1/\sqrt{2\pi}) e^{-|x|^2/2} dx$ be the error function. We further assume:

(A4) There exists a $T > 0$ such that, for each $R > 0$,

$$\lim_{r\to\infty} \mathrm{Erf}\Big(\frac{r}{\sqrt{(r+R)T}}\Big) \int_{|x|\le r+R} \rho^1(x)dx = 0.$$

From **(A4)**, we deduce the non-explosion of each tagged particle [46, Theorem 2.5]. We hence see from **(A3)** and **(A4)** that under $\mathsf{P}^{\mathsf{upr}} = \{\mathsf{P}^{\mathsf{upr}}_{\mathsf{x}}\}$ each tagged particle of $\{X^i\}_{i\in\mathbb{N}}$ neither collide each other nor hit the boundary ∂S nor explode.

We call $\mathfrak{u}$ the unlabeling map if $\mathfrak{u}((x_i)) = \sum_i \delta_{x_i}$. We call $\mathfrak{l}$ a label if $\mathfrak{l} : \mathsf{S} \to S^{\mathbb{N}}$ is a measurable map defined for μ-a.s. x, and $\mathfrak{u}\circ\mathfrak{l}(\mathsf{x}) = \mathsf{x}$. For simplicity, we take $\mathfrak{l}$ as

$$|\mathfrak{l}^i(\mathsf{x})| < |\mathfrak{l}^{i+1}(\mathsf{x})| \quad \text{for all } i \in \mathbb{N}$$

throughout the paper. Because μ has an m-point correlation function for each m, $\mathfrak{l}(\mathsf{x})$ is well defined for μ-a.s.$\,\mathsf{x}$.

Lemma 5.8 ([46, 48])**.** Assume that **(A1)**–**(A4)**. Let $\mathfrak{l}$ be a label. Then under $\mathsf{P}^{\mathsf{upr}} = \{\mathsf{P}^{\mathsf{upr}}_{\mathsf{x}}\}$ there exists a unique, labeled dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{N}} \in C([0,\infty); S^{\mathbb{N}})$ such that $\mathbf{X}_0 = \mathfrak{l}(\mathsf{X}_0)$ and that $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X^i_t}$ for all t.

Once the initial label $\mathfrak{l}$ is assigned, the particles are marked forever because they neither collide nor explode. We hence determine the labeled dynamics $\mathbf{X}$ from the unlabeled dynamics X and the label $\mathfrak{l}$ uniquely. We have thus had a natural correspondence between $\mathbf{X}$ and $(\mathsf{X}, \mathfrak{l})$ under the conditions **(A3)** and **(A4)**. We remark here that $\mathbf{X}_t \neq \mathfrak{l}(\mathsf{X}_t)$ for $t > 0$ in general.

The next lemma will be used in the proof of Lemma 5.12 and Lemma 5.20.

Lemma 5.9. Assume **(A4)**. Then for each $r, T \in \mathbb{N}$, the following holds.

$$\int_S \mathrm{Erf}\Big(\frac{|x|-r}{\sqrt{T}}\Big)\rho^1(x)dx < \infty. \tag{5.15}$$

Proof. Let $F(u) = \int_{S_u} \rho^1(x)dx$. Then from **(A4)** we deduce

$$F(u) = o\Big(\frac{1}{\mathrm{Erf}\big(\frac{u-R}{\sqrt{uT}}\big)}\Big) \quad \text{as } u \to \infty.$$

Hence we obtain

$$\begin{aligned}\int_S \mathrm{Erf}\Big(\frac{|x|-r}{\sqrt{T}}\Big)\rho^1(x)dx &= \int_0^\infty \mathrm{Erf}\Big(\frac{u-r}{\sqrt{T}}\Big)F'(u)du \\ &= \Big[\mathrm{Erf}\Big(\frac{u-r}{\sqrt{T}}\Big)F(u)\Big]_0^\infty - \int_0^\infty \frac{\partial}{\partial u}\mathrm{Erf}\Big(\frac{u-r}{\sqrt{T}}\Big)F(u)du \\ &< \infty.\end{aligned}$$

This implies (5.15). □

5.2.3 ISDE-representation: Logarithmic derivative

We next present the ISDE describing the labeled dynamics given by Lemma 5.8. The key notion for this is the logarithmic derivative of μ to be introduced below.

We first recall two new measures arising from random point field μ. The first concerns the conditioning of μ, the second its disintegration.

For $\mathbf{x} = (x_1, \dots, x_k) \in S^k$ a random point field $\mu_{\mathbf{x}}$ is called the reduced Palm measure of μ conditioned at $\mathbf{x} \in S^k$ if $\mu_{\mathbf{x}}$ is the regular conditional probability defined as

$$\mu_{\mathbf{x}} = \mu(\,\cdot\, - \sum_{i=1}^{k} \delta_{x_i} | \mathsf{s}(\{x_i\}) \geq 1 \text{ for } i = 1, \dots, k).$$

A Radon measure $\mu^{[k]}$ on $S^k \times \mathsf{S}$ is called the k-Campbell measure of μ if $\mu^{[k]}$ is given by

$$\mu^{[k]}(d\mathbf{x}d\mathsf{s}) = \rho^k(\mathbf{x})\mu_{\mathbf{x}}(d\mathsf{s})d\mathbf{x}.$$

We set $L^1_{\text{loc}}(S \times \mathsf{S}, \mu^{[1]}) = \bigcap_{r=1}^{\infty} L^1(S \times \mathsf{S}, \mu^{[1]}_r)$, where $\mu^{[1]}_r(\,\cdot\,) = \mu^{[1]}(\,\cdot\, \cap S_r \times \mathsf{S})$. We set

$$C_0^{\infty}(S) \otimes \mathcal{D}_{\circ} = \{\sum_{i=1}^{m} f_i(x)g_i(\mathsf{y})\,;\, f_i \in C_0^{\infty}(S),\ g_i \in \mathcal{D}_{\circ},\, m \in \mathbb{N}\}.$$

We now recall the notion of the logarithmic derivative of μ [47].

Definition 5.10. An $\mathbb{R}^d$-valued function $\mathsf{d}^{\mu} \in L^1_{\text{loc}}(S \times \mathsf{S}, \mu^{[1]})^d$ is called *the logarithmic derivative* of μ if, for all $\varphi \in C_0^{\infty}(S) \otimes \mathcal{D}_{\circ}$,

$$\int_{S\times\mathsf{S}} \mathsf{d}^{\mu}(x,\mathsf{y})\varphi(x,\mathsf{y})\mu^{[1]}(dxd\mathsf{y}) = -\int_{S\times\mathsf{S}} \nabla_x\varphi(x,\mathsf{y})\mu^{[1]}(dxd\mathsf{y}). \tag{5.16}$$

We make the following assumption:

(A5) μ has a logarithmic derivative d^{μ}.

The next lemma reveals the importance of logarithmic derivative.

Lemma 5.11 ([47])**.** Assume **(A1)**–**(A5)**. Let $\mathbf{X}$ and $\mathfrak{l}$ be as in Lemma 5.8. Assume

$$\mathsf{b} = \frac{1}{2}\mathsf{d}^{\mu}. \tag{5.17}$$

Then there exists $\mathsf{S}_0 \subset \mathsf{S}$ such that $\mu(\mathsf{S}_0) = 1$ and that the labeled dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ under $\mathsf{P}^{\mathsf{upr}}_{\mathsf{x}}$ solves the ISDE for each $\mathsf{x} \in \mathsf{S}_0$

$$dX^i_t = dB^i_t + \mathsf{b}(X^i_t, \mathsf{X}^{\diamond i}_t)dt, \quad i \in \mathbb{N}, \tag{5.18}$$

$$\mathbf{X}_0 = \mathfrak{l}(\mathsf{x}), \tag{5.19}$$

where $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ is the $(\mathbb{R}^d)^{\mathbb{N}}$-valued standard Brownian motion, and $\mathsf{X}^{\diamond i}_t = \sum_{j\neq i} \delta_{X^j_t}$ for $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$.

Let $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ be a solution of ISDE (5.18) and denote by $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$ the associated unlabeled process. Let μ_t be the distribution of X_t. We make the following assumptions on $\mathbf{X}$.

(μ-AC) The μ-absolutely continuous condition is satisfied. That is, if $\mathsf{X}_0 = \mu$ in law, then

$$\mu_t \prec \mu \quad \text{for all } t \geq 0, \tag{5.20}$$

where $\mu_t \prec \mu$ means μ_t is absolutely continuous with respect to μ.

(NBJ) The no-big-jump condition is satisfied. That is, if the distribution of X_0 equals to μ, then for each $r, T \in \mathbb{N}$

$$P(I_{r,T}(\mathbf{X}) < \infty) = 1, \tag{5.21}$$

where $I_{r,T}$ is the maximal label with which the particle intersects S_r defined by

$$I_{r,T}(\mathbf{X}) = \max\{i \in \mathbb{N} \cup \{\infty\}\,;\, |X_t^i| \leq r \text{ for some } 0 \leq t \leq T\}. \tag{5.22}$$

Lemma 5.12 ([53])**.** Assume **(A1)**–**(A5)**. Then under $\mathsf{P}^{\mathsf{upr}} = \{\mathsf{P}^{\mathsf{upr}}_{\mathsf{x}}\}_{\mathsf{x}\in\mathsf{S}}$ the labeled dynamics $\mathbf{X}$ satisfies the conditions **(μ-AC)**, and **(NBJ)**.

Proof. Because the unlabeled dynamics X is μ-reversible, $\mu_t = \mu$ for all t. Hence **(μ-AC)** is obvious. The second claim follows from the Lyons–Zheng decomposition and Lemma 5.9 (see [53, Lemma 9.4] for detail). $\square$

5.2.4 Finite systems in S_R of interacting Brownian motions with reflecting boundary condition

We give the SDE representation of the unlabeled process X associated with the Dirichlet form $(\mathcal{E}^{\mathsf{lwr}}_R, \mathcal{D}^{\mathsf{lwr}}_R)$ on $L^2(\mathsf{S}, \mu)$. We denote by $\mathsf{P}^{\mathsf{lwr}}_R = \{\mathsf{P}^{\mathsf{lwr}}_{R,\mathsf{x}}\}$ the family of the diffusion measures given by $(\mathcal{E}^{\mathsf{lwr}}_R, \mathcal{D}^{\mathsf{lwr}}_R)$ on $L^2(\mathsf{S}, \mu)$. The Dirichlet form $(\mathcal{E}^{\mathsf{lwr}}_R, \mathcal{D}^{\mathsf{lwr}}_R)$ is dominated by $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$, that is,

$$(\mathcal{E}^{\mathsf{lwr}}_R, \mathcal{D}^{\mathsf{lwr}}_R) \leq (\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}}). \tag{5.23}$$

Then from (5.23) we see that the capacity of $(\mathcal{E}^{\mathsf{lwr}}_R, \mathcal{D}^{\mathsf{lwr}}_R)$ is dominated by that of $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$. Hence non-collision of tagged particles under $\mathsf{P}^{\mathsf{lwr}}_R$ follows from that of the limit diffusion X given by $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$, which is assumed by **(A3)**. With the same reason, tagged particles under $\mathsf{P}^{\mathsf{lwr}}_R$ do not hit the set $(\partial S) \cap S_R$. Non-explosion of tagged particles under $\mathsf{P}^{\mathsf{lwr}}_R$ is obvious because they are reflecting diffusion on S_R and frozen outside S_R.

We now denote by $\mathbf{X} = (X^i)_{i=1}^{\infty}$ the labeled process associated with X and the label $\mathfrak{l}$. Then $\mathsf{X} = \{\mathsf{X}_t\}$ is given by $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$ from $\mathbf{X} = (X^i)_{i=1}^{\infty}$. By definition $\mathbf{X}_0 = \mathfrak{l}(\mathsf{X}_0)$. The process $\mathbf{X}$ under $\mathsf{P}^{\mathsf{lwr}}_R$ describes the system of interacting Brownian motions in which

1. each particle in S_R moves in S_R and when it hits the boundary ∂S_R, it reflects and enter the domain S_R immediately,
2. the particles out of S_R stay the initial positions forever.

We denote by $\mu^{R\mathsf{s}}$ the regular conditional distribution defined by

$$\mu^{R\mathsf{s}}(\,\cdot\,) = \mu(\,\cdot\,|\,\sigma[\pi_R^c])(\mathsf{s}) \quad \text{for } \mu\text{-a.s. } \mathsf{s}. \tag{5.24}$$

Let $\mathsf{S}^{R\mathsf{s}} = \{\mathsf{y} \in \mathsf{S}\,;\, \pi_R^c(\mathsf{y}) = \pi_R^c(\mathsf{s})\}$. Then $\mu^{R\mathsf{s}}$ is a probability measure supported on $\mathsf{S}^{R\mathsf{s}}$. Let $\mu^{R\mathsf{s},[1]}$ be the 1-Campbell measure of $\mu^{R\mathsf{s}}$. Then we have

$$\mu^{R\mathsf{s},[1]}(dxd\mathsf{y}) = \rho^{R\mathsf{s},1}(x)\mu_x^{R\mathsf{s}}(d\mathsf{y})dx \quad \text{for } x \in S_R, \tag{5.25}$$

where $\rho^{R\mathsf{s},1}$ is the one-point correlation function of $\mu^{R\mathsf{s}}$ and $\mu_x^{R\mathsf{s}}$ is the reduced Palm measure of $\mu^{R\mathsf{s}}$ conditioned at x. By the Green formula, we see for all $\varphi \in C_0^\infty(S) \otimes \mathcal{D}_\circ$,

$$\begin{aligned} -\int_{S_R\times\mathsf{S}} \nabla_x\varphi(x,\mathsf{y})\mu^{R\mathsf{s},[1]}(dxd\mathsf{y}) = &\int_{S_R\times\mathsf{S}} \mathsf{d}^\mu(x,\mathsf{y})\varphi(x,\mathsf{y})\mu^{R\mathsf{s},[1]}(dxd\mathsf{y}) \\ &+ \int_{\partial S_R\times\mathsf{S}} \varphi(x,\mathsf{y})\mathbf{n}^R(x)\mathcal{S}_R(dx)\mu_x^{R\mathsf{s}}(d\mathsf{y}), \end{aligned} \tag{5.26}$$

where $\mathcal{S}_R$ is the Lebesgue surface measure on the boundary ∂S_R and $\mathbf{n}^R(x)$ is the inward normal unit vectors at $x \in \partial S_R$. Hence for μ-a.s. s and for $\mu_x^{R\mathsf{s}}$-a.s. y, the logarithmic derivative $\mathsf{d}^{R\mathsf{s}}$ of $\mu^{R\mathsf{s}}$ coincides with the sum of

$$\mathsf{d}^\mu(x, \pi_R(\mathsf{y}) + \pi_R^c(\mathsf{s})) \quad \text{for } x \in S_R$$

and a singular part associated with the the boundary ∂S_R. We then obtain informally

$$\mathsf{d}^{R\mathsf{s}}(x,\mathsf{y}) = 1_{S_R}(x)\mathsf{d}^\mu(x, \pi_R(\mathsf{y}) + \pi_R^c(\mathsf{s})) + \mathbf{n}^R(x)1_{\partial S_R}(x)\delta_x. \tag{5.27}$$

Here we naturally extend the domain of $\mathsf{d}^{R\mathsf{s}}(x,\mathsf{y})$ to $S \times \mathsf{S}$ by taking $\mathsf{d}^{R\mathsf{s}}(x,\mathsf{y}) = 0$ for $x \notin S_R$. This is reasonable because particles outside S_R are fixed.

By definition $\mathsf{x}(S_R)$ coincides with the number of particles in S_R for a given configuration x. From the Green formula (5.26) we see that $\mathbf{X} = (X^i)_{i=1}^\infty$ is the system of infinite number of particles such that only particles in S_R move and satisfies the following SDE: For μ-a.s. $\mathsf{s} = \sum_i \delta_{s_i}$ and for $\mu^{R\mathsf{s}}$-a.s. $\mathsf{x} = \sum_i \delta_{x_i}$

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, \mathsf{X}_t^{\diamond i})dt + \frac{1}{2}\mathbf{n}^R(X_t^i)dL_t^{R,i},\ 1 \le i \le \mathsf{x}(S_R), \tag{5.28}$$

$$dL_t^{R,i} = \mathbf{1}_{\partial S_R}(X_t^i)dL_t^{R,i},\ 1 \le i \le \mathsf{x}(S_R), \tag{5.29}$$

$$X_t^i = X_0^i, \quad i > \mathsf{x}(S_R), \tag{5.30}$$

$$\mathbf{X}_0 = \mathfrak{l}(\mathsf{x}), \tag{5.31}$$

where $\mathfrak{l}(\mathsf{s}) = (s_i)_{i=1}^\infty$, $\mathfrak{l}(\mathsf{x}) = (x_i)_{i=1}^\infty$, $\mathsf{X}_t^{\diamond i} = \sum_{j\neq i} \delta_{X_t^j}$, and $L^{R,i} = \{L_t^{R,i}\}$ are non-negative increasing processes; see for instance [8]. The particles outside S_R are frozen by (5.30). Hence $L_t^{R,i} = 0$ for $i > \mathsf{x}(S_R)$. We remark that $s_i = x_i$ for all $i > \mathsf{x}(S_R)$ for μ^{s}-a.s. x.

Let $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})$ be the Dirichlet form introduced in Lemma 5.2. Then we can easily deduce from **(A1)** and **(A2)** that $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})$ is a quasi-regular Dirichlet form on $L^2(\mathsf{S}, \mu)$ and that there exists the associated diffusion X. The capacity for $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})$ is dominated by that for $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$. Hence from **(A3)** we deduce that X has also non-collision property. Clearly, each tagged particle of X does not explode because of the definition of $\mathcal{E}_R^{\mathsf{lwr}}$. We have thus obtained the labeled process $\mathbf{X}$ from the unlabeled process X and the label $\mathfrak{l}$. Moreover, using the Fukushima decomposition and taking (5.27) into account, we see that $\mathbf{X}$ is a solution of SDE (5.28)–(5.31).

Let $\mu^{R\mathsf{s}}$ be as (5.24). Then

$$\mu(\,\cdot\,) = \int_{\mathsf{S}} \mu^{R\mathsf{s}}(\,\cdot\,)\mu(d\mathsf{s}) \tag{5.32}$$

and $\mu^{R\mathsf{s}}(\mathsf{S}^{R\mathsf{s}}) = 1$, where $\mathsf{S}^{R\mathsf{s}} = \{\mathsf{y} \in \mathsf{S}\,;\, \pi_R^c(\mathsf{y}) = \pi_R^c(\mathsf{s})\}$ as before. We set

$$\begin{aligned} \mathcal{E}_R^{R\mathsf{s}}(f,g) &:= \int_{\mathsf{S}} \mathbb{D}_R[f,g]d\mu^{R\mathsf{s}} = \int_{\mathsf{S}} \mathbb{D}_R[f_{R\mathsf{s}}, g_{R\mathsf{s}}]d\mu^{R\mathsf{s}}, \\ \mathcal{D}_\circ^{R\mathsf{s}} &= \{f \in \mathcal{D}_\circ \cap L^2(\mathsf{S},\mu^{R\mathsf{s}})\,;\, \mathcal{E}_R^{R\mathsf{s}}(f,f) < \infty\}, \end{aligned} \tag{5.33}$$

where we set $h_{R\mathsf{s}}(\cdot) = h(\pi_R(\cdot) + \pi_R^c(\mathsf{s}))$ for a function h on S. The second equality in (5.33) is clear because for $\mu^{R\mathsf{s}}$-a.s. x

$$\mathbb{D}_R[f,g](\mathsf{x}) = \mathbb{D}_R[f,g]_{R\mathsf{s}}(\mathsf{x}) = \mathbb{D}_R[f_{R\mathsf{s}}, g_{R\mathsf{s}}](\mathsf{x}).$$

From **(A1)** we easily deduce that $(\mathcal{E}_R^{R\mathsf{s}}, \mathcal{D}_\circ^{R\mathsf{s}})$ is closable on $L^2(\mathsf{S},\mu^{R\mathsf{s}})$. We then denote by $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}})$ the closure of $(\mathcal{E}_R^{R\mathsf{s}}, \mathcal{D}_\circ^{R\mathsf{s}})$ on $L^2(\mathsf{S},\mu^{R\mathsf{s}})$. Furthermore, we see that $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}})$ is a quasi-regular Dirichlet form on $L^2(\mathsf{S}^{R\mathsf{s}},\mu^{R\mathsf{s}})$. Hence there exists a diffusion X associated with $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}})$ on $L^2(\mathsf{S}^{R\mathsf{s}},\mu^{R\mathsf{s}})$. Using the Fukushima decomposition, we deduce that the associated labeled diffusion is a solution of the SDE (5.28)–(5.31) for $\mu^{R\mathsf{s}}$-a.s. x for μ-a.s. s.

Lemma 5.13. Let $\mathbf{X}^{\mathsf{lwr}}$ be the solution of the SDE (5.28)–(5.31) given by the Dirichlet form $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})$ on $L^2(\mathsf{S},\mu)$. Let $\mathbf{X}^{R\mathsf{s},\mathsf{lwr}}$ be the solution of the SDE (5.28)–(5.31) given by the Dirichlet form $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}})$ on $L^2(\mathsf{S}^{R\mathsf{s}},\mu^{R\mathsf{s}})$. Recall the disintegration $\mu = \int_{\mathsf{S}} \mu^{R\mathsf{s}}\mu(d\mathsf{s})$ given by (5.32). Then, for μ-a.s. s, $\mathbf{X}^{\mathsf{lwr}} = \mathbf{X}^{R\mathsf{s},\mathsf{lwr}}$ in distribution for $\mu^{R\mathsf{s}}$-a.s. x.

Proof. Let $T_{R,t}^{\mathsf{lwr}}$ and $T_{R,t}^{R\mathsf{s},\mathsf{lwr}}$ be the semi-groups associated with the Dirichlet forms $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})$ and $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}})$ on $L^2(\mathsf{S},\mu)$ and $L^2(\mathsf{S},\mu^{R\mathsf{s}})$, respectively. Then to prove Lemma 5.13 it is sufficient to prove the coincidence of these two semi-groups.

There exists a countable subset $\mathcal{D}_\bullet$ of $L^2(\mathsf{S},\mu)$, $\mathcal{D}_R^{\mathsf{lwr}}$, and $\mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}}$ for μ-a.s. s such that $\mathcal{D}_\bullet$ is dense in $L^2(\mathsf{S},\mu)$, $\mathcal{D}_R^{\mathsf{lwr}}$, and $\mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}}$ for μ-a.s. s with respect to $L^2(\mathsf{S},\mu)$-norm, $\mathcal{E}_{R,1}^{\mathsf{lwr}}$-norm, and $\mathcal{E}_{R,1}^{R\mathsf{s},\mathsf{lwr}}$-norm for μ-a.s. s, respectively. (see Section 5.9). Here $\mathcal{E}_{R,1}^{\mathsf{lwr}}$-norm of f is given by $\mathcal{E}_R^{\mathsf{lwr}}(f,f)^{1/2} + \|f\|_{L^2(\mathsf{S},\mu)}$. We set $\mathcal{E}_{R,1}^{R\mathsf{s},\mathsf{lwr}}$-norm similarly.

From (5.9), (5.32) and (5.33) we have for $f,g \in \mathcal{D}_\bullet$

$$\mathcal{E}_R^{\mathsf{lwr}}(f,g) = \int_{\mathsf{S}} \mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}(f_{R\mathsf{s}}, g_{R\mathsf{s}})\mu(d\mathsf{s}). \tag{5.34}$$

Then we see for $f,g \in \mathcal{D}_\bullet$

$$\int_{\mathsf{S}} f(\mathsf{s})g(\mathsf{s})\mu(d\mathsf{s}) - \int_{\mathsf{S}} T_{R,t}^{\mathsf{lwr}} f(\mathsf{s})g(\mathsf{s})\mu(d\mathsf{s}) = \int_0^t \mathcal{E}_R^{\mathsf{lwr}}(T_{R,u}^{\mathsf{lwr}} f, g)du, \tag{5.35}$$

$$\int_{\mathsf{S}} f_{R\mathsf{s}}(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x}) - \int_{\mathsf{S}} T_{R,t}^{R\mathsf{s},\mathsf{lwr}}(f_{R\mathsf{s}})(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x}) = \int_0^t \mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}(T_{R,u}^{R\mathsf{s},\mathsf{lwr}}(f_{R\mathsf{s}}), g_{R\mathsf{s}})du \tag{5.36}$$

for μ-a.s. s. We recall that $\mathcal{D}_\bullet$ is countable. Hence, for μ-a.s. s, (5.36) holds for all $f,g \in \mathcal{D}_\bullet$. Because the particles outside S_R are frozen, we easily see that for $f,g \in \mathcal{D}_\bullet$ and for μ-a.s. s

$$T_{R,t}^{\mathsf{lwr}}(f_{R\mathsf{s}})\,(\mathsf{x}) = (T_{R,t}^{\mathsf{lwr}} f)_{R\mathsf{s}}\,(\mathsf{x}) \qquad \text{for } \mu^{R\mathsf{s}}\text{-a.s. } \mathsf{x}. \tag{5.37}$$

In the following we write $T_{R,t}^{\mathsf{lwr}} f_{R\mathsf{s}} = T_{R,t}^{\mathsf{lwr}}(f_{R\mathsf{s}})$ and $T_{R,t}^{R\mathsf{s},\mathsf{lwr}} f_{R\mathsf{s}} = T_{R,t}^{R\mathsf{s},\mathsf{lwr}}(f_{R\mathsf{s}})$.

From (5.32) and the definition $h_{R\mathsf{s}}(\cdot) = h(\pi_R(\cdot) + \pi_R^c(\mathsf{s}))$ we have

$$\int_{\mathsf{S}} f(\mathsf{s})g(\mathsf{s})\mu(d\mathsf{s}) = \int_{\mathsf{S}} \Big\{ \int_{\mathsf{S}} f_{R\mathsf{s}}(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x}) \Big\}\mu(d\mathsf{s}). \tag{5.38}$$

Replacing f with $T_{R,t}^{\mathsf{lwr}} f$ in (5.38) and using (5.37) we have

$$\begin{aligned} \int_{\mathsf{S}} T_{R,t}^{\mathsf{lwr}} f(\mathsf{s})g(\mathsf{s})\mu(d\mathsf{s}) &= \int_{\mathsf{S}}\int_{\mathsf{S}} (T_{R,t}^{\mathsf{lwr}} f)_{R\mathsf{s}}(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x})\mu(d\mathsf{s}) \\ &= \int_{\mathsf{S}}\int_{\mathsf{S}} T_{R,t}^{\mathsf{lwr}} f_{R\mathsf{s}}(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x})\mu(d\mathsf{s}). \end{aligned} \tag{5.39}$$

Note that $\mathcal{E}_R^{\mathsf{lwr}}(T_{R,u}^{\mathsf{lwr}} f, T_{R,u}^{\mathsf{lwr}} f) \le \mathcal{E}_R^{\mathsf{lwr}}(f,f)$. With the same reason as (5.34) and from (5.37) we have for all $f,g \in \mathcal{D}_\bullet$

$$\begin{aligned} \mathcal{E}_R^{\mathsf{lwr}}(T_{R,u}^{\mathsf{lwr}} f, g) &= \int_{\mathsf{S}} \mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}((T_{R,u}^{\mathsf{lwr}} f)_{R\mathsf{s}}, g_{R\mathsf{s}})\mu(d\mathsf{s}) \\ &= \int_{\mathsf{S}} \mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}(T_{R,u}^{\mathsf{lwr}} f_{R\mathsf{s}}, g_{R\mathsf{s}})\mu(d\mathsf{s}). \end{aligned} \tag{5.40}$$

Putting (5.38)–(5.40) into (5.35) and using the Fubini theorem we obtain

$$\begin{aligned} &\int_{\mathsf{S}} \Big\{ \int_{\mathsf{S}} f_{R\mathsf{s}}(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x}) \Big\}\mu(d\mathsf{s}) - \int_{\mathsf{S}} \Big\{ \int_{\mathsf{S}} T_{R,t}^{\mathsf{lwr}} f_{R\mathsf{s}}(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x}) \Big\}\mu(d\mathsf{s}) \\ &= \int_{\mathsf{S}} \Big\{ \int_0^t \mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}(T_{R,u}^{\mathsf{lwr}} f_{R\mathsf{s}}, g_{R\mathsf{s}})du \Big\}\mu(d\mathsf{s}). \end{aligned} \tag{5.41}$$

Hence we from (5.41) for μ-a.s. s

$$\int_{\mathsf{S}} f_{R\mathsf{s}}(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x}) - \int_{\mathsf{S}} T_{R,t}^{\mathsf{lwr}} f_{R\mathsf{s}}(\mathsf{x})g_{R\mathsf{s}}(\mathsf{x})\mu^{R\mathsf{s}}(d\mathsf{x}) = \int_0^t \mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}(T_{R,u}^{\mathsf{lwr}} f_{R\mathsf{s}}, g_{R\mathsf{s}})du. \tag{5.42}$$

Compare (5.42) with (5.36). Then we see that $f \mapsto T_{R,t}^{\mathsf{lwr}} f_{R\mathsf{s}}$ is the semi-group associated with the Dirichlet form $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}})$ on $L^2(\mathsf{S}, \mu^{R\mathsf{s}})$. We note here $f = f_{R\mathsf{s}}$ for $\mu^{R\mathsf{s}}$-a.s. Therefore, for μ-a.s. s,

$$T_{R,t}^{\mathsf{lwr}} f(\mathsf{x}) = T_{R,t}^{\mathsf{lwr}} f_{R\mathsf{s}}(\mathsf{x}) = T_{R,t}^{R\mathsf{s},\mathsf{lwr}} f(\mathsf{x}) \quad \text{for } \mu^{R\mathsf{s}}\text{-a.s. } \mathsf{x}. \tag{5.43}$$

Here we used $\mu^{R\mathsf{s}}(\mathsf{S}^{R\mathsf{s}}) = 1$, where $\mathsf{S}^{R\mathsf{s}} = \{\mathsf{y} \in \mathsf{S}\,;\, \pi_R^c(\mathsf{y}) = \pi_R^c(\mathsf{s})\}$ as before.

The solutions $\mathbf{X}^{\mathsf{lwr}}$ and $\mathbf{X}^{R\mathsf{s},\mathsf{lwr}}$ are associated with the semi-groups $T_{R,t}^{\mathsf{lwr}}$ and $T_{R,t}^{R\mathsf{s},\mathsf{lwr}}$, respectively. Hence from (5.43) we deduce that these are equivalent in distribution. We thus see that the solutions of SDE (5.28)–(5.31) given by these Dirichlet forms are the same. $\square$

5.3 Statements of the main results

We present our main results. Let $\mathcal{T}(\mathsf{S})$ be the tail σ-field of the configuration space S:

$$\mathcal{T}(\mathsf{S}) = \bigcap_{R=1}^{\infty} \sigma[\pi_R^c].$$

Let μ^{s} be a regular conditional probability conditioned by the tail σ-field defined as

$$\mu^{\mathsf{s}}(\cdot) = \mu(\cdot|\mathcal{T}(\mathsf{S}))(\mathsf{s}). \tag{5.44}$$

As S is a Polish space, such a regular conditional probability exists, and satisfies

$$\mu(\cdot) = \int_{\mathsf{S}} \mu^{\mathsf{s}}(\cdot)\mu(d\mathsf{s}). \tag{5.45}$$

From martingale convergence theorem we see that, for μ-a.s. s,

$$\lim_{R\to\infty} \mu^{R\mathsf{s}}(A) = \mu^{\mathsf{s}}(A) \quad \text{for any } A \in \mathcal{B}(\mathsf{S}). \tag{5.46}$$

This implies the weak convergence of $\{\mu^{R\mathsf{s}}\}$ to μ^{s} for μ-a.s. s.

Let $\mathsf{b} \in L^1_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]})$ be a coefficient of ISDE (5.51) below. We introduce cut-off coefficients $\mathsf{b}_{r,s,\mathsf{p}}$ of b. Let $C_b(S \times \mathsf{S})$ be the set of all bounded continuous functions on $S \times \mathsf{S}$. Then the main requirements for them are the following:

(A6) $\mathsf{b}_{r,s,\mathsf{p}} \in C_b(S \times \mathsf{S})$ for each $r, s, \mathsf{p} \in \mathbb{N}$ with $r < s$, and

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty} \sup_{R\ge r+s+1} \| \mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b} \|_{L^1_{\mathrm{loc}}(S\times\mathsf{S},\, \mu^{R\mathsf{s},[1]})} = 0 \quad \text{for } \mu\text{-a.s. } \mathsf{s}, \tag{5.47}$$

where $\mu^{R\mathsf{s},[1]}$ is the one-Campbell measure of $\mu^{R\mathsf{s}}$.

In Section 5.7, we shall present a sufficient condition of **(A6)** for coefficients $\mathsf{b} = \frac{1}{2}\mathsf{d}^{\mu}$ given by a logarithmic derivative d^{μ} with pair interaction Ψ such that $\Psi(x,y) = \Psi(y,x) = \Psi(x-y)$ and inverse temperature $\beta > 0$. We assume $\mathsf{b} \in L^p_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]})$ with $p > 1$ and

$$\mathsf{b}(x,\mathsf{y}) = \frac{\beta}{2} \lim_{s\to\infty} \Big(\Big\{ \sum_{|x-y_i|<s} \nabla\Psi(x-y_i) \Big\} - \varrho_s \Big) \quad \text{in } L^p_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]}). \tag{5.48}$$

Here $\mathsf{y} = \sum_i \delta_{y_i}$, $\Psi \in C^{\infty}(\mathbb{R}^d \backslash \{0\})$, and ϱ_s are constants. We then take $\mathsf{b}_{r,s,\mathsf{p}}$ as follows:

$$\mathsf{b}_{r,s,\mathsf{p}}(x,\mathsf{y}) = \frac{\beta}{2}\chi_r(x)\Big(\Big\{ \sum_{i=1}^{\infty} \chi_s(x-y_i)\upsilon_{\mathsf{p}}(x-y_i)\nabla\Psi(x-y_i)\Big\} - \varrho_s\Big)\varpi_{\mathsf{a}[r]}(\mathsf{y}), \tag{5.49}$$

where χ_r, υ_{p}, and $\varpi_{\mathsf{a}[r]}$ are functions defined by (5.102), (5.103), and (5.106), respectively.

Let $P^R_{\mathsf{l}(\mathsf{x})}$ be the distribution of the solution of SDE (5.28)–(5.31) given by the Dirichlet form $(\mathcal{E}^{\mathsf{lwr}}_R, \mathcal{D}^{\mathsf{lwr}}_R)$ on $L^2(\mathsf{S}, \mu)$. The first main theorem of this paper is the following.

Theorem 5.14. Assume that **(A1)**–**(A6)** hold. Then the sequence $\{P^R_{\mathsf{l}(\mathsf{x})}\}_{R\in\mathbb{N}}$ converges weakly in $C([0,\infty);S^{\mathbb{N}})$ to $P^{\infty}_{\mathsf{l}(\mathsf{x})}$ for μ-a.s. x, that is, for any $F \in C_b(C([0,\infty);S^{\mathbb{N}}))$

$$\lim_{R\to\infty}\int_{C([0,\infty);S^{\mathbb{N}})} F dP^R_{\mathsf{l}(\mathsf{x})} = \int_{C([0,\infty);S^{\mathbb{N}})} F dP^{\infty}_{\mathsf{l}(\mathsf{x})}. \tag{5.50}$$

For μ-a.s. x, the process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ under $P^{\infty}_{\mathsf{l}(\mathsf{x})}$ is a solution of the ISDE

$$dX^i_t = dB^i_t + \mathsf{b}(X^i_t, \mathsf{X}^{\diamond i}_t)dt \quad \text{for } i \in \mathbb{N}, \tag{5.51}$$

$$\mathbf{X}_0 = \mathsf{l}(\mathsf{x}) \tag{5.52}$$

satisfying conditions **(μ-AC)** and **(NBJ)**. Furthermore, $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ under $P^{\infty}_{\mathsf{l}(\mathsf{x})}$ is associated with the resolvent $\{G^{\mathsf{lwr}}_\alpha\}$ of the Dirichlet form $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ on $L^2(\mathsf{S},\mu)$ in the following sense:

$$G^{\mathsf{lwr}}_\alpha(f)(\mathsf{x}) = E^{\infty}_{\mathsf{l}(\mathsf{x})}[\int_0^\infty e^{-\alpha t} f(\mathfrak{u}(\mathbf{X}_t))dt], \tag{5.53}$$

where $f \in L^2(\mathsf{S},\mu)$, $\mathfrak{u}(\mathbf{X}_t) = \sum_{i=1}^\infty \delta_{X^i_t}$, and $E^{\infty}_{\mathsf{l}(\mathsf{x})}$ is the expectation with respect to $P^{\infty}_{\mathsf{l}(\mathsf{x})}$.

Because $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ on $L^2(\mathsf{S},\mu)$ is a Dirichlet form, there exists the associated Markovian semi-group on $L^2(\mathsf{S},\mu)$ whose resolvent is G^{lwr}_α in Theorem 5.14. We have however not yet constructed the associated diffusion. Only a stationary Markov process is thus constructed at this stage. In general, we have to prove quasi-regularity of the Dirichlet form $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ on $L^2(\mathsf{S},\mu)$ for the existence of the associated diffusion (see [42] for quasi-regularity).

The next theorem establishes the existence of the associated diffusion by proving the identity between the Dirichlet forms $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ and $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$.

We introduce another Dirichlet form $(\mathcal{E}^{+}, \mathcal{D}^{+})$. Recall that $\mathsf{b} = \frac{1}{2}\mathsf{d}^\mu$ by (5.17), where d^μ is the logarithmic derivative of μ defined by (5.16). Put

$$\begin{aligned}\mathcal{D}^{+} = \{f \in L^2(\mathsf{S},\mu);\, &\text{there exists } f' \in L^2(S\times\mathsf{S},\mu^{[1]})^d \text{ such that}\\ &-\int_{S\times\mathsf{S}} f(\delta_x + \mathsf{y})\{\nabla_x\varphi(x,\mathsf{y}) + \mathsf{d}^\mu(x,\mathsf{y})\varphi(x,\mathsf{y})\}\mu^{[1]}(dxd\mathsf{y})\\ &= \int_{S\times\mathsf{S}} f'(x,\mathsf{y})\varphi(x,\mathsf{y})\mu^{[1]}(dxd\mathsf{y}) \text{ for } \varphi \in C_0^\infty(S^{\mathrm{int}})\otimes\mathcal{D}_\circ\}.\end{aligned}$$

Let us denote the distributional derivative f' by $D_x f$ and set

$$\mathcal{E}^{+}(f,g) = \int_{S\times\mathsf{S}} \frac{1}{2} D_x f \cdot D_x g \, \mu^{[1]}(dxd\mathsf{y}), \quad f,g \in \mathcal{D}^{+}.$$

We now state our second main theorem.

Theorem 5.15. Assume **(A1)**–**(A6)**. Assume that a family of solutions of ISDE (5.18)–(5.19) defined for μ-a.s. x satisfying **(μ-AC)** and **(NBJ)** are unique in law for μ-a.s. x. Then

$$(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}}) = (\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}}) = (\mathcal{E}^{+}, \mathcal{D}^{+}). \tag{5.54}$$

In the remainder of this section, we give comments on the uniqueness of solutions of the ISDE (5.18) assumed in Theorem 5.15. There are three major examples of this. The second and third examples arise from random matrix theory, and have logarithmic interaction potentials.

• If $\Psi \in C_0^3(\mathbb{R}^d)$ or hard-core potential of the form $\Psi(x) = 1_U(x)$, where $U = \{|x| \le r\}$, and Ψ is a Ruelle-class interaction potential, then Lang [37, 38], Fritz [15], Tanemura [70], and others proved the pathwise uniqueness of stationary solutions for the associated grand canonical Gibbs measures (see Section 5.8).

• For sine$_\beta$ random point field $\mu_{\mathrm{sine},\beta}$ with $\beta \ge 1$, Tsai [76] proved the pathwise uniqueness of solutions and the existence of strong solutions for $\mu_{\mathrm{sine},\beta}$-a.s. s. In [47], the existence of solutions whose unlabeled dynamics are $\mu_{\mathrm{sine},\beta}$-reversible was proved for $\beta = 1, 2, 4$. Combining these, we see that the assumptions in Theorem 5.15 are fulfilled for $\mu_{\mathrm{sine},\beta}$ with $\beta = 1, 2, 4$ (see Section 5.8).

• In [53, Theorem 5.3 (2)], it was proved that the following uniqueness of solutions holds, which is enough for Theorem 5.15. We shall use this in Section 5.8.

Let $\mathfrak{l}$, S_0, and $\mathsf{P}_{\mathsf{x}}^{\mathsf{upr}}$ be as Lemma 5.11. Let

$$\mathsf{P}_{\mu}^{\mathsf{upr}} = \int_{\mathsf{S}} \mathsf{P}_{\mathsf{x}}^{\mathsf{upr}} \mu(d\mathsf{x}). \tag{5.55}$$

For $\mathsf{P}_{\mu}^{\mathsf{upr}}$-a.s. X with label $\mathfrak{l}$, we set $\mathsf{X}^{m*} = \{\mathsf{X}_t^{m*}\}$ is such that $\mathsf{X}_t^{m*} = \sum_{j>m} \delta_{X_t^j}$. Note that X^{m*} is determined by $m \in \mathbb{N}$, X, and $\mathfrak{l}$. Let $\mathsf{S}_{\mathrm{sde}}$ be a subset of S such that b is well defined on $\{(s, \mathsf{y}); \delta_s + \mathsf{y} \in \mathsf{S}_{\mathrm{sde}}\}$. We set

$$\mathbf{S}_{\mathrm{sde}}^m(t, \mathsf{X}) = \{\mathbf{s}_m \in S^m \, ; \, \mathfrak{u}(\mathbf{s}_m) + \mathsf{X}_t^{m*} \in \mathsf{S}_{\mathrm{sde}}\},$$

where $\mathfrak{u}(\mathbf{s}_m) = \sum_i^m \delta_{s_i}$ for $\mathbf{s}_m = (s_i)_{i=1}^m$. Let $\mathsf{H} \subset \mathsf{S}_{\mathrm{sde}}$ such that $\mu(\mathsf{H}) = 1$. We take $\mathsf{H} \subset \mathsf{S}_0 \subset \mathsf{S}_{\mathrm{sde}}$. Let $\mathbf{H} = \{\mathbf{s} \in S^{\mathbb{N}}; \, \mathfrak{u}(\mathbf{s}) \in \mathsf{H}\}$ and set

$$\mathbf{H}^m = \{(s_i)_{i=1}^m \in S^m; \, \mathbf{s} \in \mathbf{H}\}.$$

We now introduce the SDE of $\mathbf{Y}^m = (Y^{m,i})_{i=1}^m$ for each $m \in \mathbb{N}$ such that

$$dY_t^{m,i} = dB_t^i + \mathsf{b}(Y_t^{m,i}, \mathsf{Y}_t^{m,\diamond i} + \mathsf{X}_t^{m*})dt \quad \text{ for } 1 \le i \le m, \tag{5.56}$$

$$\mathbf{Y}_t^m \in \mathbf{S}_{\mathrm{sde}}^m(t, \mathsf{X}) \text{ for all } t, \tag{5.57}$$

$$\mathbf{Y}_0^m = \mathbf{y}_m \in \mathbf{H}^m, \tag{5.58}$$

where $\mathsf{Y}_t^{m,\diamond i} = \sum_{j \ne i}^m \delta_{Y_t^{m,j}}$. By definition the drift coefficient $b^{m,i}$ in (5.56) is time-inhomogeneous and is given by

$$b^{m,i}(\mathbf{y}, t) = \mathsf{b}(y_i, \sum_{j=1, \, j \ne i}^m \delta_{y_j} + \mathsf{X}_t^{m*}),$$

where $\mathbf{y} = (y_1, \ldots, y_m) \in S^m$. We emphasize the importance of (5.57). The function b is not defined on the whole space $S \times \mathsf{S}$. Hence we have to restrict the state space of the associated unlabeled dynamics as $\mathsf{S}_{\mathrm{sde}}$. The set H is the totality of the initial starting points, which are not necessarily equal to $\mathsf{S}_{\mathrm{sde}}$. Following [53], we introduce:

(IFC) There exists $(\mathsf{H}, \mathsf{S}_{\mathrm{sde}})$ such that SDE (5.56) and (5.57) has a pathwise unique strong solution for each $\mathbf{y}_m \in \mathbf{H}^m$ for each $m \in \mathbb{N}$ and for $\mathsf{P}_\mu^{\mathsf{upr}}$-a.s. X.

(TT) The tail σ-field $\mathcal{T}(\mathsf{S})$ is μ-trivial, that is, $\mu(A) \in \{0, 1\}$ for all $A \in \mathcal{T}(\mathsf{S})$.

Feasible sufficient conditions for **(IFC)** were given in [53, Section 9.3]. The importance of **(IFC)** is that it yields the pathwise uniqueness of solutions of ISDE (5.18)–(5.19) together with **(μ-AC)**, **(NBJ)**, and **(TT)** [53, Theorem 5.3 (2)]. All determinantal random point fields are tail trivial [50]. Hence **(TT)** is satisfied for sine_2, Airy_2, Bessel_2, and Ginibre random point fields. We now quote a result from [53].

Lemma 5.16 ([53, Theorem 5.3 (2)])**.** Suppose that for μ-a.s. x, ISDE (5.18)–(5.19) has solutions with conditions **(μ-AC)** and **(NBJ)** and **(IFC)**. Assume that **(TT)** holds. Then these solutions are pathwise unique for μ-a.s. x. That is, for μ-a.s. x, if there exist two such solutions $\mathbf{X}$ and $\mathbf{X}'$ defined on the same probability space with $\mathbf{X}_0 = \mathbf{X}'_0 = \mathfrak{l}(\mathsf{x})$, then $P(\mathbf{X}_t = \mathbf{X}'_t \text{ for all } t) = 1$.

Corollary 5.17. Assume **(A1)**–**(A6)**. Assume **(IFC)** and **(TT)**. Then (5.54) holds.

We next consider the case that μ is not tail trivial. We recall the decomposition of μ into μ^{s} given by (5.44)–(5.45). Then it is known that, if **(A1)** is satisfied, then μ^{s} is tail trivial for μ-a.s. s [53, Lemma 13.2].

Lemma 5.18 ([53, Theorem 5.4])**.** (1) Assume **(A1)**–**(A6)**. Then for μ-a.s. s, ISDE (5.18)–(5.19) has solutions for μ^{s}-a.s. x satisfying conditions **(μ^{s}-AC)** and **(NBJ)**.
(2) Suppose that for μ-a.s. s, ISDE (5.18)–(5.19) has solutions for μ^{s}-a.s. x satisfying **(μ^{s}-AC)**, **(NBJ)**, and **(IFC)**. Then these solutions are pathwise unique. That is, for μ-a.s. s, if there exist two such solutions $\mathbf{X}$ and $\mathbf{X}'$ defined on the same probability space with $\mathbf{X}_0 = \mathbf{X}'_0 = \mathfrak{l}(\mathsf{x})$ for μ^{s}-a.s. initial starting points x, then $P(\mathbf{X}_t = \mathbf{X}'_t \text{ for all } t) = 1$ for μ^{s}-a.s. x.

In [53], it was proved that solutions of ISDE in Lemma 5.18 (2) satisfy **(IFC)** if coefficients of ISDE comes from interaction potentials which are smooth outside the origin. See Lemma 9.7 and Section 10 in [53] for details. We then deduce that, even if μ is not tail trivial, (5.54) holds for μ^{s} such that μ^{s} satisfies the conditions mentioned in Lemma 5.18.

5.4 Proof of Theorem 5.14

In this section we prove Theorem 5.14. Let μ^{s} be as in (5.44).

Lemma 5.19. Assume that μ satisfies **(A1)**–**(A6)**. Then μ^{s} satisfies **(A1)**–**(A6)** for μ-a.s. s.

Proof. This lemma follows from disintegration of μ on μ^{s}, and also the disintegration of their correlation functions and density functions, and Fubini's theorem. □

Let $\mathbf{X}^{R\mathsf{s}} = (X^{R\mathsf{s},i})_{i=1}^{\infty}$ be the labeled diffusion process starting at $\mathfrak{l}(\mathsf{x})$ whose unlabeled process is associated with the Dirichlet form $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}})$ introduced in Section 5.2. Recall that $\mathsf{x}(S_R)$ equals the number of particles in S_R. To clarify the dependence on R

and s, we write $\mathbf{X}^{R\mathsf{s}} = (X^{R\mathsf{s},i})_{i=1}^{\infty}$ instead of $\mathbf{X} = (X^i)_{i=1}^{\infty}$. Suppose that $\mathsf{x}(S_R) \geq m$. We set the m-labeled process $\mathbf{X}^{R\mathsf{s},[m]}$ such that

$$\mathbf{X}_t^{R\mathsf{s},[m]} = (X_t^{R\mathsf{s},1}, X_t^{R\mathsf{s},2}, \dots, X_t^{R\mathsf{s},m}, \sum_{j=m+1}^{\infty} \delta_{X_t^{R\mathsf{s},j}}), \tag{5.59}$$

where we freeze particles outside S_R. Hence, $X_t^{R\mathsf{s},i} = X_0^{R\mathsf{s},i}$ for all t if $i > \mathsf{x}(S_R)$. From (5.59) we have consistency such that, if we denote by $X_t^{R\mathsf{s},[m],i}$ the i-th component of $\mathbf{X}_t^{R\mathsf{s},[m]}$ from the beginning for $1 \leq i \leq m$ to clarify the dependence on m, then

$$X_t^{R\mathsf{s},[m],i} = X_t^{R\mathsf{s},[m+1],i} = X_t^{R\mathsf{s},i} \quad (i = 1, 2, \dots, m).$$

It is known [46] that $\mathbf{X}^{R\mathsf{s},[m]}$ is the diffusion process associated with the Dirichlet form

$$\mathcal{E}^{R\mathsf{s},[m]}(f,g) = \int_{S_R^m \times \mathsf{S}} \{\frac{1}{2} \sum_{i=1}^{m} \nabla_i f \cdot \nabla_i g + \mathbb{D}_R[f,g]\}(\mathbf{x},\mathsf{s}) \mu^{R\mathsf{s},[m]}(d\mathbf{x}d\mathsf{s})$$

on $L^2(S^m \times \mathsf{S}, \mu^{R\mathsf{s},[m]})$, where the domain $\mathcal{D}^{R\mathsf{s},[m]}$ is taken as the closure of

$$\Big\{ f \in C_0^{\infty}(S^m) \otimes \mathcal{D}_{\circ};\, \mathcal{E}^{R\mathsf{s},[m]}(f,f) < \infty \Big\} \cap L^2(S^m \times \mathsf{S}, \mu^{R\mathsf{s},[m]}).$$

We set $f_i(\mathbf{x},\mathsf{s}) = x_i \otimes 1$. We can thus write for $1 \leq i \leq m$

$$X_t^{R\mathsf{s},i} - X_0^{R\mathsf{s},i} = f_i(\mathbf{X}_t^{R\mathsf{s},[m]}) - f_i(\mathbf{X}_0^{R\mathsf{s},[m]}) =: A_t^{[f_i],[m]}.$$

Because the coordinate function $x_i = x_i \otimes 1$ belongs to $\mathcal{D}^{R\mathsf{s},[m]}$, $A^{[f_i],[m]}$ is an additive functional of the m-labeled diffusion $\mathbf{X}^{R\mathsf{s},[m]}$ (see [16] for additive functional). We remark here that the m-point correlation function of $\mu^{R\mathsf{s}}$ vanishes outside S_R.

Applying the Fukushima decomposition to f_i, the additive functional $A_t^{[f_i],[m]}$ can be decomposed as a sum of a unique continuous local martingale additive functional $M^{R\mathsf{s},i}$ and an additive functional of zero energy $N^{R\mathsf{s},i}$:

$$X_t^{R\mathsf{s},i} - X_0^{R\mathsf{s},i} = M_t^{R\mathsf{s},i} + N_t^{R\mathsf{s},i}.$$

We refer to [16, Theorem 5.2.2] for the Fukushima decomposition.

We recall another decomposition of $A_t^{[f_i],[m]}$ called the Lyons–Zheng decomposition [16, Theorem 5.7.1]. Let $r_T : C([0,T];S) \to C([0,T];S)$ be such that $r_T(X)_t = X_{T-t}$. Suppose that the distribution of $\mathbf{X}_0^{R\mathsf{s},[m]}$ is $\mu^{R\mathsf{s},[m]}$, or more generally, absolutely continuous with respect to $\mu^{R\mathsf{s},[m]}$. Then from the Lyons–Zheng decomposition we obtain

$$X_t^{R\mathsf{s},i} - X_0^{R\mathsf{s},i} = \frac{1}{2} M_t^{R\mathsf{s},i} + \frac{1}{2}(M_{T-t}^{R\mathsf{s},i}(r_T) - M_T^{R\mathsf{s},i}(r_T)) \quad \text{a.s.} \tag{5.60}$$

From (5.28) and $f_i(\mathbf{x},\mathsf{s}) = x_i \otimes 1$ we see that $M^{R\mathsf{s},i} = B^i$ for $1 \leq i \leq \mathsf{x}(S_R)$, and hence (5.60) becomes a simple form. That is, for $1 \leq i \leq \mathsf{x}(S_R)$

$$X_t^{R\mathsf{s},i} - X_0^{R\mathsf{s},i} = \frac{1}{2} B_t^i + \frac{1}{2}(B_{T-t}^i(r_T) - B_T^i(r_T)). \tag{5.61}$$

For $\mathsf{x}(S_R) < i < \infty$ we have $X_t^{R\mathsf{s},i} = X_0^{R\mathsf{s},i} = \mathfrak{l}^i(\mathsf{x})$ by definition. Thus (5.61) is enough for our purpose. The decomposition (5.61) will be the main tool in this section.

We set the maximal module variable $\overline{\mathbf{X}}^{R\mathsf{s},m}$ of the first m-particles by

$$\overline{\mathbf{X}}^{R\mathsf{s},m} = \max_{1\le i\le m} \sup_{t\in[0,T]} |X_t^{R\mathsf{s},i}|. \tag{5.62}$$

Throughout this section we fix $T \in \mathbb{N}$. From (5.61) and (5.62) we obtain

Lemma 5.20. Assume that the distribution of $\mathbf{X}_0^{R\mathsf{s}}$ is $\mu^{R\mathsf{s}} \circ \mathfrak{l}^{-1}$. Then there exists a positive constant c_{33} such that for $0 \le t, u \le T$

$$\sup_{R\in\mathbb{N}} \sum_{i=1}^{m} E[|X_t^{R\mathsf{s},i} - X_u^{R\mathsf{s},i}|^4] \le c_{33} m |t-u|^2. \tag{5.63}$$

Furthermore, for each $m \in \mathbb{N}$

$$\lim_{a\to\infty} \liminf_{R\to\infty} P(\overline{\mathbf{X}}^{R\mathsf{s},m} \le a) = 1, \tag{5.64}$$

and for each $r \in \mathbb{N}$

$$\lim_{\iota\to\infty} \inf_{R\in\mathbb{N}} P(I_{r,T}(\mathbf{X}^{R\mathsf{s}}) \le \iota) = 1, \tag{5.65}$$

where $I_{r,T}$ is defined by (5.22).

Proof. From (5.61), we obtain

$$2|X_t^{R\mathsf{s},i} - X_0^{R\mathsf{s},i}| \le |B_t^i| + |B_{T-t}^i(r_T) - B_T^i(r_T)| \quad \text{a.s.} \tag{5.66}$$

From (5.66) we easily obtain (5.63).

Recall that $\mathfrak{l}(\mathsf{x}) = (\mathfrak{l}^i(\mathsf{x}))_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ is a label. From (5.46) we obtain for $A \in \mathcal{B}(S^{\mathbb{N}})$

$$\lim_{R\to\infty} \mu^{R\mathsf{s}} \circ \mathfrak{l}^{-1}(A) = \mu^{\mathsf{s}} \circ \mathfrak{l}^{-1}(A). \tag{5.67}$$

Equation (5.64) follows straightforwardly from (5.66) and (5.67).

We deduce from (5.66)

$$\begin{aligned}
&P\Big(\inf_{t\in[0,T]} |X_t^{R\mathsf{s},i}| \le r\Big) \le P\Big(|X_0^{R\mathsf{s},i}| - r \le \sup_{t\in[0,T]} |X_t^{R\mathsf{s},i} - X_0^{R\mathsf{s},i}|\Big) \\
\le& P\Big(2\{|X_0^{R\mathsf{s},i}| - r\} \le \sup_{t\in[0,T]} \{|B_t^i| + |B_{T-t}^i(r_T) - B_T^i(r_T)|\}\Big) \\
\le& P\Big(|X_0^{R\mathsf{s},i}| - r \le \sup_{t\in[0,T]} |B_t^i|\Big) + P\Big(|X_0^{R\mathsf{s},i}| - r \le \sup_{t\in[0,T]} |B_{T-t}^i(r_T) - B_T^i(r_T)|\Big) \\
=& 2P\Big(|\mathfrak{l}^i(\mathsf{x})| - r \le \sup_{t\in[0,T]} |B_t^i|\Big) \\
\le& 4d \int_S \mathrm{Erf}(\frac{|x|-r}{\sqrt{T}}) \mu \circ (\mathfrak{l}^i)^{-1}(dx).
\end{aligned} \tag{5.68}$$

Then we deduce from (5.22) and (5.68) that

$$\sup_{R\in\mathbb{N}} P\Big(I_{r,T}(\mathbf{X}^{R\mathsf{s}}) \ge \iota\Big) \le \sum_{i>\iota}^{\infty} \sup_{R\in\mathbb{N}} P\Big(\inf_{t\in[0,T]} |X_t^{R\mathsf{s},i}| \le r\Big) \tag{5.69}$$
$$\le 4d \sum_{i>\iota}^{\infty} \int_S \mathrm{Erf}(\frac{|x|-r}{\sqrt{T}})\mu\circ(\mathfrak{l}^i)^{-1}(dx).$$

From Lemma 5.9 we deduce

$$\sum_{i=1}^{\infty} \int_S \mathrm{Erf}(\frac{|x|-r}{\sqrt{T}})\mu\circ(\mathfrak{l}^i)^{-1}(dx) = \int_S \mathrm{Erf}(\frac{|x|-r}{\sqrt{T}})\rho^1(x)dx < \infty. \tag{5.70}$$

From (5.69)–(5.70) we obtain (5.65). □

From the conditions above we have the following lemma.

Lemma 5.21. Make the same assumption as Lemma 5.20. Then for each $i, a, R \in \mathbb{N}$ such that $i \le m$

$$P\big(L_T^{R\mathsf{s},i} = 0\,;\, \overline{\mathbf{X}}^{R\mathsf{s},m} \le a\big) = 1 \quad \text{for } a < R. \tag{5.71}$$

Proof. Recall that by (5.29) we have

$$L_t^{R\mathsf{s},i} = \int_0^t \mathbf{1}_{\partial S_R}(X_u^{R\mathsf{s},i})dL_u^{R\mathsf{s},i}.$$

Then $L^{R,i} = \{L_t^{R\mathsf{s},i}\}$ is non-negative and increases only when $\{X_t^{R\mathsf{s},i}\}$ touches the boundary $\partial S_R = \{|x| = R\}$. Hence $L_T^{R\mathsf{s},i} = 0$ for all $a < R$ on $\{\overline{\mathbf{X}}^{R\mathsf{s},m} \le a\}$, which implies (5.71). □

Let $\mathsf{b}_{r,s,\mathsf{p}}$ be as in **(A6)** and put

$$\mathsf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},i}(t) = \int_0^t \mathsf{b}_{r,s,\mathsf{p}}(X_u^{R\mathsf{s},i}, \mathsf{X}_u^{R\mathsf{s},\diamond i})du. \tag{5.72}$$

We set for $m \in \mathbb{N}$

$$\mathbf{X}^{R\mathsf{s},m} = (X^{R\mathsf{s},i})_{i=1}^m,\ \mathbf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},m} = (\mathsf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},i})_{i=1}^m, \text{ and } \mathbf{L}^{R\mathsf{s},m} = (L^{R\mathsf{s},i})_{i=1}^m.$$

Let $\mathbf{X}^{R\mathsf{s}} = (X^{R\mathsf{s},i})_{i=1}^{\infty}$ and consider random variables

$$\mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s},m} = (\mathbf{X}^{R\mathsf{s},m}, \mathbf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},m}, \mathbf{L}^{R\mathsf{s},m}), \tag{5.73}$$
$$\mathbb{W}_{r,s,\mathsf{p}}^{R\mathsf{s}} = \big((\mathbf{X}^{R\mathsf{s},n}, \mathbf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},n}, \mathbf{L}^{R\mathsf{s},n})_{n=1}^{\infty}, \mathbf{X}^{R\mathsf{s}}\big). \tag{5.74}$$

By construction, $\mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s},m}$ and $\mathbb{W}_{r,s,\mathsf{p}}^{R\mathsf{s}}$ are functionals of $\mathbf{X}^{R\mathsf{s}}$. Hence we can regard $\mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s},m}$ and $\mathbb{W}_{r,s,\mathsf{p}}^{R\mathsf{s}}$ are defined on a common probability space. Let

$$\sigma_a^{R\mathsf{s},m} = \inf\{0 \le t \le T;\ \max_{1\le i\le m} \big|X_t^{R\mathsf{s},i}\big| \ge a\}.$$

Let $\Xi^m = C([0,T];S^m) \times C([0,T];\mathbb{R}^{dm})^2$ and $\Xi_0^m = C([0,T];S^m) \times \mathcal{BV} \times \mathcal{C}_+$, where

$$\mathcal{BV} = \{\eta = (\eta^i)_{i=1}^m \in C([0,T];\mathbb{R}^{dm});\ \eta \text{ is bounded variation}\},$$
$$\mathcal{C}_+ = \{\zeta = (\zeta^i)_{i=1}^m \in C([0,T];\mathbb{R}^{dm});\ \zeta \text{ is non-decreasing}\}.$$

We say a sequence of random variables is tight if for any subsequence we can choose a subsequence that is convergent in law. We also remark that tightness in $C([0,T];S^{\mathbb{N}})$ for all $T \in \mathbb{N}$ is equivalent to tightness in $C([0,\infty);S^{\mathbb{N}})$ because we equip $C([0,\infty);S^{\mathbb{N}})$ with a compact uniform norm.

Lemma 5.22. Make the same assumption as Lemma 5.20. Then for μ-a.s. s, the following hold for all $T \in \mathbb{N}$.
(1) $\{\mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s},m}(\cdot \wedge \sigma_a^{R\mathsf{s},m})\}_{r,s,\mathsf{p},R\in\mathbb{N}}$ is tight in $C([0,T];\Xi^m)$ for each $m, a \in \mathbb{N}$.
(2) $\{\mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s},m}\}_{r,s,\mathsf{p},R\in\mathbb{N}}$ is tight in $C([0,T];\Xi^m)$ for each $m \in \mathbb{N}$.
(3) $\{\mathbb{W}_{r,s,\mathsf{p}}^{R\mathsf{s}}\}_{r,s,\mathsf{p},R\in\mathbb{N}}$ is tight in $\prod_{n=1}^\infty C([0,T];\Xi^n) \times C([0,T];S^{\mathbb{N}})$.

Proof. We remark that tightness of $\mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s},m}(\cdot \wedge \sigma_a^{R\mathsf{s},m})$ follows from that of each component $\mathbf{X}^{R\mathsf{s},m}(\cdot\wedge\sigma_a^{R\mathsf{s},m})$, $\mathbf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},m}(\cdot\wedge\sigma_a^{R\mathsf{s},m})$, and $\mathbf{L}^{R\mathsf{s},m}(\cdot\wedge\sigma_a^{R\mathsf{s},m})$. Tightness of $\{\mathbf{X}^{R\mathsf{s},m}(\cdot\wedge\sigma_a^{R\mathsf{s},m})\}_{R\in\mathbb{N}}$ follows from Lemma 5.20. Tightness of $\{\mathbf{L}^{R\mathsf{s},m}(\cdot \wedge \sigma_a^{R\mathsf{s},m})\}_{R\in\mathbb{N}}$ follows from Lemma 5.21.

Recall that $\mathsf{b}_{r,s,\mathsf{p}} \in C_b(S \times \mathsf{S})$ by **(A6)**. Then tightness of $\{\mathbf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},m}(\cdot \wedge \sigma_a^{R\mathsf{s},m})\}_{r,s,\mathsf{p},R\in\mathbb{N}}$ follows from (5.72) with a straightforward calculation. We thus obtain (1).

In general, a family of probability measures m_a in a Polish space is compact under the topology of weak convergence if and only if for any $\epsilon > 0$ there exists a compact set K such that $\inf_a m_a(K) \geq 1 - \epsilon$. Using this we conclude (2) from (1) combined with (5.64).

With the same reason as the proof of (1), we obtain (3) from (1) and (2). $\square$

Lemma 5.21 and Lemma 5.22 imply that for any subsequence of $\{\mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s}}(\cdot\wedge\sigma_a^{R\mathsf{s},m})\}_{r,s,\mathsf{p},R\in\mathbb{N}}$, $\{\mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s}}\}_{r,s,\mathsf{p},R\in\mathbb{N}}$, and $\{\mathbb{W}_{r,s,\mathsf{p}}^{R\mathsf{s}}\}_{r,s,\mathsf{p},R\in\mathbb{N}}$ there exist convergent-in-law subsequences, denoted by the same symbols, such that the following convergence in law holds:

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty}\lim_{R\to\infty} \mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s},m}(\cdot \wedge \sigma_a^{R\mathsf{s},m}) = \big(\mathbf{X}_a^{\mathsf{s},m}, \mathbf{B}_a^{\mathsf{s},m}, 0, \mathbf{X}_a^{\mathsf{s}}\big) \quad \text{for each } m \in \mathbb{N}, \tag{5.75}$$

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty}\lim_{R\to\infty} \mathbb{V}_{r,s,\mathsf{p}}^{R\mathsf{s},m} = \big(\mathbf{X}^{\mathsf{s},m}, \mathbf{B}^{\mathsf{s},m}, 0, \mathbf{X}^{\mathsf{s}}\big) \quad \text{for each } m \in \mathbb{N}, \tag{5.76}$$

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty}\lim_{R\to\infty} \mathbb{W}_{r,s,\mathsf{p}}^{R\mathsf{s}} = \big((\mathbf{X}^{\mathsf{s},n}, \mathbf{B}^{\mathsf{s},n}, 0)_{n=1}^\infty, \mathbf{X}^{\mathsf{s}}\big). \tag{5.77}$$

Here the subscript a in the right hand side of (5.75) denotes the dependence on a. We note that the convergence $\lim_{R\to\infty} \mathbf{L}^{R\mathsf{s},m}(\cdot \wedge \sigma_a^{R\mathsf{s},m}) = 0$ follows from Lemma 5.21. From Lemma 5.22 (3), we have consistency:

$$\mathbf{X}^{\mathsf{s},m} = (X^{\mathsf{s},1}, \dots, X^{\mathsf{s},m}).$$

Here $X^{\mathsf{s},i}$ in the right hand side is the i-th component of $\mathbf{X}^{\mathsf{s}} = (X^{\mathsf{s},i})_{i=1}^\infty$. The same holds for $\mathbf{B}^{\mathsf{s},n}$ and we write $\mathbf{B}^{\mathsf{s},n} = (\mathsf{B}^{\mathsf{s},i})_{i=1}^n$ This is the reason why we extend the state space in (3) of Lemma 5.22 from that in (1) and (2).

We next check consistency in a in the limits in (5.75) and (5.76). Without loss of generality, we can assume

$$P(\{\overline{\mathbf{X}}^{\mathsf{s},m} = a\}) = 0. \tag{5.78}$$

Indeed, if not, we can choose an increasing sequence $\{n(a)\}_{a\in\mathbb{N}}$ of positive numbers diverges to infinity such that $P(\{\overline{\mathbf{X}}^{R\mathsf{s},m} = n(a)\}) = 0$ instead of $\{a\}_{a\in\mathbb{N}}$. Let

$$\sigma_a^{\mathsf{s},m} = \inf\{0 \le t \le T; \max_{1\le i\le m} \left|X_t^{\mathsf{s},i}\right| \ge a\}.$$

Then from (5.78) we deduce that the discontinuity points of the stopping time $\sigma_a^{\mathsf{s},m}$ is probability zero. Hence from convergence in (5.75) and (5.76) we have

$$\big(\mathbf{X}_a^{\mathsf{s},m}, \mathbf{B}_a^{\mathsf{s},m}, 0, \mathbf{X}_a^{\mathsf{s}}\big)(\,\cdot\,) = \big(\mathbf{X}^{\mathsf{s},m}, \mathbf{B}^{\mathsf{s},m}, 0, \mathbf{X}^{\mathsf{s}}\big)(\,\cdot \wedge \sigma_a^{\mathsf{s},m}). \tag{5.79}$$

We set $\mathsf{X}_t^{R\mathsf{s},\diamond i} = \sum_{j\ne i} \delta_{X_t^{R\mathsf{s},j}}$ for $\mathbf{X}^{R\mathsf{s}} = (X_t^{R\mathsf{s},i})_{i=1}^{\infty}$. Using reversibility of diffusions, we obtain the following dynamic estimates from the static condition **(A6)**.

Lemma 5.23. Make the same assumption as Lemma 5.20. Furthermore, we assume **(A6)**. Then for μ-a.s. s and for each $i \in \mathbb{N}$

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty}\sup_{R\ge r+s+1} E\Big[\int_0^T 1_{S_r}(X_t^{R\mathsf{s},i})\big|\{\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b}\}(X_t^{R\mathsf{s},i}, \mathsf{X}_t^{R\mathsf{s},\diamond i})\big|dt\Big] = 0, \tag{5.80}$$

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty} E\Big[\int_0^T 1_{S_r}(X^{\mathsf{s},i})\big|\{\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b}\}(X_t^{\mathsf{s},i}, \mathsf{X}_t^{\mathsf{s},\diamond i})\big|dt\Big] = 0. \tag{5.81}$$

Proof. Let $\mathsf{X}^{R\mathsf{s}}$ be the unlabeled diffusion such that $\mathsf{X}_t^{R\mathsf{s}} = \sum_{i=1}^{\infty} \delta_{X_t^{R\mathsf{s},i}}$. Because the diffusion $\mathsf{X}^{R\mathsf{s}}$ is associated with the Dirichlet form $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}})$ introduced in Section 5.2, $\mathsf{X}^{R\mathsf{s}}$ is $\mu^{R\mathsf{s}}$-reversible. Then because of reversibility we have for all t

$$\begin{aligned} & E\big[1_{S_r}(X_t^{R\mathsf{s},i})\big|\{\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b}\}(X_t^{R\mathsf{s},i}, \mathsf{X}_t^{R\mathsf{s},\diamond i})\big|\big] \\ \le & E\big[\sum_{i=1}^{\infty} 1_{S_r}(X_t^{R\mathsf{s},i})\big|\{\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b}\}(X_t^{R\mathsf{s},i}, \mathsf{X}_t^{R\mathsf{s},\diamond i})\big|\big] \\ = & E\big[\sum_{i=1}^{\infty} 1_{S_r}(X_0^{R\mathsf{s},i})\big|\{\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b}\}(X_0^{R\mathsf{s},i}, \mathsf{X}_0^{R\mathsf{s},\diamond i})\big|\big] \\ = & \int_{\mathsf{S}} \sum_{x_i\in S_r} 1_{S_r}(x_i)\big|\{\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b}\}(x_i, \sum_{j\ne i}^{\infty} \delta_{x_j})\big|\mu^{R\mathsf{s}}(d\mathsf{x}), \end{aligned} \tag{5.82}$$

where we set $\mathsf{x} = \sum_i \delta_{x_i} \in \mathsf{S}$. Then we obtain (5.80) from (5.47) and (5.82).

Recall that $\mathsf{b} \in L^1_{\mathrm{loc}}(S \times \mathsf{S},\, \mu^{[1]})$. Then $\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b} \in L^1_{\mathrm{loc}}(S \times \mathsf{S},\, \mu^{[1]})$. Hence

$$\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b} \in L^1_{\mathrm{loc}}(S \times \mathsf{S},\, \mu^{\mathsf{s},[1]}) \text{ for } \mu\text{-a.s. } \mathsf{s}.$$

From this and martingale convergence theorem, we obtain from (5.47) that

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty} \|\, \mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b} \,\|_{L^1_{\mathrm{loc}}(S\times\mathsf{S},\, \mu^{\mathsf{s},[1]})} = 0 \quad \text{ for } \mu\text{-a.s. } \mathsf{s}.$$

Then we can prove (5.81) in the same way as (5.80). $\square$

Proof of Theorem 5.14. For $\psi \in C_0^\infty(S^m)$, let $F : \Xi_0^m \to C([0,T];\mathbb{R})$ such that

$$F(\xi,\eta,\zeta)(t) = \psi(\xi(t)) - \psi(\xi(0)) - \int_0^t \sum_{j=1}^m \nabla_j \psi(\xi(u)) \cdot d\eta^j(u) \tag{5.83}$$
$$- \int_0^t \sum_{j=1}^m \nabla_j \psi(\xi(u)) \cdot \zeta^j(du) - \int_0^t \sum_{j=1}^m \frac{1}{2} \triangle_j \psi(\xi(u)) du.$$

From Itô-Tanaka formula, (5.28)–(5.30), and $\mathsf{d}^\mu = 2\mathsf{b}$, we deduce that for each $m \in \mathbb{N}$

$$\sup_{R \ge r+s+1} E\Big[\sup_{0\le t\le T} \Big| F(\mathbf{X}^{R\mathsf{s},m}, \mathbf{B}^{R\mathsf{s},m}_{r,s,\mathsf{p}}, \mathbf{L}^{R\mathsf{s},m})(t) - \sum_{j=1}^m \int_0^t \nabla_j \psi(\mathbf{X}^{R\mathsf{s},m}_u) dB^j_u \Big|\Big] \tag{5.84}$$
$$\le c_{34}(\mathsf{s}, m, r, s, \mathsf{p}) \Big\{ \sum_{j=1}^m \sup_{x \in S^m} |\nabla_j \psi(x)| \Big\},$$

where we set

$$c_{34}(\mathsf{s}, m, r, s, \mathsf{p}) = \sup_{R \ge r+s+1} \sum_{i=1}^m E\Big[\int_0^T 1_{S_r}(X^{R\mathsf{s},i}_t) \big| \{\mathsf{b}_{r,s,\mathsf{p}} - \mathsf{b}\}(X^{R\mathsf{s},i}_t, \mathsf{X}^{R\mathsf{s},\diamond i}_t) \big| dt \Big]. \tag{5.85}$$

We deduce from (5.80) and (5.85) that c_{34} satisfy for μ-a.s. s and for each $m \in \mathbb{N}$

$$\lim_{r\to\infty} \lim_{s\to\infty} \lim_{\mathsf{p}\to\infty} c_{34}(\mathsf{s}, m, r, s, \mathsf{p}) = 0. \tag{5.86}$$

Take $\psi = \psi_Q \in C_0^\infty(S^m)$ such that $\psi_Q(x_1, \ldots, x_m) = x_i$ for $\{|x_i| \le Q\}$. Let $a, Q, R \in \mathbb{N}$ be such that $a < Q, R$. Recall that $\mathbf{L}^{R\mathsf{s},m}_t = 0$ by Lemma 5.21. Then we deduce from (5.83) and Itô-Tanaka formula that

$$F(\mathbf{X}^{R\mathsf{s},m}, \mathbf{B}^{R\mathsf{s},m}_{r,s,\mathsf{p}}, \mathbf{L}^{R\mathsf{s},m})(t \wedge \sigma_a^{R\mathsf{s},m}) - \sum_{j=1}^m \int_0^{t\wedge\sigma_a^{R\mathsf{s},m}} \nabla_j \psi_Q(\mathbf{X}^{R\mathsf{s},m}_u) dB^j_u \tag{5.87}$$
$$= X^{R\mathsf{s},i}(t \wedge \sigma_a^{R\mathsf{s},m}) - X^{R\mathsf{s},i}(0) - \mathsf{B}^{R\mathsf{s},i}_{r,s,\mathsf{p}}(t \wedge \sigma_a^{R\mathsf{s},m}) - B^i_{t\wedge\sigma_a^{R\mathsf{s},m}},$$

where we write $Y_t = Y(t)$ for a stochastic process $Y = \{Y_t\}$. We also remark that $\{B^i\}_{i=1}^\infty$ is $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion taken to be independent of R.

We write $\mathbf{X}^{\mathsf{s},m}_a = (X^{\mathsf{s},i}_a)_{i=1}^\infty$ and $X^{\mathsf{s},i}_a = \{X^{\mathsf{s},i}_{a,t}\}$. We set

$$\lim_{r,s,\mathsf{p},R} = \lim_{r\to\infty} \lim_{s\to\infty} \lim_{\mathsf{p}\to\infty} \lim_{R\to\infty}.$$

We have from (5.75), (5.87), (5.84), and (5.86) that

$$E\Big[\sup_{0\le t\le T} \big| X^{\mathsf{s},i}_{a,t} - X^{\mathsf{s},i}_{a,0} - \mathsf{B}^{\mathsf{s},i}_{a,t} - B^i_{t\wedge\sigma_a^{\mathsf{s},m}} \big|\Big]$$
$$= \lim_{r,s,\mathsf{p},R} E\Big[\sup_{0\le t\le T} \big| X^{R\mathsf{s},i}(t\wedge\sigma_a^{R\mathsf{s},m}) - X^{R\mathsf{s},i}(0) - \mathsf{B}^{R\mathsf{s},i}_{r,s,\mathsf{p}}(t\wedge\sigma_a^{R\mathsf{s},m}) - B^i_{t\wedge\sigma_a^{R\mathsf{s},m}} \big|\Big] \quad \text{by (5.75)}$$
$$= \lim_{r,s,\mathsf{p},R} E\Big[\sup_{0\le t\le T} \big| F(\mathbf{X}^{R\mathsf{s},m}, \mathbf{B}^{R\mathsf{s},m}_{r,s,\mathsf{p}}, \mathbf{L}^{R\mathsf{s},m})(t\wedge\sigma_a^{R\mathsf{s},m}) - \sum_{j=1}^m \int_0^{t\wedge\sigma_a^{R\mathsf{s},m}} \nabla_j \psi_Q(\mathbf{X}^{R\mathsf{s},m}_u) dB^j_u \big|\Big] \quad \text{by (5.87)}$$
$$= 0 \quad \text{by (5.84) and (5.86).}$$

This implies

$$X_{a,t}^{\mathsf{s},i} - X_{a,0}^{\mathsf{s},i} - \mathsf{B}_{a,t}^{\mathsf{s},i} - B_{t\wedge\sigma_a^{\mathsf{s},m}}^{i} = 0 \quad \text{for all } t. \tag{5.88}$$

Then from (5.79) and (5.88) we have for all $a \in \mathbb{N}$

$$X_{t\wedge\sigma_a^{\mathsf{s},m}}^{\mathsf{s},i} - X_{0}^{\mathsf{s},i} - \mathsf{B}_{t\wedge\sigma_a^{\mathsf{s},m}}^{\mathsf{s},i} - B_{t\wedge\sigma_a^{\mathsf{s},m}}^{i} = 0 \quad \text{for all } t. \tag{5.89}$$

From $P(\lim_{a\to\infty}\sigma_a^{\mathsf{s},m} = \infty) = 1$, (5.89) implies

$$X_{t}^{\mathsf{s},i} - X_{0}^{\mathsf{s},i} - \mathsf{B}_{t}^{\mathsf{s},i} - B_{t}^{i} = 0 \quad \text{for all } t. \tag{5.90}$$

So it only remains to calculate the representation of $\mathsf{B}^{\mathsf{s},i}$.

We now recall $\mathbf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},m} = (\mathsf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},i})_{i=1}^{m}$ and $\mathsf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},i}(t) = \int_0^t \mathsf{b}_{r,s,\mathsf{p}}(X_u^{R\mathsf{s},i},\mathsf{X}_u^{R\mathsf{s},\diamond i})du$ by definition. We then deduce from (5.73), (5.76), and (5.81) combined with $\mathsf{b}_{r,s,\mathsf{p}} \in C_b(S\times\mathsf{S})$ that

$$\begin{aligned}
\mathsf{B}_t^{\mathsf{s},i} &= \lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty}\lim_{R\to\infty} \mathsf{B}_{r,s,\mathsf{p}}^{R\mathsf{s},i}(t) && \text{by (5.73) and (5.76)} \\
&= \lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty}\lim_{R\to\infty} \int_0^t \mathsf{b}_{r,s,\mathsf{p}}(X_u^{R\mathsf{s},i},\mathsf{X}_u^{R\mathsf{s},\diamond i})du && \text{by definition} \\
&= \lim_{r\to\infty}\lim_{s\to\infty}\lim_{\mathsf{p}\to\infty} \int_0^t \mathsf{b}_{r,s,\mathsf{p}}(X_u^{\mathsf{s},i},\mathsf{X}_u^{\mathsf{s},\diamond i})du && \text{by } \mathsf{b}_{r,s,\mathsf{p}} \in C_b(S\times\mathsf{S}) \\
&= \lim_{r\to\infty} \int_0^t 1_{S_r}(X_u^{\mathsf{s},i})\mathsf{b}(X_u^{\mathsf{s},i},\mathsf{X}_u^{\mathsf{s},\diamond i})du && \text{by (5.81)} \\
&= \int_0^t \mathsf{b}(X_u^{\mathsf{s},i},\mathsf{X}_u^{\mathsf{s},\diamond i})du && \text{in law.}
\end{aligned} \tag{5.91}$$

Putting (5.90)–(5.91) together yields

$$X_{t}^{\mathsf{s},i} - X_{0}^{\mathsf{s},i} - \int_0^t \mathsf{b}(X_u^{\mathsf{s},i},\mathsf{X}_u^{\mathsf{s},\diamond i})du - B_{t}^{i} = 0.$$

We then complete the proof of Theorem 5.14. □

5.5 Proof of Theorem 5.15

In this section we prove Theorem 5.15. Let $\mu^{R\mathsf{s},[1]}(dxd\mathsf{y}) = \rho^{R\mathsf{s},1}(x)\mu_x^{R\mathsf{s}}(d\mathsf{y})dx$ as (5.25). Let

$$\begin{aligned}
\mathcal{D}_R^{R\mathsf{s},+} = \{f \in L^2(\mathsf{S},\mu^{R\mathsf{s}});&\ \text{there exists } \mathbf{f}_{R\mathsf{s}} \in L^2(S_R\times\mathsf{S},\mu^{R\mathsf{s},[1]})^d \text{ such that} \\
-\int_{S_R\times\mathsf{S}}& f(\delta_x+\mathsf{y})\{\nabla_x\varphi(x,\mathsf{y}) + \mathsf{d}^{\mu}(x,\mathsf{y})\varphi(x,\mathsf{y})\}\mu^{R\mathsf{s},[1]}(dxd\mathsf{y}) \\
&= \int_{S_R\times\mathsf{S}} \mathbf{f}_{R\mathsf{s}}(x,\mathsf{y})\varphi(x,\mathsf{y})\mu^{R\mathsf{s},[1]}(dxd\mathsf{y}) \text{ for } \varphi \in C_0^{\infty}(S_R^{\mathrm{int}})\otimes\mathcal{D}_{\circ}\}.
\end{aligned}$$

Denote $\mathbf{f}_{R\mathsf{s}}$ by $D_{R\mathsf{s},x}f$. We introduce the bilinear form $\mathcal{E}_R^{R\mathsf{s},+}$ with domain $\mathcal{D}_R^{R\mathsf{s},+}$ such that

$$\mathcal{E}_R^{R\mathsf{s},+}(f,g) = \int_{S_R\times\mathsf{S}} \frac{1}{2} D_{R\mathsf{s},x}f\cdot D_{R\mathsf{s},x}g\,\mu^{R\mathsf{s},[1]}(dxd\mathsf{y}), \tag{5.92}$$

$$\mathcal{D}_{R,\circ}^{R\mathsf{s},+} = \{f \in \mathcal{D}_{\circ}\cap L^2(\mathsf{S},\mu^{R\mathsf{s}})\,;\,\mathcal{E}_R^{R\mathsf{s},+}(f,f) < \infty\}. \tag{5.93}$$

Lemma 5.24. For μ-a.s. s, the following hold.
(1) $(\mathcal{E}_R^{R\mathsf{s},+}, \mathcal{D}_{R,\circ}^{R\mathsf{s},+})$ is closable on $L^2(\mathsf{S}, \mu^{R\mathsf{s}})$ for each $R \in \mathbb{N}$.
(2) $(\mathcal{E}_R^{R\mathsf{s},+}, \mathcal{D}_R^{R\mathsf{s},+})$ is the closure of $(\mathcal{E}_R^{R\mathsf{s},+}, \mathcal{D}_{R,\circ}^{R\mathsf{s},+})$ on $L^2(\mathsf{S}, \mu^{R\mathsf{s}})$ for each $R \in \mathbb{N}$.
(3) $(\mathcal{E}_R^{R\mathsf{s},\mathsf{lwr}}, \mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}}) = (\mathcal{E}_R^{R\mathsf{s},+}, \mathcal{D}_R^{R\mathsf{s},+})$ for each $R \in \mathbb{N}$.

Proof. We easily see from **(A1)** that $\mu^{R\mathsf{s},[1]}$ has a labeled density $m^{R\mathsf{s},[1]}(x, y_1, \ldots, y_m)$ with respect to the Lebesgue measure on S_R^{1+m} such that

$$c_{35}^{-1} e^{-\Phi(x) - \sum_{i=1}^m \Phi(y_i) - \sum_{i=1}^m \Psi(x, y_i) - \sum_{i<i'}^m \Psi(y_i, y_{i'})} \le m^{R\mathsf{s},[1]}(x, y_1, \ldots, y_m) \le c_{35} e^{-\Phi(x) - \sum_{i=1}^m \Phi(y_i) - \sum_{i=1}^m \Psi(x, y_i) - \sum_{i<i'}^m \Psi(y_i, y_{i'})}, \tag{5.94}$$

where $\mathsf{s}(S_R) = 1 + m$, $m \in \mathbb{N} \cup \{0\}$, $\pi_R^c(\mathsf{s}) = \sum_{j=m+2}^\infty \delta_{s_j}$ for $\mathfrak{l}(\mathsf{s}) = (s_i)_{i=1}^\infty$. Furthermore, c_{35} is a positive constant depending on (Φ, Ψ), $\pi_R^c(\mathsf{s})$, and m. For μ-a.s. s, $(\mathcal{E}_R^{R\mathsf{s},+}, \mathcal{D}_{R,\circ}^{R\mathsf{s},+})$ can be regarded as a form on $L^2(S_R^{1+m}, m^{R\mathsf{s},[1]} dx dy_1 \cdots dy_m)$. Hence we deduce (1) and (2) from (5.94). Once we regard $(\mathcal{E}_R^{R\mathsf{s},+}, \mathcal{D}_{R,\circ}^{R\mathsf{s},+})$ as a form in finite dimensions as above, (3) is obvious. □

We next introduce the Dirichlet form $(\mathcal{E}_R^+, \mathcal{D}_R^+)$. Let

$$\begin{aligned} \mathcal{D}_R^+ = \{f \in L^2(\mathsf{S}, \mu); &\text{there exists } \mathbf{f}_R \in L^2(S_R \times \mathsf{S}, \mu^{[1]})^d \text{ such that} \\ &- \int_{S_R \times \mathsf{S}} f(\delta_x + \mathsf{y}) \{\nabla_x \varphi(x, \mathsf{y}) + \mathsf{d}^\mu(x, \mathsf{y}) \varphi(x, \mathsf{y})\} \mu^{[1]}(dx d\mathsf{y}) \\ &\quad = \int_{S_R \times \mathsf{S}} \mathbf{f}_R(x, \mathsf{y}) \varphi(x, \mathsf{y}) \mu^{[1]}(dx d\mathsf{y}) \text{ for } \varphi \in C_0^\infty(S_R^{\mathrm{int}}) \otimes \mathcal{D}_\circ\}. \end{aligned} \tag{5.95}$$

Let us denote $\mathbf{f}_R$ by $D_{R,x} f$, and set for $f, g \in \mathcal{D}_R^+$

$$\mathcal{E}_R^+(f, g) = \int_{S_R \times \mathsf{S}} \frac{1}{2} D_{R,x} f \cdot D_{R,x} g \, \mu^{[1]}(dx d\mathsf{y}), \tag{5.96}$$

$$\mathcal{D}_{R,\circ}^+ = \{f \in \mathcal{D}_\circ \cap L^2(\mathsf{S}, \mu) \,;\, \mathcal{E}_R^+(f, f) < \infty\}. \tag{5.97}$$

Lemma 5.25. (1) $(\mathcal{E}_R^+, \mathcal{D}_{R,\circ}^+)$ is closable on $L^2(\mathsf{S}, \mu)$.
(2) $(\mathcal{E}_R^+, \mathcal{D}_R^+)$ is the closure of $(\mathcal{E}_R^+, \mathcal{D}_{R,\circ}^+)$ on $L^2(\mathsf{S}, \mu)$.
(3) $(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}}) = (\mathcal{E}_R^+, \mathcal{D}_R^+)$.

Proof. On account of the disintegration (5.32), we have

$$\mu^{[1]}(dx d\mathsf{y}) = \int_{\mathsf{S}} \mu^{R\mathsf{s},[1]}(dx d\mathsf{y}) \mu(d\mathsf{s}). \tag{5.98}$$

From (5.92)–(5.93) and the Fubini theorem, we see that $\mathcal{D}_{R,\circ}^+ \subset \mathcal{D}_{R,\circ}^{R\mathsf{s},+}$ for μ-a.s. s. From (5.92), (5.96), and (5.98) we deduce for $f, g \in \mathcal{D}_{R,\circ}^+$

$$\mathcal{E}_R^+(f, g) = \int_{\mathsf{S}} \mathcal{E}_R^{R\mathsf{s},+}(f, g) \mu(d\mathsf{s}). \tag{5.99}$$

Hence (1) follows from Lemma 5.24 (1) and the argument in [48, 45-46pp]. These together with the similar argument in Section 5.2 yield (2). We obtain (3) from (5.99) combined with (5.34) and Lemma 5.24 (3). □

Theorem 5.26. $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}}) = (\mathcal{E}^{+}, \mathcal{D}^{+})$.

Proof. Since $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ and $(\mathcal{E}^{+}, \mathcal{D}^{+})$ are the increasing limits of $\{(\mathcal{E}_R^{\mathsf{lwr}}, \mathcal{D}_R^{\mathsf{lwr}})\}$ and $\{(\mathcal{E}_R^{+}, \mathcal{D}_R^{+})\}$ as $R \to \infty$ respectively, we deduce Theorem 5.26 from Lemma 5.25 (3). $\square$

Proof of Theorem 5.15. From Lemma 5.11 and Lemma 5.12 we see that the diffusion $\mathbf{X}^{\mathsf{upr}}$ such that $\mathbf{X}_0^{\mathsf{upr}} = \mathfrak{l}(\mathsf{x})$ associated with the Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ on $L^2(\mathsf{S}, \mu)$ is a solution of (5.18)–(5.19) satisfying **(μ-AC)** and **(NBJ)**.

By Theorem 5.14 the process $\mathbf{X}$ under $P_{\mathfrak{l}(\mathsf{x})}^{\infty}$ is a solution of the ISDE (5.51)–(5.52) satisfying **(μ-AC)** and **(NBJ)**.

For μ-a.s. x, the solutions of ISDE (5.18)–(5.19) satisfying **(μ-AC)** and **(NBJ)** are unique in law by assumption. Hence for μ-a.s. x, $\mathbf{X}^{\mathsf{upr}}$ and $\mathbf{X}$ starting at $\mathfrak{l}(\mathsf{x})$ have the same distribution. Hence the associated semi-group coincides with each other. This together with (5.53) implies $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}}) = (\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$. From Theorem 5.26 we have already obtained $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}}) = (\mathcal{E}^{+}, \mathcal{D}^{+})$. Combining these we complete the proof of Theorem 5.15. $\square$

5.6 Symmetric diffusions for uniformly elliptic differential operators

In this section, we give a remark on a generalization of Theorem 5.14 and Theorem 5.15. For this purpose we introduce a function $\mathsf{a} : S \times \mathsf{S} \to \mathbb{R}^{d^2}$ and assume:

(B1) $\mathsf{a}, \nabla_x \mathsf{a} \in C_b(S \times \mathsf{S})$, $\mathsf{a} = {}^t\mathsf{a}$, and a is uniformly elliptic on $S \times \mathsf{S}$.

For $f, g \in \mathcal{D}_{\circ}$ we set $\mathbb{D}_r^{\mathsf{a},m}[f,g](\mathsf{s}) = 0$ for $\mathsf{s} \notin \mathsf{S}_r^m$ and

$$\mathbb{D}_r^{\mathsf{a},m}[f,g](\mathsf{s}) = \frac{1}{2}\sum_{i=1}^{m} \mathsf{a}(s_i, \mathsf{s}^{\diamond i}) \nabla_{s_i} f_{r,\mathsf{s}}^m(\mathbf{x}_r^m(\mathsf{s})) \cdot \nabla_{s_i} g_{r,\mathsf{s}}^m(\mathbf{x}_r^m(\mathsf{s})) \quad \text{for } \mathsf{s} \in \mathsf{S}_r^m, \tag{5.100}$$

where we set $\mathsf{s}^{\diamond i} = \sum_{j \neq i} \delta_{s_j}$ for $\mathsf{s} = \sum_i \delta_{s_i}$. Moreover, we set

$$\mathbb{D}_r^{\mathsf{a}}[f,g](\mathsf{s}) = \sum_{m=1}^{\infty} \mathbb{D}_r^{\mathsf{a},m}[f,g](\mathsf{s}). \tag{5.101}$$

If we replace the square fields $\mathbb{D}_r^m$ and $\mathbb{D}_r$ in (5.7) and (5.8) with $\mathbb{D}_r^{\mathsf{a},m}$ and $\mathbb{D}_r^{\mathsf{a}}$ in (5.100) and (5.101) and add assumption **(B1)**, then all results in Section 5.3 still hold.

5.7 Construction of cut-off coefficients $\mathsf{b}_{r,s,\mathsf{p}}$

In this section we construct $\mathsf{b}_{r,s,\mathsf{p}}$. For this purpose we prepare cut off functions.

Let $\chi_t \in C_0^{\infty}(\mathbb{R}^d)$ such that $0 \le \chi_t \le 1$ and that

$$\chi_t(x) = \begin{cases} 1 & \text{for } |x| \le t-1, \\ 0 & \text{for } |x| \ge t. \end{cases} \tag{5.102}$$

Let $\upsilon_{\mathsf{p}} \in C_0^{\infty}(\mathbb{R}^d)$ such that $0 \le \upsilon_{\mathsf{p}} \le 1$ and that

$$\upsilon_{\mathsf{p}}(x) = \begin{cases} 0 & \text{for } |x| \le 1/\mathsf{p}, \\ 1 & \text{for } 2/\mathsf{p} \le |x| < \infty. \end{cases} \tag{5.103}$$

For given increasing sequences of natural numbers $\mathbf{a} = \{a(k)\}_{k=1}^{\infty}$ we set

$$\mathsf{K}(\mathbf{a}) = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S_k) \le a(k) \ \text{ for all } k\}.$$

Note that $\mathsf{K}(\mathbf{a})$ is compact and that, furthermore, a subset A in S is relatively compact if and only if there exist such sequences $\mathbf{a}$ satisfying $\mathsf{A} \subset \mathsf{K}(\mathbf{a})$. Let $\mathbf{a}[r] = \{a[r](k)\}$ be a family of increasing sequences of natural numbers such that

$$a[r](k) < a[r+1](k) \quad \text{ for all } k.$$

Then $\mathsf{K}(\mathbf{a}[r]) \subset \mathsf{K}(\mathbf{a}[r+1])$. Because of the compactness criteria in S as above and compact regularity of probability measures on Polish spaces, we can and do take $\mathbf{a}[r]$ such that

$$\mu(\mathsf{K}(\mathbf{a}[r])^c) \downarrow 0 \quad \text{ as } r \to \infty. \tag{5.104}$$

Hence from (5.45) we have $\mu^{\mathsf{s}}(\mathsf{K}(\mathbf{a}[r])^c) \downarrow 0$ as $r \to \infty$ for μ-a.s. s. For each $R \in \mathbb{N}$ we similarly have $\mu^{R\mathsf{s}}(\mathsf{K}(\mathbf{a}[r])^c) \downarrow 0$ as $r \to \infty$ for μ-a.s. s. Then from these and (5.46) we obtain for μ-a.s. s

$$\lim_{r\to\infty} \sup_{R\in\mathbb{N}} \mu^{R\mathsf{s}}(\mathsf{K}(\mathbf{a}[r])^c) = 0. \tag{5.105}$$

Let $\mathbf{a}_+[r] = \{1 + a[r](k+1)\}_{k=1}^{\infty}$. Let $\varpi_{\mathbf{a}[r]} \in C_0(\mathsf{S})$ such that $0 \le \varpi_{\mathbf{a}[r]} \le 1$ and that

$$\varpi_{\mathbf{a}[r]}(\mathsf{s}) = \begin{cases} 1 & \text{for } \mathsf{s} \in \mathsf{K}(\mathbf{a}[r]), \\ 0 & \text{for } \mathsf{s} \in \mathsf{K}(\mathbf{a}_+[r]^c). \end{cases} \tag{5.106}$$

We refer to Section 5.9 or [44, Lemma 2.5] for the construction of such a $\varpi_{\mathbf{a}[r]}$.

We assume that $\mathsf{b} \in L^p_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]})$ with $p > 1$ and that b is given by

$$\mathsf{b}(x, \mathsf{y}) = \frac{\beta}{2} \lim_{s\to\infty} \Big(\Big\{ \sum_{|x-y_i|<s} \nabla\Psi(x - y_i) \Big\} - \varrho_s \Big) \quad \text{ in } L^p_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]}).$$

Here $\Psi \in C^{\infty}(\mathbb{R}^d \backslash \{0\})$ and ϱ_s are constants. We now take $\mathsf{b}_{r,s,\mathsf{p}}$ as follows:

$$\mathsf{b}_{r,s,\mathsf{p}}(x, \mathsf{y}) = \frac{\beta}{2} \chi_r(x) \Big(\Big\{ \sum_{i=1}^{\infty} \chi_s(x - y_i) \upsilon_{\mathsf{p}}(x - y_i) \nabla\Psi(x - y_i) \Big\} - \varrho_s \Big) \varpi_{\mathbf{a}[r]}(\mathsf{y}).$$

We easily see that **(A6)** is satisfied. Indeed, $\mathsf{b}_{r,s,\mathsf{p}} \in C_b(S \times \mathsf{S})$ follows from $\chi_t, \upsilon_{\mathsf{p}} \in C_0^{\infty}(\mathbb{R}^d)$ and $\varpi_{\mathbf{a}[r]} \in C_0(\mathsf{S})$. We next check (5.47). Similarly as (5.105), we see that for μ-a.s. s and for each $k \in \mathbb{N}$

$$\lim_{r\to\infty} \lim_{s\to\infty} \lim_{\mathsf{p}\to\infty} \sup_{R \ge r+s+1} \mu^{R\mathsf{s},[1]} \Big(\big\{ S_k \times \mathsf{K}(\mathbf{a}_+[r]) \big\} \cap \big\{ |\mathsf{b}(x, \mathsf{y}) - \mathsf{b}_{r,s,\mathsf{p}}(x, \mathsf{y})| \ne 0 \big\} \Big) = 0. \tag{5.107}$$

Because $p > 1$, (5.47) follows from the Hölder inequality, (5.105) and (5.107) combined with (5.102)–(5.103), and (5.106).

We remark that the construction of $\mathsf{b}_{r,s,\mathsf{p}}$ as above is robust and can be applied to all examples in this paper.

5.8 Examples

We now present examples of random point fields for which the following holds:

$$(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}}) = (\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}}) = (\mathcal{E}^{+}, \mathcal{D}^{+}), \tag{5.108}$$

which is the claim in Theorem 5.15. In this section we write $L^p_{\mathrm{loc}}(\mu^{[1]}) = L^p_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]})$, where $\mu^{[1]}$ is the one-Campbell measure of μ as before.

5.8.1 Sine$_\beta$ interacting Brownian motion ($\beta = 1, 2, 4$)

Let $d = 1$ and $S = \mathbb{R}$. Let $\mu_{\mathrm{sin},\beta}$ be a sine$_\beta$ random point field [43, 14], where $\beta = 1, 2, 4$. By definition, $\mu_{\mathrm{sin},2}$ is the random point field on $\mathbb{R}$ with n-point correlation function $\rho^n_{\mathrm{sin},2}$ with respect to the Lebesgue measure given by

$$\rho^n_{\mathrm{sin},2}(x_1, \ldots, x_n) = \det[\mathsf{K}_{\mathrm{sin},2}(x_i, x_j)]^n_{i,j=1}.$$

Here $\mathsf{K}_{\mathrm{sin},2}(x, y) = \sin \pi(x-y)/\pi(x-y)$ is the sine kernel. $\mu_{\mathrm{sin},1}$ and $\mu_{\mathrm{sin},4}$ are also defined by correlation functions given by quaternion determinants [43]. The random point fields $\mu_{\mathrm{sin},\beta}$ ($\beta = 1, 2, 4$) satisfy **(A1)**–**(A3)** [47, 45]. $\mu_{\mathrm{sin},\beta}$ clearly satisfy **(A4)** because their one-point correlation functions are constant.

Let $1 < p < 2$. Then **(A5)** is satisfied with the logarithmic derivative given by

$$\mathsf{b}(x, \mathsf{y}) = \frac{\beta}{2} \lim_{r\to\infty} \sum_{|x-y_i|<r} \frac{1}{x - y_i} \quad \text{in } L^p_{\mathrm{loc}}(\mu^{[1]}_{\mathrm{sin},\beta}).$$

Here $\mathsf{y} = \sum_i \delta_{y_i}$ and "in $L^p_{\mathrm{loc}}(\mu^{[1]}_{\mathrm{sin},\beta})$" means convergence in $L^p(S_r \times \mathsf{S}, \mu^{[1]}_{\mathrm{sin},\beta})$ for all $r \in \mathbb{N}$. The labeled process $\mathbf{X} = (X^i)_{i\in\mathbb{Z}}$ solves ISDE:

$$dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r\to\infty} \sum_{j\neq i,\, |X^i_t - X^j_t|<r} \frac{1}{X^i_t - X^j_t} dt \quad (i \in \mathbb{Z}),$$

with conditions **(μ-AC)** and **(NBJ)** [47, 76]. We readily verify conditions **(A6)** from Section 5.7. We have checked **(A1)**–**(A6)**. Moreover, $\mu_{\mathrm{sin},2}$ satisfies **(TT)** because $\mu_{\mathrm{sin},2}$ is a determinantal random point field [50]. Hence we apply Theorem 5.15 and obtain (5.108) for $\mu_{\mathrm{sin},2}$ and, if $\beta = 1, 4$, then for $\mu^{\mathsf{s}}_{\mathrm{sin},\beta}$ for $\mu_{\mathrm{sin},\beta}$-a.s. s.

If $\beta = 2$, then an algebraic construction of the stochastic dynamics associated with the upper Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ was known [23]. The distribution of the dynamics are determined by the space-time correlation functions, which is explicitly given by the concrete determinantal kernel. Because of the identity $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}}) = (\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ in Theorem 5.15, the same holds for the stochastic dynamics associated with the lower Dirichlet form $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$.

5.8.2 Airy$_\beta$ interacting Brownian motion ($\beta = 1, 2, 4$)

Let $d = 1$ and $S = \mathbb{R}$. Let $\mu_{\mathrm{Ai},\beta}$ be the Airy$_\beta$ random point field, where $\beta = 1, 2, 4$. By definition, $\mu_{\mathrm{Ai},2}$ is a determinantal random point field whose n-point correlation function $\rho^n_{\mathrm{Ai},2}$ with respect to the Lebesgue measure is given by

$$\rho^n_{\mathrm{Ai},2}(x_1, \ldots, x_n) = \det[\mathsf{K}_{\mathrm{Ai},2}(x_i, x_j)]^n_{i,j=1}.$$

Here $\mathsf{K}_{\mathrm{Ai},2}$ is a continuous kernel given by

$$\mathsf{K}_{\mathrm{Ai},2}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \quad \text{for } x \neq y.$$

Here the value $\mathsf{K}_{\mathrm{Ai},2}(x,x)$ is given by continuity, Ai is the Airy function, and Ai' is its derivative. The random point fields $\mu_{\mathrm{Ai},\beta}$ for $\beta = 1, 4$ are also given by a similar formula with a quaternion determinant (see [43]). $\mu_{\mathrm{Ai},\beta}$ satisfy **(A1)**–**(A3)** [49, 54, 45]. Moreover, $\mu_{\mathrm{Ai},\beta}$ clearly satisfies **(A4)**. **(A5)** is satisfied with the logarithmic derivative given by

$$\mathsf{b}(x,\mathsf{y}) = \frac{\beta}{2} \lim_{r\to\infty} \Big(\Big\{ \sum_{|y_i|<r} \frac{1}{x-y_i} \Big\} - \int_{|x|<r} \frac{\hat{\rho}(x)}{-x} dx \Big) \quad \text{in } L^p_{\mathrm{loc}}(\mu^{[1]}_{\mathrm{Ai},\beta}),$$

where $\hat{\rho}(x) = 1_{(-\infty,0)}(x)\sqrt{-x}/\pi$. The labeled process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ solves the ISDE:

$$dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r\to\infty} \Big(\Big\{ \sum_{j\neq i,\, |X^j_t|<r} \frac{1}{X^i_t - X^j_t} \Big\} - \int_{|x|<r} \frac{\hat{\rho}(x)}{-x} dx \Big) dt.$$

with conditions **(μ-AC)**, **(NBJ)** and **(IFC)** [54, 53]. We can check **(A6)** in the same fashion as Section 5.7. We have checked **(A1)**–**(A6)**. **(TT)** was proved for $\beta = 2$ [50]. Hence we apply Theorem 5.15 and obtain (5.108) for $\mu_{\mathrm{Ai},2}$. If $\beta = 1, 4$, then we obtain (5.108) for $\mu^{\mathsf{s}}_{\mathrm{Ai},\beta}$ for $\mu_{\mathrm{Ai},\beta}$-a.s. s.

Similarly as Sine_β interaction Brownian motion, an algebraic construction of the stochastic dynamics associated with the upper Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ was known if $\beta = 2$ [23]. The distribution of the dynamics are determined by the space-time correlation functions, which is explicitly given by the concrete determinantal kernel.

5.8.3 $\mathrm{Bessel}_{2,\alpha}$ interacting Brownian motion

Let $d = 1$ and $S = [0,\infty)$. Let $1 \leq \alpha < \infty$. Let $\mu_{\mathrm{Be},\alpha}$ be the $\mathrm{Bessel}_{2,\alpha}$ random point field. By definition $\mu_{\mathrm{Be},\alpha}$ is a determinantal random point field whose n-point correlation function $\rho^n_{\mathrm{Be},\alpha}$ with respect to the Lebesgue measure on $[0,\infty)$ is given by

$$\rho^n_{\mathrm{Be},\alpha}(x_1,\ldots,x_n) = \det[\mathsf{K}_{\mathrm{Be},\alpha}(x_i,x_j)]^n_{i,j=1}.$$

Here $\mathsf{K}_{\mathrm{Be},\alpha}$ is a continuous kernel given by

$$\mathsf{K}_{\mathrm{Be},\alpha}(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)} \quad \text{for } x \neq y.$$

The $\mathrm{Bessel}_{2,\alpha}$ random point fields $\mu_{\mathrm{Be},\alpha}$ satisfy **(A1)**–**(A3)** [17, 45]. **(A4)** is obvious. In [17], it was proved that **(A5)** is satisfied with the logarithmic derivative given by

$$\mathsf{b}(x,\mathsf{y}) = \frac{\alpha}{2x} + \sum_{i=1}^{\infty} \frac{1}{x-y_i} \quad \text{in } L^p_{\mathrm{loc}}(\mu^{[1]}_{\mathrm{Be},\alpha}).$$

The labeled process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ satisfies **(A3)** solves the ISDE:

$$dX^i_t = dB^i_t + \{\frac{\alpha}{2X^i_t} + \sum_{j\neq i}^{\infty} \frac{1}{X^i_t - X^j_t}\} dt \quad (i \in \mathbb{N}).$$

with conditions **(μ-AC)**, **(NBJ)** and **(IFC)** [17, 53]. We can readily verify conditions **(A6)** as in the same fashion as Section 6. **(TT)** was proved in [50]. We then apply Theorem 5.15 and obtain (5.108) for $\mu = \mu_{\mathrm{Be},\alpha}$.

An algebraic construction of the stochastic dynamics associated with the upper Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ was known [23]. The distribution of the dynamics are determined by the space-time correlation functions explicitly given by the concrete determinantal kernel.

5.8.4 Ginibre interacting Brownian motion

Let $d = 2$ and $S = \mathbb{R}^2$. Let $\beta = 2$. Let μ_{Gin} be the Ginibre random point field. By definition μ_{Gin} is a random point field on $\mathbb{R}^2$ whose n-point correlation function with respect to the Lebesgue measure is given by

$$\rho^n_{\mathrm{Gin}}(x_1, \dots, x_n) = \det[\mathsf{K}_{\mathrm{Gin}}(x_i, x_j)]^n_{i,j=1},$$

where $\mathsf{K}_{\mathrm{Gin}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ is the kernel defined by

$$\mathsf{K}_{\mathrm{Gin}}(x, y) = \frac{1}{\pi} e^{-\frac{1}{2}\{|x|^2+|y|^2\}} \cdot e^{x\bar{y}}.$$

Here we identify $\mathbb{R}^2$ as $\mathbb{C}$ by the obvious correspondence $\mathbb{R}^2 \ni x = (x_1, x_2) \mapsto x_1 + \mathrm{i}x_2 \in \mathbb{C}$, and $\bar{y} = y_1 - \mathrm{i}y_2$ is the complex conjugate in this identification, where $\mathrm{i} = \sqrt{-1}$. The random point field μ_{Gin} satisfies **(A1)**–**(A3)** [47, 48, 45]. Because the one-point correlation function is constant, μ_{Gin} clearly satisfies **(A4)**. The logarithmic derivative is given by

$$\mathsf{b}(x, \mathsf{y}) = \lim_{r\to\infty} \sum_{|x-y_j|<r} \frac{x - y_j}{|x - y_j|^2} \quad \text{in } L^p_{\mathrm{loc}}(\mu^{[1]}_{\mathrm{Gin}}) \tag{5.109}$$

and

$$\mathsf{b}(x, \mathsf{y}) = -x + \lim_{r\to\infty} \sum_{|y_j|<r} \frac{x - y_j}{|x - y_j|^2} \quad \text{in } L^p_{\mathrm{loc}}(\mu^{[1]}_{\mathrm{Gin}}). \tag{5.110}$$

It is known that (5.109) and (5.110) define the same logarithmic derivative [47]. The labeled process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ solves ISDE [47]:

$$dX^i_t = dB^i_t + \lim_{r\to\infty} \sum_{j\neq i,\, |X^i_t - X^j_t|<r} \frac{X^i_t - X^j_t}{|X^i_t - X^j_t|^2} dt \quad (i \in \mathbb{N})$$

and

$$dX^i_t = dB^i_t - X^i_t dt + \lim_{r\to\infty} \sum_{j\neq i,\, |X^j_t|<r} \frac{X^i_t - X^j_t}{|X^i_t - X^j_t|^2} dt \quad (i \in \mathbb{N})$$

with conditions **(μ-AC)**, **(NBJ)** and **(IFC)** [53]. We can readily check conditions **(A6)** in the same fashion as Section 6. **(TT)** was proved in [50]. Therefore, we can apply Theorem 5.15 and obtain (5.108) for μ_{Gin}.

5.8.5 Gibbs measures with Ruelle-class potential

Let $S = \mathbb{R}^d$ with $d \in \mathbb{N}$. Let $\Phi = 0$ and we consider ISDE (5.3). Assume that Ψ is smooth outside the origin and is a Ruelle-class potential. That is, Ψ is super-stable and regular in the sense of Ruelle [61]. Here we say Ψ is regular if there exists a positive decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}$ and R_0 such that

$$\Psi(x) \geq -\psi(|x|) \quad \text{for all } x, \quad \Psi(x) \leq \psi(|x|) \quad \text{for all } |x| \geq R_0,$$
$$\int_0^\infty \psi(t)\, t^{d-1} dt < \infty.$$

Let μ_Ψ be canonical Gibbs measures with interaction Ψ satisfying **(A2)**. We do not a priori assume the translation invariance of μ_Ψ. Instead, we assume a quantitative condition in (5.111) below, which is obviously satisfied by the translation invariant canonical Gibbs measures.

Suppose that, for each $\mathsf{p} \in \mathbb{N}$, there exist positive constants c_{36} and c_{37} satisfying

$$\sum_{r=1}^{\infty} \frac{\int_{S_r} \rho^1(x)dx}{r^{c_{36}+1}} < \infty, \quad \limsup_{r\to\infty} \frac{\int_{S_r} \rho^1(x)dx}{r^{c_{36}}} < \infty, \tag{5.111}$$
$$|\nabla\Psi(x)|,\ |\nabla^2\Psi(x)| \leq \frac{c_{37}}{(1+|x|)^{c_{36}}} \tag{5.112}$$

for all x such that $|x| \geq 1/\mathsf{p}$. Here ρ^m is the m-point correlation function of μ_Ψ.

For the non-collision property of tagged particles we assume the following. Suppose that $d \geq 2$ or that $d = 1$ with Ψ is sufficiently repulsive at the origin in the following sense [19]. There exist a positive constant c_{38} and a positive function $h : (0,\infty) \to [0,\infty]$ satisfying that

$$\int_{0<t\leq c_{38}} \frac{1}{h(t)} dt = \infty, \tag{5.113}$$
$$\rho^m(x_1,\ldots,x_m) \leq h(|x_i - x_j|) \quad \text{for all } x_i \neq x_j.$$

From the DLR equation, μ_Ψ satisfies **(A1)**. **(A2)** holds by assumption. **(A3)** follows from (5.113) [19]. **(A4)** is obvious. **(A5)** is satisfied with the logarithmic derivative given by

$$\mathsf{d}^{\mu_\Psi}(x,\mathsf{y}) = -\beta \sum_{j=1}^{\infty} \nabla\Psi(x - y^j) \quad \text{in } L^p_{\mathrm{loc}}(\mu_\Psi^{[1]}).$$

The labeled process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ solves ISDE:

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j=1,\, j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j)dt \quad (i \in \mathbb{N}).$$

with conditions **(μ-AC)**, **(NBJ)** and **(IFC)** [53]. We can readily check conditions **(A6)** in the same fashion as Section 6. We then apply Theorem 5.15 and obtain (5.108) for $\mu = \mu_\Psi^{\mathsf{s}}$ for μ_Ψ-a.s. s.

5.9 Appendix

5.9.1 Construction of $\mathcal{D}_\bullet$

In this section we construct $\mathcal{D}_\bullet$ in Section 5.2. Let $\varpi_{\mathbf{a}[r]}$ be as in (5.106). In addition to the properties stated in Section 5.7, we can take $\varpi_{\mathbf{a}[r]}$ to be

$$\mathbb{D}[\varpi_{\mathbf{a}[r]}, \varpi_{\mathbf{a}[r]}](\mathsf{s}) \le 2 \quad \text{for all } \mathsf{s} \in \mathsf{S}. \tag{5.114}$$

Indeed, we can take $\varpi_{\mathbf{a}[r]}$ as follows.

Let $\theta \in C^\infty(\mathbb{R})$ such that $0 \le \theta(t) \le 1$ for all $t \in \mathbb{R}$ and $\theta(t) = 1$ for $t \le 0$ and $\theta(t) = 0$ for $t \ge 1$. Furthermore, we assume $|\theta'(t)| \le 2$ for all t.

Let $\mathsf{s} = \sum_i \delta_{s_i}$. Recall that $\mathfrak{l}$ is a label such that $|\mathfrak{l}^i(\mathsf{s})| \le |\mathfrak{l}^{i+1}(\mathsf{s})|$ for all i. We set

$$\mathbf{d}_{\mathbf{a}[r]}(\mathsf{s}) = \Big\{ \sum_{k=1}^{\infty} \sum_{i \in J_{k,\mathsf{s}}(\mathbf{a}[r])} (k - |\mathfrak{l}^i(\mathsf{s})|)^2 \Big\}^{1/2},$$

where $J_{k,\mathsf{s}}(\mathbf{a}[r]) = \{i\,;\, i > a[r](k),\ \mathfrak{l}^i(\mathsf{s}) \in S_k\}$. Let

$$\varpi_{\mathbf{a}[r]}(\mathsf{s}) = \theta \circ \mathbf{d}_{\mathbf{a}[r]}(\mathsf{s}).$$

Then a straightforward calculation shows

$$\begin{aligned}\mathbb{D}[\varpi_{\mathbf{a}[r]}, \varpi_{\mathbf{a}[r]}](\mathsf{s}) =& \frac{1}{2} \Big\{ \frac{\theta'(\mathbf{d}_{\mathbf{a}[r]}(\mathsf{s}))}{\mathbf{d}_{\mathbf{a}[r]}(\mathsf{s})} \Big\}^2 \sum_{k=1}^{\infty} \sum_{i \in J_{k,\mathsf{s}}(\mathbf{a}[r])} (k - |\mathfrak{l}^i(\mathsf{s})|)^2 \\ =& \frac{1}{2} \big(\theta'(\mathbf{d}_{\mathbf{a}[r]}(\mathsf{s})) \big)^2 \le 2.\end{aligned}$$

We thus see that $\varpi_{\mathbf{a}[r]}$ satisfies (5.114). It is not difficult to see that $\varpi_{\mathbf{a}[r]}$ also satisfies the requirements in Section 5.7. That is, $\varpi_{\mathbf{a}[r]}$ satisfies $\varpi_{\mathbf{a}[r]} \in C_0(\mathsf{S})$, $0 \le \varpi_{\mathbf{a}[r]} \le 1$ and (5.106).

Let $\mathcal{D}_{\bullet\bullet} = \{f\varpi_{\mathbf{a}[r]};\ f \in \mathcal{D}_\circ,\ r \in \mathbb{N}\}$. Then because $\varpi_{\mathbf{a}[r]}$ has compact support and satisfies (5.114), we see $\mathcal{D}_{\bullet\bullet}$ is a subset of $L^2(\mathsf{S}, \mu)$, $\mathcal{D}_R^{\mathsf{lwr}}$, and $\mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}}$ for μ-a.s. s. Moreover, $\mathcal{D}_{\bullet\bullet}$ is dense in $L^2(\mathsf{S}, \mu)$, $\mathcal{D}_R^{\mathsf{lwr}}$, and $\mathcal{D}_R^{R\mathsf{s},\mathsf{lwr}}$ for μ-a.s. s with respect to $L^2(\mathsf{S}, \mu)$-norm, $\mathcal{E}_{R,1}^{\mathsf{lwr}}$-norm, and $\mathcal{E}_{R,1}^{R\mathsf{s},\mathsf{lwr}}$-norm for μ-a.s. s, respectively. We use here (5.104) and (5.105). We can further choose a countable subset $\mathcal{D}_\bullet$ of $\mathcal{D}_{\bullet\bullet}$ that keeps these properties. This completes the construction of $\mathcal{D}_\bullet$.

5.9.2 Proof of Lemma 5.7

In this section we prove Lemma 5.7. The assumption **(A2)** corresponds to (A.2) in [44] (we write (A.2) below). It was assumed in (A.2), in addition to **(A2)**, the boundedness of density functions of all order on each S_r. In [44], (A.2) was used only in the proof of (2.2) in Lemma 2.4 (see [44, 125p]). Moreover, (2.2) in [44] is used only to prove Lemma 2.4 (3) in [44], which is the claim such that $\mathcal{D}_\circ$ is dense in $L^2(\mathsf{S}, \mu)$. Hence our task is to prove this under **(A2)**. For this purpose we recall a mollifier on S introduced in [44].

Let $j : \mathbb{R}^d \to \mathbb{R}$ be a non-negative, smooth function such that $\int_{\mathbb{R}^d} j(x)dx = 1$, $j(x) = 0$ for $|x| \ge 1/2$. Let $j_\varepsilon(x) = \varepsilon^d j(x/\varepsilon)$ and $j_\varepsilon^i((x_1, \ldots, x_i)) = \prod_{j=1}^{i} j_\varepsilon(x_j)$.

For a $\sigma[\pi_r]$-measurable, bounded function f we set

$$\mathcal{J}_{r,\varepsilon} f(\mathsf{s}) = \begin{cases} j_\varepsilon^i * f_r^i(\mathbf{x}_r^i(\mathsf{s})) & \text{for } \mathsf{s} \in \mathsf{S}_r^i \quad (i \geq 1), \\ f(\mathsf{s}) & \text{for } \mathsf{s} \in \mathsf{S}_r^0. \end{cases} \tag{5.115}$$

Here f_r^i is S_r^i-representation of f and $\mathbf{x}_r^i(\mathsf{s})$ is the S_r^i-coordinate introduced in Section 5.2. Moreover, $*$ denotes the convolution in $(\mathbb{R}^d)^i$, that is, $j_\varepsilon^i * f_r^i(\mathbf{x}) = \int_{(\mathbb{R}^d)^i} j_\varepsilon^i(\mathbf{x}-\mathbf{y}) f_r^i(\mathbf{y}) d\mathbf{y}$, where we set $f_r^i(\mathbf{x}) = 0$ for $\mathbf{x} \notin S_r^i$.

Lemma 5.27. $\mathcal{D}_\circ$ is dense in $L^2(\mathsf{S}, \mu)$.

Proof. Let $0 < \delta < r$ ($\delta \in \mathbb{R}$) and $f \in C_b(\mathsf{S}) \cap \mathcal{B}_{r-\delta}$. Then f is a bounded continuous, $\sigma[\pi_{r-\delta}]$-measurable function on S by definition. From [44, Lemma 2.4 (2.1)] we see $\mathcal{J}_{r,\varepsilon} f \in \mathcal{D}_\circ$ for $0 < \varepsilon < \delta$. Moreover, because $f \in C_b(\mathsf{S})$, we see from (5.115) that

$$\lim_{\varepsilon \to 0} \mathcal{J}_{r,\varepsilon} f(\mathsf{s}) = f(\mathsf{s}) \quad \text{for each } \mathsf{s}, \tag{5.116}$$

$$\sup_{\mathsf{s} \in \mathsf{S}} |\mathcal{J}_{r,\varepsilon} f(\mathsf{s})| \leq \sup_{\mathsf{s} \in \mathsf{S}} |f(\mathsf{s})| < \infty. \tag{5.117}$$

From (5.116)–(5.117) we can apply the Lebesgue convergence theorem to obtain for each $r \in \mathbb{N}$

$$\lim_{\varepsilon \to 0} \int_{\mathsf{S}} |\mathcal{J}_{r,\varepsilon} f(\mathsf{s}) - f(\mathsf{s})|^2 \mu(d\mathsf{s}) = \int_{\mathsf{S}} \lim_{\varepsilon \to 0} |\mathcal{J}_{r,\varepsilon} f(\mathsf{s}) - f(\mathsf{s})|^2 \mu(d\mathsf{s}) = 0. \tag{5.118}$$

Because

$$\bigcup_{r=1}^{\infty} \bigcup_{0<\delta<r,\, \delta \in \mathbb{R}} C_b(\mathsf{S}) \cap \mathcal{B}_{r-\delta}$$

is dense in $L^2(\mathsf{S}, \mu)$, the claim follows from (5.118). □

5.10 Concluding remarks and questions

1. We have proved that the two natural Dirichlet forms $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ and $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ are equal under the assumptions in Theorem 5.15. The most important condition for this is the non-explosion property of each tagged particle that follows from **(A4)**. Indeed, this condition controls the effect of boundary ∂S_R as $R \to \infty$. We have an example of non-coincidence when tagged particles explode. We then conjecture that non-explosion is a necessary and sufficient condition of the coincidence of the upper and the lower Dirichlet forms.

Question 1. Can one prove that the upper and the lower Dirichlet forms coincide with each other if and only if each tagged particle does not explode?

2. We can naturally formulate the same problem for non-local Dirichlet forms. In particular, the case such that the associated Markov processes have big jump would be interesting.

3. In [33, 35], the uniqueness of the Silverstein extension of Dirichlet forms was studied. In particular, it was proved that the Silverstein extension is unique when the Dirichlet form is quasi-regular and equipped with a suitable exhaustion function with bounded energy measure [33, Theorem 5.1, Theorem 6.1]. Our result (5.54) however can not be derived from this because we do not a priori know whether the lower Dirichlet form $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ is a Silverstein extension of the upper Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$. As a corollary of Theorem 5.15, we see that $(\mathcal{E}^{\mathsf{lwr}}, \mathcal{D}^{\mathsf{lwr}})$ is the Silverstein extension of the upper Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$ because these two forms are equal.

4. In [69], Takeda proved the uniqueness of Markovian extension of Dirichlet forms on distorted Brownian motion in a domain in $\mathbb{R}^d$ (also called a generalized Schrödinger operator). We refer to [16, Chapter 3.3] for the Markovian extension of Dirichlet forms. This class of Dirichlet forms is a finite-dimensional counter part of the Dirichlet forms in the present paper. Hence it is natural to discuss the uniqueness of the Markovian extension of the upper Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$.

Question 2. What is the sufficient condition for the uniqueness of the Markovian extension of the upper Dirichlet form $(\mathcal{E}^{\mathsf{upr}}, \mathcal{D}^{\mathsf{upr}})$? Is it same as the condition in Question 1?

6 Dynamical universality for random matrices

6.1 Introduction

A concept of universality in strongly correlated systems was envisioned by Wigner. He conjectured that eigenvalue distribution of large random matrix behaves universal, that is, eigenvalue distribution depends only on symmetry class of matrices, not on distributions of matrix component. The universality of random matrices has been a central concept in random matrix theory, and have been studied intensively this two decades.

Let us show an example of universality results. Consider N-particles system on $\mathbb{R}$. For an analytic function $V : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{N\to\infty} V(x)/\log|x| = \infty$, let μ_V^N be the random point field whose labeled density is given by $\mathbf{m}_V^N$:

$$\mathbf{m}_V^N(d\mathbf{x}_N) = \frac{1}{Z_V^N} \prod_{i<j}^N |x_i - x_j|^2 \prod_{k=1}^N e^{-NV(x_k)}\, d\mathbf{x}_N, \tag{6.1}$$

where $\mathbf{x}_N = (x_1, \dots, x_N) \in \mathbb{R}^N$ and Z_V^N is a normalizing constant.

If $V(x) = x^2$, then μ_V^N gives the eigenvalue distribution of the Gaussian Unitary Ensemble (GUE), which is an Hermite matrix whose entries are i.i.d Gaussian distribution (see [2, 43]). Note that each particles repel by logarithmic interaction potential, which is a long-range potential.

We set $\mathsf{x}^N = \sum_{1\le i\le N} \delta_{x_i}$, where δ_a denotes the delta measure at a. Then, there exists a probability density function ϱ_V on $\mathbb{R}$ such that

$$\lim_{N\to\infty} \mathbb{E}_{\mu_V^N}[\frac{1}{N}\mathsf{x}^N((-\infty, s])] = \int_{-\infty}^{s} \varrho_V(x)\, dx. \tag{6.2}$$

The ϱ_V is called an equilibrium measure with respect to μ_V^N. When $V(x) = x^2$, ϱ_V is nothing but the Wigner semicircle law given by $\varrho_V(x) = \frac{1}{\pi}\sqrt{2 - x^2}\mathbf{1}_{\{|x|<\sqrt{2}\}}$.

The convergence in (6.2) is in macroscopic regime, then consider microscopic regime next. More precisely, we take a thermodynamical limit of (6.1) and obtain a random point field with infinitely many particles as a limit. Here, we shall take a bulk scaling limit. For θ satisfying

$$\varrho_V(\theta) > 0, \tag{6.3}$$

we set a bulk scaling at θ as

$$x \mapsto \frac{s}{N\varrho_V(\theta)} + \theta. \tag{6.4}$$

Let $\mathbf{m}_{V,\theta}^N$ be a density function of (6.1) with respect to s under the scaling (6.4), that is,

$$\mathbf{m}_{V,\theta}^N(d\mathbf{s}_N) = \frac{1}{Z_{V,\theta}^N} \prod_{i<j}^N |s_i - s_j|^2 \prod_{k=1}^N \exp\Big(-NV\Big(\frac{s_k}{N\varrho_V(\theta)} + \theta\Big)\Big)\, d\mathbf{s}_N.$$

We set $\mu_{V,\theta}^N$ as a random point field whose labeled density is given by $\mathbf{m}_{V,\theta}^N$.

Let $\mu_{\sin}$ be the sine random point field, which is a determinantal random point field whose kernel is given by

$$K_{\sin}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

Let $\{\rho^n_{\sin}\}_{n\in\mathbb{N}}$ be correlation functions of $\mu_{\sin}$. Then by definition

$$\rho^n_{\sin}(x_1,\dots,x_n) = \det[K_{\sin}(x_i,x_j)]_{1\le i\le j\le n}.$$

Bulk universality for log-gases asserts that for suitable and wide class of V and for any above θ satisfying (6.3), we expect

$$\lim_{N\to\infty} \mu^N_{V,\theta} = \mu_{\sin} \quad \text{weakly,}$$

or more strongly,

$$\lim_{N\to\infty} \rho^{N,n}_{V,\theta} = \rho^n_{\sin} \quad \text{compact uniformly for any } n\in\mathbb{N},$$

where $\{\rho^{N,n}_{V,\theta}\}_{n\in\mathbb{N}}$ are correlation functions of $\mu^N_{V,\theta}$. Here, the limit $\mu_{\sin}$ is independent of V and θ. In this sense, the sine random point field can be thought of a geometric universal object.

It is natural to ask what is the dynamical counterpart of the geometric universality results. We consider a N-dimensional SDE corresponding to $\mu^N_{V,\theta}$

$$dX^{N,i}_t = dB^i_t + \sum_{1\le j\ne i\le N} \frac{1}{X^{N,i}_t - X^{N,j}_t}dt - \frac{1}{2\varrho_V(\theta)}V'\Big(\frac{X^{N,i}_t}{N\varrho_V(\theta)} + \theta\Big)dt, \quad 1\le i\le N. \tag{6.5}$$

In fact, (unlabeled version of) a solution for (6.5) is reversible with respect to $\mu^N_{V,\theta}$.

We are interested in an infinite-dimensional stochastic differential equation (ISDE) related to (6.5). We expect that a limit of (6.5) as $N\to\infty$ is given by

$$dX^i_t = dB^i_t + \lim_{r\to\infty}\sum_{|X^i_t - X^j_t|<r} \frac{1}{X^i_t - X^j_t}dt, \quad i\in\mathbb{N}. \tag{6.6}$$

Actually, (6.6) is the ISDE related to sine random point fields. In this sense, (6.6) is a universal dynamical object.

To prove such kind of dynamical finite particle approximation, the authors established a general theory [28]. Our framework in [28] does not depend on dimension of underlying space, inverse temperature, and integrable structure, then we can apply the theory to many examples. A key point in the previous paper is the control of drift terms in finite-dimensional SDE, which is a sensitive estimate for long-range potential. Actually, we proved dynamical bulk scaling limit by completing such a estimate when $V(x) = x^2$ [29]. However, when it comes to general $V(x)$, we have to do more complicated calculation.

When we consider a ISDE related to the Airy random point field, which arises from soft-edge scaling limit of eigenvalue distribution of random matrices, this point is more sensitive problem. It is because the drift term in corresponding finite-dimensional SDE includes divergent term.

To overcome this difficulty, we construct a new method in the present paper. In this approach, we use a convergence concept of Dirichlet forms associated with finite or infinite particles systems. Except for existence of infinite particles dynamics, we assume two main conditions as follows:

(1) Uniqueness of Dirichlet forms associated with infinite particles systems (**(A3)** in Section 6.2).

(2) Strong convergence of random point fields in the sense that correlation functions of random point fields with finite particle systems converges to that of infinite particles uniformly on each compact set and capacity of zero points of density functions of the limit random point field is vanished. (**(M3)** or **(M3')** or **(M3")** in Section 6.2 for detail).

Condition (1) is issue of infinite-particle dynamics, and showing (1) comes down to uniqueness of a solution for ISDE [31].

The only condition related to finite-particle system is (2). This is purely matter of static, and remark that we do not require any assumption of estimates related to dynamics of finite-particle system such as estimate of drift term in [28]. Additionally, the method in the present paper works regardless of the dimension of underlying space, inverse temperature, and integrable structures as well as in [28].

Consequently, when there exists a unique solution for an ISDE, strong convergence of random point field derives dynamical convergence automatically. Until now, it has been proved that several ISDEs including logarithmic interaction have unique solution [53, 54, 76]. Therefore, strong universality for random matrices can be strengthened to dynamical universality with respect to not only Dyson's Brownian motion but also ISDE related to the Airy random point field, the Ginibre random point field, and so on.

The idea of the proof of the dynamical universality is the following. One of the main tools for the proof is Mosco convergence in the sense of Kuwae-Shioya [34] of Dirichlet forms. The definition of Mosco convergence consists of two limit relations related to Dirichlet forms (see Definition 6.17).

Canonically, there are two natural Dirichlet forms with respect to a random point field with infinitely many particles. Accordingly there exist two natural schemes of finite-volume Dirichlet forms and each schemes converge the limit Dirichlet forms. These two canonical Dirichlet forms are the same one because we assume the uniqueness of Dirichlet forms. Therefore, we conclude the Mosco convergence.

In the co-paper [31], we proved the uniqueness of Dirichlet forms applicable the current situation. This uniqueness theorem is robust and can be applied to random point fields arising from random matrix theory in spite of the long range interaction that these point fields have.

Generally, dynamical convergence fails under only weak convergence of measure even in one dimensional diffusion. Then we must assume stronger convergences such as (2), and (2) is thus an eligible assumption.

Recently, universality results for random matrices has been studied under extensively general assumptions as [5]. However, these universality results are weak convergence. If we improve the results to strong convergence, its dynamical version can be proved immediately.

This paper is organized as follows. In Section 6.2, we set up Dirichlet forms and state main results. There, two types of convergence of unlabeled dynamics are shown. In Section 6.4, we prove the first theorem. The second theorem, which is more convenient than first one, is proved in Section 6.5. In Section 6.6, we give examples of dynamical universality.

6.2 Set up and main results

6.2.1 Two spacial approximation schemes of Dirichlet forms and unlabeled dynamics

In this subsection, we prepare Dirichlet forms and the associated dynamics. Let $S = \mathbb{R}^d$ for any $d \in \mathbb{N}$. Let S be a configuration space over S given by

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\, s_i \in S, \mathsf{s}(K) < \infty \text{ for any compact set } K\},$$

where we regard the zero measure as an element of S. The set S is equipped with the vague topology, under which S is a Polish space. We set $S_r = \{|s| \le r\}$ and

$$\mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S_r) = m\}.$$

For a set $A \in S$, let $\pi_A : \mathsf{S} \to \mathsf{S}$ be a projection given by $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot \cap A)$. We often write $\pi_r = \pi_{S_r}$. A function f on S is called local if f is $\sigma[\pi_K]$-measurable for some compact set K in S. For such a local function f on S, f is said to be smooth if $\check{f} = \check{f}_O$ is smooth, where O is a relative compact open set in S such that $K \subset O$. Moreover, $\check{f}_O$ is a function defined on $\sum_{k=0}^{\infty} O^k$ such that $\check{f}_O(x_1, \dots x_k)$ restricted on O^k is symmetric in x_j $(j = 1, \dots, k)$ for each k such that $\check{f}_O(x_1, \dots, x_k) = f(\mathsf{x})$ for $\mathsf{x} = \sum_i \delta_{x_i}$ and that $\check{f}_O$ is smooth in $(x_1, \dots, x_k)$ for each k. Here the case $k = 0$ corresponds to a constant function. Because x is a configuration and O is relatively compact, the cardinality of the particles of x is finite in O. Note that $\check{f}_O$ has a consistency such that

$$\check{f}_O(x_1, \dots, x_k) = \check{f}_{O'}(x_1, \dots, x_k) \quad \text{for all } (x_1, \dots, x_k) \in O^k \cap O'^k.$$

We see that $f(\mathsf{x}) = \check{f}_O(x_1, \dots x_k)$ is thus well defined.

Next, we introduce carré du champs $\mathbb{D}$ and $\mathbb{D}_r^m$ on S. Let $\mathcal{D}_\circ$ be the set of all local smooth functions on S. For $f, g \in \mathcal{D}_\circ$, define $\mathbb{D}$ and $\mathbb{D}_r^m$ as

$$\begin{aligned} \mathbb{D}[f,g](\mathbf{s}) &= \frac{1}{2} \sum_i \nabla_{s_i} \check{f}(\mathbf{s}) \cdot \nabla_{s_i} \check{g}(\mathbf{s}), \\ \mathbb{D}_r^m[f,g](\mathbf{s}) &= \begin{cases} \frac{1}{2} \sum_{s_i \in S_r} \nabla_{s_i} \check{f}(\mathbf{s}) \cdot \nabla_{s_i} \check{g}(\mathbf{s}) & (\mathsf{s} \in \mathsf{S}_r^m), \\ 0 \quad (\mathsf{s} \notin \mathsf{S}_r^m), \end{cases} \end{aligned} \tag{6.7}$$

where we set $\mathsf{s} = \sum_i \delta_{s_i}$ and $\mathbf{s} = (s_i)$. Because the right-hand side of these equations are *symmetric* functions in $\mathbf{s}$, they are regarded as functions in $\mathsf{s} = \sum_i \delta_{s_i}$. We thus write

$$\mathbb{D}[f,g](\mathsf{s}) = \mathbb{D}[f,g](\mathbf{s}), \quad \mathbb{D}_r^m[f,g](\mathsf{s}) = \mathbb{D}_r^m[f,g](\mathbf{s}).$$

A probability measure μ on S is called a random point field (point process). We set $L^2(\mu) = L^2(\mathsf{S}, \mu)$. We define the bilinear form $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ on $L^2(\mu)$ as

$$\begin{aligned}\mathcal{E}(f,g) &= \int_{\mathsf{S}} \mathbb{D}[f,g](\mathsf{s}) d\mu, \\ \mathcal{D}_\circ^\mu &= \{f \in \mathcal{D}_\circ \cap L^2(\mu)\,;\, \mathcal{E}(f,f) < \infty\}.\end{aligned} \tag{6.8}$$

For each $r, m \in \mathbb{N}$ we set the bilinear form $(\mathcal{E}_r^m, \mathcal{D}_\circ^\mu)$ on $L^2(\mu)$ as

$$\mathcal{E}_r^m(f,g) = \int_{\mathsf{S}} \mathbb{D}_r^m[f,g](\mathsf{s}) d\mu. \tag{6.9}$$

We make an assumption:

(A1) $(\mathcal{E}_r^m, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mu)$ for each $r, m \in \mathbb{N}$.

Set $\mathcal{B}_r^b = \{f\,;\, f$ is bounded and $\sigma[\pi_r]$-measurable$\}$. For functions $f, g \in \mathcal{B}_r^b \cap \mathcal{D}_\circ^\mu$ being constant outside the subset $\mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S_r) = m\}$ we have

$$\mathbb{D}_r^m[f,g](\mathsf{s}) = \mathbb{D}[f,g](\mathsf{s}) \quad \text{for all } \mathsf{s} \in \mathsf{S}_r^m.$$

Here σ_r^m is the density function of μ on S_r^m with respect to the Lebesgue measure on S_r^m, that is, σ_r^m is the symmetric function such that

$$\frac{1}{m!} \int_{S_r^m} \check{f}_{S_r} \sigma_r^m d\mathbf{x}_m = \int_{\mathsf{S}_r^m} f d\mu \tag{6.10}$$

for any bounded $\sigma[\pi_r]$-measurable functions f, where $\pi_r = \pi_{S_r}$. From this we see that

$$\mathcal{E}_r^m(f,g) = \int_{S_r^m} \mathbb{D}[f,g] \sigma_r^m d\mathbf{x}_m \quad \text{for } f, g \in \mathcal{B}_r^b \cap \mathcal{D}_\circ^\mu. \tag{6.11}$$

This obvious identity is one of the key points of the argument in [44]. In the following, we quote a sequence of results from [44].

Lemma 6.1 ([44, Lemma 2.2])**.** Assume **(A1)**. Then the following holds:
(1) $(\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_r^b)$ is closable on $L^2(\mu)$.
(2) $(\mathcal{E}_r, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mu)$, where we set $\mathcal{E}_r = \sum_{m=1}^\infty \mathcal{E}_r^m$.

We write $(\mathcal{E}^1, \mathcal{D}^1) \le (\mathcal{E}^2, \mathcal{D}^2)$ if

$$\mathcal{D}^1 \supset \mathcal{D}^2 \quad \text{and} \quad \mathcal{E}^1(f,f) \le \mathcal{E}^2(f,f) \text{ for any } f \in \mathcal{D}^2,$$

and $(\mathcal{E}^1, \mathcal{D}^1) \ge (\mathcal{E}^2, \mathcal{D}^2)$ if

$$\mathcal{D}^1 \subset \mathcal{D}^2 \quad \text{and} \quad \mathcal{E}^1(f,f) \ge \mathcal{E}^2(f,f) \text{ for any } f \in \mathcal{D}^1.$$

For a sequence $\{(\mathcal{E}^n, \mathcal{D}^n)\}_{n\in\mathbb{N}}$ of positive definite, symmetric bilinear forms on $L^2(\mu)$, we say $\{(\mathcal{E}^n, \mathcal{D}^n)\}$ is increasing if $(\mathcal{E}^n, \mathcal{D}^n) \le (\mathcal{E}^{n+1}, \mathcal{D}^{n+1})$ for any $n \in \mathbb{N}$, and decreasing if $(\mathcal{E}^n, \mathcal{D}^n) \ge (\mathcal{E}^{n+1}, \mathcal{D}^{n+1})$ for any $n \in \mathbb{N}$.

By Lemma 6.1 $(\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_r^b)$ and $(\mathcal{E}_r, \mathcal{D}_\circ^\mu)$ are closable on $L^2(\mu)$. Then we denote the closures by $(\mathcal{E}_r, \mathcal{D}_r)$ and $(\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r)$, respectively.

Lemma 6.2 ([44, Lemma 2.2])**.** Assume **(A1)**. Then the following hold.
(1) $\{(\mathcal{E}_r, \mathcal{D}_r)\}_{r\in\mathbb{N}}$ is decreasing.
(2) $\{(\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r)\}_{r\in\mathbb{N}}$ is increasing.

By definition the largest closable part $((\tilde{\mathcal{E}})_{\mathrm{reg}}, (\tilde{\mathcal{D}})_{\mathrm{reg}})$ of a given positive symmetric form $(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ with a dense domain is a closable form such that $((\tilde{\mathcal{E}})_{\mathrm{reg}}, (\tilde{\mathcal{D}})_{\mathrm{reg}}) \le (\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ and that $((\tilde{\mathcal{E}})_{\mathrm{reg}}, (\tilde{\mathcal{D}})_{\mathrm{reg}})$ is the largest element of closable forms dominated by $(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$. Such a form exists uniquely.

Because $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mu)$ under **(A1)**, we define $(\mathcal{E}, \mathcal{D})$ as the closure. Let $(\mathcal{E}_\infty, \mathcal{D}_\infty)$ be the symmetric form such that

$$\mathcal{E}_\infty(f, f) = \lim_{r\to\infty} \mathcal{E}_r(f, f)$$

with the domain $\mathcal{D}_\infty = \bigcup_{r\in\mathbb{N}} \mathcal{D}_r$. Then the closure of the largest closable part $((\mathcal{E}_\infty)_{\mathrm{reg}}, (\mathcal{D}_\infty))_{\mathrm{reg}})$ of $(\mathcal{E}_\infty, \mathcal{D}_\infty)$ corresponds to $(\mathcal{E}, \mathcal{D})$ [44].

Let $(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ be the closed symmetric form such that

$$\underline{\mathcal{E}}(f, f) = \lim_{r\to\infty} \underline{\mathcal{E}}_r(f, f)$$

with the domain $\underline{\mathcal{D}} = \{f \in \bigcap_{r=1}^\infty \underline{\mathcal{D}}_r \,;\, \lim_{r\to\infty} \underline{\mathcal{E}}_r(f, f) < \infty\}$.

Summarizing above we obtain the next lemma.

Lemma 6.3. Assume **(A1)**. Then the following hold.
(1) $(\mathcal{E}, \mathcal{D})$ is the strong resolvent limit of $\{(\mathcal{E}_r, \mathcal{D}_r)\}_{r\in\mathbb{N}}$ as $r \to \infty$.
(2) $(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ is the strong resolvent limit of $\{(\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r)\}_{r\in\mathbb{N}}$ as $r \to \infty$.
(3) $(\underline{\mathcal{E}}, \underline{\mathcal{D}}) \le (\mathcal{E}, \mathcal{D})$.

Proof. The first two statements follow from Lemma 6.2 and the general theory of the monotone convergence theorem of closed forms. The third follows from $(\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r) \le (\mathcal{E}_r, \mathcal{D}_r)$ for any r, and thus we have $(\underline{\mathcal{E}}, \underline{\mathcal{D}}) \le (\mathcal{E}, \mathcal{D})$ by the monotone convergence of these forms given by Lemma 6.2. □

(A2) The random point field μ satisfies

$$\sum_{r=1}^{\infty} m\mu(\mathsf{S}_r^m) < \infty \text{ for each } m \in \mathbb{N}.$$

We refer to [42] for the quasi-regularity and the locality of Dirichlet forms and related notions. The importance of the quasi-regularity and the locality is that they guarantee the existence of diffusion associated with the Dirichlet form.

We obtain an unlabeled diffusion from [44]. The next result is one of the main theorems in [44]. To be more precise, boundedness of density functions was assumed in addition to **(A2)** in [44], this was removed in [31].

Proposition 6.4 ([44, Theorem 1, Corollary 1])**.** Assume **(A1)** and **(A2)**. Then $(\mathcal{E}, \mathcal{D})$ is a local quasi-regular Dirichlet form on $L^2(\mu)$. In particular, there exists an S-valued, μ-reversible diffusion X associated with $(\mathcal{E}, \mathcal{D})$.

Let $(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ and $(\mathcal{E}, \mathcal{D})$ be as in Lemma 6.3. We assume:

(A3) $(\underline{\mathcal{E}}, \underline{\mathcal{D}}) = (\mathcal{E}, \mathcal{D})$.

Remark 6.5. From Proposition 6.4 and **(A3)** we deduce that $(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ is a quasi-regular Dirichlet form, and there exists the associated S-valued diffusion. This diffusion is the same as that of the diffusion associated with $(\mathcal{E}, \mathcal{D})$.

6.2.2 Finite particle approximation for a random point field and main result: convergence of unlabeled dynamics.

In Section 6.2.1 we introduced two schemes of finite volume approximations related to bounded domains S_r and we take $r \to \infty$. In the present section, we introduce another approximation consisting of Dirichlet forms describing N-particles. Note that the particles in the present section move in whole S, but the number of particles at the each stage of approximating dynamics is $N \in \mathbb{N}$, and we let N go to infinity.

Let $\{\mu^N\}$ be a sequence of random point fields such that $\mu^N(\mathsf{s}(S) = N) = 1$ for any $N \in \mathbb{N}$ and $\lim_{N\to\infty} \mu^N = \mu$ weakly. For $r, m \in \mathbb{N}$, let $\sigma_r^{N,m}$ be the m-particles density of μ^N on S_r with respect to the Lebesgue measure. We set

$$\mathcal{E}_{r,k}^N(f) = \sum_{m=1}^{k} \int_{S_r^m} \mathbb{D}[f] \sigma_r^{N,m} d\mathbf{x}_m.$$

Hereafter, $\mathcal{E}(f)$ and $\mathbb{D}[f]$ denote $\mathcal{E}(f,f)$ and $\mathbb{D}[f,f]$, respectively. We remark that, if $f \in \mathcal{B}_r^b \cap \mathcal{D}_\circ^\mu$, then by (6.9) and (6.11) we have

$$\mathcal{E}_{r,k}^N(f) = \sum_{m=1}^{k} \int_{\mathsf{S}} \mathbb{D}_r^m[f](\mathsf{s}) d\mu^N = \sum_{m=1}^{k} \mathcal{E}_r^{N,m}(f).$$

From $\mu^N(\mathsf{s}(S) = N) = 1$ we have $\mathcal{D}_\circ = \mathcal{D}_\circ^{\mu^N}$, where $\mathcal{D}_\circ^{\mu^N}$ is defined by (6.8) with μ^N.

Recall that there exists a diffusion associated with a local, regular Dirichlet form. We refer to [16] for the definition of regular Dirichlet forms and related notions. To guarantee the existence of N-particles dynamics, we assume:

(M1) For any $N \in \mathbb{N}$, $(\mathcal{E}^N, \mathcal{D}_\circ)$ is closable on $L^2(\mu^N)$. Furthermore, the closure $(\mathcal{E}^N, \mathcal{D}^N)$ of $(\mathcal{E}^N, \mathcal{D}_\circ)$ is a regular Dirichlet form on $L^2(\mu^N)$.

Let X^N and X be the diffusions associated with the Dirichlet space $(\mathcal{E}^N, \mathcal{D}^N, L^2(\mu^N))$ and $(\mathcal{E}, \mathcal{D}, L^2(\mu))$, respectively. We assume the initial distributions satisfy:

(M2) The distributions of X_0^N and X_0 have densities $\xi^N \in L^2(\mu^N)$ and $\xi \in L^2(\mu)$ with respect to μ^N and μ, respectively, and satisfy

$$\lim_{N\to\infty} \xi^N = \xi$$

strongly in the sense of Definition 6.15.

We assume density functions $\sigma_r^{N,m}$ and σ_r^m of μ^N and μ defined in (6.10) satisfy:

(M3) For each $r, m \in \mathbb{N}$

$$\lim_{N\to\infty} \Big\| \frac{\sigma_r^{N,m}}{\sigma_r^m} - 1 \Big\|_{S_r^m} = 0. \tag{6.12}$$

Here $\| \cdot \|_{S_r^m}$ denotes the $L^\infty(S_r^m, d\mathbf{x})$-norm.

Theorem 6.6. Assume **(A1)**–**(A3)**. Assume **(M1)**–**(M3)**. Then we have

$$\lim_{N\to\infty} \mathsf{X}^N = \mathsf{X} \text{ in distribution in } C([0,\infty); \mathsf{S}). \tag{6.13}$$

The density σ_r^m in (6.12) may vanish in general. Then we introduce the condition

$$\mathrm{Cap}\Big(\bigcup_{m,r=1}^{\infty} \{\mathsf{s} \in \mathsf{S}_r^m \,;\, \sigma_r^m(\mathsf{s}) = 0\} \Big) = 0. \tag{6.14}$$

Here, $\sigma_r^m = \sigma_r^m(s_1, \ldots, s_m)$ is regarded as a function on $\mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S} \,;\, \mathsf{s}(S_r) = m\}$ such that $\sigma_r^m(\mathsf{s}) = \sigma_r^m(s_1, \ldots, s_m)$ for $\mathsf{s}(\cdot \cap S_r) = \sum_{i=1}^m \delta_{s_i}$, and Cap is the capacity associated with $(\mathcal{E}, \mathcal{D})$ on $L^2(\mu)$. See [16, 66p] for the definition of capacity.

We now relax the assumption **(M3)** as below. We shall use **(M3')** when we present examples in Section 6.6.

(M3') For each $r, m \in \mathbb{N}$,

$$\lim_{N\to\infty} \Big\| \sigma_r^{N,m} - \sigma_r^m \Big\|_{S_r^m} = 0. \tag{6.15}$$

Furthermore, (6.14) holds.

Theorem 6.7. Assume **(A1)**–**(A3)**. Assume **(M1)**–**(M2)** and **(M3')**. Then (6.13) holds.

A symmetric and locally integrable function $\rho^n : S^n \to [0,\infty)$ is called the n-point correlation function of a random point field μ on S with respect to the Lebesgue measure if ρ^n satisfies

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_{\mathsf{S}} \prod_{i=1}^m \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} d\mu$$

for any sequence of disjoint bounded measurable sets $A_1, \ldots, A_m \in \mathcal{B}(S)$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. If correlation functions converge compact uniformly, (6.15) is satisfied. In fact, the following relation between correlation functions and density functions hold. If for each $r \in \mathbb{N}$ there exist constants c_{39} and c_{40} satisfying $c_{39} > 0$ and $c_{40} < 1$ such that

$$\sup_{\mathbf{x}_n \in S_r^n} \rho^n(\mathbf{x}_n) \le c_{39}^n n^{c_{40} n},$$

then

$$\sigma_r^m(\mathbf{x}_m) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{S_r^m} \rho^{m+j}(\mathbf{x}_m, \mathbf{y}_j) m(d\mathbf{y}_j).$$

Let $\rho^{N,n}$ be the n-correlation function of μ^N. We shall obtain (6.15) from uniform convergence of $\rho^{N,m}$ to ρ^m on S_r^m.

(M3") Correlation functions $\rho^{N,n}$ and ρ^n satisfy

$$\lim_{N\to\infty} \left\|\rho^{N,m} - \rho^m\right\|_{S_r^m} = 0 \quad \text{for each } r, m \in \mathbb{N}, \tag{6.16}$$

$$\sup_{N\in\mathbb{N}} \sup_{\mathbf{x}_n \in S_r^n} \rho^{N,n}(\mathbf{x}_n) \le c_{39}^n n^{c_{40} n}. \tag{6.17}$$

Furthermore, (6.14) is satisfied.

Theorem 6.8. Assume **(A1)**–**(A3)**. Assume **(M1)**–**(M2)** and **(M3")**. Then (6.13) holds.

Remark 6.9. (1) If σ_r^m are bounded, then (6.12) implies (6.15).
(2) Clearly, (6.16) and (6.17) imply (6.15).
(3) Because of the variational formula of capacity, one can obtain (6.14) easily from estimates of correlation functions.

6.2.3 Convergence of labeled dynamics (SDE) and proof of Theorem 6.10

In this section, we consider labeled dynamics and formulate convergence of finite-dimensional SDEs to the limit ISDE.

Let $\mathfrak{u} : S^{\mathbb{N}} \to \mathsf{S}$ be the unlabeling map given by $\mathfrak{u}(\mathrm{s}) = \sum_i \delta_{s_i}$, where $\mathrm{s} = (s_i)_{i\in\mathbb{N}}$. We assume the following:

(A4) Each particle is non-explosion and non-collision.

Because of **(A4)**, we can construct the labeled dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{N}} \in C([0,\infty); S^{\mathbb{N}})$ such that $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$ with initial label $\mathfrak{l}(\mathsf{X}_0) = \mathbf{X}_0$. Next theorem proves dynamical convergence of labeled dynamics.

Theorem 6.10. Make the same assumptions as Theorem 6.6 or Theorem 6.7 or Theorem 6.8. Assume **(A4)** and that the initial distributions of the labeled dynamics $\mathbf{X}^N$ and $\mathbf{X}$ satisfy for each $m \in \mathbb{N}$,

$$\lim_{N\to\infty} \mu^N \circ (\mathfrak{l}^{N,1}, \dots, \mathfrak{l}^{N,m})^{-1} = \mu \circ (\mathfrak{l}^1, \dots, \mathfrak{l}^m)^{-1} \tag{6.18}$$

weakly. Then for each $m \in \mathbb{N}$,

$$\lim_{N\to\infty} (X^{N,1}, \dots, X^{N,m}) = (X^1, \dots, X^m) \tag{6.19}$$

in distribution in $C([0,\infty); S^m)$.

Proof. From **(A4)** we can construct the labeled dynamics $\mathbf{X}^N$ and $\mathbf{X}$ such that $\mathbf{X}_0^N = \mathfrak{l}^N(\mathsf{s})$ and $\mathbf{X}_0 = \mathfrak{l}(\mathsf{s})$. Note that the initial distribution is in $L^p(\mu^N)$ for some $1 < p$. Then using Lyons-Zheng decomposition, we see the tightness of $\{(X^{N,i})_{i=1}^m\}_{N\in\mathbb{N}}$ in $C([0,\infty); S^m)$ for each m.

The convergence of the finite-dimensional distributions of $\mathbf{X}^N$ follows from the weak convergence of the unlabeled processes X^N and the convergence of the labeled initial distributions (6.18).

Collecting these we obtain Theorem 6.10. □

We next present the ISDE representation of the limit labeled dynamics.

We write $f \in L^p_{\mathrm{loc}}(\mu^{[1]})$ if $f \in L^p(S_r \times \mathsf{S}, \mu^{[1]})$ for all $r \in \mathbb{N}$. Let $C_0^\infty(S) \otimes \mathcal{D}_\circ$ be the algebraic tensor product of $C_0^\infty(S)$ and $\mathcal{D}_\circ$, that is,

$$C_0^\infty(S) \otimes \mathcal{D}_\circ = \{\sum_{i=1}^N f_i(x)g_i(\mathsf{y})\, ;\, f_i \in C_0^\infty(S),\, g_i \in \mathcal{D}_\circ,\, N \in \mathbb{N}\}.$$

Definition 6.11 ([47])**.** An $\mathbb{R}^d$-valued function $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mu^{[1]})^d$ is called *the logarithmic derivative* of μ if, for all $f \in C_0^\infty(S) \otimes \{\mathcal{D}_\circ \cap L^\infty(\mu)\}$,

$$\int_{S\times\mathsf{S}} \mathsf{d}^\mu(x,\mathsf{y}) f(x,\mathsf{y}) \mu^{[1]}(dxd\mathsf{y}) = -\int_{S\times\mathsf{S}} \nabla_x f(x,\mathsf{y}) \mu^{[1]}(dxd\mathsf{y}).$$

Lemma 6.12 ([47])**.** Assume **(A1)–(A4)**. Assume the logarithmic derivatives d^μ of μ exists. Then, the following ISDE has a solution.

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, X_t^{i,\diamondsuit})dt \quad (i \in \mathbb{N}). \tag{6.20}$$

Here $X_t^{i,\diamondsuit}$ denotes $\sum_{j\neq i} \delta_{x_t^j}$

Assume the logarithmic derivative d^{μ^N} of μ^N exists. Then the finite particle dynamics $\mathbf{X}^N = (X^{N,1}, \dots, X^{N,N}) \in C([0,\infty)\, ;\, S^N)$ are solutions of SDEs such that

$$dX_t^{N,i} = dB_t^{N,i} + \frac{1}{2}\mathsf{d}^{\mu^N}(X_t^{N,i}, X_t^{N,i\diamondsuit})dt \quad (i = 1, \dots, N). \tag{6.21}$$

Combining Lemma 6.12 with Theorem 6.10, we obtain convergence in distribution of solutions of SDEs (6.21) to a solution of the ISDE (6.20).

Remark 6.13. If we assume that the ISDE (6.20) has a unique solution in distribution, then the condition **(A3)** holds [31].

6.3 The generalized Mosco convergence

Our main result needs the language of Dirichlet forms and convergence concept of it. In this section, we recall the generalized Mosco convergence in the sense of Kuwae-Shioya [34].

Definition 6.14. Let H_N ($N \in \mathbb{N}$) and H be Hilbert spaces. We say $\{H_N\}_{N\in\mathbb{N}}$ converges to H if there exists a dense subspace $\mathcal{C} \subset H$ and a sequence of operators

$$\Phi_N : \mathcal{C} \to H_N$$

such that for any $u \in \mathcal{C}$,

$$\lim_{N\to\infty} ||\Phi_N u||_{H_N} = ||u||_H.$$

Definition 6.15. (1)We say that a sequence $\{u_N\}$ with $u_N \in H_N$ strongly converges to $u \in H$ if there exists $\{\tilde{u}_M\} \subset \mathcal{C}$ such that

$$\begin{aligned}&\lim_{M\to\infty} ||\tilde{u}_M - u||_H = 0,\\ &\lim_{M\to\infty} \limsup_{N\to\infty} ||\Phi_N \tilde{u}_M - u_N||_{H_N} = 0.\end{aligned}$$

(2) We say $\{u_N\}$ with $u_N \in H_N$ weakly converges to $u \in H$ if

$$\lim_{N\to\infty} (u_N, v_N)_{H_N} = (u, v)_H.$$

for any sequence $\{v_N\}$ with $v_N \in H_N$ which strongly converges to $v \in H$.

Definition 6.16. Let $L(H)$ denote the set consisting of linear operator on H. We say that a sequence of bounded operators $\{B_N\}$ with $B_N \in L(H_N)$ strongly converges to an operator $B \in L(H)$ if for any sequence $\{u_N\}$ with $u_N \in H_N$ which strongly converges to $u \in H$, $\{B_N u_N\}$ strongly converges to Bu.

Let $(\mathcal{E}, \mathcal{D})$ be a non-negative, symmetric bilinear form $\mathcal{E} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, where $\mathcal{D}$ is a subspace of Hilbert space H. We identify a bilinear form $\mathcal{E}$ with the function on H such that

$$\mathcal{E}(u) = \begin{cases} \mathcal{E}(u,u), & u \in \mathcal{D},\\ \infty, & u \notin \mathcal{D}.\end{cases}$$

We say that $\mathcal{E}$ is a bilinear form on H if the domain of $\mathcal{E}$ is a subset of H.

Definition 6.17. We say that a sequence $\{\mathcal{E}^N\}$ of bilinear forms $\mathcal{E}$ on H_N converges in Mosco to a bilinear form $\mathcal{E}$ on H if the following two conditions hold.
(1) If a sequence $\{u_N\}$ with $u_N \in H_N$ weakly converges to $u \in H$, then

$$\mathcal{E}(u) \le \liminf_{N\to\infty} \mathcal{E}^N(u_N).$$

(2) For any $u \in H$, there exists a strongly convergent sequence $\lim_{N\to\infty} u_N = u$ with $u_N \in H_N$ such that

$$\mathcal{E}(u) = \lim_{N\to\infty} \mathcal{E}^N(u_N).$$

Let $\{T_t^N\}_{t\ge 0}$ and $\{T_t\}_{t\ge 0}$ be the subgroups on H_N and H associated with $\mathcal{E}^N$ and $\mathcal{E}$, respectively. We quote:

Proposition 6.18 ([34])**.** The following are equivalent.
(1) $\lim_{N\to\infty} \mathcal{E}^N = \mathcal{E}$ in Mosco.
(2) $\lim_{N\to\infty} T_t^N = T_t$ strongly for all $t > 0$.

We thus see that the Mosco convergence of Dirichlet forms is equivalent to the strong convergence of the associated semigroup.

6.4 Convergence of unlabeled dynamics I: Proof of Theorem 6.6

In this section, we give the proof of Theorem 6.6. Throughout this section, we always assume **(A1)**–**(A4)** and **(M1)**–**(M3)**.

We shall utilize the notion of Mosco convergence in Definition 6.17. We take $H_N = L^2(\mu^N)$, $H = L^2(\mu)$, and $\mathcal{C} = \{f \in \mathcal{D}_\circ^\mu\, ;\, f \text{ is bounded}\}$ in Definition 6.14. Furthermore, we take Φ_N as identity map. Then we have $\lim_{N\to\infty} H_N = H$ in the sense of Definition 6.14.

6.4.1 Lower schemes of Dirichlet forms.

First, we give a proof of Definition 6.17 (1). We set

$$\mathcal{E}_r^N(f,g) = \sum_{m=1}^{\infty} \int_{S_r^m} \mathbb{D}[f,g]\sigma_r^{N,m} d\mathbf{x}_m.$$

By the assumption **(M1)**, let $(\underline{\mathcal{E}}_{r,k}^N, \underline{\mathcal{D}}_{r,k}^N)$ and $(\underline{\mathcal{E}}_r^N, \underline{\mathcal{D}}_r^N)$ be closures of $(\mathcal{E}_{r,k}^N, \mathcal{D}_\circ)$ and $(\mathcal{E}_r^N, \mathcal{D}_\circ)$ on $L^2(\mu^N)$ respectively. Let $(\underline{\mathcal{E}}^N, \underline{\mathcal{D}}^N)$ be the increasing limit of $\{(\underline{\mathcal{E}}_r^N, \underline{\mathcal{D}}_r^N)\}$ as $r\to\infty$. Then $(\underline{\mathcal{E}}^N, \underline{\mathcal{D}}^N) = (\mathcal{E}^N, \mathcal{D}^N)$ because μ^N is supported on the set consisting of N-particles. The next lemma is clear by definition.

Lemma 6.19. For each $N, k, r \in \mathbb{N}$ and $f \in L^2(\mu^N)$,

$$\mathcal{E}^N(f) = \underline{\mathcal{E}}^N(f) \geq \underline{\mathcal{E}}_{r,k}^N(f). \tag{6.22}$$

Proof. By definition, we have $\underline{\mathcal{D}}^N \subset \underline{\mathcal{D}}_r^N \subset \underline{\mathcal{D}}_{r,k}^N$ and

$$\mathcal{E}^N(f) = \underline{\mathcal{E}}^N(f) \geq \underline{\mathcal{E}}_r^N(f) \geq \underline{\mathcal{E}}_{r,k}^N(f).$$

These imply (6.22). □

We consider the bilinear form $\mathcal{E}_{r,k} = \sum_{m=1}^k \mathcal{E}_r^m$ on $L^2(\mu)$. From **(A1)**, we see that $(\mathcal{E}_{r,k}, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mu)$. Then let $(\underline{\mathcal{E}}_{r,k}, \underline{\mathcal{D}}_{r,k})$ be its closure.

Lemma 6.20. Let $f_N \in L^2(\mu^N)$ and $f \in L^2(\mu)$. Assume that $f_N \to f$ weakly. Then

$$\liminf_{N\to\infty} \underline{\mathcal{E}}_{r,k}^N(f_N) \geq \underline{\mathcal{E}}_{r,k}(f). \tag{6.23}$$

Proof. If $\liminf_{N\to\infty} \underline{\mathcal{E}}_{r,k}^N(f_N) = \infty$, then (6.23) is obvious. Hence we assume

$$\liminf_{N\to\infty} \underline{\mathcal{E}}_{r,k}^N(f_N) < \infty, \tag{6.24}$$

which implies $f_N \in \underline{\mathcal{D}}_{r,k}^N$ infinitely many times. Remark that from (6.12) and (6.24) we have

$$\liminf_{N\to\infty} \underline{\mathcal{E}}_{r,k}(f_N) < \infty. \tag{6.25}$$

For $f_N \in \mathcal{D}_\circ$, by a direct calculation we have

$$\begin{aligned}\underline{\mathcal{E}}^N_{r,k}(f_N) &= \sum_{l=1}^{k} \int_{S_r^l} \mathbb{D}[f_N]\sigma_r^{N,l} d\mathbf{x}_l \\ &= \sum_{l=1}^{k} \int_{S_r^l} \mathbb{D}[f_N]\sigma_r^l \Big\{ \frac{\sigma_r^{N,l}}{\sigma_r^l} - 1 \Big\} d\mathbf{x}_l + \underline{\mathcal{E}}_{r,k}(f_N) \\ &\geq - \underline{\mathcal{E}}_{r,k}(f_N) \Big\{ \max_{l=1,\ldots,k} \Big\| \frac{\sigma_r^{N,l}}{\sigma_r^l} - 1 \Big\|_{S_r^l} \Big\} + \underline{\mathcal{E}}_{r,k}(f_N).\end{aligned} \tag{6.26}$$

From (6.12), (6.25) and (6.26) we obtain that for any $\{f_N\} \subset \mathcal{D}_\circ$ satisfying $\lim_{N\to\infty} f_N = f$ weakly,

$$\liminf_{N\to\infty} \underline{\mathcal{E}}^N_{r,k}(f_N) \geq \liminf_{N\to\infty} \underline{\mathcal{E}}_{r,k}(f_N).$$

By approximation, we obtain that for sequence $\{f_N\}$ such that $f_N \in \underline{\mathcal{D}}^N_{r,k}$ infinitely many times,

$$\liminf_{N\to\infty} \underline{\mathcal{E}}^N_{r,k}(f_N) \geq \liminf_{N\to\infty} \underline{\mathcal{E}}_{r,k}(f_N). \tag{6.27}$$

We take a subsequence of $\{\underline{\mathcal{E}}_{r,k}(f_N)\}$, denoted by the same symbol, such that

$$\lim_{N\to\infty} \underline{\mathcal{E}}_{r,k}(f_N) = \liminf_{N\to\infty} \underline{\mathcal{E}}_{r,k}(f_N).$$

Recall that $\{\underline{\mathcal{E}}_{r,k}(\cdot,\cdot) + \alpha(\cdot,\cdot)_{L^2(\mu)}\}$ is a Hilbert space for any $\alpha > 0$. By (6.24) and (6.27) we see that the subsequence $\{\underline{\mathcal{E}}_{r,k}(f_N) + \alpha||f_N||_{L^2(\mu)}\}_{N\in\mathbb{N}}$ is bounded. Hence, taking a further subsequence if necessary, $\{f_N\}$ is also a weak convergent sequence with respect to $\underline{\mathcal{E}}_{r,k}(\cdot) + \alpha||\cdot||_{L^2(\mu)}$. Then for each $\alpha > 0$ we get

$$\liminf_{N\to\infty} \{\underline{\mathcal{E}}_{r,k}(f_N) + \alpha||f_N||_{L^2(\mu)}\} \geq \underline{\mathcal{E}}_{r,k}(f) + \alpha||f||_{L^2(\mu)}.$$

Therefore we obtain

$$\liminf_{N\to\infty} \underline{\mathcal{E}}_{r,k}(f_N) \geq \underline{\mathcal{E}}_{r,k}(f) \tag{6.28}$$

Collecting (6.27) and (6.28), we obtain (6.23). □

Lemma 6.21. We have the following.

$$\lim_{k\to\infty} \underline{\mathcal{E}}_{r,k}(f) = \underline{\mathcal{E}}_r(f) \quad \text{for each } r \in \mathbb{N}, \tag{6.29}$$

$$\lim_{r\to\infty} \underline{\mathcal{E}}_r(f) = \underline{\mathcal{E}}(f). \tag{6.30}$$

Proof. The first claim is clear because the sequence $\{\underline{\mathcal{E}}_{r,k}(f)\}$ is increasing in k for each $r \in \mathbb{N}$. (6.30) follows from [44, Theorem 3]. □

Lemma 6.22. Assume that $f_N \to f$ weakly. Then

$$\liminf_{N\to\infty} \underline{\mathcal{E}}^N(f_N) \geq \mathcal{E}(f) \tag{6.31}$$

Proof. From Lemma 6.19 and Lemma 6.20 we obtain

$$\liminf_{N\to\infty} \underline{\mathcal{E}}^N(f_N) \geq \liminf_{N\to\infty} \underline{\mathcal{E}}^N_{r,k}(f_N) \geq \underline{\mathcal{E}}_{r,k}(f) \tag{6.32}$$

for any $k \in \mathbb{N}$. Combining (6.32) with (6.29), we see that

$$\liminf_{N\to\infty} \underline{\mathcal{E}}^N(f_N) \geq \underline{\mathcal{E}}_r(f). \tag{6.33}$$

Then taking $r \to \infty$ in (6.33), we obtain from (6.30)

$$\liminf_{N\to\infty} \underline{\mathcal{E}}^N(f_N) \geq \underline{\mathcal{E}}(f). \tag{6.34}$$

Recall that $\underline{\mathcal{E}} = \mathcal{E}$ by **(A3)**. Then (6.34) implies (6.31). $\square$

6.4.2 Upper schemes of Dirichlet forms.

We shall check (2) of Definition 6.14.

Let $M(\ell) = \{M_k(\ell)\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $\lim_{k\to\infty} M_k(\ell) = \infty$ for each $\ell \in \mathbb{N}$, and that for each $k, \ell \in \mathbb{N}$

$$M_k(\ell) < M_{k+1}(\ell), \quad M_k(\ell) < M_k(\ell+1).$$

We can take $M(\ell)$ in such a way that

$$\lim_{\ell\to\infty} \mu\Big(\bigcap_{k\in\mathbb{N}} \{\mathsf{s}\,;\, \mathsf{s}(S_k) \leq M_k(\ell)\}\Big) = 1. \tag{6.35}$$

Indeed, recalling that a subset A in S is relatively compact if and only if there exists an increasing sequence of natural numbers M_r such that $\mathsf{A} \subset \{\mathsf{s}; \mathsf{s}(S_r) \leq M_r \text{ for all } r\}$, we obtain (6.35).

Let $\{N_k(\ell)\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $N_1(\ell) = 1$. We set for $\ell \in \mathbb{N}$

$$\gamma_k(\ell) = \sup_{N_k(\ell)\leq N<\infty} \Big\{ \max_{1\leq j\leq M_{k+1}(\ell)+1} \Big\| \frac{\sigma_k^{N,j}}{\sigma_k^j} - 1 \Big\|_{S_k^j} \Big\}, \tag{6.36}$$

where $\|\cdot\|_{S_k^j}$ denotes the supremum norm on S_k^j.

Lemma 6.23. We can take $\{N_k(\ell)\}$ in such a way that for each $\ell \in \mathbb{N}$

$$\lim_{k\to\infty} \gamma_k(\ell) = 0.$$

Proof. This follows from **(M3)** immediately. $\square$

We next introduce the cut off functions $\{\chi_\ell\}$ as follows:

$$\chi_\ell(\mathsf{s}) = \rho \circ d_\ell(\mathsf{s}), \qquad d_\ell(\mathsf{s}) = \Big\{ \sum_{k=1}^{\infty} \sum_{j \in J_{k,\mathsf{s}}(\ell)} (k - |\mathfrak{l}_j(\mathsf{s})|)^2 \Big\}^{\frac{1}{2}}.$$

Here, $\rho \in C^\infty(\mathbb{R})$ is the function satisfying $\rho(t) \in [0,1]$ for any $t \in \mathbb{R}$, $\rho(x) = 1$ for $x \le 0$, $\rho(x) = 0$ for $x \ge 1$, and $\rho'(t) = 2$ for any $t \in \mathbb{R}$. Furthermore, $\mathfrak{l} = (\mathfrak{l}_1, \mathfrak{l}_2, \dots) : \mathsf{S} \to S^{\mathbb{N}}$ is a map satisfying $|\mathfrak{l}_j(\mathsf{s})| \le |\mathfrak{l}_{j+1}(\mathsf{s})|$ for all j, and we define

$$J_{k,\mathsf{s}}(\ell) = \{ j \,;\, j > M_k(\ell), \mathfrak{l}_j(\mathsf{s}) \in S_k \}.$$

The function $\mathfrak{l} = (\mathfrak{l}_j)$ is called a labeling map defined for μ-a.s. s. Let

$$\begin{aligned} \mathsf{S}[\ell] &= \{ \mathsf{s} \in \mathsf{S} \,;\, \mathsf{s}(S_k) \le M_k(\ell) \text{ for all } k \in \mathbb{N} \}, \\ \mathsf{S}[\ell]^+ &= \{ \mathsf{s} \in \mathsf{S} \,;\, \mathsf{s}(S_k) \le M_{k+1}(\ell) + 1 \text{ for all } k \in \mathbb{N} \}. \end{aligned} \tag{6.37}$$

By construction these are compact sets in S under the vague topology. We quote:

Lemma 6.24 ([44, Lemma 2.5])**.** For each ℓ the following hold.
(1) $\chi_\ell = 1$ on $\mathsf{S}[\ell]$ and $\chi_\ell = 0$ on $\mathsf{S} \backslash \mathsf{S}[\ell]^+$.
(2) $\chi_\ell f \in \mathcal{D}$ for each $f \in \mathcal{D}_\circ^\mu$.
(3) $\mathcal{E}_1(\chi_\ell f) \le 2\mathcal{E}_1(f)$ for each $f \in \mathcal{D}_\circ^\mu$.
(4) $0 \le \mathbb{D}[\chi_\ell](\mathsf{s}) \le 2$ on $\mathsf{S}[\ell]^+ \backslash \mathsf{S}[\ell]$, and $\mathbb{D}[\chi_\ell](\mathsf{s}) = 0$ on $(\mathsf{S}[\ell]^+ \backslash \mathsf{S}[\ell])^c$.

Lemma 6.25. For each $f \in L^2(\mu)$ there exists a sequence $\{g_N\}$ such that $g_N \in L^2(\mu^N)$ and that $\{g_N\}$ satisfies the following.

$$\lim_{N \to \infty} g_N = f \quad \text{strongly,} \tag{6.38}$$

$$\lim_{N \to \infty} \mathcal{E}^N(g_N) = \mathcal{E}(f). \tag{6.39}$$

Proof.
For any $f \in \mathcal{D}$ there exists a sequence $\{f_k\}_{k \in \mathbb{N}}$ in $\mathcal{D}_\circ^\mu$ such that

$$\lim_{k \to \infty} \mathcal{E}_1(f_k) = \mathcal{E}_1(f),$$

where $\mathcal{E}_1(f,g) = \mathcal{E}(f,g) + (f,g)_{L^2(\mu)}$ and $\mathcal{E}_1(f) = \mathcal{E}_1(f,f)$. Because μ^N is concentrated on the set consisting of N-particles, we have $\mathcal{D}_\circ \subset L^2(\mu^N)$. We then see that $f_k \in L^2(\mu^N)$ for $f_k \in \mathcal{D}_\circ^\mu$.

Recall that each element in $\mathcal{D}_\circ^\mu$ is local by definition. Then, each f_k is $\mathcal{F}_{r(k)}$-measurable for some $r(k) \in \mathbb{N}$, where $\mathcal{F}_k = \sigma[\pi_k]$ as before. Thus we can and do assume that f_k is $\mathcal{F}_k$-measurable without loss of generality. From $\{f_k\}_{k \in \mathbb{N}}$ we shall construct $\{g_N\}_{N \in \mathbb{N}}$ such that $\{g_N\}$ converges to f strongly and $\lim_{N \to \infty} \mathcal{E}_1^N(g_N) = \mathcal{E}_1(f)$.

Let $g_{\ell,N} = \chi_\ell f_k$ for $N_k(\ell) \le N < N_{k+1}(\ell)$, where $\{N_k(\ell)\}_{k \in \mathbb{N}}$ is an increasing sequence of natural numbers in (6.36). From Lemma 6.23 we can take and do $N_k(\ell)$ such that $N_k(\ell) \le N_k(\ell+1)$ for all $\ell \in \mathbb{N}$ and that for some $0 < \theta < 1$

$$\gamma_k(\ell)\mathcal{E}_1(f_k) \le \theta^k \quad \text{for each } k \in \mathbb{N}. \tag{6.40}$$

By $g_{\ell,N} = \chi_\ell f_k$, we deduce that

$$\begin{aligned}&|\mathcal{E}_1^N(g_{\ell,N}) - \mathcal{E}_1(f)| \\ \le&|\mathcal{E}_1^N(\chi_\ell f_k) - \mathcal{E}_1(\chi_\ell f_k)| + |\mathcal{E}_1(\chi_\ell f_k) - \mathcal{E}_1(\chi_\ell f)| + |\mathcal{E}_1(\chi_\ell f) - \mathcal{E}_1(f)|.\end{aligned} \tag{6.41}$$

From Lemma 6.24 and (6.40) we see that

$$|\mathcal{E}_1^N(\chi_\ell f_k) - \mathcal{E}_1(\chi_\ell f_k)| \le \gamma_k(\ell)\mathcal{E}_1(\chi_\ell f_k) \le \theta^k. \tag{6.42}$$

By the straightforward calculation we have from (6.35), (6.37), and Lemma 6.24

$$|\mathcal{E}_1(\chi_\ell f_k) - \mathcal{E}_1(\chi_\ell f)| = \Big|\int_{\mathsf{S}} \mathbb{D}[\chi_\ell f_k] - \mathbb{D}[\chi_\ell f] + \chi_\ell^2(f_k^2 - f^2)d\mu\Big| \xrightarrow[k\to\infty]{} o(\ell) \tag{6.43}$$

and

$$|\mathcal{E}_1(\chi_\ell f) - \mathcal{E}_1(f)| = \Big|\int_{\mathsf{S}} \mathbb{D}[\chi_\ell f] - \mathbb{D}[f] + (\chi_\ell^2 - 1)f^2 d\mu\Big| = o(\ell). \tag{6.44}$$

Putting (6.42), (6.43), and (6.44) into (6.41) and taking g_N from $g_{\ell,N}$ we obtain (6.38) and (6.39). $\square$

Proof of Theorem 6.6. From Lemma 6.22 and Lemma 6.25, we obtain the Mosco convergence in Definition 6.17. The Mosco convergence implies the convergence of finite-dimensional distributions of X^N to X. (see [34, Section 7]).

Because the initial distribution has a density in $L^p(\mu^N)$ for some $1 < p$, we see the tightness of $\{\mathsf{X}^N\}_{N\in\mathbb{N}}$ in $C([0,\infty);\mathsf{S})$ using Lyons-Zheng decomposition. $\square$

6.5 Convergence of unlabeled dynamics II: Proof of Theorem 6.7–Theorem 6.8.

In this section, we prove Theorem 6.7–Theorem 6.8 by using Theorem 6.6 and cut off argument for Dirichlet forms.

6.5.1 Finite volume approximation with cut off

Let $\mathbf{p} = \mathbf{p}(r,m)$ be a map $\mathbf{p} : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$. For such a map $\mathbf{p}$, we set

$$\mathsf{S}_r^m(\mathbf{p}) = \Big\{\mathsf{s} \in \mathsf{S}_r^m\,;\, \sigma_r^m(s_1,\ldots,s_m) > \frac{1}{\mathbf{p}(r,m)} \text{ for } \pi_r(\mathsf{s}) = \sum_{i=1}^m \delta_{s_i}\Big\},$$

$$\mathsf{S}_r(\mathbf{p}) = \bigcup_{m=1}^\infty \mathsf{S}_r^m(\mathbf{p}).$$

Let $\mathbb{D}$ and $\mathbb{D}_r^m$ be as in (6.7). We define new bilinear forms for $f,g \in \mathcal{D}_\circ$ as

$$\mathbb{D}_{r,\mathbf{p}}^m[f,g](\mathsf{s}) = \begin{cases}\mathbb{D}_r^m[f,g](\mathsf{s}) & \text{for } \mathsf{s} \in \mathsf{S}_r^m(\mathbf{p}),\\ 0 & \text{for } \mathsf{s} \notin \mathsf{S}_r^m(\mathbf{p}),\end{cases} \tag{6.45}$$

$$\mathcal{E}_{r,\mathbf{p}}^m(f,g) = \int_{\mathsf{S}} \mathbb{D}_{r,\mathbf{p}}^m[f,g](\mathsf{s})d\mu. \tag{6.46}$$

We set $\mathbb{D}_{r,\mathbf{p}}$ and $\mathcal{E}_{r,\mathbf{p}}$ similarly. By construction, we see that $\mathcal{E}_{r,\mathbf{p}} = \sum_{m=1}^\infty \mathcal{E}_{r,\mathbf{p}}^m$. Let

$$\mathcal{B}_{r,\mathbf{p}}^b = \{f \in \mathcal{B}_r^b\,;\, f \text{ is constant on each connected component of } \mathsf{S}_r(\mathbf{p})^c\}.$$

Lemma 6.26. (1) We have for each $r \in \mathbb{N}$ and $\mathbf{p}$

$$(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu) \le (\mathcal{E}_r, \mathcal{D}_\circ^\mu) \le (\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_r^b) \le (\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_{r,\mathbf{p}}^b). \tag{6.47}$$

(2) $(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mu)$.
(3) $(\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_{r,\mathbf{p}}^b)$ is closable on $L^2(\mu)$.

Proof. By definition, the first inequality in (6.47) is clear because $\mathsf{S}_r^m(\mathbf{p}) \subset \mathsf{S}_r^m$ and the definition of $\mathbb{D}_{r,\mathbf{p}}^m$ and $\mathcal{E}_{r,\mathbf{p}}^m$ given by (6.45) and (6.46). The second follows from $\mathcal{E}_r \le \mathcal{E}$ and $\mathcal{D}_\circ^\mu \supset \mathcal{B}_r^b$. The third follows from $\mathcal{D}_\circ^\mu \cap \mathcal{B}_r^b \supset \mathcal{D}_\circ^\mu \cap \mathcal{B}_{r,\mathbf{p}}^b$. We thus obtain (1).

We note that $\mathsf{S}_r^m(\mathbf{p})$ is an open set because σ_r^m is lower semi-continuous for each $m, r \in \mathbb{N}$. Because $\mathsf{S}_r^m(\mathbf{p})$ is an open set, $(\mathcal{E}_{r,\mathbf{p}}^m, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mu)$. Recall that

$$(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu) = (\sum_{m=1}^\infty \mathcal{E}_{r,\mathbf{p}}^m, \mathcal{D}_\circ^\mu).$$

We thus see that $(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu)$ is a countable sum of closable forms on $L^2(\mu)$. Hence $(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mu)$. We thus prove (2).

From (2) we have closability of $(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu)$ on $L^2(\mu)$. By (6.47) we have had $(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu) \le (\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_{r,\mathbf{p}}^b)$. Hence closablity of $(\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_{r,\mathbf{p}}^b)$ on $L^2(\mu)$ follows from that of $(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu)$ on $L^2(\mu)$. We thus complete the proof of (3). ☐

From Lemma 6.26, we define $(\underline{\mathcal{E}}_{r,\mathbf{p}}, \underline{\mathcal{D}}_{r,\mathbf{p}})$ and $(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_{r,\mathbf{p}})$ as the closures of $(\mathcal{E}_{r,\mathbf{p}}, \mathcal{D}_\circ^\mu)$ and $(\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_{r,\mathbf{p}}^b)$ in $L^2(\mu)$, respectively. Let $\mathbf{p}(n) = \mathbf{p}(n)(r,m)$ be a sequence of functions and satisfy the monotonicity in $n \in \mathbb{N}$ as follows: For each $(r,m) \in \mathbb{N}^2$

$$\mathbf{p}(n)(r,m) < \mathbf{p}(n+1)(r,m) \quad \text{for all } n \in \mathbb{N}.$$

Lemma 6.27. (1) $\{(\underline{\mathcal{E}}_{r,\mathbf{p}(n)}, \underline{\mathcal{D}}_{r,\mathbf{p}(n)})\}_{r\in\mathbb{N}}$ is increasing for each $n \in \mathbb{N}$.
(2) $\{(\mathcal{E}_{r,\mathbf{p}(n)}, \mathcal{D}_{r,\mathbf{p}(n)})\}_{r\in\mathbb{N}}$ is decreasing for each $n \in \mathbb{N}$.

Proof. We see that $\mathsf{S}_r(\mathbf{p}) \subset \mathsf{S}_{r+1}(\mathbf{p})$. Then (1) follows from $\mathcal{E}_{r,\mathbf{p}} \le \mathcal{E}_{r+1,\mathbf{p}}$ and (2) follows from $\mathcal{B}_{r,\mathbf{p}}^b \subset \mathcal{B}_{r+1,\mathbf{p}}^b$. ☐

By Lemma 6.27, we define the closed form $(\underline{\mathcal{E}}_{\mathbf{p}(n)}, \underline{\mathcal{D}}_{\mathbf{p}(n)})$ as the increasing limit of $(\underline{\mathcal{E}}_{r,\mathbf{p}(n)}, \underline{\mathcal{D}}_{r,\mathbf{p}(n)})$ in r, and $(\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_{\mathbf{p}(n)})$ as the decreasing limit of $(\mathcal{E}_{r,\mathbf{p}(n)}, \mathcal{D}_{r,\mathbf{p}(n)})$ in r. Moreover, $(\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_\circ^\mu)$ is closable and $(\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_{\mathbf{p}(n)})$ coincides with the closure of the maximal closable part of $(\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_\circ^\mu)$.

Lemma 6.28. (1) $\{(\underline{\mathcal{E}}_{\mathbf{p}(n)}, \underline{\mathcal{D}}_{\mathbf{p}(n)})\}_{n\in\mathbb{N}}$ is increasing.
(2) $\{(\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_{\mathbf{p}(n)})\}_{n\in\mathbb{N}}$ is decreasing.
(3) For each $n \in \mathbb{N}$ we have

$$(\underline{\mathcal{E}}_{\mathbf{p}(n)}, \underline{\mathcal{D}}_{\mathbf{p}(n)}) \le (\underline{\mathcal{E}}, \underline{\mathcal{D}}) \le (\mathcal{E}, \mathcal{D}) \le (\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_{\mathbf{p}(n)}). \tag{6.48}$$

Proof. By $\mathsf{S}_r(\mathbf{p}(n)) \subset \mathsf{S}_r(\mathbf{p}(n+1))$ and (6.45) we have $(\mathcal{E}_{r,\mathbf{p}(n)}, \mathcal{D}_\circ^\mu) \le (\mathcal{E}_{r,\mathbf{p}(n+1)}, \mathcal{D}_\circ^\mu)$. By taking closure we have $(\underline{\mathcal{E}}_{r,\mathbf{p}(n)}, \underline{\mathcal{D}}_{r,\mathbf{p}(n)}) \le (\underline{\mathcal{E}}_{r,\mathbf{p}(n+1)}, \underline{\mathcal{D}}_{r,\mathbf{p}(n+1)})$. Then taking a increasing limit in r, we obtain (1). By definition, $\mathcal{B}_{r,\mathbf{p}(n)}^b \subset \mathcal{B}_{r,\mathbf{p}(n+1)}^b$. Hence we have

$$(\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_{r,\mathbf{p}(n)}^b) \ge (\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_{r,\mathbf{p}(n+1)}^b) \quad \text{for any } r.$$

Therefore, we get $(\mathcal{E}_{r,\mathbf{p}(n)}, \mathcal{D}_{r,\mathbf{p}(n)}) \ge (\mathcal{E}_{r,\mathbf{p}(n+1)}, \mathcal{D}_{r,\mathbf{p}(n+1)})$, which shows (2). By (6.47)

$$(\underline{\mathcal{E}}_{r,\mathbf{p}(n)}, \underline{\mathcal{D}}_{r,\mathbf{p}(n)}) \le (\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r) \le (\mathcal{E}_r, \mathcal{D}_r) \le (\mathcal{E}_{r,\mathbf{p}(n)}, \mathcal{D}_{r,\mathbf{p}(n)}).$$

Taking $r \to \infty$ and recalling

$$\begin{aligned} &\lim_{r\to\infty} (\mathcal{E}_{r,\mathbf{p}(n)}, \mathcal{D}_{r,\mathbf{p}(n)}) = (\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_{\mathbf{p}(n)}) \\ &\lim_{r\to\infty} (\underline{\mathcal{E}}_{r,\mathbf{p}(n)}, \underline{\mathcal{D}}_{r,\mathbf{p}(n)}) = (\underline{\mathcal{E}}_{\mathbf{p}(n)}, \underline{\mathcal{D}}_{\mathbf{p}(n)}), \end{aligned}$$

we obtain (6.48). □

By Lemma 6.28 we obtain the limit closed forms $(\underline{\mathcal{E}}_{\mathbf{p}(\infty)}, \underline{\mathcal{D}}_{\mathbf{p}(\infty)})$ and $(\mathcal{E}_{\mathbf{p}(\infty)}, \mathcal{D}_{\mathbf{p}(\infty)})$ of $\{(\underline{\mathcal{E}}_{\mathbf{p}(n)}, \underline{\mathcal{D}}_{\mathbf{p}(n)})\}_{n\in\mathbb{N}}$ and $\{(\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_{\mathbf{p}(n)})\}_{n\in\mathbb{N}}$ as n goes to infinity, respectively. We remark that with the same reason as before $(\mathcal{E}_{\mathbf{p}(\infty)}, \mathcal{D}_{\mathbf{p}(\infty)})$ is the closed form that coincides with the closure of the maximal closable part of the decreasing limit of $(\mathcal{E}_{\mathbf{p}(n)}, \mathcal{D}_{\mathbf{p}(n)})$ as $n \to \infty$. Then from Lemma 6.28 (1) we have

$$(\underline{\mathcal{E}}_{\mathbf{p}(\infty)}, \underline{\mathcal{D}}_{\mathbf{p}(\infty)}) \le (\underline{\mathcal{E}}, \underline{\mathcal{D}}) \le (\mathcal{E}, \mathcal{D}) \le (\mathcal{E}_{\mathbf{p}(\infty)}, \mathcal{D}_{\mathbf{p}(\infty)}). \tag{6.49}$$

Lemma 6.29. Assume (6.14). Then the following hold.

(1) $\mathcal{D}_{\mathbf{p}(\infty)}$ is dense in $\mathcal{D}$ with respect to $\mathcal{E}$.

(2) $\underline{\mathcal{D}}$ is dense in $\underline{\mathcal{D}}_{\mathbf{p}(\infty)}$ with respect to $\underline{\mathcal{E}}_{\mathbf{p}(\infty)}$

Proof. Let $\mathcal{N} = \bigcup_{m,r=1}^{\infty} \{\mathsf{s} \in \mathsf{S}_r^m \,;\, \sigma_r^m(\mathsf{s}) = 0\}$. Then by (6.15) we have

$$\mathrm{Cap}(\mathcal{N}) = 0. \tag{6.50}$$

From (6.50) we see that $\mu(\mathcal{N}) = 0$. Thus, there exists a sequence $\{\varphi_q\}$ in $\mathcal{D}$ satisfying

$$\begin{aligned} &\varphi_q = 1 \quad \text{on } \mathcal{N}, \\ &0 \le \varphi_q(\mathsf{s}) \le 1 \quad \text{for all } \mathsf{s} \in \mathsf{S}, \\ &\lim_{q\to 0} \varphi_q(\mathsf{s}) = 0 \quad \text{for } \mu\text{-a.s. } \mathsf{s}, \\ &\lim_{q\to 0} \mathcal{E}_1(\varphi_q) = 0. \end{aligned} \tag{6.51}$$

Because of (6.51), we have $\varphi_q \in \mathcal{D}_{\mathbf{p}(\infty)}$. Hence

$$(1 - \varphi_q) \in \mathcal{D}_{\mathbf{p}(\infty)}. \tag{6.52}$$

Let $f \in \mathcal{D} \cap L^\infty(\mu)$. Then from (6.51) and (6.52) we easily deduce that

$$f(1 - \varphi_q) \in \mathcal{D}_{\mathbf{p}(\infty)} \cap L^\infty(\mu).$$

Moreover, when $f, \varphi_q \in \mathcal{D}_\circ^\mu$, we obtain

$$\begin{aligned}
\mathcal{E}(f - f(1-\varphi_q)) &= \mathcal{E}_{\mathbf{p}(\infty)}(f\varphi_q) \\
&= \lim_{n\to\infty}\lim_{r\to\infty}\int_{\mathsf{S}_r(\mathbf{p}(n))} \mathbb{D}[f\varphi_q]d\mu \\
&\le \lim_{n\to\infty}\lim_{r\to\infty} 2\int_{\mathsf{S}_r(\mathbf{p}(n))} \mathbb{D}[f]|\varphi_q|^2 + |f|^2\mathbb{D}[\varphi_q]d\mu \\
&= \lim_{n\to\infty}\lim_{r\to\infty} 2\int_{\mathsf{S}_r(\mathbf{p}(n))} \mathbb{D}[f]|\varphi_q|^2 + |f|^2\mathbb{D}[\varphi_q]d\mu \\
&= o(q) \quad \text{ as } q\to\infty.
\end{aligned}$$

Hence by approximation, $\mathcal{D}_{\mathbf{p}(\infty)} \cap L^\infty(\mu)$ is dense in $\mathcal{D}\cap L^\infty(\mu)$ with respect to $\mathcal{E}$. This completes the proof of (1) because $\mathcal{D}\cap L^\infty(\mu)$ is dense in $\mathcal{D}$ with respect to $\mathcal{E}$.

Remark that $\varphi_q \in \underline{\mathcal{D}}$ since $\mathcal{D}\subset\underline{\mathcal{D}}$. Then for $f \in \underline{\mathcal{D}}_{\mathbf{p}(\infty)} \cap L^\infty(\mu)$, we have

$$f(1-\varphi_q) \in \underline{\mathcal{D}} \cap L^\infty(\mu).$$

We conclude (2) in a similar way to the proof of (1). $\square$

Lemma 6.30. Assume (6.14). Then

$$(\underline{\mathcal{E}}_{\mathbf{p}(\infty)}, \underline{\mathcal{D}}_{\mathbf{p}(\infty)}) = (\underline{\mathcal{E}}, \underline{\mathcal{D}}) \le (\mathcal{E}, \mathcal{D}) = (\mathcal{E}_{\mathbf{p}(\infty)}, \mathcal{D}_{\mathbf{p}(\infty)}). \tag{6.53}$$

Proof. (6.53) follows from (6.49) and Lemma 6.29 immediately.

$\square$

6.5.2 N-particle approximation and the Mosco convergence II.

Next, we define cut off Dirichlet forms associated with μ^N. Let $(\underline{\mathcal{E}}^N_{\mathbf{p}(n)}, \underline{\mathcal{D}}^N_{\mathbf{p}(n)})$ be the increasing limit $\lim_{r\to\infty}(\underline{\mathcal{E}}^N_{r,\mathbf{p}(n)}, \underline{\mathcal{D}}^N_{r,\mathbf{p}(n)})$, here $(\underline{\mathcal{E}}^N_{r,\mathbf{p}(n)}, \underline{\mathcal{D}}^N_{r,\mathbf{p}(n)})$ is the closure of $(\mathcal{E}^N_{r,\mathbf{p}(n)}, \mathcal{D}_\circ)$ on $L^2(\mu^N)$. Define $(\mathcal{E}^N_{\mathbf{p}(n)}, \mathcal{D}^N_{\mathbf{p}(n)})$ as the maximal closable part less than $\lim_{r\to\infty}(\mathcal{E}^N_{r,\mathbf{p}(n)}, \mathcal{D}^N_{r,\mathbf{p}(n)})$, here $(\mathcal{E}^N_{r,\mathbf{p}(n)}, \mathcal{D}^N_{r,\mathbf{p}(n)})$ is the closure of $(\mathcal{E}^N, \mathcal{D}_\circ\cap\mathcal{B}^b_{r,\mathbf{p}(n)})$ on $L^2(\mu^N)$. Clearly, we obtain

$$(\underline{\mathcal{E}}^N_{\mathbf{p}(n)}, \underline{\mathcal{D}}^N_{\mathbf{p}(n)}) \le (\mathcal{E}^N, \mathcal{D}^N) \le (\mathcal{E}^N_{\mathbf{p}(n)}, \mathcal{D}^N_{\mathbf{p}(n)}). \tag{6.54}$$

Lemma 6.31. Assume **(M3')**. For any $f\in L^2(\mu)$ and $n\in\mathbb{N}$, the following holds.
(1) Assume that $\lim_{N\to\infty} f_N = f$ weakly. Then

$$\underline{\mathcal{E}}_{\mathbf{p}(n)}(f) \le \liminf_{N\to\infty} \underline{\mathcal{E}}^N_{\mathbf{p}(n)}(f_N). \tag{6.55}$$

(2) There exist $f_{n,N}\in L^2(\mu^N)$ such that $\lim_{N\to\infty} f_{n,N} = f$ strongly and that

$$\mathcal{E}_{\mathbf{p}(n)}(f) = \lim_{N\to\infty} \mathcal{E}^N_{\mathbf{p}(n)}(f_{n,N}) \tag{6.56}$$

Proof. Recalling that densities of μ are positive on $\mathsf{S}(\mathbf{p}(n))$, **(M3)** holds on $\mathsf{S}(\mathbf{p}(n))$ from (6.15) in **(M3')**. Therefore we obtain (1) by the same argument as in Section 6.4.1. Similarly, we obtain (2) by the same argument as in Section 6.4.2. $\square$

Lemma 6.32. Assume **(M3')** and **(A3)**. Assume that $f_N \to f$ weakly. Then

$$\mathcal{E}(f) \le \liminf_{N\to\infty} \mathcal{E}^N(f_N) \tag{6.57}$$

Proof. By (6.54) we have $\underline{\mathcal{E}}^N_{\mathbf{p}(n)}(f_N) \le \mathcal{E}^N(f_N)$. From this, (6.53), and (6.55),

$$\begin{aligned}\underline{\mathcal{E}}(f) = \underline{\mathcal{E}}_{\mathbf{p}(\infty)}(f) &= \lim_{n\to\infty} \underline{\mathcal{E}}_{\mathbf{p}(n)}(f) \\ &\le \liminf_{n\to\infty} \liminf_{N\to\infty} \underline{\mathcal{E}}^N_{\mathbf{p}(n)}(f_N) \\ &\le \liminf_{n\to\infty} \liminf_{N\to\infty} \mathcal{E}^N(f_N) = \liminf_{N\to\infty} \mathcal{E}^N(f_N).\end{aligned} \tag{6.58}$$

By **(A3)** we see $\mathcal{E}(f) = \underline{\mathcal{E}}(f)$, which together with (6.58) implies (6.57). □

In the next lemma, we prove Definition 6.17 (2).

Lemma 6.33. Assume **(M3')** and **(A3)**. Then for any $f \in L^2(\mu)$ there exists a sequence $g_N \in L^2(\mu^N)$ $(N \in \mathbb{N})$ such that g_N converges to f strongly and

$$\lim_{N\to\infty} \mathcal{E}^N(g_N) = \mathcal{E}(f). \tag{6.59}$$

Proof. Let $f_{n,N}$ denote the sequence in (6.56). Note that from (6.53) we obtain

$$\mathcal{E}(f) = \lim_{n\to\infty} \mathcal{E}_{\mathbf{p}(n)}(f) \tag{6.60}$$

Combining (6.56) and (6.60), we can take g_N satisfying (6.59) by choosing a subsequence of $\{f_{n,N}\}_{n,N\in\mathbb{N}}$. □

Proof of Theorem 6.7. Lemma 6.32 and Lemma 6.33 implies the Mosco convergence. The rest of the proof is same as that of Theorem 6.6. □

Proof of Theorem 6.8. **(M3")** implies **(M3')**. Hence Theorem 6.8 follows from Theorem 6.7. □

6.6 Examples.

In this section, we give examples of dynamical universality. All examples satisfy the assumptions in Theorem 6.6–Theorem 6.10, and main theorems are thus applicable to these examples.

6.6.1 Universality of Airy interacting Brownian motions.

The first example is Airy random point field with $\beta = 2$. Let μ_{Ai} be the Airy random point field with $\beta = 2$, that is, μ_{Ai} is the determinantal random point field on $S = \mathbb{R}$ whose kernel with respect to the Lebesgue measure is given by

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(x)'\mathrm{Ai}(y)}{x-y}.$$

Here, $\mathrm{Ai}(x)$ is the Airy function. The corresponding ISDE is [54]

$$dX_t^i = dB_t^i + \lim_{r\to\infty}\Big\{\Big(\sum_{j\neq i,\ |X_t^j|<r}\frac{1}{X_t^i - X_t^j}\Big) - \int_{|x|<r}\frac{\varsigma_{Ai}(x)}{-x}dx\Big\}dt. \tag{6.61}$$

Here we set

$$\varsigma_{\mathrm{Ai}}(x) = \frac{1_{(-\infty,0)}(x)}{\pi}\sqrt{-x}.$$

It is known that ISDE (6.61) has a pathwise unique, strong solution $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ for μ_{Ai}-a.s. $\mathsf{s} = \sum_{i=1}^{\infty}\delta_{X_0^i}$ [54].

To introduce the N-particle system, we set $l \in \mathbb{N}$ and $\kappa_{2l} > 0$. Let

$$V(x) = \sum_{i=0}^{2l}\kappa_i x^i.$$

Consider $\mu_{\mathrm{Ai},V}^N$ such that

$$\mu_{V,\mathrm{Ai}}^N(d\mathbf{s}^N) = \frac{1}{Z}\prod_{i<j}^N|s_i - s_j|^2\prod_{k=1}^N\exp(-NV(N^{-\frac{1}{2l}}(c_N\Big(1+\frac{s}{\alpha_N N^{\frac{2}{3}}}\Big)+d_N)))\,d\mathbf{s}^N.$$

Here, α_N, c_N, d_N are constants depending only on V and N. By integration by parts, we can easily calculate the logarithmic derivative of $\mu_{\mathrm{Ai},V}^N$ and show that the corresponding N-dimensional SDE is

$$\begin{aligned}dX_t^{N,i} = dB_t^i &+ \sum_{1\le j\neq i\le N}\frac{1}{X_t^{N,i} - X_t^{N,j}}dt \\ &- \frac{N^{\frac{1}{3}-\frac{1}{2l}}c_N}{2\alpha_N}V'(N^{-\frac{1}{2l}}(c_N(1+\frac{X_t^{N,i}}{\alpha_N N^{\frac{2}{3}}})+d_N))dt.\end{aligned}$$

Let $\rho_{\mathrm{Ai},V}^{N,n}$ be the n-correlation function of $\mu_{\mathrm{Ai},V}^N$. Then the soft edge universality result is proved by [9], which asserts

$$\lim_{N\to\infty}\rho_{\mathrm{Ai},V}^{N,n} = \rho_{\mathrm{Ai}}^n \quad \text{compact uniformly.}$$

In [31] **(A3)** is proved. We can thus apply Theorem 6.8 and Theorem 6.10, which show the convergences (6.13) and (6.19).

6.6.2 Universality of Ginibre interacting Brownian motions at strong non-Hermiticity.

In this section we apply our result to random matrix model with strong non-Hermiticity introduced in [1].

Let $S = \mathbb{R}^2$. We naturally regard $\mathbb{R}^2$ as $\mathbb{C}$ by $(x,y)\mapsto x+\sqrt{-1}y$. The Ginibre random point field μ_{gin} is the determinantal random point field with kernel with respect to the Lebesgue measure such that

$$K_{\mathrm{gin}}(x,y) = \frac{1}{\pi}\exp[-\frac{|x|^2+|y|^2}{2}+x\bar{y}].$$

Then by definition the k-point correlation function ρ^k_{gin} is given by

$$\rho^k_{\mathrm{gin}}(x_1,\dots,x_k) = \det[K_{\mathrm{gin}}(x_i,x_j)]_{i.j=1}^k.$$

It is known that the Ginibre random point field μ_{gin} satisfies **(A3)** (see [31]).

Let $\mathcal{J}(N)$ be the space of the normal matrices of order N. For constants $\gamma \ge 0$, $K_p \in \mathbb{R}$, and $\tau \in [0,1)$, we consider probability on $\mathcal{J}(N)$ whose density is given by

$$\sigma(J) = \frac{1}{\mathcal{Z}} \exp\Big[-\frac{N}{1-\tau^2}\mathrm{Tr}(JJ^* - \frac{\tau}{2}(J^2+J^{*2})) - \gamma(\mathrm{Tr}JJ^* - NK_p)^2\Big]$$

Then the joint density of eigen values is proportional to

$$\prod_{i\neq j}^N |z_i - z_j|^2 \times \exp[-\frac{N}{1-\tau^2}(\sum_{i=1}^N |z_i|^2 - \frac{\tau}{2}\sum_{i=1}^N (z_i^2 + \bar{z}_i{}^2)) - \gamma(\sum_{i=1}^N |z_i|^2 - NK_p)^2] \tag{6.62}$$

Let ρ^k_N be the k-point correlation function of the eigen value density corresponding to (6.62). Then there exists c_{41}, c_{42}, $c_{43} > 0$ depending on K_p, γ, τ such that with $E = \{z \in \mathbb{C}; c_{41}(\Re z)^2 + c_{42}(\Im z)^2 < 1\}$ the following universality holds:

Theorem 6.34 ([1, Theorem 1])**.** *For any $\zeta \in E$, $k \in \mathbb{N}$*

$$\lim_{N\to\infty} \frac{1}{N}\rho^1_N(\zeta) = \frac{c_{43}}{\pi} 1_E(\zeta).$$

As for local densities we have

$$\frac{1}{(c_{43}N)^k}\rho^k_N(\zeta + \frac{z_1}{\sqrt{c_{43}N}},\dots,\zeta+\frac{z_k}{\sqrt{c_{43}N}}) = \rho^k_{\mathrm{gin}}(z_1,\dots,z_k) + \mathcal{O}(\frac{1}{\sqrt{N}}).$$

From this theorem we obtain **(M3")**. Hence we obtain (6.14) and (6.19).

From (6.62) we easily calculate the logarithmic derivative of N-particle system, and obtain the N-dimensional SDE corresponding to μ^N_{Gin} as follows:

$$\begin{aligned} dX^i_t = dB^i_t &+ \frac{1}{2}\Big\{ \sum_{1\le j\neq i\le N} \frac{2(X^i_t - X^j_t)}{|X^i_t - X^j_t|^2}\Big\} - \frac{\tau N}{1-\tau^2}(\zeta + \frac{X^i_t}{\sqrt{c_{43}N}})\frac{1}{\sqrt{c_{43}N}} \\ &+ \frac{\tau N}{1-\tau^2}(\zeta + \frac{X^i_t}{\sqrt{c_{43}N}})^\dagger \frac{1}{\sqrt{c_{43}N}} \\ &- (\zeta + \frac{X^i_t}{\sqrt{c_{43}N}})\frac{2\gamma}{\sqrt{c_{43}N}}\Big\{\sum_{k=1}^N \Big|\zeta + \frac{X^k_t}{\sqrt{c_{43}N}}\Big|^2 - NK_p\Big\}dt \quad \text{(i=1,...,N)}. \end{aligned}$$

Here we set $(x,y)^\dagger = (x,-y) \in \mathbb{R}^2$. The ISDE corresponding to μ_{Gin} is the following [47]:

$$dX^i_t = dB^i_t + \lim_{r\to\infty} \sum_{|X^i_t - X^j_t|<r, j\neq i} \frac{X^i_t - X^j_t}{|X^i_t - X^j_t|^2}dt \quad (i \in \mathbb{N}).$$

We thus see the representation of N-particle SDE is quite complicated, and the limit is very simple and universal.

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