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Braid zeta functions and certain q-identities

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Contents

Preface

For a finite set X, an automorphism $\sigma \in \mathrm{Aut}(X)$, we define the zeta function of the dynamical system (σ, X) as the following generating function.

$$\zeta_\sigma(s, X) := \exp\left\{\sum_{m=1}^{\infty} \frac{|\mathrm{Fix}(\sigma^m, X)|}{m} s^m\right\}.$$

Here, $\mathrm{Fix}(\sigma, X)$ is the fixed point set of σ in X. Kim, Koyama and Kurokawa showed that this dynamical zeta function has some properties which are analogous to the ones of Riemann zeta function such as functional equation, Euler product ([9]). Furthermore the zeta function of (σ, X) satisfies the analogue of the Riemann hypothesis. The primary reason for this property is that this dynamical zeta function has the determinant expression. Then focusing on this expression, we define the representation zeta function by using the determinant expression. More precisely, let G be a group and $\rho : G \longrightarrow \mathrm{GL}(V)$ be a finite-dimensional representation over $\mathbb{C}$, then we define the representation zeta function of $g \in G$ by

$$\zeta(s, g; \rho) := \det(I_{\dim \rho} - \rho(g)s)^{-1}.$$

Our motivation is to describe the invariants or information of the geometric objects such as braids and knots by using the zeta function. Then in this paper, we study the properties of the zeta function associated with a representation of the braid group B_n

In Chapter 2, we will introduce the braid zeta functions associated with three famous braid representations: Burau representation $\beta_{n,q}$, Jones representation $\chi_{n,q}$, and HOMFLY representation $\tau_{n,q}^{(N)}$. Especially, the zeta function associated with the Burau representation, for simplicity, we call "Burau zeta function" here, is a q-analogue of the zeta function of the finite dynamical system (σ, X). Furthermore, Burau zeta function has similar properties of dynamical zeta function except the Euler product. In particular, like the class number formula of the Dedekind zeta function which is a generalization of the Riemann zeta function, the residue at $s = 1$ of the Burau zeta function encodes an important geometric invariant such as the Alexander polynomial. Similar to the Burau zeta function, we will show that the Jones polynomial and HOMFLY polynomial which are famous knot invariants are expressed as the special value of the logarithmic derivative of the zeta functions associated with the remaining two representations respectively. Furthermore, we show that these zeta functions can be written by the Burau zeta function. Then considering the relationship between two different braid zeta functions, we can obtain the relationship between knot invariants. For example, in the last section of Chapter 2, by using the zeta function, we show that the HOMFLY polynomial of the closure of $\sigma \in B_3$ can be expressed by the Burau representation.

In Chapter 3, we consider the relation between the zeta function of a braid which is made by the two braids σ and τ, and two zeta functions of σ and τ.

$$\zeta(s, (\sigma, \tau)) \underset{?}{\longleftrightarrow} \zeta(s, \sigma), \zeta(s, \tau)$$

In this chapter, we give an example by defining the special product of braids and explicit formula of the Burau zeta function of a braid which is expressed as the special product of two braids. As an application, for two braids σ, τ, we show that the Alexander polynomial of the closure of special product $\sigma * \tau$ can be expressed by two Alexander polynomials of σ and τ.

In Chapter 4, we mainly study the zeta function of the torus type braid, which is defined in Chapter 1, and its arithmetic properties. First we introduce the Kosyak's braid representation of B_3 which is a q-deformation of the symmetric power of the reduced Burau representation. By simple calculation, we can see that the zeta function associated with this representation for the torus type braid in B_3 is related to the following q-identity which is known to be the Euler's pentagonal number theorem.

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k(3k-1)}{2}}.$$

In this chapter, we define the representation for B_n, denoted by $\rho_{n,q,t}^{(N)}$, which is a generalization of Kosyak's representation for B_3 and consider the relation between torus type braid and related q-series in general case. The main result of Chapter 4 shows that the zeta function associated with $\rho_{n,q,t}^{(N)}$ for the torus type braid can be expressed by the zeta function associated with a representation of the symmetric group $\mathfrak{S}_n$. As a corollary of this formula, we obtain some related q-identities. In the last section, for $\sigma \in B_n$, we define the trace generating function $Z_{q,t}(s,\sigma)$ as

$$Z_{q,t}(s,\sigma):=1+\sum_{N=1}^{\infty}\operatorname{tr}\rho_{n,q,t}^{(N)}(\sigma)s^N.$$

By the definition of $\rho_{n,q,t}^{(N)}$, and MacMahon Master Theorem, we can see that $Z_{q,t}(s,\sigma)$ is the q-deformation of the zeta function associated with the reduced Burau representation. Then, we show that $Z_{q,t}(s,\sigma)$ turns to the Alexander polynomial essentially when $q\to 1, s=1$. On the other hand when σ is torus type braid, we have the related q-series by substituting $t\to 1, s=1$. Thus, these limit formulas bring a new prospect for the relation between invariant of torus knots and arithmetic properties of q-series.

1 Introduction

1.1 Braid group

Let B_n be the braid group on n strands. It is known that B_n has the following presentation.

$$B_n := \langle \sigma_i \ (1 \le i \le n-1) \mid \sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j| \ge 2), \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ (1 \le i \le n-2)\rangle.$$

The generator σ_i can be identified with the crossing between the i-th and $(i+1)$-st strands as Figure 1, and the multiplication of generators implies that the braid obtained by attaching the

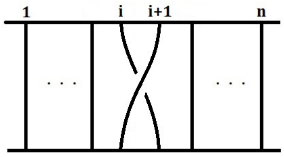

Figure 1: generator σ_i

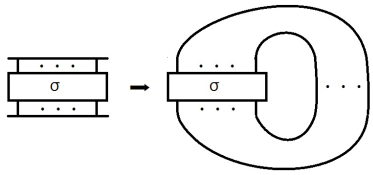

Figure 2: the closure of a braid σ

generators from the top to the bottom. The *closure* of a braid is the link obtained from the braid by connecting upper ends and lower ends as Figure 2. The closure of σ is denoted by $\widehat{\sigma}$. For a pair $(n, m) \in \mathbb{N}^2$, we define the *torus type braid* as follows.

$$\sigma_{n,m} := (\sigma_1 \cdots \sigma_{n-1})^m \in B_n.$$


Figure 3: torus tybe braid $\sigma_{5,3}$

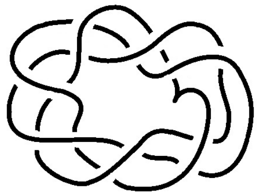
Figure 4: Torus knot $T(3, 7)$

When (n, m) is coprime, the closure of $\sigma_{n,m}$ turns to the torus knot which is denoted by $T(n, m)$ as usual. Next we define a map $\varepsilon : B_n \longrightarrow \mathbb{Z}$. We assume that $\sigma \in B_n$ can be expressed as $\sigma_{i_1}^{e_1}\sigma_{i_2}^{e_2}\cdots\sigma_{i_r}^{e_r}$, Then $\varepsilon(\sigma)$ is defined as

$$\varepsilon(\sigma) := e_1 + e_2 + \cdots + e_r.$$

Let $\beta_{n,q}$ be the *Burau representation*, which is defined by

$$\beta_{n,q} : B_n \longrightarrow GL(W_n),$$

$$\beta_{n,q}(\sigma_i) := I_{i-1} \oplus \begin{pmatrix} 1-q & 1 \\ q & 0 \end{pmatrix} \oplus I_{n-i-1} \qquad (i = 1, 2, \ldots, n-1).$$

Here W_n is n-dimensional vector space over $\mathbb{C}$ spanned by $\{f_1, f_2, \ldots, f_n\}$, and q is complex parameter. For simplicity, we assume that q is generic. Since $\det \beta_{n,q}(\sigma_i) = (-q)$, then we have

$$\det \beta_{n,q}(\sigma) = (-q)^{\varepsilon(\sigma)}.$$

It is well-known that the Burau representation $\beta_{n,q}$ can be decomposed into the trivial representation $\mathbf{1}$ and an $(n-1)$-dimensional irreducible representation $\beta^r_{n,q}$.

$$\beta_{n,q} = \mathbf{1} \oplus \beta^r_{n,q},$$

where $\beta^r_{n,q}$ is defined by

$$\beta^r_{n,q} : B_n \longrightarrow GL(W^r_n),$$

$$\beta^r_{n,q}(\sigma_i) := \begin{cases} \begin{pmatrix} -q & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3} & (i=1), \\ I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ q & -q & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} & (2 \le i \le n-2), \\ I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ q & -q \end{pmatrix} & (i = n-1). \end{cases}$$

Here, W^r_n is $(n-1)$-dimensional vector space spanned by $\{\widetilde{f}_1, \widetilde{f}_2, \dots, \widetilde{f}_{n-1}\}$. $\beta^r_{n,q}$ is called the *reduced Burau representation.*

1.2 The zeta function of a finite dynamical system

Let X_n be a finite set $X_n := \{1, 2, \dots, n\}$. For an automorphism $\sigma \in \mathrm{Aut}(X_n) \simeq \mathfrak{S}_n$, the pair (σ, X_n) is called *dynamical system.* Furthermore, for a dynamical system (σ, X_n), we define

$$\zeta_\sigma(s, X_n) := \exp\left\{ \sum_{m=1}^{\infty} \frac{|\mathrm{Fix}(\sigma^m, X_n)|}{m} s^m \right\},$$

where $\mathrm{Fix}(\sigma^m, X_n) := \{x \in X_n \mid \sigma^m x = x\}$. We call $\zeta_\sigma(s, X_n)$ the *dynamical zeta function of* (σ, X_n).

Example 1.2.1. Put $\sigma = (12)(345) \in \mathrm{Aut}(X_5) \simeq \mathfrak{S}_5$, then the number of fixed points is calculated as

$$|\mathrm{Fix}(\sigma^m, X_5)| = \begin{cases} 5 & (m \equiv 0 \mod 6), \\ 0 & (m \equiv \pm 1 \mod 6), \\ 2 & (m \equiv \pm 2 \mod 6), \\ 3 & (m \equiv 3 \mod 6). \end{cases}$$

Then we have

$$\begin{aligned} \zeta_\sigma(s, X_5) &= \exp\left\{ \sum_{m=1}^{\infty} \frac{|\mathrm{Fix}(\sigma^m, X_5)|}{m} s^m \right\} \\ &= \exp\left\{ \sum_{k=1}^{\infty} \frac{5}{6k} s^{6k} + \sum_{k=1}^{\infty} \frac{2}{6k-4} s^{6k-4} + \sum_{k=1}^{\infty} \frac{2}{6k-2} s^{6k-2} + \sum_{k=1}^{\infty} \frac{3}{6k-3} s^{6k-3} \right\} \\ &= \exp\left\{ \sum_{k=1}^{\infty} \frac{2}{2k} s^{2k} + \sum_{k=1}^{\infty} \frac{3}{3k} s^{3k} \right\} = \exp\left\{ \log(1-s^2)^{-1} + \log(1-s^3)^{-1} \right\} = \frac{1}{(1-s^2)(1-s^3)}. \end{aligned}$$

It is well-known that the dynamical zeta function $\zeta_\sigma(s, X_n)$ has the following properties (see [9] Proposition 1).

Proposition 1.2.1. *We regard* $\mathrm{Aut}(X_n) \simeq \mathfrak{S}_n$.
(1) *Let* $\mathrm{Cycle}(\sigma)$ *be the set of primitive cycles of* $\sigma \in \mathfrak{S}_n$, *and* $l(P)$ *be the length of cycle* $P \in \mathrm{Cycle}(\sigma)$. *Then,* $\zeta_\sigma(s, X_n)$ *has the following expression.*

$$\zeta_\sigma(s) = \prod_{P \in \mathrm{Cycle}(\sigma)} \frac{1}{1 - s^{l(P)}}.$$

(2) $\zeta_\sigma(s, X_n)$ *satisfies the following functional equation.*

$$\zeta_\sigma(s, X_n) = \mathrm{sgn}(\sigma)(-s)^{-n}\zeta_\sigma(s^{-1}, X_n),$$

where $\mathrm{sgn} : \mathfrak{S}_n \longrightarrow \{\pm 1\}$ *is the signature of the permutation.*
(3) *We have the following expression.*

$$\zeta_\sigma(s, X_n) = \det(I_n - p_n(\sigma)s)^{-1}.$$

Here, $p_n : \mathfrak{S}_n \longrightarrow GL_n(\mathbb{Z})$ *is the permutation representation.*
(4) *All poles of* $\zeta_\sigma(e^{-s}, X_n)$ *satisfy*

$$\mathrm{Re}(s) = 0.$$

Note that the formula (1) is the analogue of Euler product expression of the Riemann zeta function. Furthermore, (4) is the analogue of the Riemann hypothesis. Since the zeta function $\zeta_\sigma(s, X_n)$ has the determinant expression (3), we can prove easily the analogue of the Riemann hypothesis (4).

Next we focus on the determinant expression, and consider the generalization of the dynamical zeta function.

1.3 Representation zeta function

Let G be a group, and $\rho : G \longrightarrow GL(V)$ be a finite-dimensional representation of G over $\mathbb{C}$. Then we define the representation zeta function of $g \in G$ as

$$\zeta(s, g; \rho) := \det(I_{\dim \rho} - \rho(g)s)^{-1}.$$

Note that $\zeta(s, g; \rho)$ has the generating function expression as follows.

$$\zeta(s, g; \rho) = \exp\left\{\sum_{m=1}^{\infty} \frac{\mathrm{tr}\,\rho(g^m)}{m} s^m\right\}. \tag{1.1}$$

By the definition, if $\rho_1 \simeq \rho_2$, we have $\zeta(s, g; \rho_1) = \zeta(s, g; \rho_2)$. Furthermore if $\rho = \rho_1 \oplus \rho_2$, we have $\mathrm{tr}\,\rho(g) = \mathrm{tr}\,\rho_1(g) + \mathrm{tr}\,\rho_2(g)$ for all $g \in G$. Then, by (1.1), we have the following formula.

$$\zeta(s, g; \rho_1 \oplus \rho_2) = \zeta(s, g; \rho_1)\zeta(s, g; \rho_2). \tag{1.2}$$

When $\rho = p_n$ which is the permutation representation of the symmetric group $\mathfrak{S}_n$, the zeta function $\zeta(s, \sigma; p_n)$ coincides with the dynamical zeta function $\zeta_\sigma(s, X_n)$.

$$\zeta_\sigma(s, X_n) = \det(I_n - p_n(\sigma)s)^{-1} = \zeta(s, \sigma; p_n).$$

Hence, the representation zeta function can be regarded as the generalization of the dynamical zeta function.

2 Braid zeta functions

In this section we define the zeta function of a braid by using the braid group representation. That is, if there are some braid representations, then we can define the zeta functions associated with each representation.

2.1 The Burau representation $\beta_{n,q}$

First, we introduce the case of the Burau representation. Since the Burau representation is a q-deformation of the permutation representation, we can regard the zeta function associated with the Burau representation as a q-analogue of the dynamical zeta function. This is the simplest, but the most important braid zeta function. For a braid $\sigma \in B_n$, we can define the zeta function associated with the Burau representation as follows.

$$\zeta(s, \sigma; \beta_{n,q}) := \det(I_n - \beta_{n,q}(\sigma)s)^{-1}.$$

Theorem 2.1.1. (1) *For any $\sigma \in B_n$, we have the following limit formula.*

$$\lim_{q \to 1} \zeta(s, \sigma; \beta_{n,q}) = \zeta(s, \pi_n(\sigma); p_n) = \zeta_{\pi_n(\sigma)}(s, X_n).$$

Here, $\pi_n : B_n \longrightarrow \mathfrak{S}_n$ is natural projection defined by $\pi_n(\sigma_i) := (i, i+1) \in \mathfrak{S}_n$.
(2) *For any $\sigma \in B_n$, we have*

$$\zeta(s, \sigma; \beta_{n,q}) = (-q)^{-\varepsilon(\sigma)}(-s)^{-n}\zeta(1/s, \sigma^{-1}; \beta_{n,q}).$$

(3) *We assume that the closure of $\sigma \in B_n$ is a knot. Then we have the following residue formula.*

$$\operatorname*{Res}_{s=1} \zeta(s, \sigma; \beta_{n,q}) = -\frac{1}{[n]_q}\Delta_{\widehat{\sigma}}(q)^{-1}.$$

Here $\Delta_{\widehat{\sigma}}(q)$ is the Alexander polynomial of a knot $\widehat{\sigma}$, and $[n]_q$ is defined by

$$[n]_q := \frac{1 - q^n}{1 - q}.$$

(4) *Assume that q is a point of the unit circle on $\mathbb{C}$, in other words, q is expressed by $e^{i\theta}(\theta \in \mathbb{R})$, and that the argument of q satisfies $|\theta| < 2\pi/n$. Then for any $\sigma \in B_n$, all poles of $\zeta(e^{-s}, \sigma; \beta_{n,q})$ satisfy*

$$\mathrm{Re}(s) = 0.$$

(5) *For a coprime pair $(n, m) \in \mathbb{N}^2$, we have the following explicit formula.*

$$\zeta(s, \sigma_{n,m}; \beta_{n,q}) = \frac{(1 - q^m s)}{(1 - s)(1 - q^{nm}s^n)}.$$

Remark that $\zeta(s, \sigma; \beta_{n,q})$ does not have the Euler product expression in general. However, (2) and (4) are analogous to Proposition 1.2.1. Furthermore (3) is the characteristic property of the braid zeta function associated with the Burau representation. This property can be regarded as the analogue of the residue formula of the Dedekind zeta function which is the generalization of the Riemann zeta function.

Proof. (1) When $q \to 1$, the Burau representation turns to the permutation representation. Then we have the formula (1).
(2) By the definition, we have

$$\begin{aligned}\zeta(s,\sigma;\beta_{n,q}) = \det(I_n - \beta_{n,q}(\sigma)s)^{-1} &= \det(\beta_{n,q}(\sigma)s)^{-1}(-1)^n \det(I_n - \beta_{n,q}^{-1}(\sigma)s^{-1})^{-1} \\ &= (-q)^{-\varepsilon(\sigma)}(-s)^{-n}\zeta(s^{-1},\sigma^{-1};\beta_{n,q}).\end{aligned}$$

(3) When the closure of σ is knot, it is well-known that the Alexander polynomial of $\widehat{\sigma}$ can be obtained by using the reduced Burau representation as follows (see [4, Theorem 3.11]).

$$\det(I_{n-1} - \beta_{n,q}^r(\sigma)) = (1+q+q^2+\cdots+q^{n-1})\Delta_{\widehat{\sigma}}(q).$$

Since $\pi_n(\sigma)$ is the simple cycle, $\zeta(s,\pi_n(\sigma);p_n)$ has a simple pole at $s=1$. On the other hand, by the decomposition of the Burau representation, $\zeta(s,\sigma;\beta_{n,q})$ must have a pole at $s=1$. By the limit formula (1), the order of this pole is smaller or equal to the order of the pole of $\zeta(s,\pi_n(\sigma);p_n)$ at $s=1$. Thus $\zeta(s,\sigma;\beta_{n,q})$ has a simple pole at $s=1$. Moreover the residue of $\zeta(s,\sigma;\beta_{n,q})$ can be calculated as follows.

$$\begin{aligned}\operatorname*{Res}_{s=1}\zeta(s,\sigma;\beta_{n,q}) = \lim_{s\to 1}\det(I_n - \mathbf{1}\oplus\beta_{n,q}(\sigma)s)^{-1} &= \lim_{s\to 1}\frac{s-1}{1-s}\det(I_{n-1}-\beta_{n,q}(\sigma)s)^{-1} \\ &= -\frac{1}{[n]_q}\Delta_{\widehat{\sigma}}(q)^{-1}.\end{aligned}$$

(4) If the absolute values of the eigenvalues of $\beta_{n,q}^r(\sigma)$ are all equal to 1, then all poles of $\zeta(e^{-s},\sigma;\beta_{n,q})$ satisfy

$$e^{-\mathrm{Re}(s)} = |e^{-s}| = |\alpha_q|^{-1} = 1,$$

where α_q is one of the eigenvalues of $\beta_{n,q}^r(\sigma)$. Then, the real part of s is equal to 0.

$$\mathrm{Re}(s) = 0.$$

Hence, it is sufficient to show that the absolute values of the eigenvalues of $\beta_{n,q}^r(\sigma)$ are all equal to 1. In [18], Squier proved that the reduced Burau representation is unitary in the following sense. We put

$$\Omega_n^r = \begin{pmatrix} q^{\frac{1}{2}}+q^{-\frac{1}{2}} & -q^{\frac{1}{2}} & & O \\ -q^{-\frac{1}{2}} & \ddots & \ddots & \\ & \ddots & \ddots & -q^{\frac{1}{2}} \\ O & & -q^{-\frac{1}{2}} & q^{\frac{1}{2}}+q^{-\frac{1}{2}} \end{pmatrix}.$$

Then, the following equation holds for any braid $\sigma \in B_n$.

$${}^t\beta_{n,q}^r(\sigma)\cdot\Omega_n^r\cdot\beta_{n,q^{-1}}^r(\sigma) = \Omega_n^r.$$

When $q \in \{z \in \mathbb{C} \mid |z| = 1\}$, we can regard $q \mapsto q^{-1}$ as the arbitrary conjugation of the matrix with complex entries. Furthermore, we can regard that B_n acts on $\mathbb{C}^{n-1}$ by using the reduced Burau representation.

$$\beta_{n,q}^r : B_n \longrightarrow GL_{n-1}(\mathbb{C}).$$

Now, we define the following sesquilinear form for $x, y \in \mathbb{C}^{n-1}$

$$\langle x, y\rangle_{B_n} := {}^t x\cdot\Omega_n^r\cdot\bar{y}.$$

Here $\bar{y}$ is the complex conjugation of y. Then, we have

$$\begin{aligned}\langle \beta^r_{n,q}(\sigma)x, \beta^r_{n,q}(\sigma)y\rangle_{B_n} &= {}^t x {}^t\beta^r_{n,q}(\sigma)\cdot \Omega^r_n \cdot \beta^r_{n,q^{-1}}(\sigma)\bar{y} \\ &= {}^t x \cdot \Omega^r_n \cdot \bar{y} \\ &= \langle x, y\rangle_{B_n}.\end{aligned}$$

Since Ω^r_n is the Hermitian matrix, the sesquilinear form $\langle\cdot,\cdot\rangle_{B_n}$ is positive definite if and only if the eigenvalues of Ω^r_n are all positive. In this case, the eigenvalues of Ω^r_n can be computed explicitly by using the formula for the tridiagonal matrix (see [19]). Then the set of eigenvalues of Ω^r_n can be expressed by

$$\left\{q^{\frac{1}{2}} + q^{-\frac{1}{2}} - 2\cos\frac{\pi j}{n} \mid 1 \le j \le n-1\right\}.$$

Hence, consequently we can say that $\langle\cdot,\cdot\rangle_{B_n}$ is positive definite if and only if

$$|\theta| < \frac{2\pi}{n}. \tag{2.1}$$

We assume that x is an eigenvector of $\beta^r_{n,q}(\sigma)$ with the eigenvalue α_q for $\sigma \in B_n$. Then we have

$$\langle x, x\rangle_{B_n} = \langle \beta^r_{n,q}(\sigma)x, \beta^r_{n,q}(\sigma)x\rangle_{B_n} = \langle \alpha_q x, \alpha_q x\rangle_{B_n} = |\alpha_q|^2\langle x, x\rangle_{B_n}.$$

Under the condition (2.1), all eigenvalues of $\beta^r_{n,q}(\sigma)$ satisfy $|\alpha_q| = 1$. Therefore we complete the proof of (4).
(5) We compute the eigenvalues of $\beta_{n,q}(\sigma_{n,m})$. By the definition of the Burau representation, we have

$$\beta_{n,q}(\sigma_{n,1}) = \begin{pmatrix} 1-q & 1-q & \cdots & 1-q & 1 \\ q & 0 & & & \\ & q & \ddots & & \\ & & \ddots & 0 & \\ & & & q & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned}\det(I_n - \beta_{n,q}(\sigma_{n,1})s) &= \det\begin{pmatrix} 1-(1-q)s & -(1-q)s & \cdots & -(1-q)s & -s \\ -qs & 1 & & & \\ & -qs & \ddots & & \\ & & \ddots & 1 & \\ & & & -qs & 1 \end{pmatrix} \\ &= (1-(1-q)s) - (1-q)s\sum_{j=1}^{n-2}(qs)^j - s(qs)^{n-1} \\ &= -s\frac{1-(qs)^n}{1-qs} + \frac{1-(qs)^n}{1-qs} \\ &= \frac{(1-s)(1-(qs)^n)}{1-qs}.\end{aligned}$$

Putting $\xi_n = e^{\frac{2\pi i}{n}}$, the eigenvalues of matrix $\beta_{n,q}(\sigma_{n,1})$ are presented by $1, q^{-1}\xi_n, \dots, q^{-1}\xi_n^{n-1}$. When a pair (n,m) is coprime, the eigenvalues of $\beta_{n,q}(\sigma_{n,m})$ coincide with $1, q^{-m}\xi_n, \dots, q^{-m}\xi_n^{n-1}$. Then we have the formula (5) by replacing $q \mapsto q^m$. □

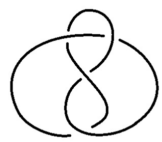

Figure 5: $\widehat{(\sigma_1\sigma_2^{-1})^2}$

Example 2.1.1. We consider the braid $\sigma = (\sigma_1\sigma_2^{-1})^2 \in B_3$. The closure of this braid turns to a famous knot which is said to be *Figure-eight knot* (Figure 5).

We have

$$\zeta(s, (\sigma_1\sigma_2^{-1})^2; \beta_{3,q}) = \frac{q^2}{(1-s)(q^2 - (1-2q+q^2-2q^3+q^4)s + q^2s^2)}. \tag{2.2}$$

By Theorem 2.1.1,

$$\begin{aligned}\operatorname*{Res}_{s=1} \zeta(s, (\sigma_1\sigma_2^{-1})^2; \beta_{3,q}) &= \lim_{s\to 1} \frac{q^2}{q^2 - (1-2q+q^2-2q^3+q^4)s + q^2s^2} \\ &= -\frac{q^2}{(1+q+q^2)(-1+3q-q^2)}.\end{aligned}$$

Then we have $\Delta_{\widehat{(\sigma_1\sigma_2^{-1})^2}}(q) = q^{-2}(-1+3q-q^2)$. Moreover, by (2.2), the non-trivial poles of $\zeta(s, (\sigma_1\sigma_2^{-1})^2; \beta_{3,q})$ are equal to the solutions of the following quadratic equation.

$$1 - (q^{-2} - 2q^{-1} + 1 - 2q + q^2)\alpha + \alpha^2 = 0. \tag{2.3}$$

Since $\alpha \neq 0$, by (2.3), we have

$$\alpha + \alpha^{-1} = q^{-2} - 2q^{-1} + 1 - 2q + q^2 = (q+q^{-1})^2 - 2(q+q^{-1}) - 1.$$

If the argument of q, denoted by θ, satisfies $|\theta| < \frac{2\pi}{3}$, we have $-1 < q + q^{-1} \leq 2$. Then under this condition, we have

$$\alpha + \alpha^{-1} = (q+q^{-1})^2 - 2(q+q^{-1}) - 1 < 2.$$

Since $|\alpha| = 1$ if and only if $\alpha + \alpha^{-1} \leq 2$, we conclude that the analogue of Riemann hypothesis holds under the condition $|\theta| < \frac{2\pi}{3}$.

2.2 The Jones representation $\chi_{n,q}$

The zeta function associated with the Burau representation is connected with the Alexander polynomial. Next we introduce the braid zeta function connected with the Jones polynomial which is also famous knot invariant. Let V_2 be a 2-dimensional vector space and R_q be a linear map on $V_2^{\otimes 2}$ defined by

$$R_q := \begin{pmatrix} 1 & & & \\ & 0 & q & \\ & 1 & 1-q & \\ & & & 1 \end{pmatrix}.$$

By directly computing, we have the following equation.

$$(I_2 \otimes R_q)(R_q \otimes I_2)(I_2 \otimes R_q) = (R_q \otimes I_2)(I_2 \otimes R_q)(R_q \otimes I_2). \tag{2.4}$$

Here, I_2 is the identity map on V_2, and $\otimes$ is the Kronecker product. The equation (2.4) is said to be the *Yang-Baxter equation*.

Definition 2.2.1. We define the linear map $\chi_{n,q} : B_n \longrightarrow GL(V_2^{\otimes n})$ as follows.

$$\chi_{n,q}(\sigma_i) := I_2^{\otimes(i-1)} \otimes R_q \otimes I_2^{\otimes(n-i-1)}.$$

Since R_q satisfies the equation (2.4), $\chi_{n,q}$ is the braid representation. We call $\chi_{n,q}$ *the Jones representation.* Then we can define the braid zeta function associated with $\chi_{n,q}$. Here, we define the "weighted" braid zeta function. For $\sigma \in B_n$,

$$\zeta_t(s, \sigma; \chi_{n,q}) := \det(I_{2^n} - \chi_{n,q}(\sigma)\mu_2(t)^{\otimes n} s)^{-1}.$$

Here, $\mu_2(t) := \mathrm{diag}(1, t)$, and t is complex parameter. When we let $t = 1$, we have the usual representation zeta function $\zeta(s, \sigma; \chi_{n,q})$.

Theorem 2.2.1. (1) *For any $\sigma \in B_n$, we have the following functional equation.*

$$\zeta_t(s, \sigma; \chi_{n,q}) = s^{-2^n}(-q)^{-2^{n-2}\varepsilon(\sigma)} t^{-n2^{n-1}} \zeta_{t^{-1}}(s^{-1}, \sigma^{-1}; \chi_{n,q}).$$

(2) *We assume that the closure of $\sigma \in B_n$ is a knot, then we have*

$$\left.\frac{d}{ds} \log \zeta_q(s, \sigma; \chi_{n,q})\right|_{s=0} = q^{\frac{1}{2}(n-\varepsilon(\sigma)-1)}(1+q)J_{\widehat{\sigma}}(q). \tag{2.5}$$

Here, $J_{\widehat{\sigma}}(q)$ is the Jones polynomial of $\widehat{\sigma}$ which is famous knot invariant.

Proof. (1) From the definition, we have

$$\begin{aligned}\zeta_t(s, \sigma; \chi_{n,q}) &= \det(-s\chi_{n,q}(\sigma)\mu_2(t)^{\otimes n})^{-1} \det(I_{2^n} - \chi_{n,q}(\sigma^{-1})\mu_2(t^{-1})^{\otimes n} s^{-1}) \\ &= (-s)^{-2^n} \det(\chi_{n,q}(\sigma))^{-1} \det(\mu_2(t)^{\otimes n})^{-1} \zeta_{t^{-1}}(s^{-1}, \sigma^{-1}; \chi_{n,q}).\end{aligned}$$

To compute $\det(\chi_{n,q}(\sigma))$ and $\det(\mu_2(t)^{\otimes n})$, we use the following property of the Kronecker product. Let A be an $n \times n$ matrix, and B be an $m \times m$ matrix, then

$$\det(A \otimes B) = \det(A)^m \det(B)^n. \tag{2.6}$$

Since $\det(R_q) = -q$, we have

$$\begin{aligned}\det(\chi_{n,q}(\sigma_i)) &= \det(I_2^{\otimes(i-1)})^{2^{n-i+1}} \det(R_q \otimes I_2^{\otimes(n-i-1)})^{2^{i-1}} \\ &= \{\det(R_q)^{2^{n-i-1}} \det(I_2)^{2^2}\}^{2^{i-1}} \\ &= (-q)^{2^{n-2}}.\end{aligned}$$

Hence we have $\det(\chi_{n,q}(\sigma)) = (-q)^{2^{n-2}\varepsilon(\sigma)}$. Similarly, we have $\det(\mu_2(t)^{\otimes n}) = t^{n2^{n-1}}$. Then (1) is hold.

(2) It is well-known that the Jones polynomial of a knot $\widehat{\sigma}$ is given as following formula (see [14], Chapter 12).

$$J_{\widehat{\sigma}}(q) = \frac{q^{-\frac{1}{2}(n-\varepsilon(\sigma)-1)}}{1+q} \mathrm{tr}\,(\chi_{n,q}(\sigma)\mu_2(q)^{\otimes n}). \tag{2.7}$$

By using the generating function expression, we have

$$\begin{aligned}\frac{d}{ds} \log \zeta_q(s, \sigma; \chi_{n,q}) &= \frac{d}{ds} \sum_{m=1}^{\infty} \frac{\mathrm{tr}\,((\chi_{n,q}(\sigma)\mu_2(q)^{\otimes n})^m)}{m} s^m \\ &= \sum_{m=1}^{\infty} \mathrm{tr}\,((\chi_{n,q}(\sigma)\mu_2(q)^{\otimes n})^m) s^{m-1}.\end{aligned}$$

Hence

$$\frac{d}{ds}\log\zeta_q(s,\sigma;\chi_{n,q})\bigg|_{s=0} = \mathrm{tr}\,(\chi_{n,q}(\sigma)\mu_2(q)^{\otimes n}).$$

By (2.7), we have the formula (2.12) $\square$

Next we introduce the q-analogue of the exterior algebra to show that the zeta function $\zeta_t(s,\sigma;\chi_{n,q})$ can be expressed by using only the information of the Burau representation.

Definition 2.2.2. Let U_n be an n-dimensional vector space spanned by $\{u_1,u_2,\dots,u_n\}$. Then the *q-exterior algebra* of U_n is defined as

$$\bigwedge\nolimits_q(U_n) := T(U_n)/\langle u_i\otimes u_j + qu_j\otimes u_i (1\le i<j\le n), u_i\otimes u_i (1\le i\le n)\rangle.$$

Here $T(U_n)$ is the tensor algebra of U_n defined by

$$T(U_n) := \bigoplus_{m=0}^{\infty} U_n^{\otimes m}.$$

The product of $\bigwedge_q(U_n)$ is written by $\wedge$.

The k-th q-exterior power of U_n, denoted by $\bigwedge_q^k(U_n)$, is the vector subspace of $\bigwedge_q(U_n)$ spanned by the elements of the form $x_1\wedge\cdots\wedge x_k$ for $x_i\in U_n$. Then the set $\{u_{i_1}\wedge u_{i_2}\wedge\cdots\wedge u_{i_k} \mid 1\le i_1<i_2<\cdots<i_k\le n\}$ is a basis of $\bigwedge_q^k(U_n)$. Furthermore we have

$$\bigwedge\nolimits_q(U_n) = \bigoplus_{k=0}^{n}\bigwedge\nolimits_q^k(U_n).$$

Suppose that F is the linear map on U_n, then we can construct the linear map on $\bigwedge_q(U_n)$ denoted by $\wedge_q F$. More concretely, for each $0\le k\le n$, we define the map $\wedge_q^k F$ as

$$\wedge_q^k F(x_1\wedge\cdots\wedge x_k) := F(x_1)\wedge\cdots\wedge F(x_k).$$

Theorem 2.2.2. *For $\sigma\in B_n$, we have*

$$\zeta_t(s,\sigma;\chi_{n,q}) = \prod_{k=0}^{n}\zeta(t^k s,\sigma;\varphi_{n,q}^k). \tag{2.8}$$

Here, $\varphi_{n,q}^k := \wedge_{-q}^k\beta_{n,q}$.

Proof. Let $\beta_{n,q}: B_n \longrightarrow GL(W_n)$ be the Burau representation and $\{f_1,\dots,f_n\}$ be the basis of W_n. Then we consider the $(-q)$-exterior algebra of W_n. First, we define the map $\mathfrak{H}_n$ inductively as follows.

$$\mathfrak{H}_n : V_2^{\otimes n} \xrightarrow{\mathfrak{H}_{n-1}\otimes id_{V_2}} \bigwedge\nolimits_{-q}(W_{n-1})\otimes V_2 \longrightarrow \bigwedge\nolimits_{-q}(W_n). \tag{2.9}$$

Here, $\mathfrak{H}_1(e_0) := 1, \mathfrak{H}_1(e_1) := f_1$, and the second map is defined by

$$\alpha_0\otimes e_0 + \alpha_1\otimes e_1 \mapsto \alpha_0 + \alpha_1\wedge f_n, \tag{2.10}$$

for $\alpha_0,\alpha_1\in\bigwedge_{-q}(W_{n-1})$. Then $\mathfrak{H}_n$ is an isomorphism. We next show that $\mathfrak{H}_n$ is equivariant with respect to the actions of B_n on $V_2^{\otimes}$ and $\bigwedge_{-q}(W_n)$ by induction on n. The case of $n=2$ is trivial, then we consider the general n. We assume that $\mathfrak{H}_{n-1}$ is equivariant with respect to the action of B_{n-1}. Then by construction (2.9), we can see that $\mathfrak{H}_n$ is equivariant with respect to the action of B_{n-1}. Thus, it is sufficient to consider the action of $\sigma_{n-1} \in B_n$, since we can regard $B_{n-1} \subset B_n$. Computation of $\mathfrak{H}_n$ can be expressed as follows.

$$\mathfrak{H}_n : V_2^{\otimes n} \xrightarrow{\mathfrak{H}_{n-2} \otimes id_{V_2} \otimes id_{V_2}} \bigwedge{}_{-q}(W_{n-2}) \otimes V_2 \otimes V_2 \longrightarrow \bigwedge{}_{-q}(W_n).$$

Here the second map is computed by

$$\begin{aligned} \alpha_{00} \otimes e_0 \otimes e_0 + &\alpha_{01} \otimes e_0 \otimes e_1 + \alpha_{10} \otimes e_1 \otimes e_0 + \alpha_{11} \otimes e_1 \otimes e_1 \\ &\mapsto \alpha_{0,0} + \alpha_{0,1} \wedge f_n + \alpha_{1,0} \wedge f_{n-1} + \alpha_{1,1} \wedge f_{n-1} \wedge f_n. \end{aligned} \tag{2.11}$$

Here, $\alpha_{i,j} \in \bigwedge_{-q}(W_{n-2})$ $(i, j = 0, 1)$. By the definition of R_q, the action of σ_{n-1} on $V_2^{\otimes}$ takes the left hand side of (2.11) to the image

$$\alpha_{0,0} \otimes e_0 \otimes e_0 + \alpha_{0,1} \otimes (qe_1 \otimes e_0) + \alpha_{1,0} \otimes (e_0 \otimes e_1 + (1-q)e_1 \otimes e_0) + \alpha_{1,1} \otimes e_1 \otimes e_1.$$

On the other hand, from the definition of the Burau representation, the action of σ_{n-1} on W_n takes $f_{n-1} \mapsto (1-q)f_{n-1} + f_n$, $f_n \mapsto qf_{n-1}$. Then the action of σ_{n-1} on $\bigwedge_{-q}(W_n)$ takes the right hand side of (2.11) to the image

$$\begin{aligned} &\alpha_{0,0} + \alpha_{0,1} \wedge (qf_{n-1}) + \alpha_{1,0} \wedge ((1-q)f_{n-1} + f_n) + \alpha_{1,1} \wedge ((1-q)f_{n-1} + f_n) \wedge (qf_{n-1}) \\ &= \alpha_{0,0} + \alpha_{0,1} \wedge (qf_{n-1}) + \alpha_{1,0} \wedge ((1-q)f_{n-1} + f_n) + \alpha_{1,1} \wedge (f_n) \wedge (qf_{n-1}) \\ &= \alpha_{0,0} + \alpha_{0,1} \wedge (qf_{n-1}) + \alpha_{1,0} \wedge ((1-q)f_{n-1} + f_n) + \alpha_{1,1} \wedge (f_{n-1}) \wedge (f_n). \end{aligned}$$

Hence we can conclude that $\mathfrak{H}_n$ is equivariant with respect to the action of σ_{n-1}, and we have

$$\chi_{n,q} \simeq \wedge_{-q}\beta_{n,q} = \bigoplus_{k=0}^{n} \wedge_{-q}^{k}\beta_{n,q} = \bigoplus_{k=0}^{n} \varphi_{n,q}^{k}.$$

Then, by using the formula (1.2), we have

$$\zeta(s, \sigma; \chi_{n,q}) = \prod_{k=0}^{n} \zeta(s, \sigma; \varphi_{n,q}^{k}).$$

By (2.10), the map $\mu_2(t)^{\otimes n}$ on $\bigwedge_{-q}^{k}(W_n)$ can be regarded as scalar multiplication $t^k id_{\bigwedge_{-q}^{k}(W_n)}$. Then we have the formula (2.8). $\square$

Remark 2.2.1. From the formula (2.12), we have

$$J_{\widehat{\sigma}}(q) = \frac{q^{-\frac{1}{2}(n-\varepsilon(\sigma)-1)}}{1+q} \sum_{k=0}^{n} q^k \mathrm{tr}(\varphi_{n,q}^{k}(\sigma)).$$

Furthermore, $\varphi_{n,q}^{0}$ is the trivial representation. Moreover, $\varphi_{n,q}^{1}$ coincides with the Burau representation.

2.3 The HOMFLY representation $\tau_{n,q}^{(N)}$

In this section, we introduce the HOMFLY representation $\tau_{n,q}^{(N)}$ which is a generalization of the Jones representation.

Definition 2.3.1. ([14], Chapter 12) Let V_N be an N-dimensional vector space spanned by $\{e_0, e_1, \dots, e_{N-1}\}$. Then we define the map $\tau_{n,q}^{(N)}$ as

$$\tau_{n,q}^{(N)} : B_n \longrightarrow \mathrm{GL}(V_N^{\otimes n}),$$
$$\tau_{n,q}^{(N)}(\sigma_i) := id_{V_N}^{\otimes(i-1)} \otimes R_q^{(N)} \otimes id_{V_N}^{\otimes(n-i-1)}.$$

Here, $R_q^{(N)}$ is defined by

$$R_q^{(N)}(e_i \otimes e_j) \begin{cases} q e_j \otimes e_i & (i < j), \\ e_i \otimes e_i & (i = j), \\ e_j \otimes e_i + (1-q) e_i \otimes e_j & (i > j). \end{cases}$$

First we show the following proposition.

Proposition 2.3.1. $R_q^{(N)}$ *satisfies the following Yang-Baxter equation.*

$$(R_q^{(N)} \otimes id_{V_N})(id_{V_N} \otimes R_q^{(N)})(R_q^{(N)} \otimes id_{V_N}) = (id_{V_N} \otimes R_q^{(N)})(R_q^{(N)} \otimes id_{V_N})(id_{V_N} \otimes R_q^{(N)}),$$

Proof. For simplicity, we consider only 3 basis e_1, e_2, e_3. Furthermore, we put $R_1 := (R_q^{(N)} \otimes id_{V_N})(id_{V_N} \otimes R_q^{(N)})(R_q^{(N)} \otimes id_{V_N})$, $R_2 := (id_{V_N} \otimes R_q^{(N)})(R_q^{(N)} \otimes id_{V_N})(id_{V_N} \otimes R_q^{(N)})$. Then it is sufficient to show the equation

$$R_1(e_i \otimes e_j \otimes e_k) = R_2(e_i \otimes e_j \otimes e_k)$$

for following 13 cases: (1) $e_1 \otimes e_1 \otimes e_1$, (2) $e_1 \otimes e_1 \otimes e_2$, (3) $e_1 \otimes e_2 \otimes e_1$, (4) $e_1 \otimes e_2 \otimes e_2$, (5) $e_1 \otimes e_2 \otimes e_3$, (6) $e_1 \otimes e_3 \otimes e_2$, (7) $e_2 \otimes e_1 \otimes e_1$, (8) $e_2 \otimes e_1 \otimes e_2$, (9) $e_2 \otimes e_1 \otimes e_3$, (10) $e_2 \otimes e_2 \otimes e_1$, (11) $e_2 \otimes e_3 \otimes e_1$, (12) $e_3 \otimes e_1 \otimes e_2$, (13) $e_3 \otimes e_2 \otimes e_1$. By the definition of $R_q^{(N)}$, we heve

(1) $e_1 \otimes e_1 \otimes e_1$

$$R_1(e_1 \otimes e_1 \otimes e_1) = e_1 \otimes e_1 \otimes e_1,$$
$$R_2(e_1 \otimes e_1 \otimes e_1) = e_1 \otimes e_1 \otimes e_1.$$

(2) $e_1 \otimes e_1 \otimes e_2$

$$R_1(e_1 \otimes e_1 \otimes e_2) = q^2 e_2 \otimes e_1 \otimes e_1,$$
$$R_2(e_1 \otimes e_1 \otimes e_2) = q^2 e_2 \otimes e_1 \otimes e_1.$$

(3) $e_1 \otimes e_2 \otimes e_1$

$$R_1(e_1 \otimes e_2 \otimes e_1) = q e_1 \otimes e_2 \otimes e_1 + q(1-q) e_2 \otimes e_1 \otimes e_1,$$
$$R_2(e_1 \otimes e_2 \otimes e_1) = q e_1 \otimes e_2 \otimes e_1 + q(1-q) e_2 \otimes e_1 \otimes e_1.$$

(4) $e_1 \otimes e_2 \otimes e_2$

$$R_1(e_1 \otimes e_2 \otimes e_2) = q^2 e_2 \otimes e_2 \otimes e_1,$$
$$R_2(e_1 \otimes e_2 \otimes e_2) = q^2 e_2 \otimes e_2 \otimes e_1.$$

(5) $e_1 \otimes e_2 \otimes e_3$

$$R_1(e_1 \otimes e_2 \otimes e_3) = q^3 e_3 \otimes e_2 \otimes e_1,$$
$$R_2(e_1 \otimes e_2 \otimes e_3) = q^3 e_3 \otimes e_2 \otimes e_1.$$

(6) $e_1 \otimes e_3 \otimes e_2$

$$R_1(e_1 \otimes e_3 \otimes e_2) = q^2 e_2 \otimes e_3 \otimes e_1 + q^2(1-q)e_3 \otimes e_2 \otimes e_1,$$
$$R_2(e_1 \otimes e_3 \otimes e_2) = q^2 e_2 \otimes e_3 \otimes e_1 + q^2(1-q)e_3 \otimes e_2 \otimes e_1.$$

(7) $e_2 \otimes e_1 \otimes e_1$

$$R_1(e_2 \otimes e_1 \otimes e_1) = e_1 \otimes e_2 \otimes e_1 + (1-q)(e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_1 \otimes e_1),$$
$$R(e_2 \otimes e_1 \otimes e_1) = e_1 \otimes e_2 \otimes e_1 + (1-q)(e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_1 \otimes e_1).$$

(8) $e_2 \otimes e_1 \otimes e_2$

$$R_1(e_2 \otimes e_1 \otimes e_2) = qe_2 \otimes e_1 \otimes e_1 + q(1-q)e_2 \otimes e_2 \otimes e_1,$$
$$R_2(e_2 \otimes e_1 \otimes e_2) = qe_2 \otimes e_1 \otimes e_1 + q(1-q)e_2 \otimes e_2 \otimes e_1.$$

(9) $e_2 \otimes e_1 \otimes e_3$

$$R_1(e_2 \otimes e_1 \otimes e_3) = q^2 e_3 \otimes e_1 \otimes e_2 + q^2(1-q)e_3 \otimes e_2 \otimes e_1,$$
$$R_2(e_2 \otimes e_1 \otimes e_3) = q^2 e_3 \otimes e_1 \otimes e_2 + q^2(1-q)e_3 \otimes e_2 \otimes e_1.$$

(10) $e_2 \otimes e_2 \otimes e_1$

$$R_1(e_2 \otimes e_2 \otimes e_1) = e_1 \otimes e_2 \otimes e_2 + (1-q)(e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1),$$
$$R_2(e_2 \otimes e_2 \otimes e_1) = e_1 \otimes e_2 \otimes e_2 + (1-q)(e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1).$$

(11) $e_2 \otimes e_3 \otimes e_1$

$$\begin{aligned} &R_1(e_2 \otimes e_3 \otimes e_1) \\ &\qquad = qe_1 \otimes e_3 \otimes e_2 + q(1-q)(e_3 \otimes e_1 \otimes e_2 + e_2 \otimes e_3 \otimes e_1) + q(1-q)^2 e_3 \otimes e_2 \otimes e_1, \\ &R_2(e_2 \otimes e_3 \otimes e_1) \\ &\qquad = qe_1 \otimes e_3 \otimes e_2 + q(1-q)(e_3 \otimes e_1 \otimes e_2 + e_2 \otimes e_3 \otimes e_1) + q(1-q)^2 e_3 \otimes e_2 \otimes e_1. \end{aligned}$$

(12) $e_3 \otimes e_1 \otimes e_2$

$$\begin{aligned} &R_1(e_3 \otimes e_1 \otimes e_2) \\ &\qquad = qe_2 \otimes e_1 \otimes e_3 + q(1-q)(e_2 \otimes e_3 \otimes e_3 + e_3 \otimes e_1 \otimes e_2) + q(1-q)^2 e_3 \otimes e_2 \otimes e_1, \\ &R_2(e_3 \otimes e_1 \otimes e_2) \\ &\qquad = qe_2 \otimes e_1 \otimes e_3 + q(1-q)(e_2 \otimes e_3 \otimes e_3 + e_3 \otimes e_1 \otimes e_2) + q(1-q)^2 e_3 \otimes e_2 \otimes e_1. \end{aligned}$$

(13) $e_3\otimes e_2\otimes e_1$

$$\begin{aligned}R_1(e_3\otimes e_2\otimes e_1)=&e_1\otimes e_2\otimes e_3+(1-q)(e_1\otimes e_3\otimes e_2+e_2\otimes e_1\otimes e_3)\\&+(1-q)^2(e_2\otimes e_3\otimes e_1+e_3\otimes e_1\otimes e_2)+(1-q)(1-q+q^2)e_3\otimes e_2\otimes e_1,\\R_2(e_3\otimes e_2\otimes e_1)=&e_1\otimes e_2\otimes e_3+(1-q)(e_1\otimes e_3\otimes e_2+e_2\otimes e_1\otimes e_3)\\&+(1-q)^2(e_2\otimes e_3\otimes e_1+e_3\otimes e_1\otimes e_2)+(1-q)(1-q+q^2)e_3\otimes e_2\otimes e_1.\end{aligned}$$

Hence $R_q^{(N)}$ satisfies the Yang-Baxter equation. □

By Proposition 2.3.1, we can see that $\tau_{n,q}^{(N)}$ is the braid representation which is called *N-th degree HOMFLY representation*. Furthermore we remark that the representation $\tau_{n,q}^{(N)}$ coincides with the Jones representation when $N=2$. Similar to the Jones representation, we can define the braid zeta function associated with $\tau_{n,q}^{(N)}$ and the weighted braid zeta function as follows.

$$\zeta_t(s,\sigma;\tau_{n,q}^{(N)}):=\det(I_{N^n}-\tau_{n,q}^{(N)}(\sigma)\mu_N(t)^{\otimes n}s)^{-1},$$

where, $\mu_N(t):=\mathrm{diag}(1,t,\dots,t^{N-1})$.

Theorem 2.3.1. (1) *For any $\sigma\in B_n$, we have the following functional equation.*

$$\zeta_t(s,\sigma;\tau_{n,q}^{(N)})=(-s)^{-N^n}(-q)^{-\frac{1}{2}(N-1)N^{n-1}\varepsilon(\sigma)}t^{-\frac{n}{2}N^n(N-1)}\zeta_{t^{-1}}(s^{-1},\sigma^{-1};\tau_{n,q}^{(N)}).$$

(2) *We assume that the closure of $\sigma\in B_n$ is a knot, then we have*

$$\frac{d}{ds}\log\zeta_q(s,\sigma;\tau_{n,q}^{(N)})\bigg|_{s=0}=q^{\frac{1}{2}(N-1)(n-\varepsilon(\sigma)-1)}[N]_qH_{\widehat{\sigma}}^{(N)}(q). \tag{2.12}$$

Here, $H_{\widehat{\sigma}}^{(N)}(q)$ is called N-th degree HOMFLY polynomial of $\widehat{\sigma}$ which is famous knot invariant.

Proof. (1) By similar calculation of the proof of Theorem 2.2.1,

$$\begin{aligned}\zeta_t(s,\sigma;\tau_{n,q}^{(N)})&=\det(-s\tau_{n,q}^{(N)}(\sigma)\mu_N(t)^{\otimes n})^{-1}\det(I_{N^n}-\tau_{n,q}^{(N)}(\sigma^{-1})\mu_N(t^{-1})^{\otimes n}s^{-1})\\&=(-s)^{-N^n}\det(\tau_{n,q}(\sigma))^{-1}\det(\mu_N(t)^{\otimes n})^{-1}\zeta_{t^{-1}}(s^{-1},\sigma^{-1};\tau_{n,q}^{(N)}).\end{aligned}$$

Then we calculate the determinant of $R_q^{(N)}$ by sorting the basis of $V_N\otimes V_N$ as

$$\underbrace{e_0\otimes e_0,\dots e_0\otimes e_{N-2},e_1\otimes e_0,\dots,e_1\otimes e_{N-2},\dots,e_{N-2}\otimes e_0,\dots,e_{N-2}\otimes e_{N-2}}_{(A_1)},$$

$$\underbrace{e_0\otimes e_{N-1},e_{N-1}\otimes e_0}_{(A_2)},\underbrace{e_2\otimes e_{N-1},e_{N-1}\otimes e_2}_{(A_2)},\dots,\underbrace{e_{N-2}\otimes e_{N-1},e_{N-1}\otimes e_{N-2}}_{(A_2)},\underbrace{e_{N-1}\otimes e_{N-1}}_{(A_3)}.$$

Here, the action on the basis (A_1) corresponds to $R_q^{(N-1)}$, and (A_2) corresponds to the following 2×2 matrix.

$$B(q):=\begin{pmatrix}0&q\\1&1-q\end{pmatrix}.$$

Since the action on (A_3) is trivial and $\det B(q)=(-q)$, then we have

$$\det R_q^{(N)}=\det R_q^{(N-1)}(\det B(q))^{N-1}=\det R_q^{(N-1)}(-q)^{N-1}=(-q)^{\frac{N(N-1)}{2}}.$$

By the formula (2.6),

$$\det(\tau_{n,q}^{(N)}(\sigma_i)) = \det(I_{N^{i-1}} \otimes R_q^{(N)} \otimes I_{N^{n-i-1}}) = (\det R_q^{(N)})^{N^{n-2}}.$$

Thus, for $\sigma \in B_n$,

$$\det(\tau_{n,q}^{(N)}(\sigma)) = (-q)^{\frac{1}{2}(N-1)N^{n-1}\varepsilon(\sigma)}.$$

Similarly, $\det(\mu_N(t)^{\otimes n})$ can be calculated as follows.

$$\det(\mu_N(t)^{\otimes n}) = (\det(\mu_N(t)^{\otimes(n-1)}))^N (t^{\frac{1}{2}N(N-1)})^{N^{n-1}} = t^{\frac{n}{2}N^n(N-1)}.$$

Hence (1) holds.

(2) It is well-known that the N-th degree HOMFLY polynomial of a knot $\widehat{\sigma}$ is given as the following formula (see [14], Chapter 12).

$$H_{\widehat{\sigma}}^N(q) = \frac{q^{-\frac{1}{2}(N-1)(n-\varepsilon(\sigma)-1)}}{[N]_q} \mathrm{tr}\,(\tau_{n,q}^{(N)}(\sigma)\mu_N(q)^{\otimes n}).$$

By the same calculation of the proof of Theorem 2.2.1, we have the formula (2.12). □

Next we introduce the product formula of the zeta function $\zeta_t(s,\sigma;\tau_{n,q}^{(N)})$ which is generalization of Theorem 2.2.2. To state the product formula, we define some notations.

For $I = (i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, N-1\}^n$, we define

$$e_I := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \in V_N^{\otimes n}.$$

Then $\{e_I \mid I \in \{0, 1, \ldots, N-1\}^n\}$ is the basis of $V_N^{\otimes}$. Furthermore we assume that I and J are equivalent if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $J = \sigma(I)$, where $\sigma(I) = \sigma((i_1, i_2, \cdots, i_n))$ is defined as

$$\sigma((i_1, i_2, \cdots, i_n)) := (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}).$$

Thus we can define the quotient set $\{0, 1, \ldots, N-1\}^n/\mathfrak{S}_n$. For simplicity, we regard this quotient set as the following set.

$$\mathfrak{I}_n := \{(i_1, i_2, \cdots, i_n) \mid 0 \le i_1 \le i_2 \le \cdots \le i_n \le N-1\}.$$

For $I \in \mathfrak{I}_n$, we can define a vector subspace V_I spanned by all permutations of I, furthermore, since the action of the braid group B_n preserves V_I for all $I \in \mathfrak{I}_n$, then we can obtain the braid representation with respect to $I \in \mathfrak{I}_n$ as

$$\varphi_{n,q}^I : B_n \longrightarrow GL(V_I).$$

Moreover we have the following decomposition.

$$V_N^{\otimes n} = \bigoplus_{I \in \mathfrak{I}_n} V_I. \tag{2.13}$$

Theorem 2.3.2. *Let $N \ge 2$. Then, for any $\sigma \in B_n$, we have the following formula.*

$$\zeta_t(s,\sigma;\tau_{n,q}^{(N)}) = \prod_{I \in \mathfrak{I}_n} \zeta(t^{|I|}s, \sigma; \varphi_{n,q}^I).$$

Here, $|I| := i_1 + i_2 + \cdots + i_n$ for $I = (i_1, i_2, \ldots, i_n) \in \mathfrak{I}_n$.

Proof. By (2.13), we immediately have

$$\zeta(s,\sigma;\tau_{n,q}^{(N)}) = \prod_{I \in \mathfrak{I}_n} \zeta(s,\sigma;\varphi_{n,q}^I).$$

Since $\mu_N(t)e_{i_k} = t^{i_k}e_{i_k}$ for $i_k \in \{0, 1, \ldots, N-1\}$, then the action of $\mu_N(t)^{\otimes n}$ on V_I turns to the scalar multiplication $t^{|I|}id_{V_I}$. Hence the formula of Theorem 2.3.2 holds. □

2.4 Classical limit and dynamical zeta function

Let 2^{X_n} be the power set of X_n. Then $\sigma \in \mathfrak{S}_n$ acts on 2^{X_n} naturally. Then the pair $(\sigma,\, 2^{X_n})$ is dynamical system.

Example 2.4.1. The transposition $s_1 := (1,2) \in \mathfrak{S}_3$ acts on the power set 2^{X_3} as follows.

$$s_1(\phi) = \phi, \qquad s_1(\{1\}) = \{2\}, \qquad s_1(\{2\}) = \{1\}, \qquad s_1(\{3\}) = \{3\},$$

$$s_1(\{1,2\}) = \{1,2\}, \qquad s_1(\{2,3\}) = \{1,3\}, \qquad s_1(\{3,1\}) = \{2,3\}, \qquad s_1(\{1,2,3\}) = \{1,2,3\}.$$

Furthermore, we consider the generalization of the power set of the finite set.

Definition 2.4.1. For $N \geq 2$, we define

$$N^{X_n} := \{\{1_{(i_1)}, 2_{(i_2)}, \ldots, n_{(i_n)}\} \mid 0 \leq i_j \leq N-1\}.$$

We call the set N^{X_n} *N-colored power set.*

Note that 2-colored power set coincides with usual power set 2^{X_n}. Moreover, similar to the power set 2^{X_n}, the action of $\sigma \in \mathrm{S}_n$ on N^{X_n} can be defined naturally. Then, for $N \geq 2$, we can define the zeta function of the dynamical system $(\sigma,\, N^{X_n})$ as follows.

$$\zeta_\sigma(s, N^{X_n}) := \exp\left\{\sum_{m=1}^{\infty} \frac{|\mathrm{Fix}(\sigma^m, N^{X_n})|}{m} s^m\right\}.$$

Here, $\mathrm{Fix}(\sigma^m, N^{X_n}) := \{x \in N^{X_n} \mid \sigma^m x = x\}$. Next we consider the classical limit $q \to 1$ for the braid zeta function associated with $\tau_{n,q}^{(N)}$.

Theorem 2.4.1. *For any $\sigma \in B_n$, we have the following limit formula.*

$$\lim_{q \to 1} \zeta(s, \sigma; \tau_{n,q}^{(N)}) = \zeta_{\pi_n(\sigma)}(s, N^{X_n}). \tag{2.14}$$

Proof. When $q \to 1$, $R_q^{(N)}$ turns to the following operator.

$$R_1^{(N)}(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & (i < j), \\ e_i \otimes e_i & (i = j), \\ e_j \otimes e_i & (i > j). \end{cases}$$

Then we obtain the representation $p_n^{(N)} \circ \pi_n := \tau_{n,1}^{(N)} : B_n \longrightarrow GL(V_N^{\otimes n})$. Furthermore, we identify $\{1_{(i_1)}, 2_{(i_2)}, \ldots, n_{(i_n)}\} \in N^{X_n}$ with a vector $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \in V_N^{\otimes n}$. Then we have

$$\tau_{n,1}^{(N)}(\sigma) = \mathrm{tr}\, p_n^{(N)}(\pi_n(\sigma)) = \mathrm{tr}\, |\mathrm{Fix}(\pi_n(\sigma), N^{X_n})|.$$

Hence the formula (2.14) holds. □

If $\sigma \in \mathfrak{S}_n$ is the cycle element with length n, we have the following explicit formula.

Theorem 2.4.2. *Let $c_n \in \mathfrak{S}_n$ be a cycle element with length n, then we have*

$$\zeta_{c_n}(s, N^{X_n}) = \prod_{d|n} \frac{1}{(1-s^d)^{M(N,d)}}. \tag{2.15}$$

Here, $M(N,d)$ is defined by

$$M(N,d) := \frac{1}{d} \sum_{k|d} \mu\left(\frac{d}{k}\right) N^k,$$

where $\mu(n)$ is the Möbius function.

Proof. Since $p_n^{(N)}(c_n^n) = I_{N^n}$, the eigenvalues of $p_n^{(N)}(c_n)$ are d-th roots of unity, where $d \mid n$. Moreover, since $\mathrm{tr}(p_n^{(N)}(c_n))$ is an integer value, the zeta function can be written by the following form.

$$\zeta_{c_n}(s, N^{X_n}) = \prod_{d|n} \frac{1}{(1-s^d)^{f_{N,d}}}.$$

Considering the degree of $\zeta_{c_n}(s, N^{X_n})^{-1}$, we have

$$\sum_{d|n} d \cdot f_{N,d} = N^n.$$

By using the Möbius inversion formula,

$$f_{N,d} = \frac{1}{d} \sum_{k|d} \mu\left(\frac{d}{k}\right) N^k.$$

Then, Theorem 2.4.2 holds. □

As an application of Theorem 2.4.1, and Theorem 2.4.2, for $\sigma \in B_3$, we obtain the explicit formula of the HOMFLY polynomial $H_{\widehat{\sigma}}^{(N)}(q)$ by using only the Burau representation.

Corollary 2.4.1. *We assume that the closure of $\sigma \in B_3$ is a knot, Then for $N \geq 2$, we have*

$$H_{\widehat{\sigma}}^{(N)}(q) = q^{\frac{1}{2}(N-1)(\varepsilon(\sigma)-2)} \frac{(1-q)^2(1+q^N+q^{2N}) + q(1-q^{N-1})(1-q^{N+1})\mathrm{tr}\beta_{3,q}(\sigma)}{(1-q)(1-q^3)}.$$

Proof. From the definition of $R_q^{(N)}$, we have the following equation for arbitrary n.

$$\tau_{n,q}^{(N)}(\sigma_i)^2 = (1-q)\tau_{n,q}^{(N)}(\sigma_i) + q \cdot id_{V_N^{\otimes n}}.$$

Then, the HOMFLY representation $\tau_{n,q}^{(N)}$ has the structure of the Iwahori-Hecke algebra (see [8], Chapter 4). Then from the Appendix A.2, we can see that the representation $\tau_{n,q}^{(N)}$ can be decomposed into the following three types: (1) reduced Burau representation $\beta_{3,q}^r$, (2) trivial representation $\mathbf{1}$, (3) 1-dimensional representation $\sigma \mapsto (-q)^{\varepsilon(\sigma)}$ which is equivalent to the composition $\det \circ \beta_{3,q}^r$. On the other hand, by Theorem 2.4.2, we have

$$\lim_{q \to 1} \zeta(s, \sigma; \tau_{3,q}^{(N)}) = \prod_{d|3} \frac{1}{(1-s^d)^{M(N,d)}} = \frac{1}{(1-s)^N (1-s^3)^{N(N^2-1)/3}}. \tag{2.16}$$

Then we can see that the type (3) does not appear in $\varphi_{3,q}^I$ for any $I \in \mathfrak{I}_n$. Hence we can conclude that $\tau_{3,q}^{(N)}$ is decomposed into N trivial representations $\mathbf{1}$ and $\frac{1}{3}N(N^2-1)$ Burau representations $\mathbf{1} \oplus \beta_{3,q}^r$. Next we classify the set $\mathfrak{I}_3$ as the following 3 patterns.

$$C_1 := \{(k,k,k) \in \mathfrak{I}_3\}, \qquad C_2 := \{(k_1,k_1,k_2),(k_1,k_2,k_2) \in \mathfrak{I}_3 \mid k_1 < k_2\},$$
$$C_3 := \{(l_1,l_2,l_3) \in \mathfrak{I}_3 \mid l_1 < l_2 < l_3\}$$

For $I \in \mathfrak{I}_3$, the dimension of V_I is equal to the number of the permutations of I. Then we obtain the following correspondences.

$$\mathbf{1} \longleftrightarrow I \in C_1, \qquad \beta_{n,q} \longleftrightarrow I \in C_2, \qquad \beta_{n,q} \oplus \beta_{n,q} \longleftrightarrow I \in C_3.$$

By Theorem 2.3.2, we have

$$
\begin{aligned}
\zeta_t(s,\sigma;\tau_{3,q}^{(N)}) &= \prod_{I\in\mathfrak{I}_3} \zeta(t^{|I|}s,\sigma;\varphi_{3,q}^{I}) \\
&= \frac{\displaystyle\prod_{0\le k_1\neq k_2\le N-1} \zeta(t^{2k_1+k_2}s,\sigma;\beta_{3,q}) \prod_{0\le l_1<l_2<l_3\le N-1} \zeta(t^{l_1+l_2+l_3}s,\sigma;\beta_{3,q})^2}{\displaystyle\prod_{k=0}^{N-1} 1-t^{3k}s} \\
&= \prod_{k=0}^{N-1} \exp\Big\{\sum_{m=1}^{\infty} \frac{t^{3km}}{m}s^m\Big\} \prod_{0\le k_1\neq k_2\le N-1} \exp\Big\{\sum_{m=1}^{\infty} \frac{t^{(2k_1+k_2)m}\mathrm{tr}\,\beta_{3,q}(\sigma^m)}{m}s^m\Big\} \\
&\qquad \times \prod_{0\le l_1<l_2<l_3\le N-1} \exp\Big\{\sum_{m=1}^{\infty} \frac{2t^{(l_1+l_2+l_3)m}\mathrm{tr}\,\beta_{3,q}(\sigma^m)}{m}s^m\Big\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{tr}(\tau_{3,q}^{(N)}(\sigma)\cdot\mu_N(t)^{\otimes 3}) &= \sum_{k=0}^{N-1} t^{3k} + \Big(\sum_{0\le k_1\neq k_2\le N-1} t^{2k_1+k_2} + 2\sum_{0\le l_1<l_2<l_3\le N-1} t^{l_1+l_2+l_3}\Big)\mathrm{tr}(\beta_{3,q}(\sigma)) \\
&= \sum_{k=0}^{N-1} t^{3k} + \Bigg\{\Big\{\Big(\sum_{k=0}^{N-1} t^{2k}\Big)\Big(\sum_{k=0}^{N-1} t^{k}\Big) - \sum_{k=0}^{N-1} t^{3k}\Big\} \\
&\qquad + \frac{2}{6}\Big\{\Big(\sum_{k=0}^{N-1} t^{k}\Big)^3 - 3\Big\{\Big(\sum_{k=0}^{N-1} t^{2k}\Big)\Big(\sum_{k=0}^{N-1} t^{k}\Big) - \sum_{k=0}^{N-1} t^{3k}\Big\} - \sum_{k=0}^{N-1} t^{3k}\Big\}\Bigg\}\mathrm{tr}(\beta_{3,q}(\sigma)) \\
&= [N]_{t^3} + \frac{1}{3}([N]_t^3 - [N]_{t^3})\mathrm{tr}(\beta_{3,q}(\sigma)).
\end{aligned}
$$

Here, $[N]_t := 1+t+\cdots+t^{N-1}$. By Theorem 2.3.1, we have

$$
\begin{aligned}
H_{\widehat{\sigma}}^{(N)}(q) &= \frac{q^{-\frac{1}{2}(N-1)(2-\varepsilon(\sigma))}}{[N]_q}\mathrm{tr}\,(\tau_{3,q}^{(N)}(\sigma)\mu_N(q)^{\otimes 3}) \\
&= q^{\frac{1}{2}(N-1)(\varepsilon(\sigma)-2)} \frac{[N]_{q^3} + \frac{1}{3}([N]_q^3 - [N]_{q^3})\mathrm{tr}(\beta_{3,q}(\sigma))}{[N]_q} \\
&= q^{\frac{1}{2}(N-1)(\varepsilon(\sigma)-2)} \Big\{\frac{1+q^N+q^{2N}}{1+q+q^2} + \frac{(1-q^3)(1-q^N)^2 - (1-q)^3(1+q^N+q^{2N})}{3(1-q)^2(1-q^3)}\mathrm{tr}(\beta_{3,q}(\sigma))\Big\} \\
&= q^{\frac{1}{2}(N-1)(\varepsilon(\sigma)-2)} \frac{(1-q)^2(1+q^N+q^{2N}) + q(1-q^{N-1})(1-q^{N+1})\mathrm{tr}\beta_{3,q}(\sigma)}{(1-q)(1-q^3)}.
\end{aligned}
$$

□

We give an example.

Example 2.4.2. When $\sigma = (\sigma_1\sigma_2^{-1})^2 \in B_3$, the trace of the Burau representation can be calculated as

$$
\mathrm{tr}\beta_{3,q}(\sigma) = (1-q)^2 + (1-q^{-1})^2 = q^{-2}(1+q^2)(1-q)^2.
$$

Since $\varepsilon(\sigma) = 0$, the N-degree HOMFLY polynomial of $\widehat{\sigma}$ can be calculated as

$$\begin{aligned} H^{(N)}_{\widehat{\sigma}}(q) &= q^{-(N-1)} \frac{(1-q)^2(1+q^N+q^{2N}) + q^{-1}(1-q^{N-1})(1-q^{N+1})(1+q^2)(1-q)^2}{(1-q)(1-q^3)} \\ &= q^{-N} - q^{-1} + 1 - q + q^N. \end{aligned}$$

3 Special product and decomposition formula

In this section, we consider the special product of braids. The main result of this section is decomposition formula of the braid zeta function associated with the Burau representation for a braid which is expressed by the special product. Namely, for two braids σ and τ, we define a new braid written by $\sigma * \tau$. Then we will show that the zeta function of $\sigma * \tau$ can be expressed by the zeta function of σ and zeta function of τ.

3.1 Special product

Definition 3.1.1. We define the homomorphism $w_m : B_n \longrightarrow B_{nm}$ by

$$w_m(\sigma_i) := (\sigma_{im} \cdots \sigma_{im-(m-1)})(\sigma_{im+1} \cdots \sigma_{im-(m-2)}) \cdots (\sigma_{im+(m-1)} \cdots \sigma_{im}).$$

We call $w_m(\sigma)$ *m-cabling braid.*

Here is a example of m-cabling of $\sigma_1 \in B_2$.

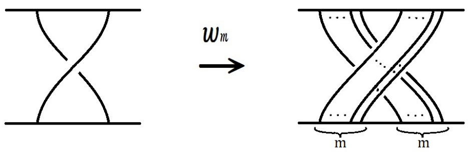

Figure 6: m-cabling braid $w_m(\sigma_1) \in \mathrm{B}_{2m}$

From the definition of w_m, for any positive integers m, n, we have

$$w_m \circ w_n = w_n \circ w_m = w_{mn}.$$

Furthermore we define the special product of the braid.

Definition 3.1.2. For $\sigma \in B_m$ and $\tau \in B_n$, we define the *special product* $\sigma * \tau$ by

$$\sigma * \tau := \iota_{m,mn}(\sigma) \cdot w_m(\tau) \in B_{mn}.$$

Here, $\iota_{n,n+r} : B_n \longrightarrow B_{n+r}$ is a natural inclusion defined by $\iota_{n,n+r}(\sigma_i) := \sigma_i \in B_{n+r}$ $(i = 1, 2, \ldots, n-1)$.

Example 3.1.1. For $\sigma := (\sigma_1\sigma_2^{-1})^2 \in B_3$, $\tau := \sigma_1^3 \in B_2$, the special product and its closure can be expressed by Figure 7, and Figure 8.

Lemma 3.1.1. *For $\sigma \in B_l$, $\tau \in B_m$, $\mu \in B_n$, we have*

$$(\sigma * \tau) * \mu = \sigma * (\tau * \mu).$$

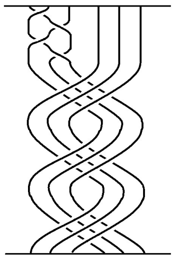

Figure 7: Special product $\sigma * \tau$

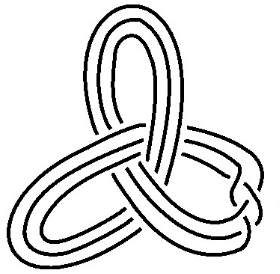

Figure 8: The closure of $\sigma * \tau$

Proof. From the definition of the special product,

$$\begin{aligned}(\sigma * \tau) * \mu &= (\iota_{l,lm}(\sigma) w_l(\tau)) * \mu \\ &= \iota_{lm,lmn}(\iota_{l,lm}(\sigma) w_l(\tau)) w_{lm}(\mu) \\ &= \iota_{l,lmn}(\sigma) \iota_{lm,lmn}(w_l(\tau)) w_{lm}(\mu).\end{aligned}$$

Since $\iota_{lm,lmn}(w_l(\tau)) = w_l(\iota_{m,mn}(\tau))$,

$$(\sigma * \tau) * \mu = \iota_{l,lmn}(\sigma)(w_l(\iota_{m,mn}(\tau)) w_{lm}(\mu).$$

On the other hand,

$$\begin{aligned}\sigma * (\tau * \mu) &= \sigma * (\iota_{m,mn}(\tau) w_m(\mu)) \\ &= \iota_{l,lmn}(\sigma) w_l(\iota_{m,mn}(\tau) w_m(\mu)) \\ &= \iota_{l,lmn}(\sigma) w_l(\iota_{m,mn}(\tau)) w_{lm}(\mu).\end{aligned}$$

Then Lemma 3.1.1 holds. □

Next we calculate the formula of the Burau representation of m-cabling braids.

Lemma 3.1.2. *For $m \in \mathbb{N}$, $\sigma_i \in B_n$, we have*

$$\beta_{nm,q}(w_m(\sigma_i)) = I_{m(i-1)} \oplus \begin{pmatrix} (1-q)A_m(q) & I_m \\ q^m I_m & O \end{pmatrix} \oplus I_{m(n-i-1)}.$$

Here, $A_n(q)$ is an $m \times m$ matrix defined by

$$A_m(q) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q & q & \cdots & q \\ q^2 & q^2 & \cdots & q^2 \\ \vdots & \vdots & & \vdots \\ q^{m-1} & q^{m-1} & \cdots & q^{m-1} \end{pmatrix}.$$

Proof. It is sufficient to show the case of $\sigma_1 \in B_2$ as follows.

$$\beta_{2m,q}(w_m(\sigma_1)) = \begin{pmatrix} (1-q)A_m(q) & I_m \\ q^m I_m & O \end{pmatrix}. \tag{3.1}$$

We show the formula (3.1) by using the induction on m. When $m = 1$, since $w_1(\sigma_1) = \sigma_1 \in B_2$, the formula (3.1) holds. We suppose that (3.1) is true for $m = k \geq 2$. By Figure 9, $w_{k+1}(\sigma_1)$ can be expressed by

$$w_{k+1}(\sigma_1) = (\sigma_{k+1}\sigma_{k+2} \cdots \sigma_{2k}) w_k(\sigma_1)(\sigma_{2k+1}\sigma_{2k} \cdots \sigma_{k+1}).$$

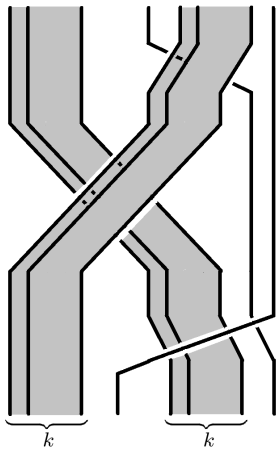

Figure 9: $w_{k+1}(\sigma_1)$ expressed by using $w_k(\sigma_1)$

Hence, we can calculate $\beta_{2(k+1),q}(w_{k+1}(\sigma_1))$ as follows.

$$\beta_{2(k+1),q}(w_{k+1}(\sigma_1)) = \begin{pmatrix} I_k & & \\ & \beta_{k+1,q}(\sigma_{k+1,1}) & \\ & & 1 \end{pmatrix} \begin{pmatrix} \beta_{2k,q}(w_k(\sigma_1)) & \\ & I_2 \end{pmatrix} \begin{pmatrix} I_k & \\ & \beta_{k+2,q}(\sigma_{k+1}\cdots\sigma_1) \end{pmatrix}.$$

Since

$$\beta_{n,q}(\sigma_{1,n}) = \begin{pmatrix} 1-q & 1 & & & \\ q(1-q) & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ q^{n-2}(1-q) & & & 0 & 1 \\ q^{n-1} & & & & 0 \end{pmatrix}, \quad \beta_{n,q}(\sigma_{n-1}\cdots\sigma_1) = \begin{pmatrix} 1-q & 1-q & \cdots & 1-q & 1 \\ q & 0 & & & \\ & \ddots & \ddots & & \\ & & q & 0 & \\ & & & q & 0 \end{pmatrix},$$

we obtain

$$\beta_{2(k+1),q}(w_{k+1}(\sigma_1)) = \begin{pmatrix} (1-q)A_{k+1}(q) & I_{k+1} \\ q^{k+1}I_{k+1} & O \end{pmatrix}.$$

Hence (3.1) is true for any positive integer m. Then, Lemma 3.1.2 holds. □

3.2 Component number

In this section, we calculate the component number of the link $\widehat{\sigma}$ by using the dynamical zeta function. We write the component number of $\widehat{\sigma}$ by $c(\sigma)$. Since $c(\sigma)$ is equal to the number of the primitive cycle of $\pi_n(\sigma) \in \mathfrak{S}_n$, then we have the following proposition.

Proposition 3.2.1. *For $\sigma \in B_n$, the component number $c(\sigma)$ is equal to the order of the pole $s = 1$ of the dynamical zeta function $\zeta_{\pi_n(\sigma)}(s, X_n)$.*

Proof. By the Euler product expression of dynamical zeta function, we have

$$\zeta_\sigma(s) = \prod_{P \in \mathrm{Cycle}(\sigma)} \frac{1}{1 - s^{l(P)}} = \frac{1}{(1-s)^{|\mathrm{Cycle}(\sigma)|}} \prod_{P \in \mathrm{Cycle}(\sigma)} \frac{1}{1 + s + \cdots + s^{l(P)-1}}.$$

Then, Proposition 3.2.1 holds. □

For $m \in \mathbb{N}$ and $\sigma \in B_n$, we obtain the formula of the dynamical zeta function of $\pi_{nm}(w_m(\sigma))$.

Proposition 3.2.2. *Let m be a positive integer. For $\sigma \in B_n$, we have*

$$\zeta_{\pi_{nm}(w_m(\sigma))}(s, X_{nm}) = \zeta_{\pi_n(\sigma)}(s, X_n)^m.$$

In particular,

$$c(w_m(\sigma)) = mc(\sigma).$$

Proof. First, we compute $p_{nm}(\pi_{nm}(w_m(\sigma)))$. When we let $q \to 1$, Burau representation turns to the permutation representation. Then by Lemma 3.1.2 we have

$$\begin{aligned} p_{nm}(\pi_{nm}(w_m(\sigma_i))) = \lim_{q\to 1} \beta_{nm,q}(w_m(\sigma_i)) &= I_{m(i-1)} \oplus \begin{pmatrix} O & I_m \\ I_m & O \end{pmatrix} \oplus I_{m(n-i-1)} \\ &= (I_{(i-1)} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{(n-i-1)}) \otimes I_m \\ &= p_n(\pi_n(\sigma_i)) \otimes I_m. \end{aligned}$$

By generating function expression of the dynamical zeta function, we have

$$\begin{aligned} \zeta_{\pi_{nm}(w_m(\sigma))}(s, X_{nm}) &= \exp\left\{\sum_{j=1}^{\infty} \frac{\mathrm{tr}(\, p_{nm}(\pi_{nm}(w_m(\sigma^j))))}{j} s^j\right\} \\ &= \exp\left\{\sum_{j=1}^{\infty} \frac{\mathrm{tr}\,(p_n(\pi_n(\sigma^j)) \otimes I_m)}{j} s^j\right\} \\ &= \exp\left\{\sum_{j=1}^{\infty} \frac{m \cdot \mathrm{tr}\,(p_n(\pi_n(\sigma^j)))}{j} s^j\right\} = \zeta_{\pi_n(\sigma)}(s, X_n)^m. \end{aligned}$$

Since the order of the pole at $s = 1$ of $\zeta_{\pi_{nm}(w_m(\sigma))}(s, X_{nm})$ is equal to $mc(\sigma)$, then we have $c(w_m(\sigma)) = mc(\sigma)$ by Proposition 3.2.1. □

Next we consider the component number of the special product of braids.

Proposition 3.2.3. *For $\sigma \in B_m$, $\tau \in B_n$, we have*

$$c(\sigma * \tau) = c(\sigma) + m(c(\tau) - 1).$$

Proof. First, we calculate the zeta function $\zeta_{\pi_{nm}(\sigma*\tau)}(s, X_{nm})$. $\pi_n(\tau) \in \mathfrak{S}_n$ can be expressed as follows.

$$\pi_n(\tau) = (1, i_2, \ldots, i_{l_{(1)}})(i_{l_{(1)}+1}, \ldots, i_{l_{(1)}+l_{(2)}}) \cdots (i_{l_{(1)}+\cdots+l_{(r-1)}}, \ldots, i_{l_{(1)}+\cdots+l_{(r)}}).$$

Then there exists a permutation $\mu \in \mathfrak{S}_{n-1}$ such that

$$\begin{pmatrix} 1 & \\ & p_{n-1}(\mu) \end{pmatrix} p_n(\pi_n(\tau)) \begin{pmatrix} 1 & \\ & p_{n-1}(\mu) \end{pmatrix}^{-1} = \bigoplus_{k=1}^{c(\tau)} C_{l_{(k)}}.$$

Here, $C_{l_{(k)}}$ is defined as

$$C_{l_{(k)}} := \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in GL_{l_{(k)}}(\mathbb{Z}).$$

For simplicity, we put

$$Q := \begin{pmatrix} 1 & \\ & p_{n-1}(\mu) \end{pmatrix}.$$

Then, we compute

$$\begin{aligned}
&(Q \otimes I_m) \cdot p_{nm}(\pi_{nm}(\sigma * \tau)) \cdot (Q \otimes I_m)^{-1} \\
&\qquad = (Q \otimes I_m) \cdot p_{nm}(\pi_{nm}(\iota_{m,nm}(\sigma))) \cdot p_{nm}(w_m(\tau)) \cdot (Q \otimes I_m)^{-1} \\
&\qquad = (Q \otimes I_m) \cdot (p_m(\pi_m(\sigma)) \oplus I_{(n-1)m}) \cdot (p_n(\pi_n(\tau)) \otimes I_m) \cdot (Q \otimes I_m)^{-1} \\
&\qquad = p_m(\pi_m(\sigma) \oplus I_{(n-1)m}) \cdot (Q \otimes I_m) \cdot (p_n(\pi_n(\tau)) \otimes I_m) \cdot (Q \otimes I_m)^{-1} \\
&\qquad = p_m(\pi_m(\sigma) \oplus I_{(n-1)m}) \cdot ((\bigoplus_{k=1}^{c(\tau)} C_{l_{(k)}}) \otimes I_m) \\
&\qquad = p_m(\pi_m(\sigma) \oplus I_{(n-1)m}) \cdot (C_{l_{(1)}} \otimes I_m) \oplus \bigoplus_{k=2}^{c(\tau)} (C_{l_{(k)}} \otimes I_m) \\
&\qquad = (p_m(\pi_m(\sigma) \oplus I_{(l_{(1)}-1)m})(C_{l_{(1)}} \otimes I_m)) \oplus \bigoplus_{k=2}^{c(\tau)} (C_{l_{(k)}} \otimes I_m).
\end{aligned}$$

We write $C_1(\sigma) := p_m(\pi_m(\sigma) \oplus I_{(l_{(1)}-1)m})(C_{l_{(1)}} \otimes I_m) \in GL_{ml_{(1)}}(\mathbb{Z})$. Then the zeta function $\zeta_{\pi_{nm}(\sigma*\tau)}(s, X_{nm})$ can be expressed as follows.

$$\zeta_{\pi_{nm}(\sigma*\tau)}(s, X_{nm}) = \det(I_{ml_{(1)}} - C_1(\sigma)s)^{-1} \prod_{k=2}^{c(\tau)} \frac{1}{(1 - s^{l_{(k)}})^m}.$$

Since the form of $C_1(\sigma)$ is

$$C_1(\sigma) = \begin{pmatrix} O & & & p_m(\pi_m(\sigma)) \\ I_m & O & & \\ & \ddots & \ddots & \\ & & I_m & O \end{pmatrix} \in GL_{ml_{(1)}}(\mathbb{Z}),$$

then we have

$$\mathrm{tr}\,(C_1(\sigma)^j) = \begin{cases} 0 & (j \not\equiv 0 \ (\bmod\ l_{(1)})), \\ l_{(1)} \cdot \mathrm{tr}\,(p_m(\pi_m(\sigma^k))) & (j = kl_{(1)}, k \in \mathbb{N}). \end{cases}$$

Hence,

$$\begin{aligned}
\det(I_{ml_{(1)}} - C_1(\sigma)s)^{-1} &= \exp\Big\{ \sum_{j=1}^{\infty} \frac{\mathrm{tr}\,(C_1(\sigma)^j)}{j} s^j \Big\} \\
&= \exp\Big\{ \sum_{j \not\equiv 0\ (\bmod\ l_{(1)})} \frac{0}{j} s^j + \sum_{k=1}^{\infty} \frac{l_{(1)} \cdot \mathrm{tr}\,(p_m(\pi_m(\sigma^k)))}{kl_{(1)}} s^{kl_{(1)}} \Big\} \\
&= \zeta_{\pi_m(\sigma)}(s^{l_{(1)}}, X_m).
\end{aligned}$$

Thus,

$$\begin{aligned}
\zeta_{\pi_{nm}(\sigma*\tau)}(s, X_{nm}) &= \zeta_{\pi_m(\sigma)}(s^{l_{(1)}}, X_m) \prod_{k=2}^{c(\tau)} \frac{1}{(1-s^{l_{(k)}})^m} \\
&= \frac{1}{(1-s)^{m(c(\tau)-1)}} \prod_{k=2}^{c(\tau)} \frac{1}{[l_{(k)}]_s^m} \prod_{P\in \mathrm{Cycle}(\pi_m(\sigma))} \frac{1}{1-s^{l_{(1)}l(P)}} \\
&= \frac{1}{(1-s)^{m(c(\tau)-1)+c(\sigma)}} \prod_{k=2}^{c(\tau)} \frac{1}{[l_{(k)}]_s^m} \prod_{P\in \mathrm{Cycle}(\pi_m(\sigma))} \frac{1}{[l_{(1)}]_s \cdot [l(P)]_{s^{l_{(1)}}}}.
\end{aligned}$$

By Proposition 3.2.1, we have

$$c(\sigma * \tau) = c(\sigma) + m(c(\tau) - 1).$$

□

From Proposition 3.2.3, we obtain the following Corollary.

Corollary 3.2.1. *For $\sigma \in B_m$, $\tau \in B_n$, $\widehat{\sigma * \tau}$ is a knot if and only if both $\widehat{\sigma}$ and $\widehat{\tau}$ are knots.*

Moreover, for $\sigma \in B_n$, we define the *N-th special product* as follows.

$$\sigma^{*N} := \sigma * \sigma * \cdots * \sigma \in B_{n^N}.$$

We calculate the component number of the N-th special product of σ.

Proposition 3.2.4. *For $\sigma \in B_n$ $(n > 2)$, $N \in \mathbb{N}$, we have*

$$c(\sigma^{*N}) = [N]_n(c(\sigma) - 1) + 1,$$

where $[N]_n := 1 + n + n^2 + \cdots + n^{N-1}$, and we put $[0]_n = 0$. In particular, if the closure of $\sigma \in B_n$ is a knot, then the closure of σ^{*N} is a knot.

Proof. By Proposition 3.2.3,

$$\begin{aligned}
c(\sigma^{*N}) &= c(\sigma^{*(N-1)} * \sigma) \\
&= n(c(\sigma^{*(N-1)}) - 1) + c(\sigma) \\
&= nc(\sigma^{*(N-1)}) + c(\sigma) - n.
\end{aligned}$$

Then we have the following recurrence relation.

$$c(\sigma^{*N}) - \frac{n - c(\sigma)}{n-1} = n\{c(\sigma^{*(N-1)}) - \frac{n-c(\sigma)}{n-1}\}.$$

Thus,

$$\begin{aligned}
c(\sigma^{*N}) &= n^{N-1}\{c(\sigma) - \frac{n-c(\sigma)}{n-1}\} + \frac{n-c(\sigma)}{n-1} \\
&= \frac{1}{n-1}\{n^N(c(\sigma)-1) + n - 1 - (c(\sigma)-1)\} \\
&= [N]_n(c(\sigma)-1) + 1.
\end{aligned}$$

If $c(\sigma) = 1$, we have $c(\sigma^{*N}) = 1$. Then Proposition 3.2.4 holds. □

3.3 Braid zeta function of the cabling braid

We consider the formula of the braid zeta function of the cabling braid.

Definition 3.3.1. *We define the following braid representation.*

$$b_{n,q} : B_n \longrightarrow GL(W_n),$$

$$b_{n,q}(\sigma_i) := I_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \oplus I_{n-i-1} \qquad (i = 1, 2, \ldots, n-1).$$

From the definition, for $\sigma \in B_n$, we have

$$\det b_{n,q}(\sigma) = (-q)^{\varepsilon(\sigma)}.$$

Lemma 3.3.1. *If the closure of $\sigma \in B_n$ is knot, we have*

$$b_{n,q}(\sigma^n) = q^{\varepsilon(\sigma)} \cdot I_n.$$

Proof. Since $\widehat{\sigma}$ is knot, $\pi_n(\sigma)$ is simple cycle. In other word, there exists $\mu \in \mathfrak{S}_n$ such that

$$p_n(\mu)p_n(\pi_n(\sigma))p_n(\mu)^{-1} = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

On the other hand, $b_{n,q}(\sigma)$ has exactly one entry of 1, or the power of q in each row and each column and 0's elsewhere. Since $\lim_{q \to 1} b_{n,q}(\sigma) = p_n(\pi_n(\sigma))$, we obtain

$$p_n(\mu)b_{n,q}(\sigma)p_n(\mu)^{-1} = \begin{pmatrix} 0 & & & q^{t_1} \\ q^{t_2} & 0 & & \\ & \ddots & \ddots & \\ & & q^{t_n} & 0 \end{pmatrix} =: C_{n,q}.$$

By cofactor expansion, we can calculate the zeta function associated with $b_{n,q}$ as follows.

$$\begin{aligned} \zeta(s, \sigma; b_{n,q}) = \det(I_n - b_{n,q}(\sigma))^{-1} &= \det(I_n - p_n(\mu)b_{n,q}(\sigma)p_n(\mu)^{-1})^{-1} \\ &= \frac{1}{1 + (-1)^n \det(C_{n,q})s^n} \\ &= \frac{1}{1 + (-1)^{n+\varepsilon(\sigma)}q^{\varepsilon(\sigma)}s^n}. \end{aligned}$$

Since $\pi_n(\sigma)$ is simple cycle,

$$(-1)^{\varepsilon(\sigma)} = \mathrm{sgn}(\pi_n(\sigma)) = (-1)^{n-1}.$$

Then we have

$$\zeta(s, \sigma; b_{n,q}) = \frac{1}{1 - q^{\varepsilon(\sigma)}s^n} = \exp\left\{ \sum_{m=1}^{\infty} \frac{1}{m}(q^{\varepsilon(\sigma)}s)^m \right\}.$$

Thus we obtain the following equation.

$$\mathrm{tr}\, b_{n,q}(\sigma^{nk}) = n \cdot q^{\varepsilon(\sigma)k}.$$

Since all non-zero entries of $b_{n,q}(\sigma)$ are the power of q, we can conclude

$$b_{n,q}(\sigma^{nk}) = q^{\varepsilon(\sigma)k} I_n.$$

□

For $m \in \mathbb{N}$, we have the following formula.

Theorem 3.3.1. *We assume that the closure of $\sigma \in B_n$ is a knot. Then, we have*

$$\zeta(s, w_m(\sigma); \beta_{nm,q}) = \frac{1}{(1 - q^{m\varepsilon(\sigma)} s^n)^{m-1}} \zeta(s, \sigma; \beta_{n,q^m}).$$

Proof. First, we compute the trace of $w_m(\sigma)$. For a generator $\sigma_i \in B_n$, the Burau representation can be expressed by

$$\beta_{n,q}(\sigma_i) = O_{i-1} \oplus \begin{pmatrix} 1-q & 0 \\ 0 & 0 \end{pmatrix} \oplus O_{n-i-1} + b_{n,q}(\sigma_i),$$

$$\beta_{n,q}(\sigma_i^{-1}) = O_{i-1} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1-q^{-1} \end{pmatrix} \oplus O_{n-i-1} + b_{n,q}(\sigma_i^{-1}).$$

Here, O_k is a square zero matrix of size $k \times k$. Moreover for $\sigma \in B_n$ we put T by

$$\beta_{n,q}(\sigma) = T(\sigma, q) + b_{n,q}(\sigma).$$

To examine the (i,j)-entry of $T(\sigma, q)$, we introduce the following matrices.

$$X_+(x, i) := O_{i-1} \oplus \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \oplus O_{n-i-1} + b_{n,q}(\sigma_i),$$

$$X_-(y, i) := O_{i-1} \oplus \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \oplus O_{n-i-1} + b_{n,q}(\sigma_i^{-1}).$$

For non-zero integer e, we define

$$X_i^e := \begin{cases} X_+(x,i)^e & (e > 0), \\ X_-(y,i)^{-e} & (e < 0). \end{cases}$$

We fix an expression $\sigma = \sigma_{i_1}^{e_1} \sigma_{i_2}^{e_2} \cdots \sigma_{i_r}^{e_r}$. For this expression, we define the matrix U by

$$U := X_{i_1}^{e_1} X_{i_2}^{e_2} \cdots X_{i_r}^{e_r} \in M_n(\mathbb{Z}[x, y, q^{\pm 1}]).$$

Furthermore we define $\widetilde{T}(x, y, q)$ as

$$\widetilde{T}(x, y, q) := U - b_{n,q}(\sigma).$$

Then (i,j)-entry of $\widetilde{T}(x, y, q)$ can be written by the following form.

$$\widetilde{T}(x, y, q)_{i,j} = \sum_{a+b \geq 1, c \in Z} \omega_{a,b,c} x^a y^b q^c. \tag{3.2}$$

By replacing $x = 1 - q$, $y = 1 - q^{-1}$, $\widetilde{T}(x, y, q)$ turns to $T(\sigma, q)$. Then the (i,j)-entry of $T(\sigma, q)$ can be written by

$$T(\sigma, q)_{i,j} = \sum_{a+b \geq 1, c \in \mathbb{Z}} \omega_{a,b,c} (1-q)^a (1-q^{-1})^b q^c. \tag{3.3}$$

Next we consider the cabling braid. For $m \in \mathbb{N}$, we define

$$T_m(\sigma, q) := \beta_{nm,q}(w_m(\sigma)) - b_{n,q^m}(\sigma) \otimes I_m.$$

For generator $\sigma_i \in B_n$, we can calculate

$$\beta_{nm,q}(w_m(\sigma_i)) = O_{m(i-1)} \oplus \begin{pmatrix} (1-q)A_m(q) & O \\ O & O \end{pmatrix} \oplus O_{m(n-i-1)} + b_{n,q^m}(\sigma_i) \otimes I_m,$$

$$\beta_{nm,q}(w_m(\sigma_i^{-1})) = O_{m(i-1)} \oplus \begin{pmatrix} O & O \\ O & (1-q^{-1})q^{-m+1}A_m(q) \end{pmatrix} \oplus O_{m(n-i-1)} + b_{n,q^m}(\sigma_i^{-1}) \otimes I_m.$$

We assume that the (i,j)-entry of $\widetilde{T}(x,y,q)$ can be expressed as (3.2), Then we have

$$T_m(\sigma,q)_{[i,j]} = \sum_{a+b\geq 1, c\in\mathbb{Z}} \omega_{a,b,c}\{(1-q)A_m(q)\}^a\{(1-q^{-1})q^{-m+1}A_m(q)\}^b\{q^m I_m\}^c.$$

Here, $T_m(\sigma,q)_{[i,j]}$ is the (i,j)-block entry of $T_m(\sigma,q)$. Moreover for $N \in \mathbb{N}$, we immediately have

$$A_m(q)^N = [m]^{N-1} \cdot A_m(q).$$

Then the trace of $A_m(q)^N$ can be computed as

$$\mathrm{tr}\,(A_m(q)^N) = [m]^{N-1}\mathrm{tr}\,A_m(q) = [m]^N.$$

Thus we have

$$\begin{aligned}
\mathrm{tr}\,T_m(\sigma,q)_{[i,j]} &= \sum_{a+b\geq 1, c\in\mathbb{Z}} \omega_{a,b,c}\mathrm{tr}\left\{\{(1-q)A_m(q)\}^a\{(1-q^{-1})q^{-m+1}A_m(q)\}^b\{q^m I_m\}^c\right\} \\
&= \sum_{a+b\geq 1, c\in\mathbb{Z}} \omega_{a,b,c}(1-q)^a(1-q^{-1})^b q^{(-m+1)b}q^{mc}\mathrm{tr}\,(A_m(q)^{a+b}) \\
&= \sum_{a+b\geq 1, c\in\mathbb{Z}} \omega_{a,b,c}(1-q)^a(1-q^{-1})^b q^{(-m+1)b}q^{mc}[m]^{a+b} \\
&= \sum_{a+b\geq 1, c\in\mathbb{Z}} \omega_{a,b,c}(1-q^m)^a(1-q^{-m})^b q^{mc}.
\end{aligned}$$

Hence

$$\mathrm{tr}\,T_m(\sigma,q)_{[i,j]} = T(\sigma,q^m)_{i,j}.$$

We suppose that the closure of $\sigma \in B_n$ is a knot. When $j \not\equiv 0 \pmod n$, diagonal components of $b_{n,q}(\sigma^j)$ are all 0 because $\pi_n(\sigma)$ is simple cycle. Then we have

$$\begin{aligned}
\mathrm{tr}\,\beta_{nm,q}(w_m(\sigma^j)) &= \mathrm{tr}\,T_m(\sigma^j,q) + \mathrm{tr}\,(b_{n,q^m}(\sigma^j)\otimes I_m) \\
&= \sum_{i=1}^{n} \mathrm{tr}\,T(\sigma^j,q^m)_{[i,i]} + 0 = \mathrm{tr}\,\beta_{n,q^m}(\sigma^j).
\end{aligned}$$

If $j = nk\ (\in \mathbb{N})$, by Lemma 3.3.1,

$$\mathrm{tr}\,\beta_{n,q}(\sigma^{nk}) = \mathrm{tr}\,T(\sigma^{nk},q) + n\cdot q^{k\varepsilon(\sigma)}.$$

Then for $m \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{tr}\,\beta_{nm,q}(w_m(\sigma^{nk})) &= \mathrm{tr}\,T_m(\sigma^{nk},q) + q^{mk\varepsilon(\sigma)}\mathrm{tr}\,I_{mn} \\
&= \mathrm{tr}\,T(\sigma^{nk},q^m) + n\cdot q^{mk\varepsilon(\sigma)} + n(m-1)\cdot q^{mk\varepsilon(\sigma)} \\
&= \mathrm{tr}\,\beta_{n,q^m}(\sigma^{nk}) + n(m-1)\cdot q^{mk\varepsilon(\sigma)}.
\end{aligned}$$

Hence, by generating function expression, we have

$$\zeta(s, w_m(\sigma); \beta_{nm,q}) = \exp\left\{ \sum_{j \not\equiv 0 \pmod n} \frac{\operatorname{tr} \beta_{n,q^m}(\sigma^j)}{j} s^j + \sum_{k=1}^{\infty} \frac{\operatorname{tr} \beta_{n,q^m}(\sigma^{nk}) + n(m-1)q^{mk\varepsilon(\sigma)}}{nk} s^{nk} \right\}$$

$$= \exp\left\{ \sum_{j=1}^{\infty} \frac{\operatorname{tr} \beta_{n,q^m}(\sigma^j)}{j} s^j + \sum_{k=1}^{\infty} \frac{(m-1)q^{mk\varepsilon(\sigma)}}{k} s^{nk} \right\}$$

$$= \zeta(s, \sigma; \beta_{n,q^m}) \frac{1}{(1 - q^{m\varepsilon(\sigma)} s^n)^{m-1}}.$$

□

We give an example.

Example 3.3.1. We calculate the zeta function of $w_m(\sigma_1^3) \in B_{2m}$. By using the formula of torus type braid (Theorem 2.1.1, (5)) and Theorem 3.3.1, we have

$$\zeta(s, w_m(\sigma_1^3); \beta_{2m,q}) = \frac{1}{(1 - q^{3m}s^2)^{m-1}} \zeta(s, \sigma_1^3; \beta_{2,q^m}) = \frac{1}{(1-s)(1+q^{3m}s)(1-q^{3m}s^2)^{m-1}}.$$

The closure of $w_m(\sigma_1^3)$ turns to m-cabling version of the trefoil knot.

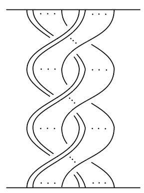

Figure 10: m-cabling braid $w_m(\sigma_1^3) \in B_{2m}$

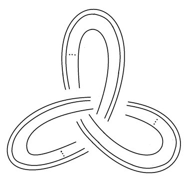

Figure 11: The closure of $w_m(\sigma_1^3)$

3.4 Decomposition formula

The main goal of this section is to show the following decomposition formula.

Theorem 3.4.1. *Let $\sigma \in B_m, \tau \in B_n$ be the braids whose closures are both knots, then we have the following formula.*

$$\zeta(s, \sigma * \tau; \beta_{nm,q}) = (1 - q^{m\varepsilon(\tau)} s^n) \zeta(q^{m\varepsilon(\tau)} s^n, \sigma; \beta_{m,q}) \zeta(s, \tau; \beta_{n,q^m}).$$

Proof. We calculate the trace of $\beta_{nm,q}((\sigma * \tau)^j)$. By using T_m, we have

$$\begin{aligned}\beta_{nm,q}(\sigma * \tau) &= \beta_{nm,q}(\iota_{m,nm}(\sigma)) \cdot \beta_{nm,q}(w_m(\tau)) \\ &= (\beta_{m,q}(\sigma) \oplus I_{m(n-1)})(T_m(\tau, q) + b_{n,q^m}(\tau) \otimes I_m).\end{aligned}$$

As with the proof of Theorem 3.3.1, we express the (i, j)-block entries of $T_m(\tau, q)$ as

$$T_m(\tau, q)_{i,j} = \sum_{a+b \geq 1, c \in \mathbb{Z}} \omega_{a,b,c} \{(1-q)A_m(q)\}^a \{(1-q^{-1})q^{-m+1}A_m(q)\}^b \{q^m I_m\}^c.$$

By easy calculation, we have

$$\begin{pmatrix} 1-q & 1 \\ q & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ q & q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ q & q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ q & q \end{pmatrix}\begin{pmatrix} 1-q & 1 \\ q & 0 \end{pmatrix}.$$

Then, for any $\sigma \in B_m$, the matrix $A_m(q)$ satisfies

$$\beta_{m,q}(\sigma)A_m(q) = A_m(q) = A_m(q)\beta_{m,q}(\sigma).$$

Hence we have

$$T_m(\tau, q)(\beta_{m,q}(\sigma) \oplus I_{m(n-1)}) = T_m(\tau, q) = (\beta_{m,q}(\sigma) \oplus I_{m(n-1)})T_m(\tau, q).$$

Thus $\beta_{nm,q}((\sigma * \tau)^j)$ can be computed as

$$\begin{aligned}\beta_{nm,q}((\sigma * \tau)^j) &= \{(\beta_{m,q}(\sigma) \oplus I_{m(n-1)})(b_{n,q^m}(\tau) \otimes I_m) + T_m(\tau, q)\}^j \\ &= \{(\beta_{m,q}(\sigma) \oplus I_{m(n-1)})(b_{n,q^m}(\tau) \otimes I_m)\}^j + T_m(\tau^j, q).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\beta_{nm,q}(w_m(\tau^j)) &= \{T_m(\tau, q) + b_{n,q^m}(\tau) \otimes I_m\}^j \\ &= T_m(\tau^j, q) + (b_{n,q^m}(\tau) \otimes I_m)^j.\end{aligned}$$

Since the closure of τ is a knot, there is the permutation $\mu \in \mathfrak{S}_n$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & p_{n-1}(\mu) \end{pmatrix} p_n(\pi_n(\sigma)) \begin{pmatrix} 1 & 0 \\ 0 & p_{n-1}(\mu) \end{pmatrix}^{-1} = C_n.$$

For simplicity, we set

$$Q := \begin{pmatrix} 1 & 0 \\ 0 & p_{n-1}(\mu) \end{pmatrix}.$$

Furthermore Q satisfies

$$(Q \otimes I_m)(\beta_{m,q}(\sigma) \oplus I_{m(n-1)}) = (\beta_{m,q}(\sigma) \oplus I_{m(n-1)})(Q \otimes I_m).$$

Then we have

$$\begin{aligned}&(Q \otimes I_m)\beta_{nm,q}((\sigma * \tau)^j)(Q \otimes I_m)^{-1} \\ &\quad = (Q \otimes I_m)\{T_m(\tau^j, q) + \{(\beta_{m,q}(\sigma) \oplus I_{m(n-1)})(b_{n,q^m}(\tau) \otimes I_m)\}^j\}(Q \otimes I_m)^{-1} \\ &\quad = (Q \otimes I_m)T_m(\tau^j, q)(Q^{-1} \otimes I_m) + (Q \otimes I_m)\{(\beta_{m,q}(\sigma) \oplus I_{m(n-1)})(b_{n,q^m}(\tau) \otimes I_m)\}^j(Q^{-1} \otimes I_m) \\ &\quad = (Q \otimes I_m)T_m(\tau^j, q)(Q^{-1} \otimes I_m) + \{(Q \otimes I_m)(\beta_{m,q}(\sigma) \oplus I_{m(n-1)})(b_{n,q^m}(\tau) \otimes I_m)(Q^{-1} \otimes I_m)\}^j \\ &\quad = (Q \otimes I_m)T_m(\tau^j, q)(Q^{-1} \otimes I_m) + \{(\beta_{m,q}(\sigma) \oplus I_{m(n-1)}(Q \otimes I_m))(b_{n,q^m}(\tau) \otimes I_m)(Q^{-1} \otimes I_m)\}^j \\ &\quad = (Q \otimes I_m)T_m(\tau^j, q)(Q^{-1} \otimes I_m) + \begin{pmatrix} O & & & q^{mi_1}\beta_{m,q}(\sigma) \\ q^{mi_2}I_m & \ddots & & \\ & \ddots & \ddots & \\ & & q^{mi_n}I_m & O \end{pmatrix}^j.\end{aligned}$$

Here, $i_1 + i_2 + \cdots + i_n = \varepsilon(\tau)$. When $j \not\equiv 0 \,(\mathrm{mod}\ n)$, we have

$$\begin{aligned}
\mathrm{tr}\,\beta_{nm,q}((\sigma * \tau)^j) &= \mathrm{tr}\,\{(Q \otimes I_m)\beta_{nm,q}((\sigma * \tau)^j)(Q \otimes I_m)^{-1}\} \\
&= \mathrm{tr}\,\{(Q \otimes I_m)T_m(\tau^j, q)(Q \otimes I_m)^{-1}\} + 0 \\
&= \mathrm{tr}\,T_m(\tau^j, q) \\
&= \mathrm{tr}\,T(\tau^j, q^m) \\
&= \mathrm{tr}\,\beta_{n,q^m}(\tau^j).
\end{aligned}$$

On the other hand, when $j = nk (k \in \mathbb{N})$, from Lemma 3.3.1,

$$\begin{aligned}
\begin{pmatrix} O & & & q^{mi_1}\beta_{m,q}(\sigma) \\ q^{mi_2}I_m & \ddots & & \\ & \ddots & \ddots & \\ & & q^{mi_n}I_m & O \end{pmatrix}^j &= \begin{pmatrix} q^{m(i_1+\cdots+i_n)}\beta_{m,q}(\sigma) & & \\ & \ddots & \\ & & q^{m(i_1+\cdots+i_n)}\beta_{m,q}(\sigma) \end{pmatrix}^k \\
&= \begin{pmatrix} q^{mk\varepsilon(\tau)}\beta_{m,q}(\sigma^k) & & \\ & \ddots & \\ & & q^{mk\varepsilon(\tau)}\beta_{m,q}(\sigma^k) \end{pmatrix} \\
&= q^{mk\varepsilon(\tau)}(I_n \otimes \beta_{n,q}(\sigma^k)).
\end{aligned}$$

Hence we can calculate the trace as follows.

$$\begin{aligned}
\mathrm{tr}\,\beta_{nm,q}((\sigma * \tau)^{nk}) &= \mathrm{tr}\,T_m(\tau^{nk}, q) + \mathrm{tr}\,(q^{mk\varepsilon(\tau)}(I_n \otimes \beta_{n,q}(\sigma^k))) \\
&= \mathrm{tr}\,T(\tau^{nk}, q^m) + n \cdot q^{mk\varepsilon(\tau)}\mathrm{tr}\,\beta_{n,q}(\sigma^k) \\
&= \mathrm{tr}\,\beta_{n,q^m}(\tau^{nk}) + nq^{mk\varepsilon(\tau)}(\mathrm{tr}\,\beta_{n,q}(\sigma^k) - 1).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\zeta(s, \sigma * \tau; \beta_{nm,q}) &= \exp\Bigg\{\sum_{j=1}^{\infty} \frac{\mathrm{tr}\,\beta_{nm,q}((\sigma * \tau)^j)}{j} s^j\Bigg\} \\
&= \exp\Bigg\{\sum_{j \not\equiv 0\ (\bmod\ n)} \frac{\mathrm{tr}\,\beta_{n,q^m}(\tau^j)}{j} s^j + \sum_{k=1}^{\infty} \frac{\mathrm{tr}\,\beta_{n,q^m}(\tau^{nk}) + nq^{m\varepsilon(\tau)k}(\mathrm{tr}\,\beta_{m,q}(\sigma^k) - 1)}{nk} s^{nk}\Bigg\} \\
&= \exp\Bigg\{\sum_{j=1}^{\infty} \frac{\mathrm{tr}\,\beta_{n,q^m}((\tau)^j)}{j} s^j + \sum_{k=1}^{\infty} \frac{\mathrm{tr}\,\beta_{m,q}((\sigma)^k)}{k} (q^{m\varepsilon(\tau)}s^n)^k - \sum_{k=1}^{\infty} \frac{1}{k}(q^{m\varepsilon(\tau)}s^n)^k\Bigg\} \\
&= \zeta(s, \tau; \beta_{n,q^m}) \cdot \zeta(q^{m\varepsilon(\tau)}s^n, \sigma; \beta_{m,q}) \cdot (1 - q^{m\varepsilon(\tau)}s^n).
\end{aligned}$$

Hence we finish the proof of Theorem 3.4.1. □

Corollary 3.4.1. *For $\sigma \in B_m, \tau \in B_n$, we assume that both $\widehat{\sigma}$ and $\widehat{\tau}$ are knots and $\varepsilon(\tau) = 0$. Then we have*

$$\Delta_{\widehat{\sigma * \tau}}(q) = \Delta_{\widehat{\sigma}}(q)\Delta_{\widehat{\tau}}(q^m).$$

Proof. Since $\widehat{\sigma * \tau}$ is a knot, by Theorem 2.1.1, we have

$$\operatorname*{Res}_{s=1} \zeta(s, \sigma * \tau; \beta_{nm,q}) = -\frac{1}{[nm]_q}\Delta_{\widehat{\sigma * \tau}}(q)^{-1}.$$

On the other hand, by Theorem 3.4.1,

$$\begin{aligned}\operatorname*{Res}_{s=1}\zeta(s,\sigma*\tau;\beta_{nm,q}) &= \operatorname*{Res}_{s=1}\{\zeta(s,\tau;\beta_{n,q^m})\cdot\zeta(s^n,\sigma;\beta_{m,q})\cdot(1-s^n)\}\\ &= \lim_{s\to 1}(s-1)(1-s^n)\zeta(s^n,\sigma;\beta_{m,q})\zeta(s,\tau;\beta_{n,q^m})\\ &= -\frac{1}{[n]_{q^m}}\Delta_{\widehat{\tau}}(q^m)^{-1}\frac{1}{[m]_q}\Delta_{\widehat{\sigma}}(q)^{-1}.\end{aligned}$$

Since $[nm]_q=[m]_q[n]_{q^m}$, Corollary 3.4.1 holds. □

We now compute the zeta function of the N-th special product of $\sigma\in B_n$.

Theorem 3.4.2. *We assume that the closure of $\sigma\in B_n$ is a knot, then we have*

$$\zeta(s,\sigma^{*N};\beta_{n^N,q})=\zeta(s,\sigma;\beta_{n,q^{n^{N-1}}})\prod_{j=1}^{N-1}(1-q^{n^{N-j}\varepsilon(\sigma^{*j})}s^{n^j})\zeta(q^{n^{N-j}\varepsilon(\sigma^{*j})}s^{n^j},\sigma;\beta_{n,q^{n^{N-j-1}}}). \quad (3.4)$$

Proof. We prove Theorem 3.4.2 by induction. The equation (3.4) is obviously hold when $N=1$. Then we suppose that the formula holds for σ^{*k} $(k\geq 2)$. By Theorem 3.4.1,

$$\begin{aligned}&\zeta(s,\sigma^{*(k+1)};\beta_{n^{k+1},q})\\ &=(1-q^{n\varepsilon(\sigma^{*k})}s^{n^k})\zeta(q^{n\varepsilon(\sigma^{*k})}s^{n^k},\sigma;\beta_{n,q})\zeta(s,\sigma^{*k};\beta_{n^k,q^n})\\ &=(1-q^{n\varepsilon(\sigma^{*k})}s^{n^k})\zeta(q^{n\varepsilon(\sigma^{*k})}s^{n^k},\sigma;\beta_{n,q})\zeta(s,\sigma;\beta_{n,q^{n^k}})\prod_{j=1}^{k-1}(1-q^{n^{k+1-j}\varepsilon(\sigma^{*j})}s^{n^j})\zeta(q^{n^{k+1-j}\varepsilon(\sigma^{*j})}s^{n^j},\sigma;\beta_{n,q^{n^{k-j}}})\\ &=\zeta(s,\sigma;\beta_{n,q^{n^k}})\prod_{j=1}^{k}(1-q^{n^{k+1-j}\varepsilon((\sigma)^{*j})}s^{n^j})\zeta(q^{n^{k+1-j}\varepsilon((\sigma)^{*j})}s^{n^j},\sigma;\beta_{n,q^{n^{k-j}}}).\end{aligned}$$

Then the equation (3.4) holds for any $N\in\mathbb{N}$. □

Example 3.4.1. We calculate the zeta function of 3-rd special product of $\sigma_1^3\in B_2$ as follows.

$$\begin{aligned}\zeta(s,\sigma_1^3*\sigma_1^3*\sigma_1^3;\beta_{8,q}) &= \zeta(s,\sigma_1^3;\beta_{2,q^4})\prod_{j=1}^{2}(1-q^{2^{3-j}\varepsilon(\sigma^{*j})}s^{2^j})\zeta(q^{2^{3-j}\varepsilon(\sigma^{*j})}s^{2^j},\sigma_1^3;\beta_{2,q^{2^{2-j}}})\\ &= \zeta(s,\sigma_1^3;\beta_{2,q^4})(1-q^{12}s^2)\zeta(q^{12}s^2,\sigma_1^3;\beta_{2,q^2})(1-q^{30}s^4)\zeta(q^{30}s^4,\sigma;\beta_{2,q})\\ &= \frac{1}{(1-s)(1+q^{12}s)(1+q^{18}s^2)(1+q^{33}s^4)}.\end{aligned}$$

Then, we obtain the Alexander polynomial of $\widehat{(\sigma_1^3)^{*3}}$.

$$\begin{aligned}\Delta_{\widehat{(\sigma_1^3)^{*3}}}(q) &= \frac{(1-q)(1+q^{12})(1+q^{18})(1+q^{33})}{1-q^8}\\ &= 1-q+q^8-q^9+q^{12}-q^{13}+q^{16}-q^{17}+q^{18}-q^{19}+q^{20}\\ &\quad -q^{21}+q^{24}-q^{25}+q^{26}-q^{27}+q^{28}-q^{29}+q^{30}-q^{31}+q^{32}\\ &\quad -q^{35}+q^{36}-q^{37}+q^{38}-q^{39}+q^{40}-q^{43}+q^{44}-q^{47}+q^{48}\\ &\quad -q^{55}+q^{56}.\end{aligned}$$

Next we consider the N-th special product of torus type braid.

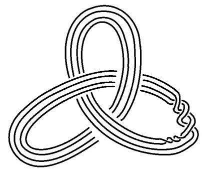

Figure 12: The closure of $(\sigma_1^3)^{*3}$

Theorem 3.4.3. *We assume that (n,m) is coprime, then for any $N \in \mathbb{N}$, we have*

$$\Delta_{\widehat{\sigma^{*N}_{(n,m)}}}(q) = \prod_{j=0}^{N-1} \frac{(1-q^{n^{N-j-1}})(1-q^{n^{N-j}a^{(j)}_{n,m}})}{(1-q^{n^{N-j}})(1-q^{n^{N-j-1}a^{(j)}_{n,m}})} = \prod_{j=0}^{N-1} \frac{[a^{(j)}_{n,m}]_{q^{N-j}}}{[a^{(j)}_{n,m}]_{q^{N-j-1}}}.$$

Here $a^{(j)}_{n,m} := m(n^{2j+1}+1)/(n+1)$.

Proof. From the definition, we have $\varepsilon(\sigma_{(n,m)}) = m(n-1)$. Then for $j \in \mathbb{N}$ we compute

$$\varepsilon(\sigma^{*j}_{(n,m)}) = \sum_{k=0}^{j-1} \varepsilon(w_{n^k}(\sigma_{(n,m)})) = m(n-1)\sum_{k=0}^{j-1} n^{2k} = \frac{m(n^{2j}-1)}{n+1}.$$

By Theorem 3.4.2 and formula of the zeta function of the torus type braid, we have

$$\begin{aligned}
\zeta(s,\sigma^{*N}_{(n,m)};\beta_{n^N,q}) &= \zeta(s,\sigma^{*N}_{(n,m)};\beta_{n,q^{N-1}}) \prod_{j=1}^{N-1} (1-q^{n^{N-j}\varepsilon(\sigma^{(j)}_{(n,m)})}s^{n^j})\zeta(q^{n^{N-j}\varepsilon(\sigma^{(j)}_{(n,m)})}s^{n^j},\sigma_{(n,m)};\beta_{n,q^{n^{N-j-1}}}) \\
&= \frac{1-q^{mn^{N-1}}s}{(1-s)(1-q^{mn^N}s^n)} \prod_{j=1}^{N-1} \frac{1-q^{mn^{N-j-1}+n^{N-j}\varepsilon(\sigma^{*j}_{(n,m)})}s^{n^j}}{1-q^{mn^{N-j}+n^{N-j+1}\varepsilon(\sigma^{*j}_{(n,m)})}s^{n^j+1}} \\
&= \frac{1}{1-s}\prod_{j=0}^{N-1} \frac{1-q^{n^{N-j-1}a^{(j)}_{n,m}}s^{n^j}}{1-q^{n^{N-j}a^{(j)}_{n,m}}s^{n^{j+1}}}.
\end{aligned}$$

By Theorem 2.1.1, we have

$$\begin{aligned}
\Delta_{\widehat{\sigma^{*N}_{(n,m)}}}(q) &= \frac{1}{[n^N]_q}\prod_{j=0}^{N-1} \frac{1-q^{n^{N-j}a^{(j)}_{n,m}}}{1-q^{n^{N-j-1}a^{(j)}_{n,m}}} \\
&= \prod_{j=0}^{N-1} \frac{(1-q^{n^{N-j-1}})(1-q^{n^{N-j}a^{(j)}_{n,m}})}{(1-q^{n^{N-j}})(1-q^{n^{N-j-1}a^{(j)}_{n,m}})} \\
&= \prod_{j=0}^{N-1} \frac{[a^{(j)}_{n,m}]_{q^{N-j}}}{[a^{(j)}_{n,m}]_{q^{N-j-1}}}.
\end{aligned}$$

Thus, we finish the proof of Theorem 3.4.3. □

Example 3.4.2. We have the formula of the Alexander polynomial of $(\sigma_{2,3})^{*N}$ as

$$\begin{aligned}\Delta_{\widehat{(\sigma_{2,3})^{*N}}}(q) &= \prod_{j=0}^{N-1} \frac{(1-q^{2^{N-j-1}})(1-q^{2^{N-j}(2^{2j+1}+1)})}{(1-q^{2^{N-j}})(1-q^{2^{N-j-1}(2^{2j+1}+1)})} \\ &= \prod_{j=0}^{N-1} \frac{1+q^{2^{N-j-1}(2^{2j+1}+1)}}{1+q^{2^{N-j-1}}} \\ &= \prod_{j=0}^{N-1} \sum_{k=0}^{2^{2j+1}} (-1)^k q^{k\cdot 2^{N-j-1}}.\end{aligned}$$

4 q-analogue of symmetric Burau representation

A. Kosyak introduced the new braid representation by quantizing the symmetric power of the Burau representation for the case of B_3 (see [1]). The goal of this section is to generalize the case of B_n, and to calculate the zeta function associated with this representation for the torus type braid. Furthermore, calculating the zeta function of the torus type braid by different two ways, we obtain a certain q-identities.

4.1 Construction of the representation $\rho^{(N)}_{n,q,t}$

In this section, we define the braid representation of B_n which is a generalization of the Kosyak' representation of B_3. Let $\beta^r_{n,q}$ be the reduced Burau representation as we already defined. Then we decompose $\beta^r_{n,a}(\sigma_i)$ into two or three matrices and replace q by $-t$ as follows.

$$\beta^r_{n,q}(\sigma_i) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3} & (i=1), \\ I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} & (2 \le i \le n-2), \\ I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & (i=n-1). \end{cases}$$

We can obtain the braid representation by considering the N-th symmetric power of $\beta^r_{n,-t}$ for $N \in \mathbb{N}$. We write this representation by $\mathrm{Sym}^{(\mathrm{N})}\beta^r_{n,-t} : B_n \longrightarrow GL(\mathrm{Sym}^{(\mathrm{N})}(W^r_n))$. We define the set $\mathbb{I}^r_n(N)$ as

$$\mathbb{I}^r_n(N) := \{I = (i_1, i_2, \dots, i_N) \in X^N_{n-1} \mid i_1 \le i_2 \le \cdots \le i_N\}.$$

Here, $X_{n-1} := \{1, 2, \dots, n-1\}$. For $I = (i_1, i_2, \dots, i_N) \in \mathbb{I}^r_n(N)$, we put

$$\widetilde{f}_I := \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_N}.$$

Then, we can regard $\{\widetilde{f}_I \mid I \in \mathbb{I}^r_n(N)\}$ as the basis of $\mathrm{Sym}^{(\mathrm{N})}(W^r_n)$. We denote $N_i(I)$ as the number of i in $I \in \mathrm{I}^r_n(N)$, and put $N_i(I) = 0$ for $i \notin X_{n-1}$. For $i \in X_{n-1}$, we define the maps

τ_i^+ and τ_i^- as follows.

$$\tau_i^+ : \mathbb{I}_n^r(N) \longrightarrow \mathbb{I}_n^r(N),$$
$$\tau_i^+ : I = (\dots, *, \underbrace{i, \dots, i}_{N_i(I)}, *, \dots) \mapsto (\dots, *, \underbrace{i, \dots, i}_{N_i(I)-1}, i+1, *, \dots),$$
$$\tau_i^- : \mathbb{I}_n^r(N) \longrightarrow \mathbb{I}_n^r(N),$$
$$\tau_i^- : I = (\dots, *, \underbrace{i, \dots, i}_{N_i(I)}, *, \dots) \mapsto (\dots, *, i-1, \underbrace{i, \dots, i}_{N_i(I)-1}, *, \dots).$$

If $N_i(I) = 0$, we assume that $\tau_i^\pm = id$. By using these maps, we define the following sets.

$$T_i^+(I) := \{\tau_i^{+k}(I) \mid 0 \le k \le N_i(I)\}, \qquad T_i^-(I) := \{\tau_i^{-k}(I) \mid 0 \le k \le N_i(I)\}.$$

Here, $\tau_i^{\pm k} := \underbrace{\tau_i^\pm \circ \cdots \circ \tau_i^\pm}_{k}$. Then, using these notations, the (I,J)-entry of the representation $\mathrm{Sym}^{(N)}\beta^r_{n,-t}$ can be expressed by

$$\mathrm{Sym}^{(N)}\beta^r_{n,-t}(\sigma_i) = \begin{cases} \binom{N_1(I)}{N_1(J)} t^{N_1(J)} \delta_{T_1^+(I)}(J) & (i=1), \\ (L_i \cdot R_i(t))_{I,J} & (2 \le i \le n-2), \\ (-1)^{N_{n-1}(I)-N_{n-1}(J)} \binom{N_{n-1}(I)}{N_{n-1}(J)} t^{N_{n-1}(I)} \delta_{T_{n-1}^-(I)}(J) & (i = n-1). \end{cases}$$

Here, L_i and $R_i(t)$ are matrices defined as

$$(L_i)_{I,J} := \binom{N_i(I)}{N_i(J)} \delta_{T_i^+(I)}(J), \qquad R_i(t)_{I,J} := (-1)^{N_i(J)-N_i(J)} \binom{N_i(I)}{N_i(J)} t^{N_i(I)} \delta_{T_i^-(I)}(J).$$

For a set X and its subset A, we define the functionδ_A on X by

$$\delta_A(x) := \begin{cases} 1 & (x \in A), \\ 0 & (x \notin A). \end{cases}$$

Then we consider the following q-deformation for all entries of $\mathrm{Sym}^{(N)}\beta^r_{n,-t}(\sigma_i)$ except for L_i.

$$\binom{n}{m} \mapsto q^{t(m)} \binom{n}{m}_q.$$

Here, $t(m) := \frac{m(m-1)}{2}$, and $\binom{n}{m}_q$ is q-binomial coefficient defined as

$$\binom{n}{m}_q := \begin{cases} \frac{[n]_q!}{[m]_q! \cdot [n-m]_q!} & (0 \le m \le n), \\ 0 & (\text{otherwise}), \end{cases} \qquad [n]_q! := \begin{cases} [n]_q [n-1]_q \cdots [1]_q & (n \ge 1), \\ 1 & (n = 0). \end{cases}$$

Finally, we replace the binomial coefficient of the (I,J)-entries of L_i to the usual q-binomial coefficient as follows.

$$\binom{n}{m} \mapsto \binom{n}{m}_q.$$

Hence we can get the matrix $A_i(q,t)$ with respect to the generator $\sigma_i \in B_n$ as follows.

$$A_i(q,t) := \begin{cases} q^{t(N_i(J))} \binom{N_1(I)}{N_1(J)}_q t^{N_1(J)} \delta_{T_1^+(I)}(J) & (i=1), \\ (L_i(q) \cdot R_i(q,t))_{I,J} & (2 \le i \le n-2), \\ (-1)^{N_{n-1}(I)-N_{n-1}(J)} q^{t(N_{n-1}(J))} \binom{N_{n-1}(I)}{N_{n-1}(J)}_q t^{N_{n-1}(I)} \delta_{T_{n-1}^-(I)}(J) & (i=n-1). \end{cases}$$

Here $L_i(q)$ and $R_i(q,t)$ are the following matrices.

$$L_i(q)_{I,J} := \binom{N_i(I)}{N_i(J)}_q \delta_{T_i^+(I)}(J), \qquad R_i(q,t)_{I,J} := (-1)^{N_i(J)-N_i(J)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(J)}_q t^{N_i(I)} \delta_{T_i^-(I)}(J).$$

Now we show some properties of $A_i(q,t)$.

Proposition 4.1.1. *For $i \in X_{n-1}$, the matrix $A_i(q,t)$ can be expressed by*

$$A_i(q,t) = (-1)^{N_{i-1}(I)-N_{i-1}(J)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(J)}_q \binom{N_i(I)-N_i(J)}{N_{i+1}(J)-N_{i+1}(I)}_q t^{N_i(I)+N_{i+1}(I)-N_{i+1}(J)} \chi_i(I,J).$$

Here,

$$\chi_i(I,J) := \begin{cases} 1 & N_k(I) = N_k(J), (k \neq i-1, i, i+1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First we show the case of $2 \le i \le n-2$. By definition,

$$\begin{aligned} A_i(q,t)_{I,J} &= \sum_{K \in \mathbb{I}_n^r(N)} \delta_{T_i^+(I)}(K) \delta_{T_i^-(K)}(J) (-1)^{N_i(K)-N_i(J)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(K)}_q \binom{N_i(K)}{N_i(J)}_q t^{N_i(K)} \\ &= \sum_{K \in T_i^+(I) \cap T_{i-1}^+(J)} (-1)^{N_i(K)-N_i(J)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(K)}_q \binom{N_i(K)}{N_i(J)}_q t^{N_i(K)}. \end{aligned}$$

Since $N_k(I) = N_k(J)$ for $k \in X_{n-1} \setminus \{i-1, i, i+1\}$, there exists $K \in T_i^+(I) \cap T_{i-1}^+(J)$ uniquely. Then we can get m and l such that $K' = \tau_i^{+m}(I) = \tau_{i-1}^{+l}(J)$. Since $N_i(K') = N_i(I) + N_{i+1}(I) - N_{i+1}(J)$, then we have

$$\begin{aligned} A_i(q,t) &= (-1)^{N_i(K')-N_i(J)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(K')}_q \binom{N_i(K')}{N_i(J)}_q t^{N_i(K')} \\ &= (-1)^{N_i(K')-N_i(J)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(J)}_q \binom{N_i(I)-N_i(J)}{N_i(I)-N_i(K')}_q t^{N_i(K')} \\ &= (-1)^{N_{i-1}(I)-N_{i-1}(J)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(J)}_q \binom{N_i(I)-N_i(J)}{N_{i+1}(J)-N_{i+1}(I)}_q t^{N_i(I)+N_{i+1}(I)-N_{i+1}(J)} \chi_i(I,J). \end{aligned}$$

When $i=1$, we can immediately see that $N_0(I) = N_0(J) = 0$, and $N_1(I) - N_J(I) = N_2(J) - N_2(I)$ from the definition of $N_i(I)$ and $\chi_1(I,J)$. On the other hand, when $i = n-1$, we have $N_n(I) = N_n(J)$ and $N_{n-2}(I) - N_{n-2}(J) = N_{n-1}(J) - N_{n-1}(I)$. Hence we can show that all $A_i(q,t)$ are expressed by the form of Proposition 4.1.1. □

Now, we introduce the following useful lemma.

Lemma 4.1.1. *For $n \geq 1$, we have the following formulas.*

$$(1+z)_q^n := (1+z)(1+qz)\cdots(1+q^{n-1}z) = \sum_{k=0}^{n} q^{t(k)} \binom{n}{k}_q z^k,$$

$$\frac{1}{(1+z)_q^n} = \frac{1}{(1+z)(1+qz)\cdots(1+q^{n-1}z)} = \sum_{l=0}^{\infty} (-1)^l \binom{n+l-1}{l}_q z^l.$$

If $n=0$, we put $(1+z)_q^0 := 1$.

Proposition 4.1.2. *The matrices $A_1(q,t), \ldots, A_{n-1}(q,t)$ satisfy the following relations.*

$$A_i(q,t)A_j(q,t) = A_j(q,t)A_i(q,t) \quad (|i-j|>1),$$
$$A_i(q,t)A_{i+1}(q,t)A_i(q,t) = A_{i+1}(q,t)A_i(q,t)A_{i+1}(q,t) \quad (1 \leq i \leq n-2).$$

Proof. To show these relations, we use another expression of $A_i(q,t)$ as follows.

$$A_i(q,t) = (-1)^{N_i(K')-N_i(J)} q^{t(N_i(J))} \binom{N_i(I)}{m, l}_q t^{N_i(K')} \chi_i(I,J).$$

Here, l and m are numbers which satisfy $K' = \tau_i^{+m}(I) = \tau_{i-1}^{+l}(J)$, and $\binom{n}{m_1, m_2}_q$ is the q-multinomial coefficient defined by

$$\binom{n}{m_1, m_2}_q := \begin{cases} \frac{[n]_q!}{[m_1]_q![m_2]_q![n-m_1-m_2]_q!} & (0 \leq m_1, m_2 \leq n, 0 \leq m_1+m_2 \leq n), \\ 0 & (\text{othetwise}). \end{cases}$$

Now we show the first relation. For simplicity, we calculate the case of $j = i+2$.

$$A_i(q,t)A_{i+2}(q,t)_{I,J} = \sum_K (-1)^{l_1} q^{t(N_i(K))} \binom{N_i(I)}{m_1, l_1}_q t^{N_i(K_1)} (-1)^{l_2} q^{t(N_{i+2}(J))} \binom{N_{i+2}(K)}{m_2, l_2}_q t^{N_{i+2}(K_2)}$$
$$\times \chi_i(I,K)\chi_{i+2}(K,J).$$

Here, $K_1 = \tau_i^{+m_1}(I) = \tau_{i-1}^{+l_1}(K)$, and $K_2 = \tau_{i+2}^{+m_2}(K) = \tau_{i+1}^{+l_2}(J)$. The following table helps our calculation.

	$i-1$	i	$i+1$	$i+2$	$i+3$
I	$N_{i-1}(I)$	$N_i(I)$	$N_{i+1}(I)$	$N_{i+2}(I)$	$N_{i+3}(I)$
K_1	$N_{i-1}(I)$	$N_i(I)-m_1$	$N_{i+1}(I)+m_1$	$N_{i+2}(I)$	$N_{i+3}(I)$
K_1	$N_{i-1}(K)-l_1$	$N_i(K)+l_1$	$N_{i+1}(K)$	$N_{i+2}(K)$	$N_{i+3}(K)$
K	$N_{i-1}(K)$	$N_i(K)$	$N_{i+1}(K)$	$N_{i+2}(K)$	$N_{i+3}(K)$
K_2	$N_{i-1}(K)$	$N_i(K)$	$N_{i+1}(K)$	$N_{i+2}(K)-m_2$	$N_{i+3}(K)+m_2$
K_2	$N_{i-1}(J)$	$N_i(J)$	$N_{i+1}(J)-l_2$	$N_{i+2}(J)+l_2$	$N_{i+3}(J)$
J	$N_{i-1}(J)$	$N_i(J)$	$N_{i+1}(J)$	$N_{i+2}(J)$	$N_{i+3}(J)$

From this table, we have

$$l_1 = N_{i-1}(J) - N_{i-1}(I), \qquad m_1 = N_{i-1}(I) + N_i(I) - N_{i-1}(J) - N_i(J),$$
$$l_2 = N_{i+2}(I) + N_{i+3}(I) - N_{i+2}(J) - N_{i+3}(J), \qquad m_2 = N_{i+3}(J) - N_{i+3}(I).$$

Hence

$$A_i(q,t)A_{i+2}(q,t)_{I,J}$$

$$= (-1)^{N_{i-1}(J)-N_{i-1}(I)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(J)}_q \binom{N_i(I)-N_i(J)}{N_{i-1}(J)-N_{i-1}(I)}_q t^{N_{i-1}(J)+N_i(J)-N_{i-1}(I)}$$

$$\times (-1)^{N_{i+2}(I)+N_{i+3}(I)-N_{i+2}(J)-N_{i+3}(J)} q^{t(N_{i+2}(J))} \binom{N_{i+2}(I)}{N_{i+2}(J)}_q \binom{N_{i+2}(I)-N_{i+2}(J)}{N_{i+3}(J)-N_{i+3}(I)}_q t^{N_{i+2}(I)+N_{i+3}(I)-N_{i+3}(J)}$$

$$\times \chi_{i,i+2}(I,J).$$

Here,

$$\chi_{i,j}(I,J) := \begin{cases} 1 & N_k(I)=N_k(J) \text{ for } k\in X_{n-1}\setminus(\{i-1,i,i+1\}\cup\{j-1,j,j+1\}), \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $A_{i+2}(q,t)A_i(q,t)_{I,J}$ can be computed similarly.

$$A_{i+2}(q,t)A_i(q,t)_{I,J} = \sum_K (-1)^{l_1} q^{t(N_{i+2}(K))} \binom{N_{i+2}(I)}{m_1, l_1}_q t^{N_{i+2}(K_1)} (-1)^{l_2} q^{t(N_i(J))} \binom{N_i(K)}{m_2, l_2}_q t^{N_i(K_2)}$$

$$= (-1)^{N_{i+2}(I)+N_{i+3}(I)-N_{i+2}(J)-N_{i+3}(J)} q^{t(N_{i+2}(J))} \binom{N_{i+2}(I)}{N_{i+2}(J)}_q \binom{N_{i+2}(I)-N_{i+2}(J)}{N_{i+3}(J)-N_{i+3}(I)}_q t^{N_{i+2}(I)+N_{i+3}(I)-N_{i+3}(J)}$$

$$\times (-1)^{N_{i-1}(J)-N_{i-1}(I)} q^{t(N_i(J))} \binom{N_i(I)}{N_i(J)}_q \binom{N_i(I)-N_i(J)}{N_{i-1}(J)-N_{i-1}(I)}_q t^{N_{i-1}(J)+N_i(J)-N_{i-1}(I)}$$

$$\times \chi_{i+2,i}(I,J).$$

Then we have $A_i(q,t)A_{i+2}(q,t) = A_{i+2}(q,t)A_i(q,t)$. If $j > i+2$, we can get the first relation by calculating the same way.

Then we next show the second relation. First we calculate $A_i(q,t)A_{i+1}(q,t)$.

$$A_i(q,t)A_{i+1}(q,t) = \sum_K (-1)^{l_1} q^{t(N_i(K))} \binom{N_i(I)}{l_1, m_1}_q t^{N_i(I)-m_1} (-1)^{l_2} q^{t(N_{i+1}(J))} \binom{N_{i+1}(K)}{m_2, l_2}_q t^{N_{i+1}(K)-m_2}$$
$$\times \chi_i(I,K)\chi_{i+1}(K,J).$$

In order to calculate $A_i(q,t)A_{i+1}(q,t)$ we use the following table.

	$i-1$	i	$i+1$	$i+2$
I	$N_{i-1}(I)$	$N_i(I)$	$N_{i+1}(I)$	$N_{i+2}(I)$
K_1	$N_{i-1}(I)$	$N_i(I)-m_1$	$N_{i+1}(I)+m_1$	$N_{i+2}(I)$
K_1	$N_{i-1}(K)-l_1$	$N_i(K)+l_1$	$N_{i+1}(K)$	$N_{i+2}(K)$
K	$N_{i-1}(K)$	$N_i(K)$	$N_{i+1}(K)$	$N_{i+2}(K)$
K_2	$N_{i-1}(K)$	$N_i(K)$	$N_{i+1}(K)-m_2$	$N_{i+2}(K)+m_2$
K_2	$N_{i-1}(J)$	$N_i(J)-l_2$	$N_{i+1}(J)+l_2$	$N_{i+2}(J)$
J	$N_{i-1}(J)$	$N_i(J)$	$N_{i+1}(J)$	$N_{i+2}(J)$

From the above table, we have

$$l_1 = N_{i-1}(J)-N_{i-1}(I), \qquad m_2 = N_{i+2}(J)-N_{i+2}(I),$$

$$m_1 = N_{i-1}(I)+N_i(I)-N_{i-1}(J)-N_i(J)+l_2.$$

Then $A_i(q,t)A_{i+1}(q,t)_{I,J}$ can be calculated by

$$
\begin{aligned}
&A_i(q,t)A_{i+1}(q,t)_{I,J} \\
&= \sum_{l_2=0}^{N_i(J)} (-1)^{N_{i-1}(J)-N_{i-1}(I)} q^{t(N_i(J)-l_2)} \binom{N_i(I)}{N_{i-1}(J)-N_{i-1}(I)}_q \binom{N_{i-1}(I)+N_i(I)-N_{i-1}(J)}{N_i(J)-l_2}_q \\
&\times (-1)^{l_2} q^{t(N_{i+1}(J))} \binom{N_{i+1}(J)+N_{i+2}(J)-N_{i+2}(I)+l_2}{l_2}_q \binom{N_{i+1}(J)+N_{i+2}(J)-N_{i+2}(I)}{N_{i+1}(J)}_q \\
&\times t^{N_i(I)-m_1+N_{i+1}(I)+m_1-m_2} \chi_{i,i+1}(I,J) \\
&= (-1)^{N_{i-1}(J)-N_{i-1}(I)} q^{t(N_{i+1}(J))} t^{N_i(I)+N_{i+1}(I)+N_{i+2}(I)-N_{i+2}(J)} \\
&\times \binom{N_i(I)}{N_{i-1}(J)-N_{i-1}(I)}_q \binom{N_{i+1}(J)+N_{i+2}(J)-N_{i+2}(I)}{N_{i+1}(J)}_q \chi_{i,i+1}(I,J) \\
&\times \sum_{l_2} q^{t(N_i(J)-l_2)} \binom{N_{i-1}(I)+N_i(I)-N_{i-1}(J)}{N_i(J)-l_2}_q (-1)^{l_2} \binom{N_{i+1}(J)+N_{i+2}(J)-N_{i+2}(I)+l_2}{l_2}_q .
\end{aligned}
$$

By Lemma 4.1.1, we can see that the last sum coincides with the coefficient of $z^{N_i(J)}$ in the following function.

$$
\frac{(1+z)_q^{N_{i-1}(I)+N_i(I)-N_{i-1}(J)}}{(1+z)_q^{N_{i+1}(J)+N_{i+2}(J)-N_{i+2}(I)+1}} = \sum_{k=0}^{\infty} (-1)^k \binom{N_{i+1}(I)-N_i(J)+k}{k}_q (q^{N_{i-1}(I)+N_i(I)-N_{i-1}(J)} z)^k.
$$

Hence we have

$$
\begin{aligned}
A_i(q,t)A_{i+1}(q,t)_{I,J} =& (-1)^{N_{i-1}(J)+N_i(J)-N_{i-1}(I)} q^{t(N_{i+1}(J))+N_i(J)(N_{i-1}(I)+N_i(I)-N_{i-1}(J))} \\
&\times \binom{N_i(I)}{N_{i-1}(J)-N_{i-1}(I)}_q \binom{N_{i+1}(J)+N_{i+2}(J)-N_{i+2}(I)}{N_{i+1}(J)}_q \binom{N_{i+1}(I)}{N_i(J)}_q \\
&\times t^{N_i(I)+N_{i+1}(I)+N_{i+2}(I)-N_{i+2}(J)} \chi_{i,i+1}(I,J).
\end{aligned}
$$

Then we next calculate $A_i(q,t)A_{i+1}(q,t)A_i(q,t)_{I,J}$.

$$
\begin{aligned}
A_i(q,t)A_{i+1}(q,t)A_i(q,t)_{I,J} &= \sum_K A_i(q,T)A_{i+1}(q,t)_{I,K} \cdot A_i(q,t)_{K,J} \\
&= \sum_K (-1)^{N_i(K)+N_{i-1}(J)-N_{i-1}(I)} q^{t(N_{i+1}(K))+(N_{i-1}(I)+N_i(I)-N_{i-1}(K))N_i(K)+t(N_i(J))} \\
&\quad \times \binom{N_i(I)}{N_{i-1}(K)-N_{i-1}(I)}_q \binom{N_{i+1}(K)+N_{i+2}(J)-N_{i+2}(I)}{N_{i+1}(K)}_q \\
&\quad \times \binom{N_{i+1}(I)}{N_{N_i(K)}}_q \binom{N_i(K)}{N_{i-1}(J)-N_{i-1}(K), N_{i+1}(J)-N_{i+1}(K)}_q \\
&\quad \times t^{N_i(I)+N_{i+1}(I)+N_{i+2}(I)+N_{i-1}(J)+N_i(J)-N_{i+2}(J)-N_{i-1}(K)} \\
&\quad \times \chi_{i,i+1}(I,K)\chi_i(K,J).
\end{aligned}
$$

By substituting $N_{i-1}(K) = N_{i-1}(J) - l, N_i(K) = N_i(J) + l + m$ and $N_{i+1}(K) = N_{i+1(J)} - m$, we can express the $A_i(q,t)A_{i+1}(q,t)A_i(q,t)_{I,J}$ as the sum of l and m.

$$\begin{aligned}A_i(q,t)A_{i+1}(q,t)A_i(q,t)_{I,J} &= (-1)^{N_{i-1}(J)-N_{i-1}(I)} q^{t(N_i(J))} t^{N_i(I)+N_{i+1}(I)+N_{i+2}(I)+N_i(J)+N_{i-1}(J)-N_{i+2}(J)} \\ &\times \sum_{l,m} (-1)^{N_i(J)+l+m} q^{t(N_{i+1}(J)-m)+(N_{i-1}(I)+N_i(I)-N_{i-1}(J)+l)(N_i(J)+l+m)} \\ &\times t^{l-N_{i-1}(J)} \binom{N_i(I)}{N_{i-1}(J)-N_{i-1}(I)-l}_q \binom{N_{i+2}(J)+N_{i+1}(J)-N_{i+2}(I)-m}{N_{i+1}(J)-m}_q \\ &\times \binom{N_{i+1}(I)}{N_i(J)+l+m}_q \binom{N_i(J)+l+m}{N_i(J)}_q \binom{l+m}{m}_q \chi_{i,i+1}(I,J).\end{aligned}$$

The last three binomial coefficients becomes

$$\begin{aligned}&\binom{N_{i+1}(I)}{N_i(J)+l+m}_q \binom{N_i(J)+l+m}{N_i(J)}_q \binom{l+m}{m}_q \\ &= \binom{N_{i+1}(I)}{N_i(J)}_q \binom{N_{i+1}(I)-N_i(J)-l}{m}_q \binom{N_{i+1}(I)-N_i(J)}{l}_q.\end{aligned}$$

Thus we have

$$\begin{aligned}A_i(q,t)A_{i+1}(q,t)A_i(q,t)_{I,J} &= (-1)^{N_{i-1}(J)-N_{i-1}(I)} q^{t(N_i(J))} t^{N_i(I)+N_{i+1}(I)+N_{i+2}(I)+N_i(J)-N_{i+2}(J)} \\ &\times \sum_{l} (-1)^{N_i(J)+N_{i+1}(J)+l} q^{t(N_{i+1}(J))+(N_{i-1}(I)+N_i(I)-N_{i-1}(J)+l)(N_i(J)+l)} \\ &\times t^{l} \binom{N_i(I)}{N_{i-1}(J)-N_{i-1}(I)-l}_q \binom{N_{i+1}(I)-N_i(J)}{l}_q \binom{N_{i+1}(I)}{N_i(J)}_q \\ &\times \sum_{m} (-1)^{N_{i+1}(J)-m} \binom{N_{i+2}(J)+N_{i+1}(J)-N_{i+2}(I)-m}{N_{i+1}(J)-m}_q \\ &\times q^{t(m)} \binom{N_{i+1}(I)-N_i(J)-l}{m}_q q^{(N_{i-1}(I)+N_i(I)-N_{i-1}(J)-N_{i+1}(J)+l+1)m}.\end{aligned}$$

The last sum coincides with the coefficient of $z^{N_{i+1}(J)}$ in the following function.

$$\begin{aligned}&\frac{(1+q^{N_{i-1}(I)+N_i(I)-N_{i-1}(J)-N_{i+1}(J)+l+1}z)_q^{N_{i+1}(I)-N_i(J)-l}}{(1+z)_q^{N_{i+2}(J)-N_{i+2}(I)+1}} \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{N_{i-1}(I)+N_i(I)-N_{i-1}(J)-N_{i+1}(J)+l+k}{k}_q z^k.\end{aligned}$$

Hence we have

$$\begin{aligned}A_i(q,t)A_{i+1}(q,t)A_i(q,t)_{I,J} &= (-1)^{N_{i-1}(J)+N_i(J)-N_{i-1}(I)} q^{t(N_i(J))+t(N_{i+1}(J))+N_i(J)(N_{i-1}(I)+N_i(I)-N_{i-1}(J))} \\ &\times t^{N_i(I)+N_{i+1}(I)+N_{i+2}(I)+N_i(J)-N_{i+2}(J)} \binom{N_i(I)}{N_{i+1}(J)}_q \binom{N_{i+1}(I)}{N_i(J)}_q \\ &\times \sum_{l} (-t)^l \binom{N_{i+1}(I)-N_i(J)}{l}_q \binom{N_i(I)-N_{i+1}(J)}{N_{i-1}(J)-N_{i-1}(I)-l}_q \\ &\times q^{(N_{i-1}(I)+N_i(I)-N_{i-1}(J)+N_i(J)+l)l}.\end{aligned}$$

On the other hand, $A_{i+1}(q,t)A_i(q,t)A_{i+1}(q,t)$ can be calculated similarly as follows.

$$\begin{aligned} A_{i+1}(q,t)A_i(q,t)A_{i+1}(q,t)_{I,J} &= (-1)^{N_{i-1}(I)+N_i(J)-N_{i-1}(J)} q^{t(N_{i+1}(J))+t(N_i(J))+N_i(J)(N_{i-1}(I)+N_i(I)-N_{i-1}(J))} \\ &\times t^{N_i(I)+N_{i+1}(I)+N_{i+2}(I)+N_i(J)-N_{i+2}(J)} \binom{N_i(I)}{N_{i+1}(J)}_q \binom{N_{i+1}(I)}{N_i(J)}_q \\ &\times \sum_m (-t)^{N_{i+1}(I)-N_i(J)-m} \binom{N_{i+1}(I)-N_i(J)}{N_{i+1}(I)-N_i(J)-m}_q \\ &\times \binom{N_i(I)-N_{i+1}(J)}{N_{i-1}(J)-N_{i-1}(I)-N_{i+1}(I)+N_i(J)+m}_q \\ &\times q^{(N_{i-1}(I)+N_i(I)-N_{i-1}(J)+N_i(J)+N_{i+1}(I)-N_i(J)-m)(N_{i+1}(I)-N_i(J)-m)}. \end{aligned}$$

Putting $N_{i+1}(I)-N_i(J)-m=l$, we have

$$A_i(q,t)A_{i+1}(q,t)A_i(q,t)_{I,J} = A_{i+1}(q,t)A_i(q,t)A_{i+1}(q,t)_{I,J}.$$

Then we complete the proof of Proposition 4.1.2. □

From Proposition 4.1.2, we obtain the following braid group representation.

Theorem 4.1.1. *For $i \in X_{n-1}$, we define*

$$\rho^{(N)}_{n,q,t}(\sigma_i) := A_i(q,t).$$

Then, $\rho^{(N)}_{n,q,t} : B_n \longrightarrow GL(\mathrm{Sym}^{(N)}(W^r_n))$ is the braid group representation.

4.2 Braid zeta functions of the torus type braid

In this section, we calculate the zeta function associated with $\rho^{(N)}_{n,q,t}$ for a torus type braid. Our main result is the following formula.

Theorem 4.2.1. *For coprime pair $(n,m) \in \mathbb{N}^2$, we have*

$$\zeta(s,\sigma_{n,m};\rho^{(N)}_{n,q,t}) = \zeta((-t)^{mN} q^{\frac{2mt(N)}{n}} s, c_n; \mathrm{Sym}^{(N)} p^r_n).$$

Here, $c_n \in \mathfrak{S}_n$ is a cycle with length n, and p^r_n is $(n-1)$-dimensional irreducible representation of $\mathfrak{S}_n$ which is called standard representation.

Lemma 4.2.1. *Put $S_i(K) := N_1(K)+\cdots+N_i(K)$ for $K \in \mathbb{I}^r_n(N)$, then (I,J)-entry of $\rho^{(N)}_{n,q,t}(\sigma_{n,1})$ can be expressed as follows.*

$$\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J} = (-1)^{N-N_{n-1}(J)} t^N q^{t(N_{n-1}(J))} \prod_{i=1}^{n-2} \binom{N_{i+1}(I)}{N_i(J)}_q q^{(S_i(I)-S_{i-1}(J))N_i(J)}.$$

Proof. $\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J}$ can be expressed by the following sum for $K_1,\dots,K_{n-2} \in \mathbb{I}^r_n(N)$.

$$\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J} = \sum_{K_1,K_2,\dots,K_{n-2}} \rho^{(N)}_{n,q,t}(\sigma_1)_{I,K_1} \cdot \rho^{(N)}_{n,q,t}(\sigma_2)_{K_1,K_2} \cdots \rho^{(N)}_{n,q,t}(\sigma_{n-1})_{K_{n-2},J}.$$

By Proposition 4.1.1, we have

$$\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J} = \sum_{K_1,K_2,\ldots,K_{n-2}} \prod_{i=1}^{n-1} (-1)^{l_{i-1,i}} q^{t(N_i(K_i))} \binom{N_i(K_{i-1})}{N_i(K_i)} t^{N_i(K_i)} \chi_i(K_{i-1},K_i)$$

$$= \sum_{K_1,K_2,\ldots,K_{n-2}} \prod_{i=1}^{n-1} (-1)^{N_i(K_{i-1,i})-N_i(K_i)} q^{t(N_i(K_i))} \binom{N_i(K_{i-1})}{m_{i-1.i}, l_{i-1,i}}_q t^{N_i(K_{i-1,i})} \chi_{i(K_{i-1},K_i)}.$$

Here, $K_{i-1,i} = \tau_i^{+m_{i-1,i}}(K_{i-1}) = \tau_{i-1}^{+l_{i-1,i}}(K_i)$ for $i = 1, \ldots, n-1$, and we put $K_0 = I, K_{n-1} = J$. Then we can see that $l_{0,1} = m_{n-2,n-1} = 0$ by the definition of $A_i(q,t)$. The power of t can be calculated by

$$\sum_{i=1}^{n-1} N_i(K_{i-1,i}) = N_1(I) - m_{0,1} + \sum_{i=2}^{n-2}(N_i(I) + m_{i-2,i-1} - m_{i-1,i}) + N_{n-1}(I) - m_{n-3,n-2}$$

$$= N_1(I) - m_{0,1} + \sum_{i=2}^{n-2} N_i(I) + m_{0,1} - m_{n-3,n-2} + N_{n-1}(I) - m_{n-3,n-2}$$

$$= \sum_{i=1}^{n-1} N_i(I) = N.$$

$m_{i-1,i}$ can be expressed as

$$m_{i-1,i} = S_i(I) - S_i(J) + l_{i,i+1}.$$

Then we have

$$\binom{N_i(K_{i-1})}{m_{i-1,i}, l_{i-1,i}}_q = \binom{S_i(I) - S_{i-1}(J) + l_{i-1,i}}{l_{i-1,i}}_q \binom{S_i(I) - S_{i-1}(J)}{N_i(J) - l_{i,i+1}}_q.$$

Hence,

$$\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J}$$

$$= t^N \sum_{l_{1,2},\ldots,l_{n-3,n-2},l} \prod_{i=1}^{n-1} (-1)^{l_{i-1,i}} q^{t(N_i(J)-l_{i,i+1})} \binom{S_i(I) - S_{i-1}(J) + l_{i-1,i}}{l_{i-1,i}}_q \binom{S_i(I) - S_{i-1}(J)}{N_i(J) - l_{i,i-1}}_q$$

$$= t^N q^{t(N_{n-1}(J))} \sum_{l_{1,2},\ldots,l_{n-3,n-2},l} \prod_{i=1}^{n-2} q^{t(N_i(J)-l_{i,1+1})} \binom{S_i(I) - S_{i-1}(J)}{N_i(J) - l_{i,i+1}}_q (-1)^{l_{i,i+1}} \binom{S_{i+1}(I) - S_i(J) + l_{i,i+1}}{l_{i,i+1}}_q.$$

Here, each sum of

$$q^{t(N_i(J)-l_{i,i+1})} \binom{S_i(I) - S_{i-1}(J)}{N_i(J) - l_{i,i+1}}_q (-1)^{l_{i,i+1}} \binom{S_{i+1}(I) - S_i(J) + l_{i,i+1}}{l_{i,i+1}}_q$$

is equal to the coefficient of $z^{N_i(J)}$ in the following function

$$\frac{(1+z)_q^{S_i(I)-S_{i-1}(J)}}{(1+z)_q^{S_{i+1}(I)-S_i(J)}} = \sum_{l=0}^{\infty} (-1)^l \binom{N_{i+1}(I) - N_i(J) + l}{l}_q (q^{S_i(I)-S_{i-1}(J)} z)^l.$$

Then we have

$$\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J} = t^N q^{t(N_{n-1}(J))} \prod_{i=1}^{n-2} (-1)^{N_i(J)} \binom{N_{i+1}(I)}{N_i(J)}_q q^{(S_i(I)-S_{i-1}(J))N_i(J)}.$$

Thus, we finish the proof of Lemma 4.2.1. □

By simple calculation, we have the another expression of $\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J}$ as follows.

$$\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J} = t^N q^{t(N)} \prod_{i=1}^{n-2} (-1)^{N_i(J)} q^{-t(N_i(J))} \binom{N_{i+1}(I)}{N_i(J)}_{q^{-1}} q^{-(N-S_{i+1}(I))N_i(J)}. \tag{4.1}$$

The formula (4.1) is useful to compute the zeta function. Now we define the following array and the corresponding polynomial.

Definition 4.2.1. For $m \in X_{n-1}$, we define *m-shifted array* as

$$\mathcal{A}_{n,m} := \begin{bmatrix} 1 & 2 & \cdots & m-1 & m & m+1 & \cdots & n-1 & \bullet \\ n-m+1 & n-m+2 & \cdots & n-1 & \bullet & 1 & \cdots & n-m-1 & n-m \end{bmatrix}.$$

We write the set of pair of above and below of $\mathcal{A}_{n,m}$ except two pairs which include a dot $\bullet$ by $\mathcal{P}(n,m)$. Moreover, we define the following subsets of $\mathcal{P}(n,m)$.

$$\mathcal{P}_-(n,m) := \{(i,j) \in \mathcal{P}(n,m) \mid 1 \le i \le m-1\},$$
$$\mathcal{P}_+(n,m) := \{(i,j) \in \mathcal{P}(n,m) \mid m+1 \le i \le n-1\}.$$

For $I, J \in \mathbb{I}^r_n(N)$, we define the following polynomial.

$$\mathcal{S}_{n,m}(q;(I,J)) := \prod_{(i,j)\in\mathcal{P}(n,m)} q^{t(N_j(J))} \binom{N_i(I)}{N_j(J)}_q.$$

By using the above symbols, we can express (I,J)-entry of $\rho^{(N)}_{n,q,t}(\sigma_{n,1})$ as follows.

$$\rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,J} = (-t)^N (-1)^{N_{n-1}(J)} q^{t(N)} \mathcal{S}_{n,1}(q^{-1};(I,J)) q^{a_{n,1}(I,J)-a(I,J)}.$$

Here, $a_{n,1}(I,J)$ and $a(I,J)$ are defined by

$$a_{n,1}(I,J) := \sum_{i=1}^{n-2} N_{i+1}(I)N_i(J), \qquad a(I,J) := \sum_{i>j} N_i(I)N_j(J).$$

Furthermore, for $m = 2, \dots, n-1$, we define

$$a_{n,m}(I,J) := \sum_{(i,j)\in\mathcal{P}(n,m)} N_i(I)N_j(J) - N_{n-m+1}(J), \qquad b_{n,m}(I,J) := \sum_{l=1}^{m} a_{n,l}(I,J).$$

Remark 4.2.1. Since $N_n(J) = 0$, the definition of $a_{n.m}(I,J)$ is consistent for all $m \in X_{n-1}$.

Lemma 4.2.2. *For $m \in X_{n-1}$, and $I, J \in \mathbb{I}^r_n(N)$, we can express the (I,J)-entry of $\rho^{(N)}_{n,q,t}(\sigma_{n,m})$ as follows.*

$$\rho^{(N)}_{n,q,t}(\sigma_{n,m})_{I,J} = (-t)^{mN} (-1)^{N_{n-m}(J)} q^{t(N)} \mathcal{S}_{n,m}(q^{-1};(I,J)) q^{b_{n,m}(I,J)-a(I,J)}. \tag{4.2}$$

Proof. We put

$$P^{(m)}_{I,J} = (-t)^{mN}(-1)^{N_{n-m}(J)} q^{t(N)} \mathcal{S}_{n,m}(q^{-1};(I,J)) q^{b_{n,m}(I,J)-a(I,J)}.$$

Since $P^{(1)}_{I,J} = \rho^{(N)}_{n,q,t}(\sigma_{n,1})$, it is sufficient to show that

$$\sum_K P^{(m)}_{I,K} P^{(1)}_{K,J} = P^{(m+1)}_{I,J},$$

for $1 \leq m \leq n-2$. Then we have

$$\sum_K P^{(m)}_{I,K} P^{(1)}_{K,J} = (-t)^{(m+1)N}(-1)^{N_{n-1}(J)} q^{2t(N)} \sum_K (-1)^{N_{n-m}(K)} \mathcal{S}_{n,m}(q^{-1};(I,K))$$
$$\times \mathcal{S}_{n,1}(q^{-1};(K,J)) q^{b_{n,m}(I,K)-a(I,K)+a_{n,1}(K,J)-a(K,J)}.$$

For simplicity, we put $k_i := N_i(K)$. Then the part of summation becomes

$$\sum_K (-1)^{k_{n-m}} \mathcal{S}_{n,m}(q^{-1};(I,K)) \mathcal{S}_{n,1}(q^{-1};(K,J)) q^{b_{n,m}(I,K)-a(I,K)+a_{n,1}(K,J)-a(K,J)}$$

$$= q^{-t(N_{n-m-1}(J))-N_{n-m-1}(J)S_{n-m-2}(J)} \binom{N_1(I)}{N_{n-m}(J)}_{q^{-1}} \cdots \binom{N_{m-1}(I)}{N_{n-2}(J)}_{q^{-1}} \binom{N_{m+2}(I)}{N_1(J)}_{q^{-1}} \cdots \binom{N_{n-1}(I)}{N_{n-m-2}(J)}_{q^{-1}}$$

$$\times \sum_K q^{-t(k_1)} \binom{N_{m+1}(I)}{k_1}_{q^{-1}} q^{-(N-S_{m+1}(I))k_1} (-1)^{k_{n-m}} \binom{k_{n-m}}{k_{n-m}-N_{n-m-1}(J)}_{q^{-1}} q^{-(k_{n-m}-N_{n-m-1}(J))S_{n-m-2}(J)}$$

$$\times q^{-t(k_{n-m+1}-N_{n-m}(J))} \binom{N_1(I)-N_{n-m}(J)}{k_{n-m+1}-N_{n-m}(J)}_{q^{-1}} q^{-N_{n-m}(J)(k_{n-m+1}-1)} q^{-k_{n-m+1}(S_{n-m-1}(J)-S_1(I)+1)}$$

$$\vdots$$

$$\times q^{-t(k_{n-1}-N_{n-2}(J))} \binom{N_{m-1}(I)-N_{n-2}(J)}{k_{n-1}-N_{n-2}(J)}_{q^{-1}} q^{-N_{n-2}(J)(k_{n-1}-1)} q^{-k_{n-1}(S_{n-3}(J)-S_{m-1}(I)+1)}$$

$$\times q^{-t(k_2-N_1(J))} \binom{N_{m+2}(I)-N_1(J)}{k_2-N_1(J)}_{q^{-1}} q^{-N_1(J)(k_2-1)} q^{-k_2(N-S_{m+2}(I))}$$

$$\vdots$$

$$\times q^{-t(k_{n-m-1}-N_{n-m-2}(J))} \binom{N_{n-1}(I)-N_{n-m-2}(J)}{k_{n-m-1}-N_{n-m-2}(J)}_{q^{-1}} q^{-N_{n-m-2}(J)(k_{n-m-1}-1)} q^{-k_{n-m-1}S_{n-m-3}(I)}$$

$$= q^{-t(N_{n-m-1}(J))-N_{n-m-1}(J)S_{n-m-2}(J)} \prod_{(i,j)\in\mathcal{P}(n,m+1)\setminus\{(m,n-1)\}} \binom{N_i(I)}{N_j(J)}_{q^{-1}}$$

$$\times \sum_K q^{-t(k_1)} \binom{N_{m+1}(I)}{k_1}_{q^{-1}} q^{-(N-S_{m+1}(I))k_1} (-1)^{k_{n-m}} \binom{k_{n-m}}{k_{n-m}-N_{n-m-1}(J)}_{q^{-1}} q^{-(k_{n-m}-N_{n-m-1}(J))S_{n-m-2}(J)}$$

$$\times \prod_{(i,j)\in\mathcal{P}_-(n,m+1)\setminus\{(m,n-1)\}} q^{-t(k_{j+1}-N_j(J))} \binom{N_i(I)-N_j(J)}{k_{j+1}-N_j(J)}_{q^{-1}} q^{-N_j(J)(k_{j+1}-1)} q^{-k_{j+1}(S_{j-1}(J)-S_i(I)+1)}$$

$$\times \prod_{(i,j)\in\mathcal{P}_+(n,m+1)} q^{-t(k_{j+1}-N_j(J))} \binom{N_i(I)-N_j(J)}{k_{j+1}-N_j(J)}_{q^{-1}} q^{-N_j(J)(k_{j+1}-1)} q^{-k_{j+1}(N-S_i(I)+S_{j-1}(J))}.$$

Here, first equation is given by considering the coefficient of k_i in $b_{n,m}(I,K) - a(I,K) + a_{n,1}(K,J) - a(K,J)$. Concretely, the coefficient of k_1 is equal to $-(N - S_{m+1}(I))$, when $2 \le i \le n-m$, the coefficient of k_i is equal to $-(N - S_{m+i}(I) + S_{i-2}(J))$. On the other hand, when $n-m+1 \le i \le n-1$, the coefficient of k_i is equal to $-(S_{i-2}(J) - S_{i-n+m}(I) + 1)$. Thus

$$\sum_K (-1)^{k_{n-m}} \mathcal{S}_{n,m}(q^{-1};(I,K)) \mathcal{S}_{n,1}(q^{-1};(K,J)) q^{b_{n,m}(I,K)-a(I,K)+a_{n,1}(K,J)-a(K,J)}$$

$$= q^{-t(N_{n-m-1}(J))-N_{n-m-1}(J)S_{n-m-2}(J)} \prod_{(i,j)\in\mathcal{P}(n,m+1)\setminus\{(m,n-1)\}} \binom{N_i(I)}{N_j(J)}_{q^{-1}}$$

$$\times \sum_K q^{-t(k_1)} \binom{N_{m+1}(I)}{k_1}_{q^{-1}} q^{-(N-S_{m+1}(I))k_1} (-1)^{k_{n-m}} \binom{k_{n-m}}{k_{n-m} - N_{n-m-1}(J)}_{q^{-1}} q^{-(k_{n-m}-N_{n-m-1}(J))S_{n-m-2}(J)}$$

$$\times \prod_{(i,j)\in\mathcal{P}_-(n,m+1)\setminus\{(m,n-1)\}} q^{-t(k_{j+1}-N_j(J))} \binom{N_i(I)-N_j(J)}{k_{j+1}-N_j(J)}_{q^{-1}} q^{-(k_{j+1}-N_j(J))(S_j(J)-S_i(I)+1)-N_j(J)(S_j(J)-S_i(I))}$$

$$\times \prod_{(i,j)\in\mathcal{P}_+(n,m+1)} q^{-t(k_{j+1}-N_j(J))} \binom{N_i(I)-N_j(J)}{k_{j+1}-N_j(J)}_{q^{-1}} q^{-(k_{j+1}-N_j(J))(N-S_i(I)+S_j(J))-N_j(J)(N-S_i(I)+S_j(J)-1)}$$

$$= (-1)^{N_{n-m-1}(J)} q^{-t(N_{n-m-1}(J))-N_{n-m-1}(J)S_{n-m-2}(J)-(A_-(I,J)+A_+(I,J))} X \prod_{(i,j)\in\mathcal{P}(n,m+1)\setminus\{(m,n-1)\}} \binom{N_i(I)}{N_j(J)}_{q^{-1}}.$$

Here,

$$A_-(I,J) := \sum_{(i,j)\in\mathcal{P}_-(n,m+1)\setminus\{(m,n-1)\}} N_j(J)(S_j(J) - S_i(I)),$$

$$A_+(I,J) := \sum_{(i,j)\in\mathcal{P}_+(n,m+1)} N_j(J)(N - S_i(I) + S_j(J) - 1).$$

Furthermore, by using Lemma 4.1.1, we can see that X is the coefficient of $z^{N_{n-1}(J)}$ in the function $F(z,q^{-1})$. Here, $F(z,q)$ is defined by

$$\begin{aligned}
F(z,q) &:= F_+(z,q)F_1(z,q)F_-(z,q),\\
F_+(z,q) &:= \prod_{(i,j)\in\mathcal{P}_+(n,m+1)} (1+q^{N-S_i(I)+S_j(J)}z)_q^{N_i(I)-N_j(J)}\\
&= (1+q^{S_{n-m-2}(J)}z)_q^{N-S_{m+1}(I)-S_{n-m-2}(J)},\\
F_1(z,q) &:= \frac{(1+q^{N-S_{m+1}(I)}z)_q^{N_{m+1}(I)}}{(1+q^{S_{n-m-2}(J)}z)_q^{N_{n-m-1}(J)+1}},\\
F_-(z,q) &:= \prod_{(i,j)\in\mathcal{P}_-(n,m+1)\setminus\{(m,n-1)\}} (1+q^{S_j(J)-S_i(I)+1}z)_q^{N_i(I)-N_j(J)}\\
&= (1+q^{S_{n-2}(J)-S_{m-1}(I)+1}z)_q^{S_{n-m-1}(J)-S_{n-2}(J)+S_{m-1}(I)}.
\end{aligned}$$

Then we have

$$F(z,q) = \frac{1}{(1+q^{N-S_m(I)}z)_q^{N_m(I)-N_{n-1}(J)}} = \sum_{l=0}^{\infty} (-1)^l \binom{N_m(I)-N_{n-1}(J)+l}{l}_q (q^{N-S_m(I)}z)^l.$$

Hence

$$\sum_K P^{(m)}_{I,K} P^{(1)}_{K,J}$$
$$= (-t)^{(m+1)N}(-1)^{N_{n-m-1}(J)} q^{2t(N)}$$
$$\times q^{-t(N_{n-m-1}(J))-N_{n-m-1}(J)S_{n-m-2}(J)-(N-S_m(I))N_{n-1}(J)-(A_-(I,J)+A_+(I,J))} \prod_{(i,j)\in\mathcal{P}(n,m+1)} \binom{N_i(I)}{N_j(J)}_{q^{-1}}.$$

We next compute the power of q as follows.

$$2t(N) - t(N_{n-m-1}(J)) - N_{n-m-1}(J)S_{n-m-2}(J) - N_{n-1}(J)(S_{n-1}(J) - S_m(I)) - (A_-(I,J) + A_+(I,J))$$
$$= 2t(N) - t(N_{n-m-1}(J)) - N_{n-m-1}(J)S_{n-m-2}(J) - \sum_{i\in X_{n-1}\setminus\{n-m-1\}} (S_i(J)N_i(J) - N_i(J))$$
$$= t(N) + b_{n,m+1}(I,J) + t(N) - t(N_{n-m-1}(J)) - \sum_{i=1}^{n-2} S_i(J)N_{i+1}(J) - \sum_{i\in X_{n-1}\setminus\{n-m-1\}} 2t(N_i(J))$$
$$= t(N) + b_{n,m+1}(I,J) - \sum_{i\in X_{n-1}\setminus\{n-m-1\}} t(N_i(J)).$$

Thus,

$$\sum_K P^{(m)}_{I,K} P^{(1)}_{K,J} = (-t)^{(m+1)N}(-1)^{N_{n-m-1}(J)} q^{t(N)} \mathcal{S}_{n,m+1}(q^{-1};(I,J)) q^{b_{n,m+1}(I,J)} = P^{(m+1)}_{I,J}.$$

Then we finish the proof of Lemma 4.2.2. □

Lemma 4.2.3. *We have*

$$\rho^{(N)}_{n,q,t}(\sigma_{n,n})_{I,J} = (-t)^{nN} q^{2t(N)} \delta_{I,J}.$$

Here, $\delta_{I,J}$ is defined as

$$\delta_{I,J} := \begin{cases} 1 & (I = J), \\ 0 & (I \neq J). \end{cases}$$

Proof. By Lemma 4.2.2, we have

$$\rho^{(N)}_{n,q,t}(\sigma_{n,n-1})_{I,J} = (-t)^{(n-1)N}(-1)^{N_1(J)} q^{t(N)} \mathcal{S}_{n,n-1}(q^{-1};(I,J)) q^{b_{n,n-1}(I,J)-a(I,J)}$$
$$= (-t)^{(n-1)N}(-1)^{N_1(J)} q^{t(N)+b_{n,n-1}(I,J)-a(I,J)} \prod_{i=1}^{n-2} q^{-t(N_{i+1}(J))} \binom{N_i(I)}{N_{i+1}(J)}_{q^{-1}}.$$

We can obtain $\rho^{(N)}_{n,q,t}(\sigma_{n,n})_{I,J}$ by the following two computations.

$$\rho^{(N)}_{n,q,t}(\sigma_{n,n})_{I,J} = \sum_K \rho^{(N)}_{n,q,t}(\sigma_{n,n-1})_{I,K} \rho^{(N)}_{n,q,t}(\sigma_{n,1})_{K,J}$$
$$= \sum_M \rho^{(N)}_{n,q,t}(\sigma_{n,1})_{I,M} \rho^{(N)}_{n,q,t}(\sigma_{n,n-1})_{M,J}.$$

From the first computation and the definition of q-binomial coefficient, we have

$$\rho^{(N)}_{n,q,t}(\sigma_{n,n})_{I,J} \neq 0 \Longleftrightarrow N_i(J) \leq N_{i+1}(K) \leq N_i(I) \text{ for } 1 \leq i \leq n-2.$$

On the other hand, by the second computation,

$$\rho^{(N)}_{n,q,t}(\sigma_{n,n})_{I,J} \neq 0 \Longleftrightarrow N_i(J) \leq N_{i+1}(K) \leq N_i(I) \text{ for } 2 \leq i \leq n-1.$$

Hence we have

$$\rho^{(N)}_{n,q,t}(\sigma_{n,n})_{I,J} \neq 0 \iff I = J.$$

Then $\rho^{(N)}_{n,q,t}(\sigma_{n,n})$ is diagonal matrix. we next calculate the (I,I)-entry of $\rho^{(N)}_{n,q,t}(\sigma_{n,n})$. By replacing $N_1(K) = N_{n-1}(I)$ and $N_{i+1}(K) = N_i(I), (1 \leq i \leq n-2)$, we have

$$\begin{aligned}
\rho^{(N)}_{n,q,t}(\sigma_{n,n})_{I,I} &= (-t)^{(n-1)N}(-1)^{N_1(K)}q^{t(N)}\mathcal{S}_{n,n-1}(q^{-1};(I,K))q^{b_{n,n-1}(I,K)-a(I,K)} \\
&\quad \times (-t)^N(-1)^{N_{n-1}(I)}q^{t(N)}\mathcal{S}_{n,1}(q^{-1};(K,I))q^{a_{n,1}(K,I)-a(K,J)} \\
&= (-t)^{nN}q^{2t(N)}q^{-2(t(N_1(I))+\cdots+t(N_{n-2}(I)))}q^{b_{n,n-1}(I,K)-a(I,K)-a(K,I)+a_{n,1}(K,I)}.
\end{aligned}$$

Here we calculate $b_{n,n-1}(I,K) - a(I,K) - a(K,I) + a_{n,1}(K,I)$ as follows.

$$\begin{aligned}
&b_{n,n-1}(I,K) - a(I,K) - a(K,I) + a_{n,1}(K,I) \\
&= \sum_{m=1}^{n-1}\sum_{(i,j)\in\mathcal{P}(n,m)} N_i(I)N_j(K) - \sum_{i>j} N_i(I)N_j(K) - \sum_{i<j} N_i(I)N_j(K) \\
&\quad + \sum_{i=1}^{n-2}(N_i(I)N_{i+1}(K) - N_{i+1}(K)) \\
&= 0 + \sum_{i=1}^{n-2}(N_i(I)N_{i+1}(K) - N_{i+1}(K)) = 2\sum_{i=1}^{n-2} t(N_i(I)).
\end{aligned}$$

Hence,

$$\rho^{(N)}_{n,q,t}(\sigma_{n,n})_{I,J} = (-t)^{nN}q^{2t(N)}\delta_{I,J}.$$

□

Then we next show the Theorem 4.2.1.

***Proof of Theorem* 4.2.1**. By the construction of $\rho^{(N)}_{n,q,t}$, and $\beta^r_{n,1} = p^r_n$, we have $\rho^{(N)}_{n,1,-1} = \mathrm{Sym}^{(N)}p^r_n$. Then, from the result of Lemma 4.2.3, we have

$$\zeta(s, \sigma_{n,1}; \rho^{(N)}_{n,q,t}) = \zeta((-t)^N q^{\frac{2t(N)}{n}} s, c_n; \mathrm{Sym}^{(N)}p^r_n).$$

Let P be the set of poles of $\zeta(s, c_n; \mathrm{Sym}^{(N)}p^r_n)$ which consists of d-th roots of unity for $d \mid n$. Then the set of poles of $\zeta(s, \sigma_{n,1}; \rho^{(N)}_{n,q,t})$ can be expressed by $(-t)^{-N}q^{-\frac{2t(N)}{n}}P$. Since (n,m) is coprime, the set of poles of $\zeta(s, \sigma_{n,m}; \rho^{(N)}_{n,q,t})$ coincides with $(-t)^{-mN}q^{-\frac{2mt(N)}{n}}P$. Then we have

$$\zeta(s, \sigma_{n,m}; \rho^{(N)}_{n,q,t}) = \zeta((-t)^{mN} q^{\frac{2mt(N)}{n}} s, c_n; \mathrm{Sym}^{(N)}p^r_n).$$

□

4.3 q-identies and q-series

As an application of Theorem 4.2.1, we have the following q-identity.

Corollary 4.3.1. *We assume that (n,m) is coprime for $n,m \in \mathbb{N}^2$, then we have*

$$\sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_{n-1})\\ \lambda_1+\cdots+\lambda_{n-1}=N}} (-1)^{\lambda_{n-m}} q^{t(\lambda_{n-m})+s_{n,m}(\lambda)} \prod_{(i,j)\in\mathcal{P}(n,m)} q^{\lambda_j^2} \binom{\lambda_i}{\lambda_j}_q = \begin{cases} q^{mk(nk-1)} & (N=nk), \\ -q^{mk(nk+1)} & (N=nk+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Here, $s_{n,1}(\lambda) := 0$ and for $2 \le m \le n-1$, we define

$$s_{n,m}(\lambda) := \sum_{l=1}^{m-1} \Bigg(\sum_{(i,j)\in\mathcal{P}(n,l)} \lambda_i\lambda_j - \lambda_{n-l} \Bigg).$$

Proof. By Theorem 4.2.1, we have

$$\sum_{k=1}^{\infty} \frac{\operatorname{tr} \rho_{n,q,t}^{(N)}(\sigma_{n,m}^k)}{k} s^k = \sum_{k=1}^{\infty} \frac{\operatorname{tr} \operatorname{Sym}^{(N)} p_n^r(c_n^m)}{m} (-t)^{mNk} q^{\frac{2mt(N)k}{n}} s^k. \tag{4.3}$$

Then comparing the coefficient of s in (4.3), we have

$$(-t)^{mN} \sum_I (-1)^{\lambda_{n-m}} q^{t(N)+b_{n,m}(I,I)-a(I,I)} \mathcal{S}_{n,m}(q^{-1};(I,I)) = \operatorname{tr} \operatorname{Sym}^{(N)} p_n^r(c_n) \cdot (-t)^{mN} q^{\frac{2mt(N)}{n}}.$$

Putting $N_i(I) = \lambda_i$, the above sum can be regarded as the sum of the partitions of N. Moreover,

$$\mathcal{S}_{n,m}(q^{-1},(I,I)) = \prod_{(i,j)\in\mathcal{P}(n,m)} q^{-t(\lambda_j)} \binom{\lambda_i}{\lambda_j}_{q^{-1}} = \prod_{(i,j)\in\mathcal{P}(n,m)} q^{\frac{1}{2}\lambda_j(\lambda_j+1)-\lambda_i\lambda_j} \binom{\lambda_i}{\lambda_j}_q.$$

Then, the power of q can be computed as

$$\begin{aligned} & t(N) + b_{n,m}(I,I) - a(I,I) + \sum_{(i,j)\in\mathcal{P}(n,m)} \left\{ \frac{1}{2}\lambda_j(\lambda_j+1) - \lambda_i\lambda_j \right\} \\ &= \sum_{k=1}^{n-1} t(\lambda_k) + \sum_{i<j} \lambda_i\lambda_j + \sum_{l=1}^{m} \sum_{(i,j)\in\mathcal{P}(n,l)} \lambda_i\lambda_j - \sum_{l=1}^{m} \lambda_{n-l+1} - \sum_{i<j} \lambda_i\lambda_j + \sum_{(i,j)\in\mathcal{P}(n,m)} \left\{ \frac{1}{2}\lambda_j(\lambda_j+1) - \lambda_i\lambda_j \right\} \\ &= t(\lambda_{n-m}) + \sum_{(i,j)\in\mathcal{P}(n,m)} \lambda_j^2 + s_{n,m}(\lambda). \end{aligned}$$

From MacMahon Master Theorem (see [13]),

$$\zeta(s, c_n; p_n^r) = \sum_{k=0}^{\infty} \operatorname{Sym}^{(k)} p_n^r(c_n) s^k. \tag{4.4}$$

On the other hand, we compute the left-hand side of (4.4),

$$\zeta(s, c_n; p_n^r) = \frac{1-s}{1-s^n} = \sum_{k=0}^{\infty} (s^{nk} - s^{nk+1}).$$

Hence we have

$$\mathrm{Sym}^{(k)} p_n^r(c_n) = \begin{cases} 1 & (N = nk), \\ -1 & (N = nk+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then Corollary 4.3.1 holds. □

Remark 4.3.1. When $m = 1$, Corollary 4.3.1 turns to the following simple identity.

$$\sum_{\lambda_1+\cdots+\lambda_{n-1}=N} (-1)^{\lambda_{n-1}} q^{\lambda_1^2+\cdots+\lambda_{n-2}^2+t(\lambda_{n-1})} \prod_{k=1}^{n-2} \binom{\lambda_{k+1}}{\lambda_k}_q = \begin{cases} q^{k(nk-1)} & (N = nk), \\ -q^{k(nk+1)} & (N = nk+1), \\ 0 & (\text{otherwise}). \end{cases}$$

4.4 Generating functions

We introduce the following trace generating function.

Definition 4.4.1. *For $\sigma \in B_n$, we define the following series.*

$$Z_{q,t}(s, \sigma) := 1 + \sum_{N=1}^{\infty} \operatorname{tr} \rho_{n,q,t}^{(N)}(\sigma) s^N.$$

The trace generating function $Z_{q,t}(s, \sigma)$ has the following properties.

Proposition 4.4.1. (1) *For any $\sigma \in B_n$, we have*

$$\lim_{q \to 1} Z_{q,t}(s, \sigma) = \zeta(s, \sigma; \beta_{n,-t}^r).$$

(2) *We assume that (n, m) is coprime, then we have*

$$\lim_{t \to 1} Z_{q,t}(1, \sigma_{n,m}) = \sum_{k=0}^{\infty} (-1)^{nmk} (q^{mk(nk-1)} - (-1)^m q^{mk(nk+1)}).$$

Proof. If we let $q \to 1$, then $\rho_{n,q,t}^{(N)}$ turns to the symmetric power representation of reduced Burau representation denoted by $\mathrm{Sym}^{(N)} \beta_{n,-t}^r$. Then we can show the first statement (1) by MacMahon Master Theorem. Furthermore from the formula of Corollary 4.3.1, the second statement (2) immediately holds. □

Remark 4.4.1. By Proposition 4.4.1, $Z_{q,t}(s, \sigma)$ can be regarded as a q-analogue of $\zeta(s, \sigma; \beta_{n,-t}^r)$. Furthermore, if $m = 1$, we have the following expression by using Jacobi triple product.

$$\lim_{t \to 1} Z_{q,t}(1, \sigma_{n,1}) - 1 = \prod_{k \equiv 0, \pm(n-1) \pmod{2n}} (1 - (-q)^k). \tag{4.5}$$

In summary, we obtain the relationship diagram in Figure 13.

We consider the case of $n = 3$. Here we calculate the trace of $\rho_{3,q,t}^{(N)}(\sigma_{3,1})$ by another way. From Proposition 4.2.1,

$$\rho_{3,q,t}^{(N)}(\sigma_{3,1})_{I,J} = (-1)^{N-N_2(J)} t^N q^{t(N_2(J))+N_1(I)N_1(J)} \binom{N_2(I)}{N_1(J)}_q.$$

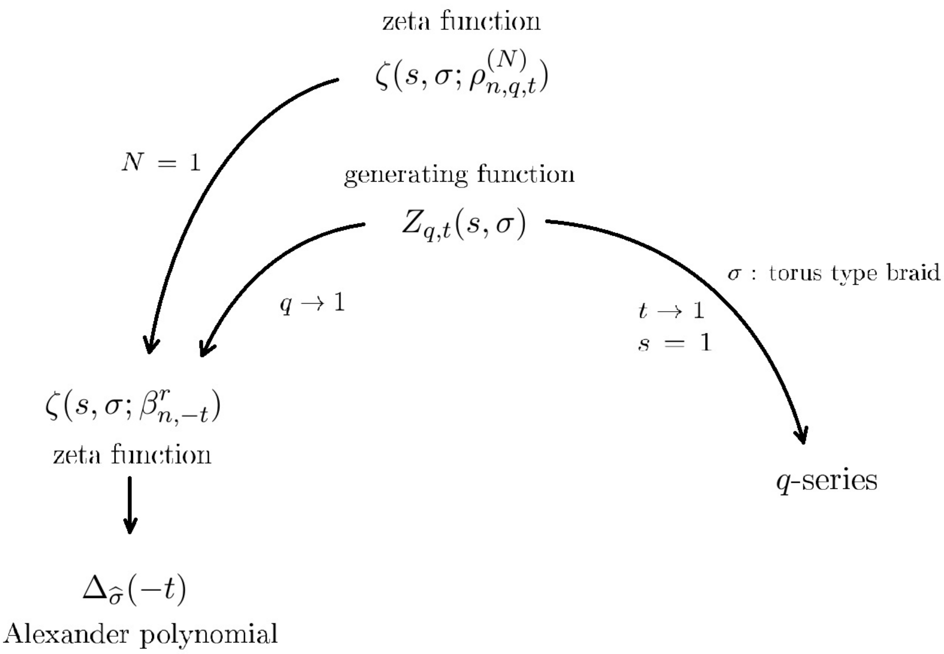

Figure 13: braid zeta functions, generating function, and q-series

Putting $N_2(I)=i, N_2(J)=j$ $(i,j=0,1,2,\ldots,N)$, we can express $\rho^{(N)}_{3,q,t}(\sigma_{3,1})_{I,J}$ as follows

$$\begin{aligned}\rho^{(N)}_{3,q,t}(\sigma_{3,1})_{i,j} &= (-1)^{N-j}t^N q^{t(j)+(N-i)(N-j)}\binom{i}{N-j}_q \\ &= (-1)^{N-j}t^N q^{t(N)-t(N-j)}\binom{i}{N-j}_{q^{-1}}.\end{aligned}$$

Next, we compute the trace.

$$\operatorname{tr}\rho^{(N)}_{3,q,t}(\sigma_{3,1}) = t^N q^{t(N)}\sum_{i=0}^{N}(-1)^{N-i}q^{-t(N-i)}\binom{i}{N-i}_{q^{-1}} = t^N q^{t(N)}c^{(3)}_N(q^{-1}).$$

Here we define

$$c^{(3)}_N(q) := \sum_{i=0}^{N}(-1)^{N-i}q^{t(N-i)}\binom{i}{N-i}_q.$$

To get the formula of $\operatorname{tr}\rho^{(N)}_{3,q,t}(\sigma_{3,1})$, we need to calculate $c^{(3)}_N(q)$. In order to compute $c^{(3)}_N(q)$, we consider the following generating function.

$$f_3(z,q) := \sum_{N=0}^{\infty}c^{(3)}_N(q)z^N.$$

Here, we set $c^{(3)}_0(q)=1$. Since $(-1)^{N-i}q^{t(N-i)}\binom{i}{N-i}_q$ is equal to the coefficient of z^{N-i} in $(1-z)^i_q$, the generating function $f_3(z,q)$ can be expressed by

$$f_3(z,q) = \sum_{n=0}^{\infty}z^n(1-z)^n_q = (1-z)F(z,0,z;q).$$

Here, $F(a,b;z,q)$ is a kind of q-hypergeometric series defined by

$$F(a,b;z,q) := \sum_{n=0}^{\infty} \frac{(1-a)_q^n}{(1-b)_q^n} z^n.$$

By using the properties of q-hypergeometric series (see [7], Chapter 1), we have the following q-difference equation.

$$f_3(z,q) = 1 + qz - qz^3 f_3(qz,q). \tag{4.6}$$

Thus, we have

$$f_3(z,q) = \sum_{k=0}^{\infty} (-1)^k \{q^{\frac{n(3k-1)}{2}} z^{3k} + q^{\frac{k(3k+1)}{2}} z^{3k+1}\}. \tag{4.7}$$

Hence we obtain the formula of Corollary 4.3.1 in the case of $n=3, m=1$. Conversely, we can obtain the identity (4.7) by using the result of Corollary 4.3.1. Moreover, we remark that the equation (4.7) includes a famous identity which is called Euler's pentagonal number theorem as

$$\prod_{k=1}^{\infty}(1-q^k) = \sum_{n\in\mathbb{Z}} (-1)^k q^{\frac{n(3n-1)}{2}}.$$

We next consider the case of $n=4$. By Proposition 4.2.1,

$$\rho_{4,q,t}^{(N)}(\sigma_{4,1})_{I,J} = (-1)^{N_1(J)+N_2(J)} t^N q^{t(N)} q^{-t(N_1(J))} \binom{N_2(I)}{N_1(J)}_{q^{-1}} q^{-t(N_2(J))} \binom{N_3(I)}{N_2(J)}_{q^{-1}} q^{-N_3(I)N_1(J)}.$$

Then we have

$$\operatorname{tr} \rho_{4,q,t}^{(N)}(\sigma_{4,1}) = t^N q^{t(N)} c_N^{(4)}(q^{-1}).$$

Here $c_N^{(4)}(q)$ is defined by

$$c_N^{(4)}(q) := \sum_{\lambda_1+\lambda_2+\lambda_3=N} (-1)^{\lambda_1} q^{t(\lambda_1)} \binom{\lambda_2}{\lambda_1}_q (-1)^{\lambda_2} q^{t(\lambda_2)} \binom{\lambda_2}{\lambda_3}_q q^{\lambda_3\lambda_1}.$$

We put $c_0^{(4)}(q) = 1$. Then the generating function of $c_N^{(4)}(q)$ can be expressed by

$$f_4(z,q) := \sum_{N=0}^{\infty} c_N^{(4)}(q) z^N = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} (-1)^k q^{t(k)} \binom{n}{k}_q (1-q^n z)_q^k z^k.$$

It is difficult to calculate the function $f_4(z,q)$ directly, then we use the result of Corollary 4.3.1.

$$\operatorname{tr} \rho_{4,q,t}^{(N)}(\sigma_{4,1}) = \begin{cases} (-t)^{4k} q^{k(4k-1)} & (N=4k), \\ (-t)^{4k+1}(-1) q^{k(4k+1)} & (N=4k+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Hence,

$$c_N^{(4)}(q) = \begin{cases} q^{k(4k-1)} & (N=4k), \\ q^{k(4k+1)} & (N=4k+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we obtain the following expansion.

$$f_4(z,q) = \sum_{k=0}^{\infty} \{q^{k(4k-1)} z^{4k} + q^{k(4k+1)} z^{4k+1}\}.$$

A Appendix

In this section, we give some propositions and formula which are used in this paper.

A.1 Möbius inversion formula

We introduce the Möbius inversion formula.

Definition A.1.1. If the domain of the map f is the positive integers, and its range is subset of $\mathbb{C}$, then f is called *arithmetic function*.

Definition A.1.2. For positive integer n, the function μ is defined as follows.

$$\mu(n) := \begin{cases} (-1)^k & (n = p_1 p_2 \cdots p_k,\ (p_i \neq p_j)), \\ 1 & (n = 1), \\ 0 & (n \text{ is not square free}). \end{cases}$$

We call this arithmetic function *Möbius function*.

Proposition A.1.1. *For any $n \geq 1$, we assume that arithmetic function f and g satisfy*

$$g(n) = \sum_{d|n} f(n).$$

Then,

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right). \tag{A.1}$$

The formula $(A.1)$ is called *Möbius inversion formula*.

A.2 Iwahori-Hecke algebra

In this section, we introduce the foundation of the theory of Iwahori-Hecke algebra. For more detail, see [8], Chapter 5. We assume that $n \geq 1$, R is a commutative ring, and q is invertible element of R.

Definition A.2.1. $H_n^R(q)$ is the universal associative R-algebra generated by $T_1, T_2, \ldots, T_{n-1}$ which satisfy the following relations.

$$\begin{aligned} T_i T_j &= T_j T_i \quad (|i-j| > 1), \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad (i = 1, 2, \ldots, n-2), \\ T_i^2 &= (q-1) T_i + q \cdot 1 \quad (i = 1, 2, \ldots, n-1). \end{aligned}$$

$H_n^R(q)$ is called (*one-parameter*) *Iwahori-Hecke algebra*.

Remark A.2.1. When $n = 1$, we can see $H_n^R(q) \simeq R$ by the definition. Furthermore, it is well-know that $H_n^R(1) \simeq R[\mathfrak{S}_n]$ which is group ring.

Next we state some famous facts.

Theorem A.2.1. *We assume that $q \in \mathbb{C}$ is neither 1 nor root of unity, then the Iwahori-Hecke algebra $H_n^{\mathbb{C}}(q)$ is semisimple. Thus, there is a finite family of simple subalgebras $\{A_\lambda\}_{\lambda \in \Lambda}$ of $H_n^{\mathbb{C}}(q)$ such that*

$$H_n^{\mathbb{C}}(q) = \bigoplus_{\lambda \in \Lambda} A_\lambda.$$

In other words, we can decompose any representation of $H_n^{\mathbb{C}}(q)$ into some irreducible representations of $H_n^{\mathbb{C}}(q)$. In fact, we can describe all irreducible representations of $H_n^{\mathbb{C}}(q)$ by using the Young diagram of the partition of n.

Definition A.2.2. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of a nonnegative integer n. We assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. Then we define the set

$$D(\lambda) = \{(r, s) \mid 1 \leq r \leq l, 1 \leq s \leq \lambda_r\}$$

and define the corresponding diagram which is called *Young diagram* as Figure 14.

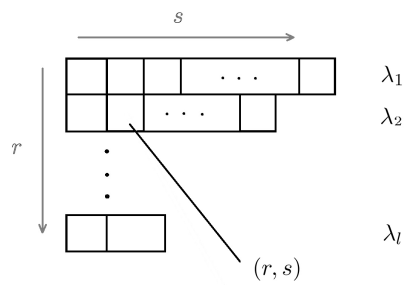

Figure 14: Young diagram

Definition A.2.3. For a partition λ of $n \geq 1$, we define the bijection $T : D(\lambda) \longrightarrow \{1, 2, \dots, n\}$. We call T *labeling*. In particular, if T satisfies

$$T(r, s) \leq T(r', s')$$

for all (r, s), $(r', s') \in D(\lambda)$ such that $r \leq r'$, $s \leq s'$, the labeling T is said to be *standard*. We set $\mathcal{T}_\lambda$ as the set of standard labeling of $D(\lambda)$.

Let T be a label of $D(\lambda)$. We assume that $T(r, s) = i, T(r', s') = i + 1$, and we set $d(r, s) := s - r$. Then we define

$$d_T(i) := d(T^{-1}(i + 1)) - d(T^{-1}(i)) \in \mathbb{Z}.$$

Furthermore, we set

$$a_T(i) := \frac{q^{d_T(i)}}{[d_T(i)]_q}, \qquad b_T(i) := a_T(i) - q.$$

Here, q is a complex parameter and suppose that q is generic. Using $a_T(i)$ and $b_T(i)$, we define the representation of $H_n^{(\mathbb{C})}(q)$.

Definition A.2.4. Let V_λ be the complex vector space with basis $\{v_T\}_{T \in \mathcal{T}_\lambda}$. Then we let the generators $T_1, \dots, T_{n-1}$ of $H_n^{\mathbb{C}}(q)$ act on the basis of V_λ by

$$T_i(v_T) := a_T(i) v_T + b_T(i) v_{s_i \cdot T}. \tag{A.2}$$

Here, s_i is the transposition $(i, i+1) \in \mathfrak{S}_n$ and $s_i T$ is given by switching the labels i and $i + 1$ of T. If $s_i T$ is not standard, we set $s_i T = 0$.

Proposition A.2.1. *The action on V_λ defined by (A.2) has the structure of $H_n^{\mathbb{C}}(q)$. Then we obtain the representation*

$$\rho_\lambda : H_n^{\mathbb{C}}(q) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_\lambda).$$

ρ_λ is called *seminormal representation* of $H_n^{\mathbb{C}}(q)$. Furthermore, we can define the group homomorphism $\omega_n : B_n \longrightarrow H_n^{\mathbb{C}}(q)^\times$ by sending the generator σ_i to T_i. Then we have a braid representation with respect to the partition λ as

$$\rho_\lambda^{br} := \rho_\lambda \circ \omega_n : B_n \longrightarrow \mathrm{GL}(V_\lambda).$$

Proposition A.2.2. *ρ_λ is irreducible. Furthermore, for any irreducible representation ρ which has Iwahori-Hecke algebra, there exists a unique partition λ of n such that $\rho \simeq \rho_\lambda$.*

Proposition A.2.3. *The partition $\lambda = (n)$ corresponds to the trivial representation $\mathbf{1}$ of B_n, and $\lambda = (\underbrace{1, 1, \ldots, 1}_{n})$ corresponds to the one dimensional representation $\mathrm{sgn}_q : \sigma \mapsto (-q)^{\varepsilon(\sigma)}$. Moreover, the partition $\lambda = (2, \underbrace{1, \ldots, 1}_{n-2})$ corresponds to the reduced Burau representation $\beta_{n,q}^r$.*

The Figure 15 is called *Bratteli diagram* which is the oriented graph of Young diagrams. From this Figure and Proposition $A.2.3$, we can see that irreducible representation of B_3 which has the structure of Iwahori-Hecke algebra is classified into only 3 patterns: $\mathbf{1}, \mathrm{sgn}_q, \beta_{3,q}^r$.

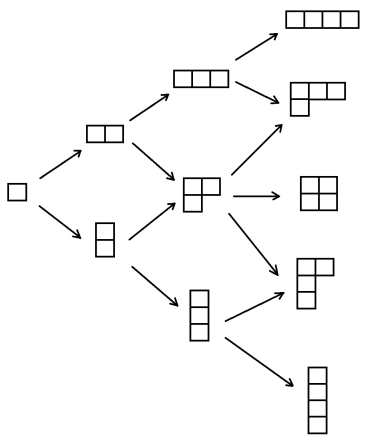

Figure 15: Bratteli diagram

A.3 MacMahon Master Theorem

We state the simplest version of MacMahon Master Theorem which is useful in this paper.

Proposition A.3.1. *Let A be an $n \times n$ matrix. Then the following identity holds.*

$$\det(I_n - As)^{-1} = \sum_{k=0}^{\infty} \mathrm{tr}\,(S^k A) s^k.$$

Here, $S^k A$ is the symmetric power of A.

A.4 Jacobi triple product

Finally, we state the Jacobi triple product.

Proposition A.4.1. *The following identity holds.*

$$\sum_{n \in \mathbb{Z}} q^{n^2} z^n = \prod_{k=1}^{\infty} (1 - q^{2k})(1 + q^{2k-1} z)(1 + q^{2k+1} z^{-1}).$$

Replacing $q \mapsto q^n$ and $z \mapsto (-1)^n q^{-1}$, we have the identity (4.5).

References

[1] S. Albeverio, A. Kosyak, q-Pascal's triangle and irreducible representations of the braid group B_3 in arbitrary dimension, arXiv:math.QA(RT)/0803.2778v2.

[2] J. W. Alexander. Topological invariants of knots and links. Trans. Amer. Math. Soc. 20 (1928) 275–306.

[3] E. Artin. Theory of braids, Ann. of Math. (2) 48 (1947) 101–126.

[4] J. S. Birman. Braids, links, and mapping class groups. Ann. Math. Studies 82. Princeton University. Press, 1974.

[5] J. S. Birman, On the Jones polynomial of closed 3-braids. Inventiones mathematicae 81 (1985) 287–294.

[6] A. Deitmar and S. Koyama and N. Kurokawa. Absolute zeta functions, Proc. Japan Acad. Ser. A Math. Sci. 84 (2008) no. 8, 138–142.

[7] Nathan J. Fine. Basic Hypergeometric Series and Applications. Mathematical Surveys and Monographs, number 27, Amer. Math. Soc. 1988.

[8] C. Kassel and V. Turaev. Braid Groups. Springer, 2008.

[9] S. Kim and S. Koyama and N. Kurokawa. The Riemann hypothesis and functional equations for zeta functions over $\mathbf{F}_1$, Proc. Japan Acad. Ser. A Math. Sci. 85, (2009) no. 6, 75–80.

[10] A Kosyak. The Lawrence-Krammer representation is a quantization of the symmetric square of the Burau representation, arXiv:math. QA(RT)/1609.06663.

[11] S. Koyama and S. Nakajima. Zeta functions of generalized permutations with application to their factorization formulas, Proc. Japan Acad. Ser. A Math. Sci. Volume 88, (2012) no. 8, 115–120.

[12] B. I. Kurpita and K. Murasugi. A study of braids . Mathematics and its Applications, 484. Kluwer Academic Publishers, Dordrecht, 1999.

[13] P. A. MacMahon, Combinatory Analysis vol.1, Cambridge University Press, 1917. Reprinted by Chelsea, 1984

[14] K. Murasugi. Knot Theory and Its Applications, Birkhäuser Basel, 1993.

[15] T. Ohtsuki. Quantum invariants, — A study of knots, 3-manifolds, and their sets, Series on Knots and Everything, 29. World Scientific Publishing Co., Inc., 2002.

[16] K. Okamoto. Braid zeta function and some formulae for the torus type, arXiv:math. QA(RT)/1611.08370.

[17] L. Rozansky. The universal R-matrix, Burau representation, and the Melvin-Morton expansion of the colored Jones polynomial, Adv. Math. 134 (1998) 1–31.

[18] C. C. Squier. The Burau representation is unitary. Proceedings of the Amer. Math. Soc. 90(2) (1984)199–202. MR 85b:20056.

[19] T. Yamamoto. Inversion formulas for tridiagonal matrices with applications to boundary value problems, Numerical Functional Analysis and Optimization 22, (2001) 357–385.