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1 Introduction

The hyperbolic Eisenstein series is the Eisenstein series associated to hyperbolic fixed points, or equivalently a primitive hyperbolic element of Fuchsian groups of the first kind. It was first introduced by S. S. Kudla and J. J. Millson [11] in 1979 as an analogue of the ordinary Eisenstein series associated to a parabolic fixed point. They established an explicit construction of the harmonic 1-form dual to an oriented closed geodesic on an oriented Riemann surface M of genus greater than 1. Furthermore, they proved the meromorphic continuation of the hyperbolic Eisenstein series to all of $\mathbb{C}$ and gave the location of the possible poles when M is compact. After that, they generalized the results of [11] to compact n-dimensional hyperbolic manifold and its totally geodesic hyperbolic $(n-k)$-manifolds. In [11, 12], they constructed the hyperbolic Eisenstein series by averaging certain smooth closed k-form.

Following Kudla and Millson's point of view, the scalar-valued analogue of the hyperbolic Eisenstein series is defined in [3, 4], and [10]. It is defined as follows. Let $\mathbb{H}^2 := \{z = x+iy \in \mathbb{C} \mid x, y \in \mathbb{R},\, y > 0\}$ be the upper-half plane and $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ a Fuchsian group of the first kind acting on $\mathbb{H}^2$ by the fractional linear transformations. Then the quotient $\Gamma\backslash\mathbb{H}^2$ is a hyperbolic Riemann surface of finite volume. Let $\gamma \in \Gamma$ be a primitive hyperbolic element and $\Gamma_\gamma = \langle\gamma\rangle$ be its centralizer group in Γ. Consider the coordinates $x = e^\rho \cos\theta$ and $y = e^\rho \sin\theta$. In this setting, the hyperbolic Eisenstein series associated to γ is defined by the series

$$E_{\mathrm{hyp},\gamma}(z,s) := \sum_{\eta\in\Gamma_\gamma\backslash\Gamma} (\sin\theta(A\eta z))^s, \tag{1}$$

where $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$ and A is an element in $\mathrm{PSL}(2,\mathbb{R})$ such that $A\gamma A^{-1} = \left(\begin{smallmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1}\end{smallmatrix}\right)$ for some $a(\gamma) \in \mathbb{R}$ with $|a(\gamma)| > 1$. The hyperbolic Eisenstein series (1) converges for any $z \in \mathbb{H}^2$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and defines a Γ-invariant function where it converges. Furthermore, it is known that the hyperbolic Eisenstein series $E_{\mathrm{hyp},\gamma}(z,s)$ satisfies the following differential equation

$$(-\Delta + s(s-1))E_{\mathrm{hyp},\gamma}(z,s) = s^2 E_{\mathrm{hyp},\gamma}(z,s+2),$$

where Δ is the hyperbolic Laplace-Beltrami operator.

There are many researches on this hyperbolic Eisenstein series. M. S. Risager [16] studied the hyperbolic Eisenstein series twisted by modular symbols. T. Falliero [2] and D. Garbin, J. Jorgenson and M. Munn [3] studied the asymptotic behavior of the hyperbolic Eisenstein series for a degenerating family of finite volume hyperbolic Riemann surfaces. They proved that the limit of the hyperbolic Eisenstein series associated to a pinching geodesic for degenerating family is equal to the parabolic Eisenstein series associated to the newly formed cusp on the limit surface. D. Garbin and A.-M. v. Pippich [4] studied the asymptotic behavior of parabolic, hyperbolic and elliptic Eisenstein series for a elliptic degenerating family of finite volume hyperbolic Riemann surfaces.

Furthermore, J. Jorgenson, J. Kramer and A.–M. v. Pippich [10], in 2010, proved that the hyperbolic Eisenstein series is a square integrable function on $\Gamma\backslash\mathbb{H}^2$ and obtained the spectral expansion associated to the hyperbolic Laplace-Beltrami operator $-\Delta$ precisely. It is given as follows. Let

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

be the eigenvalues of $-\Delta$ and e_m the eigenfunction corresponding to λ_m. Let $\mathfrak{D} \subset \mathbb{N}$ is an index set for a complete orthogonal system of eigenfunctions $\{e_m\}_{m\in\mathfrak{D}}$. We denote a cusp of $\Gamma\backslash\mathbb{H}^2$ by ν and the ordinary Eisenstein series associated to the cusp ν by $E_\nu(z,s)$. Then the spectral expansion of the hyperbolic Eisenstein series $E_{\mathrm{hyp},\gamma}(z,s)$ is given by

$$E_{\mathrm{hyp},\gamma}(z,s) = \sum_{m\in\mathfrak{D}} a_{m,\gamma}(s)e_m(z) + \frac{1}{4\pi}\sum_{\nu:\mathrm{cusps}}\int_{-\infty}^{\infty} a_{1/2+i\mu,\gamma}(s)E_\nu(z,1/2+i\mu)d\mu. \quad (2)$$

Then this series converges absolutely and locally uniformly. The coefficients $a_{m,\gamma}(s)$ and $a_{1/2+i\mu,\gamma}(s)$ are given by

$$a_{m,\gamma}(s) = \sqrt{\pi}\cdot\frac{\Gamma((s-1/2+\mu_m)/2)\Gamma((s-1/2-\mu_m)/2)}{\Gamma(s/2)^2} \times \int_{\widetilde{L}_\gamma} e_m(z)d\sigma \quad (3)$$

and

$$a_{1/2+i\mu,\gamma}(s) = \sqrt{\pi} \cdot \frac{\Gamma((s-1/2+i\mu)/2)\Gamma((s-1/2-i\mu)/2)}{\Gamma(s/2)^2} \times \int_{\widetilde{L}_\gamma} E_\nu(z, 1/2+i\mu) d\sigma, \quad (4)$$

where $\mu_m^2 = \frac{1}{4} - \lambda_m$ and $\widetilde{L}_\gamma$ is the closed geodesic corresponding to γ. Furtheremore, they proved the meromorphic continuation of $E_{\mathrm{hyp},\gamma}(z,s)$ to the whole complex plane $\mathbb{C}$. They also derived the location of the possible poles and residues from the spectral expansion (2) and the meromorphic continuation.

In our previous paper [5], we defined the hyperbolic Eisenstein series for a loxodromic element of the cofinite Kleinian groups acting on 3-dimensional hyperbolic space and proved the results analogous to [10]. We also in [7] consider the asymptotic behavior of the hyperbolic Eisenstein series for the degeneration of 3-dimensional hyperbolic manifolds and obtain the results corresponding to [2, 3].

Our purpose in this article is to obtain a generalization of the hyperbolic Eisenstein series (1) for the n-dimensional hyperbolic spaces. Let

$$\mathbb{H}^n := \{\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

be the n-dimensional upper-half space of $\mathbb{R}^n$. With the hyperbolic metric $\frac{|d\mathbf{x}|}{x_n}$, $\mathbb{H}^n$ is the n-dimensional hyperbolic space. Let $G := O(n,1)$ be the orthogonal group of signature $(n,1)$ and G_0 the connected component of G containing the unit element. Then G_0 acts on $\mathbb{H}^n$ transitively and any element of G_0 determines an element of the group of orientation preserving isometries of $\mathbb{H}^n$. Let $\Gamma \subset G_0$ be a torsion-free cofinite discrete subgroup. Then the quotient $\Gamma\backslash\mathbb{H}^n$ is a n-dimensional hyperbolic manifold. Let $\sigma \in O(n+1)$ be the involution which is invariant on a hyperbolic $(n-k)$-plane $D_\sigma \subset \mathbb{H}^n$. We denote by G_σ and Γ_σ the centralizer group of σ in G_0 and the intersection $\Gamma \cap G_\sigma$ respectively. We assume the following assumption

$$\sigma\Gamma\sigma = \Gamma.$$

Then the quotient $\Gamma_\sigma\backslash D_\sigma$ is naturally identified $(n-k)$-submanifold of $\Gamma\backslash\mathbb{H}^n$. In addition, we assume $\Gamma_\sigma\backslash D_\sigma$ is compact. Without loss of generality, we may identify D_σ with the subset

$$\{\mathbf{x} = (x_1, ..., x_n) \in \mathbb{H}^n \mid x_i = 0,\ 1 \le i \le k\} \subset \mathbb{H}^n.$$

Then, for $\mathbf{x} \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$, the hyperbolic Eisenstein series associated to the involution σ is defined as follows.

$$E_\sigma(\mathbf{x}, s) := \sum_{\eta \in \Gamma_\sigma \backslash \Gamma} \sin \varphi_0(\eta \mathbf{x})^s, \tag{5}$$

where φ_0 is given by $\sin \varphi_0(\eta \mathbf{x}) = \cosh(d_{\mathrm{hyp}}(\eta \mathbf{x}, D_\sigma))^{-1}$ and $d_{\mathrm{hyp}}(\eta \mathbf{x}, D_\sigma)$ denotes the hyperbolic distance from $\eta \mathbf{x}$ to D_σ. The hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ converges absolutely and locally uniformly for any $\mathbf{x} \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n - 1$ and satisfies the following differential shift equation

$$(-\Delta + s(s - n + 1))E_\sigma(\mathbf{x}, s) = s(s - n + k + 1)E_\sigma(\mathbf{x}, s + 2), \tag{6}$$

where $-\Delta$ denotes the Laplace-Beltrami operator associated to the hyperbolic metric. In this aritcle, about this hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$, we obtain the following results:

Main Theorem. (See Theorems 4.3 and 4.4). Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$ be the eigenvalues of $-\Delta$ and e_m the eigenfunction corresponding to λ_m. Let $\mathfrak{D} \subset \mathbb{N}$ be an index set for a complete orthogonal system of eigenfunctions $\{e_m\}_{m \in \mathfrak{D}}$. Then, for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n - 1$, the Eisenstein series $E_\sigma(\mathbf{x}, s)$ admits the following spectral expansion.

$$\begin{aligned} E_\sigma(\mathbf{x}, s) = {} & \sum_{m \in \mathfrak{D}} a_{m,\sigma}(s) e_m(\mathbf{x}) \\ & + \frac{1}{4\pi} \sum_{\nu:\mathrm{cusps}} \int_{-\infty}^{\infty} a_{\frac{n-1}{2}+i\mu,\sigma}(s) E_\nu\left(\mathbf{x}, \frac{n-1}{2} + i\mu\right) d\mu, \end{aligned} \tag{7}$$

where E_ν is the ordinary Eisenstein series associated to the cusp ν. Then this series converges absolutely and locally uniformly. The coefficients $a_{m,\sigma}(s)$ and $a_{\frac{n-1}{2}+i\mu,\sigma}(s)$ are given by

$$\begin{aligned} a_{m,\sigma} = {} & \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \Gamma\left(\frac{k}{2}\right) \\ & \times \frac{\Gamma\left(\left(s - \frac{n-1}{2} + \mu_m\right)/2\right) \Gamma\left(\left(s - \frac{n-1}{2} - \mu_m\right)/2\right)}{\Gamma\left(s/2\right) \Gamma\left(\left(s - n + k + 1\right)/2\right)} \\ & \times \int_{\Gamma_\sigma \backslash D_\sigma} e_m dv_2 \end{aligned} \tag{8}$$

and

$$a_{\frac{n-1}{2}+i\mu,\sigma} = \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \Gamma\left(\frac{k}{2}\right) \times \frac{\Gamma\left(\left(s-\frac{n-1}{2}+i\mu\right)/2\right)\Gamma\left(\left(s-\frac{n-1}{2}-i\mu\right)/2\right)}{\Gamma\left(s/2\right)\Gamma\left(\left(s-n+k+1\right)/2\right)} \times \int_{\Gamma_\sigma\backslash D_\sigma} E_\nu\left(\mathbf{x}, \frac{n-1}{2}+i\mu\right) dv_2, \quad (9)$$

where $\mu_m^2 = (\frac{n-1}{2})^2 - \lambda_m$ and dv_2 is the hyperbolic volume element restricted on $\Gamma_\sigma\backslash D_\sigma$. In addition, $\mathrm{vol}(S^{k-1})$ denotes the Euclidean volume of the unit $(k-1)$-dimensional sphere

$$S^{k-1} := \{\mathbf{x} = (x_1, ..., x_k) \in \mathbb{R}^k \mid |x|^2 = x_1^2 + \cdots + x_k^2 = 1\}.$$

Besides, we derive the meromorphic continuation of $E_\sigma(\mathbf{x}, s)$ to all complex plane $\mathbb{C}$ and the possible poles and residues from the spectral expansion. (See Theorem 4.4).

We finally introduce the outline of this article. In Section 2, we establish basic notations and recall known results needed later in this article. Section 3 is devoted to the definition of the hyperbolic Eisenstein series on n-dimensional hyperbolic spaces and its fundamental properties. In Section 4, we state details of Main Theorem and its proof.

2 Preliminaries

2.1 The hyperboloid model of the hyperbolic n-space

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional real vector space and $\mathbf{e}_i$ $(1 \leq i \leq n+1)$ be the standard basis of $\mathbb{R}^{n+1}$. For any vector $\mathbf{x} \in \mathbb{R}^{n+1}$, we write the coordinate representation in standard basis of $\mathbb{R}^{n+1}$ as

$$\mathbf{x} = (x_1, x_2, ..., x_{n+1}).$$

We consider the Lorentzian inner product $(\ ,\ )$ on $\mathbb{R}^{n+1}$. It is defined for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+1}$ as follows.

$$(\mathbf{x}, \mathbf{y}) := x_1y_1 + x_2y_2 + ... + x_ny_n - x_{n+1}y_{n+1}.$$

The inner product space $\mathbb{R}^{n+1}$ together with the Lorentzian inner product $(\ ,\ )$ is called Lorentzian $(n+1)$-space and is also denoted by $\mathbb{R}^{n,1}$. The norm in $\mathbb{R}^{n+1}$ associated with $(\ ,\ )$ is defined to be the complex number

$$||\mathbf{x}|| = (\mathbf{x}, \mathbf{x})^{\frac{1}{2}}.$$

Here $||\mathbf{x}||$ is either positive real number, zero, or positive imaginary. This norm is also called the Lorentzian norm. A function $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Lorentz transformation if and only if

$$(\phi(\mathbf{x}), \phi(\mathbf{y})) = (\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^{n+1}$.

Definition 2.1. A vector $\mathbf{x} \in \mathbb{R}^{n+1}$ is called

(1) *space-like* if $||\mathbf{x}|| > 0$,

(2) *light-like* if $||\mathbf{x}|| = 0$, or

(3) *time-like* if $||\mathbf{x}||$ is imaginary.

If a vector $\mathbf{x} \in \mathbb{R}^{n+1}$ is light-like or time-like, $\mathbf{x}$ is said to be *positive* (resp. *negative*) if and only if $x_{n+1} > 0$ (resp. $x_{n+1} < 0$).

Definition 2.2. A vector subspace V of $\mathbb{R}^{n+1}$ is called

(1) *space-like* if and only if every nonzero vector in V is space-like,

(2) *time-like* if and only if V has a time-like vector, or

(3) *light-like* otherwise.

Two vector $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+1}$ is called *Lorentz orthogonal* if and only if $(\mathbf{x}, \mathbf{y}) = 0$. Then if both $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors, the one of $\mathbf{x}$ or $\mathbf{y}$ is time-like and the other is space-like. In addition, Lorentz orthogonal two vectors $\mathbf{x}$ and $\mathbf{y}$ called *Lorentz orthonormal* if and only if $||x||^2 = \pm 1$ and $||y|| = \mp 1$.

We define the hyperbolic n-space as the hyperboloid model. Let $\mathcal{F}^n \subset \mathbb{R}^{n+1}$ be the sphere of unit imaginary radius, i.e.

$$\mathcal{F}^n := \{ \ \mathbf{x} \in \mathbb{R}^{n+1} \mid ||\mathbf{x}|| = -1 \ \}.$$

Then $\mathcal{F}^n$ is disconnected. The subset of all $\mathbf{x} \in \mathcal{F}^n$ such that $x_{n+1} > 0$ (resp. $x_{n+1} < 0$) is called the *positive* (resp. *negative*) sheet of $\mathcal{F}^n$. The *hyperboloid model* of hyperbolic n-space is defined as the positive sheet of $\mathcal{F}^n$. We denote it by $\mathcal{F}^n_+$. Then, for two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{F}^n_+$, the hyperbolic distance between $\mathbf{x}$ and $\mathbf{y}$ is written as follows.

$$\cosh d_{\mathcal{F}^n_+}(\mathbf{x}, \mathbf{y}) = -(\mathbf{x}, \mathbf{y}),$$

where $(,)$ is the Lorentzian inner product. This hyperbolic distance function defines a hyperbolic metric on $\mathcal{F}^n_+$. Then the hyperbolic geodesic and the hyperbolic m-plane in $\mathcal{F}^n_+$ are defined as follows.

Definition 2.3. A hyperbolic geodesic of $\mathcal{F}^n_+$ is the intersection of $\mathcal{F}^n_+$ with a 2-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$.

Definition 2.4. A hyperbolic m-plane of $\mathcal{F}^n_+$ is the intersection of $\mathcal{F}^n_+$ with a $(m+1)$-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$.

Let V be the $(m+1)$-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$. Then the intersection $\mathcal{F}^n_+ \cap V$ is a hyperbolic m-plane. Then any $\mathbf{x} \in \mathcal{F}^n_+$ there exist orthonormal vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ such that

$$\mathbf{x} = \cosh(t)\mathbf{x}_1 + \sinh(t)\mathbf{x}_2,$$

where $\mathbf{x}_1 \in \mathcal{F}^n_+ \cap V$, $||\mathbf{x}_2|| = 1$ and t is the hyperbolic distance from $\mathbf{x}$ to $\mathbf{x}_1$.

2.2 The conformal ball model

Next, we define the conformal ball model of the hyperbolic n-space. Let B^n be the open unit ball in $\mathbb{R}^n$.

$$B^n := \{\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n \mid |\mathbf{x}| = x_1^2 + \cdots + x_n^2 < 1\}.$$

Let P_1 be the stereographic projection of B^n onto $\mathcal{F}_+^n$ defined by

$$P_1(\mathbf{x}) = \mathbf{x} + \frac{1 + |\mathbf{x}|^2}{1 - |\mathbf{x}|^2}(\mathbf{x} + \mathbf{e}_{n+1}),$$

where $|\ |$ is the Euclidean norm of $\mathbf{x}$. The metric d_{B^n} on B^n is defined by

$$d_{B^n}(\mathbf{x}, \mathbf{y}) = d_{\mathcal{F}_+^n}(P_1(\mathbf{x}), P_1(\mathbf{y})).$$

Then P_1 is an isometry from B^n to $\mathcal{F}_+^n$ with this metric d_{B^n}. The hyperbolic line element and the hyperbolic volume element of B^n associated to d_{B^n} are given as

$$\frac{2|d\mathbf{x}|}{1 - |\mathbf{x}|^2} \qquad \text{and} \qquad \frac{2^n dx_1 \cdots dx_n}{(1 - |\mathbf{x}|^2)^n}.$$

Then the hyperbolic Laplace-Beltrami operator associated with the hyperbolic line element is given by

$$\Delta = \frac{1}{4}(1 - |\mathbf{x}|^2)^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{n-2}{2}(1 - |\mathbf{x}|^2) \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

2.3 The upper-half space model

We introduce another model of hyperbolic n-space, namely upper-half space model. Let U^n be the upper-half space of $\mathbb{R}^n$ i.e.

$$U^n = \{\ \mathbf{x} \in \mathbb{R}^n \mid x_n > 0\ \}.$$

The isometry from B^n to U^n is given by the stereographic projection P_2 of $\mathbb{R}^n$. The metric d_{U^n} on U^n is defined

$$d_{U^n}(\mathbf{x}, \mathbf{y}) = d_{B^n}(P_2^{-1}(\mathbf{x}), P_2^{-1}(\mathbf{y})).$$

The hyperbolic line element and the hyperbolic volume element of U^n associated to d_{U^n} are given as

$$\frac{|d\mathbf{x}|}{x_n} \quad \text{and} \quad \frac{dx_1 \cdots dx_n}{{x_n}^n}.$$

Then the hyperbolic Laplace-Beltrami operator associated with the hyperbolic line element is given by

$$\Delta = x_n^2 \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) - (n-2) x_n \frac{\partial}{\partial x_n}.$$

2.4 Orthogonal group $O(n, 1)$

A real $(n+1) \times (n+1)$ matrix A is said to be Lorentzian if and only if the corresponding linear transformation $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is Lorentzian. A Lorentzian matrix A is said to be positive (resp. negative) if and only if A transforms positive time-like vectors into positive (resp. negative) time-like vectors. The set of all Lorentzian matrices forms a group with the ordinary matrix multiplication. We let

$$G = O(n,1) := \left\{ g \in GL(n+1, \mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} 1_n & \\ & -1 \end{pmatrix} g = \begin{pmatrix} 1_n & \\ & -1 \end{pmatrix} \right\}$$

be the orthogonal group of signature $(n, 1)$. Here 1_n denotes the $n \times n$ unit matrix. Then any element of G is a Lorentzian matrix and G is naturally isomorphic to the group of all Lorentz transformations of $\mathbb{R}^{n+1}$. Immediately, G acts $\mathcal{F}^n$ transitively and preserves the Lorentz inner product so that we can naturally identify G with the isometry group of $\mathcal{F}^n$. Let $PO(n,1)$ be the set of all positive matrices in G. Then $PO(n, 1)$ acts on $\mathcal{F}^n_+$ transitively.

Let K be the stabilizer of $\mathbf{e}_{n+1}$ in G. Then K is a maximal compact subgroup of G. By definition, the determinant of $g \in G$ is equal to $+1$ or -1. We denote the connected component of G (resp. K) containing the unit element by G_0 (resp. K_0). Then G_0 acts on $\mathcal{F}^n_+$ transitively and naturally identifies the orientation preserving isometries on $\mathcal{F}^n_+$. K_0 is the stabilizer of $\mathbf{e}_{n+1}$ in G_0 and a maximal compact subgroup of G_0. Then the quotient space G_0/K_0 is naturally identified with $\mathcal{F}^n_+$.

2.5 Eisenstein series associated to cusps

Let $\Gamma \subset G_0$ be a cofinite discrete subgroup of G_0 and $\zeta \in \mathbb{R}^{n-1} \cup \{\infty\}$ be a cusp. We define the stabilizer-group of ζ by

$$\Gamma_\zeta := \{M \in \Gamma \mid M\zeta = \zeta\}$$

Choose $A \in G_0$ such that $A\zeta = \infty$. For any $\mathbf{x} \in U^n$, we write its coordinates

$$\mathbf{x} = (x_1, ..., x_n).$$

Then, for any $\mathbf{x} \in U^n$ and $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$, the Eisenstein series associated to ζ is defined as

$$E_\zeta(\mathbf{x}, s) := \sum_{M \in \Gamma_\zeta \backslash \Gamma} x_n(AM\mathbf{x})^s.$$

The Eisenstein series $E_\zeta(\mathbf{x}, s)$ converges absolutely and locally uniformly for any $\mathbf{x} \in U^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n - 1$ and it defines a Γ-invariant function where it converges.

We set $B_1, \dots, B_h \in G_0$ so that

$$\eta_1 = B_1^{-1}\infty \ , \dots , \ \eta_h = B_h^{-1}\infty \in \mathbb{R}^{n-1} \cup \infty$$

are the representatives for Γ-classes of cusps of Γ. For $\nu = 1, ..., h$ and $\mathbf{x} \in U^n$, the Eisenstein series associated to η_ν is defined

$$E_\nu(\mathbf{x}, s) = \sum_{M \in \Gamma_{\eta_\nu} \backslash \Gamma} x_n(B_\nu M\mathbf{x})^s.$$

Then $E_\nu(\mathbf{x}, s)$ has the Fourier expansions at the cusp of the form

$$E_\nu(B_\nu^{-1}\mathbf{x}, s) = x_n^s + \phi_{\nu\nu}(s)x_n^{n-1-s} + \cdots$$

for $\nu = 1, ..., h$ and the case $\nu \neq \mu$

$$E_\nu(B_\mu^{-1}\mathbf{x}, s) = \phi_{\nu\mu}(s)x_n^{n-1-s} + \cdots ,$$

where the functions $\phi_{\nu\mu}(s)$ are described as certain Dirichlet series. Using the above notation, we define

$$\mathcal{E}(\mathbf{x}, s) := \begin{pmatrix} E_1(\mathbf{x}, s) \\ \vdots \\ E_h(\mathbf{x}, s) \end{pmatrix},$$

$$\Phi(s) := (\phi_{\nu\mu}(s)),$$

where ν is the row index and μ the column index. The matrix $\Phi(s)$ is called the scattering matrix for $\mathcal{E}(\mathbf{x}, s)$. Then the following facts are known about $\mathcal{E}(\mathbf{x}, s)$ and $\Phi(s)$. Both $\Phi(s)$ and $\mathcal{E}(\mathbf{x}, s)$ have meromorphic continuations to all of $\mathbb{C}$ in the following sense. There is a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ with $g \neq 0$ such that for every $\nu = 1, ..., h$ the product $g(s)E_\nu(\mathbf{x}, s)$ can be continued to a function on $U^n \times \mathbb{C}$ which is real analytic in $\mathbf{x}$ and holomorphic in $s \in \mathbb{C}$. Then the continued functions satisfy the following functional equation

$$\mathcal{E}(\mathbf{x}, n-1-s) = \Phi(n-1-s)\mathcal{E}(\mathbf{x}, s)$$

and

$$\Phi(s)\Phi(n-1-s) = 1_h,$$

where 1_h is the $h \times h$ unit matrix. Furthermore, the continued components $E_\nu(\mathbf{x}, s)$ of $\mathcal{E}(\mathbf{x}, s)$ satisfy the following differential equation

$$(-\Delta - s(n-1-s))E_\nu(\mathbf{x}, s) = 0,$$

if s is not a pole of $E_\nu(\mathbf{x}, s)$.

The Eisenstein series $E_\nu(\mathbf{x}, s)$ and the entries of scattering matrix $\Phi(s)$ have no poles in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \frac{n-1}{2}\}$ except possibly finitely many points in the semi-open interval $(\frac{n-1}{2}, n-1]$ on the real line. These poles are simple. Furthermore, $E_\nu(\mathbf{x}, s)$ has a simple pole at $s = n-1$. Both the entries of $\mathcal{E}(\cdot, s)$ and of $\Phi(s)$ has no poles for $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \frac{n-1}{2}$ and $\Phi(s)$ is a unitary matrix on this line.

2.6 Domain of Laplace-Beltrami operator

Let $\Gamma \subset G_0$ be a cofinite subgroup of G_0. We denote by $L^2(\Gamma\backslash U^n)$ the set of all Γ-invariant measurable functions $f : U^n \to \mathbb{C}$ which satisfy

$$\int_{\mathcal{F}_\Gamma} |f|^2 \, dv < \infty,$$

where $\mathcal{F}_\Gamma$ denotes a fundamental domain of Γ. For $f, g \in L^2(\Gamma\backslash U^n)$, the function $f\bar{g}$ is Γ-invariant. Hence the definition

$$\langle f, g\rangle := \int_{\mathcal{F}_\Gamma} f\bar{g} \, dv \tag{10}$$

makes sense and $\langle \cdot, \cdot \rangle$ is an inner product on $L^2(\Gamma\backslash U^n)$. The space $L^2(\Gamma\backslash U^n)$ is a Hilbert space through the inner product $\langle \cdot, \cdot \rangle$. For any $f \in L^2(\Gamma\backslash U^n)$, we have the following lemma.

Proposition 2.5. Every $f \in L^2(\Gamma\backslash U^n)$ has the following spectral expansion associated to $-\Delta$

$$\begin{aligned} f(\mathbf{x}) &= \sum_{m\in\mathfrak{D}} \langle f, e_m \rangle e_m(\mathbf{x}) \\ &+ \frac{1}{4\pi} \sum_{\nu\,:\,\text{cusps}} \int_{-\infty}^{\infty} \left\langle f, E_\nu \left(\cdot, \frac{n-1}{2} + it \right) \right\rangle \cdot E_\nu \left(\mathbf{x}, \frac{n-1}{2} + it \right) dt, \end{aligned} \quad (11)$$

where $\mathfrak{D} \subset \mathbb{N}$ is an index set for a complete orthonormal set of eigenfunctions $(e_n)_{n\in\mathfrak{D}}$ for $-\Delta$ in $L^2(\Gamma\backslash U^n)$ and $\langle f, E_\nu(\cdot, \frac{n-1}{2} + it) \rangle$ is defined by $\int_{\mathcal{F}_\Gamma} f(\mathbf{y}) \overline{E_\nu(\mathbf{y}, \frac{n-1}{2} + it)} dv(\mathbf{y})$. The series of the right hand side of (11) converges in the norm of the $L^2(\Gamma\backslash U^n)$.

Besides, if $f \in C^{l_0}(\Gamma\backslash U^n) \cap L^2(\Gamma\backslash U^n)$ for a positive integer $l_0 > 0$ such that $l_0 > \frac{n}{2}$ and $-\Delta^l f \in L^2(\Gamma\backslash U^n)$ for any $0 \leq l \leq \lfloor \frac{n+1}{4} \rfloor + 1$, the spectral expansion (11) of f converges uniformly and locally uniformly on $\Gamma\backslash U^n$. Especially, if f and $-\Delta^l f$ are smooth and bounded on $\Gamma\backslash U^n$ for any $0 \leq l \leq \lfloor \frac{n+1}{4} \rfloor + 1$, the spectral expansion (11) of f converges uniformly and locally uniformly on $\Gamma\backslash U^n$.

Proof. See [9] p. 103 in Chapter 7, [1] p. 268 in Chapter 6 or [17]. □

3 Hyperbolic Eisenstein series

Let V be the space-like vector subgroup of $\dim V = k$ and $V^\perp$ be the orthogonal complement space of V. The dimension of $V^\perp$ is $n-k+1$. Then $\mathcal{F}^n_+ \cap V^\perp$ is the hyperbolic $(n-k)$-plane.

Let $\sigma = \sigma_V \in O(n+1)$ be the involution such that

$$\sigma = \begin{cases} -1 & \text{on} \quad V \\ \ \ 1 & \text{on} \quad V^\perp. \end{cases}$$

Then $\mathcal{F}^n_+ \cap V^\perp$ is the fixed point set of σ in $\mathcal{F}^n_+$. Let G_σ be the centralizer of σ in G i.e.

$$G_\sigma = \{ \ g \in G \mid \sigma g \sigma = g \ \}.$$

Let $\Gamma \subset G$ be a cofinite discrete subgroup of G i.e. the quotient $\Gamma\backslash\mathcal{F}^n_+$ has finite volume and Γ_σ be the intersection of Γ with G_σ. We assume the following assumption.

$$\sigma\Gamma\sigma = \Gamma. \tag{12}$$

In addition, we assume the quotient $\Gamma\backslash(\mathcal{F}^n_+ \cap V^\perp)$ is compact.

Without loss of generality, we may assume the vector subspace V and $V^\perp$ in $\mathbb{R}^{n+1}$ as follows.

$$\begin{aligned} V &= \{ \ \mathbf{x} \in \mathbb{R}^{n+1} \mid x_i = 0, \quad k+1 \le i \le n+1 \ \} \\ V^\perp &= \{ \ \mathbf{x} \in \mathbb{R}^{n+1} \mid x_i = 0, \quad 1 \le i \le k \ \}. \end{aligned}$$

Then the intersection $\mathcal{F}^n_+ \cap V^\perp$ is identified with

$$D_\sigma = \{ \ \mathbf{x} \in U^n \mid \mathbf{x} = (0, \cdots, 0, x_{k+1}, \ldots, x_n), \ x_n > 0 \ \}.$$

We introduce the partial polar coordinate on U^n. It is defined as follows.

If $2 \leq k \leq n-1$,

$$\begin{cases} x_1 = e^{\rho} \cos \varphi_0 \sin \varphi_1, \\ \vdots \\ x_i = e^{\rho} \cos \varphi_0 \cdots \cos \varphi_{i-1} \sin \varphi_i, \ \ 2 \leq i \leq k-1, \\ \vdots \\ x_k = e^{\rho} \cos \varphi_0 \cdots \cos \varphi_{k-2} \cos \varphi_{k-1}, \\ x_{k+1} = x_{k+1}, \\ \vdots \\ x_{n-1} = x_{n-1}, \ \text{ and} \\ x_n = e^{\rho} \sin \varphi_0, \end{cases} \tag{13}$$

where

$$\left(\begin{array}{r} \rho = \log \sqrt{x_1^2 + \cdots + x_k^2 + x_n^2}, \\ 0 < \varphi_0 < \dfrac{\pi}{2}, \\ -\dfrac{\pi}{2} < \varphi_i < \dfrac{\pi}{2}, \ 1 \leq i \leq k-2, \ \text{and} \\ 0 \leq \varphi_{k-1} < 2\pi. \end{array} \right.$$

If $k = 1$,

$$\begin{cases} x_1 = e^{\rho} \cos \varphi_0, \\ x_2 = x_2, \\ \vdots \\ x_{n-1} = x_{n-1}, \ \text{ and} \\ x_n = e^{\rho} \sin \varphi_0, \end{cases} \tag{14}$$

where

$$\left(\begin{array}{l} \rho = \log \sqrt{x_1^2 + x_n^2}, \\ 0 < \varphi_0 < \pi. \end{array} \right.$$

Under this coordinates, the hyperbolic line element $d\sigma$ and the hyperbolic volume element dv are given by

$$d\sigma^2 = \frac{1}{(\sin\varphi_0)^2}\left\{d\rho^2 + d\varphi_0^2 + \sum_{i=1}^{k-1}\left(\prod_{j=0}^{i-1}(\cos\varphi_j)^2\right)d\varphi_i^2 + \sum_{i=k+1}^{n-1}\frac{dx_i^2}{e^{2\rho}}\right\},$$

$$dv = \frac{1}{e^{(n-k)\rho}(\sin\varphi_0)^n}\cdot\prod_{i=0}^{k-2}(\cos\varphi_i)^{k-1-i}\cdot d\rho d\varphi_0\cdots d\varphi_{k-1}dx_{k+1}\cdots dx_{n-1}.$$

We define restricted volume element dv_1 and dv_2 as follows.

$$dv_1 = \left(\prod_{i=1}^{k-2}(\cos\varphi_i)^{k-1-i}\right)\cdot d\varphi_1\cdots d\varphi_{k-1},$$

$$dv_2 = \frac{1}{e^{(n-k)\rho}}\cdot d\rho dx_{k+1}\cdots dx_{n-1}.$$

Then

$$dv = \frac{(\cos\varphi_0)^{k-1}}{(\sin\varphi_0)^n}d\varphi_0 dv_1 dv_2.$$

Furthermore, the Laplace-Beltrami operator is written by

$$\Delta = (\sin\varphi_0)^2\left\{\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\varphi_0^2} + \sum_{i=1}^{k-1}\frac{1}{\prod_{j=0}^{i-1}(\cos\varphi_j)^2}\cdot\frac{\partial^2}{\partial\varphi_i^2} + \sum_{i=k+1}^{n-1}e^{2\rho}\frac{\partial^2}{\partial x_i^2}\right)\right.$$
$$-\left((n-k-1)\frac{\partial}{\partial\rho} + ((k-1)\tan\varphi_0 + (n-2)\cot\varphi_0)\frac{\partial}{\partial\varphi_0}\right.$$
$$\left.\left.+\sum_{i=1}^{k-1}(k-i-1)\tan\varphi_i\frac{1}{\prod_{j=0}^{i-1}\cos\varphi_j}\frac{\partial}{\partial\varphi_i}\right)\right\}.$$

Under above coordinates, we define the generalized hyperbolic Eisenstein series associated to σ as follows.

Definition 3.1. Let $\mathbf{x} \in U^n$ and $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$. Then the hyperbolic Eisenstein series associated to the involution σ is defined as follows.

$$E_\sigma(\mathbf{x}, s) := \sum_{\eta\in\Gamma_\sigma\backslash\Gamma}(\sin\varphi_0(\eta\mathbf{x}))^s. \tag{15}$$

Let $d_{\text{hyp}}(\mathbf{x}, D_\sigma)$ be the hyperbolic distance from $\mathbf{x}$ to D_σ. Then we have

$$\sin\varphi_0(\mathbf{x}) \cdot \cosh(d_{\text{hyp}}(\mathbf{x}, D_\sigma)) = 1$$

for any $\mathbf{x} \in U^n$. Using this formula, we can write the Eisenstein series associated to σ as

$$E_\sigma(\mathbf{x}, s) = \sum_{\eta \in \Gamma_\sigma \backslash \Gamma} \cosh(d_{\text{hyp}}(\eta\mathbf{x}, D_\sigma))^{-s}. \tag{16}$$

Proposition 3.2. The Eisenstein series associated to σ converges absolutely and locally uniformly for any $\mathbf{x} \in U^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ and satisfies the following differential shift equation

$$(-\Delta + s(s-n+1))E_\sigma(\mathbf{x}, s) = s(s-n+k+1)E_\sigma(\mathbf{x}, s+2). \tag{17}$$

Definition 3.3. Let $T > 0$ be a positive real number. Then we define the counting function associated to σ as follows.

$$N_\sigma(T; \mathbf{x}, D_\sigma) := \sharp\{\eta \in \Gamma_\sigma \backslash \Gamma \mid d_{\text{hyp}}(\eta\mathbf{x}, D_\sigma) < T\}, \tag{18}$$

where $\sharp$ is the cardinality of the set.

By using the counting function defined above, we can write the hyperbolic Eisenstein series associated to σ as the Stieltjes integrals, namely

$$E_\sigma(\mathbf{x}, s) = \int_0^\infty \cosh(u)^{-s} dN_\sigma(u; \mathbf{x}, D_\sigma). \tag{19}$$

Lemma 3.4. Let $r > 0$ be the injective radius at $\mathbf{x}$, i.e. for any $\gamma_1, \gamma_2 \in \Gamma$, $B_{\text{hyp}}(\gamma_1\mathbf{x}, r) \cap B_{\text{hyp}}(\gamma_2\mathbf{x}, r) = \emptyset$. For positive real numbers u, T_0 and r such that $u > T_0 > r$, we have the following inequality of the counting function

$$\begin{aligned} N_\sigma(u; \mathbf{x}, D_\sigma) \leq\ & N_\sigma(T_0; \mathbf{x}, D_\sigma) \\ & + \frac{1}{n-1} \cdot \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\text{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}_{\text{hyp}}(B_{\text{hyp}}(\mathbf{x}, r))} \\ & \times \{(\cosh(u+r))^{n-1} - (\cosh(T_0 - r))^{n-1}\}, \end{aligned}$$

where S^{k-1} is the unit $(k-1)$-dimensional sphere.

Proof. We define the subset $V(T) \subset \Gamma_\sigma \backslash U^n$ by

$$V(T) := \{\mathbf{x} \in \Gamma_\sigma \backslash U^n \mid d_{\text{hyp}}(\mathbf{x}, \Gamma_\sigma \backslash D_\sigma) < T\}.$$

The hyperbolic distance from any $\mathbf{x} \in U^n$ to $\Gamma_\sigma \backslash D_\sigma$ is given by $\sin\varphi_0(\mathbf{x})$. Hence the hyperbolic volume of $V(T)$ is given by following integral

$$\int_{\Gamma_\sigma \backslash D_\sigma} \int_{\varphi_1 = -\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{\varphi_{k-2} = -\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\varphi_{k-1}=0}^{2\pi} \int_{\varphi_0 = \varphi_0(\mathbf{x})}^{\frac{\pi}{2}} \frac{(\cos\varphi_0)^{k-1}}{(\sin\varphi_0)^n} d\varphi_0 dv_1 dv_2.$$

Since the restricted volume element dv_2 is the hyperbolic volume element on hyperbolic $n-k$-plane D_σ, we have

$$\int_{\Gamma_\sigma \backslash D_\sigma} dv_2 = \text{vol}_{\text{hyp}}(\Gamma_\sigma \backslash D_\sigma).$$

In addition,

$$\begin{aligned}
&\int_{\varphi_1 = -\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{\varphi_{k-2} = -\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\varphi_{k-1}=0}^{2\pi} dv_1 \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} (\cos\varphi_1)^{k-2} (\cos\varphi_2)^{k-3} \cdots \cos\varphi_{k-2} d\varphi_1 \cdots d\varphi_{k-2} d\varphi_{k-1} \\
&= \text{vol}(S^{k-1}),
\end{aligned}$$

where $\text{vol}(S^{k-1})$ is the Euclidean volume of the $(k-1)$-dimensional unit sphere. It is given explicitly as follows.

$$\begin{cases}
2^{k-2} \cdot \left(\frac{1}{k-2} \cdot \frac{1}{k-4} \cdots \frac{1}{2}\right) \cdot \left(\frac{\pi}{2}\right)^{\frac{k-2}{2}} & k : \text{even}, \\
2^{k-2} \cdot \left(\frac{1}{k-2} \cdot \frac{1}{k-4} \cdots \frac{1}{3}\right) \cdot \left(\frac{\pi}{2}\right)^{\frac{k-3}{2}} & k : \text{odd}.
\end{cases}$$

Therefore, the hyperbolic volume of $V(T)$ is given by

$$\text{vol}_{\text{hyp}}(V(T)) = \text{vol}(S^{k-1}) \cdot \text{vol}_{\text{hyp}}(\Gamma_\sigma \backslash D_\sigma) \cdot \int_{\varphi_0(T)}^{\frac{\pi}{2}} \frac{(\cos\varphi_0)^{k-1}}{(\sin\varphi_0)^n} d\varphi_0,$$

where $\sin\varphi_0(T)\cosh(T) = 1$. Let $r > 0$ be the injective radius at $\mathbf{x}$ and $B_{\text{hyp}}(\mathbf{x}, r) := \{\mathbf{y} \in U^n \mid d_{\text{hyp}}(\mathbf{x}, \mathbf{y}) < r\}$ be the hyperbolic ball with center

$\mathbf{x}$ and hyperbolic radius r in U^n. For positive real numbers u and T_0 such that $u > T_0 > r$, we define $\{\eta_k\}_k \subset \Gamma_\sigma \backslash \Gamma$ as the maximal set such that $\eta\mathbf{x} \in \mathrm{V}(u) \setminus \mathrm{V}(T_0)$. Then

$$\bigcup_k B_{\mathrm{hyp}}(\eta_k \mathbf{x}, r) \subset V(u+r) \setminus V(T_0 - r).$$

From this inclusion of the set, the following inequality holds

$$\begin{aligned}
\mathrm{vol}_{\mathrm{hyp}}\left(\bigcup_k B_{\mathrm{hyp}}(\eta_k \mathbf{x}, r)\right) &= \mathrm{vol}_{\mathrm{hyp}}\left(\sum_k B_{\mathrm{hyp}}(\eta_k \mathbf{x}, r)\right) \\
&= \sum_k \mathrm{vol}_{\mathrm{hyp}}(B_{\mathrm{hyp}}(\eta_k \mathbf{x}, r)) \\
&= \sum_k \mathrm{vol}_{\mathrm{hyp}}(B_{\mathrm{hyp}}(\mathbf{x}, r)) \\
&\leq \mathrm{vol}_{\mathrm{hyp}}\left(V(u+r) \setminus V(T_0 - r)\right) \\
&= \mathrm{vol}_{\mathrm{hyp}}(V(u)) - \mathrm{vol}_{\mathrm{hyp}}(V(T_0)). \qquad (20)
\end{aligned}$$

For $u > T_0 > 0$, we have

$$\begin{aligned}
&\int_{\varphi(u)}^{\frac{\pi}{2}} \frac{(\cos\varphi_0)^{k-1}}{(\sin\varphi_0)^n} d\varphi_0 - \int_{\varphi(T_0)}^{\frac{\pi}{2}} \frac{(\cos\varphi_0)^{k-1}}{(\sin\varphi_0)^n} d\varphi_0 \\
&= \int_{\varphi(u)}^{\varphi(T_0)} \frac{(\cos\varphi_0)^{k-1}}{(\sin\varphi_0)^n} d\varphi_0 \\
&\leq \int_{\varphi(u)}^{\varphi(T_0)} \frac{\cos\varphi_0}{(\sin\varphi_0)^n} d\varphi_0 \\
&= \frac{1}{n-1}\left(\frac{1}{(\sin\varphi(u))^{n-1}} - \frac{1}{(\sin\varphi(T_0))^{n-1}}\right) \\
&= \frac{1}{n-1}\left((\cosh(u))^{n-1} - (\cosh(T_0))^{n-1}\right).
\end{aligned}$$

Hence we have the following inequality

$$\begin{aligned}
&\mathrm{vol}_{\mathrm{hyp}}(V(u)) - \mathrm{vol}_{\mathrm{hyp}}(V(T_0)) \\
&\quad \leq \mathrm{vol}(S^{k-1}) \cdot \mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma) \cdot \frac{1}{n-1} \cdot \left\{(\cosh u)^{n-1} - (\cosh T_0)^{n-1}\right\}. \qquad (21)
\end{aligned}$$

From (20) and (21), we have

$$
\begin{aligned}
\sharp\{\eta \in \Gamma_\sigma\backslash\Gamma \mid T_0 \le d_{\text{hyp}}(\eta\mathbf{x}, D_\sigma) < u\ \} = k = \frac{\sum_k \text{vol}_{\text{hyp}}\left(B_{\text{hyp}}(\mathbf{x}, r)\right)}{\text{vol}\left(B_{\text{hyp}}(\mathbf{x}, r)\right)} \\
\le \text{vol}(S^{k-1}) \cdot \frac{\text{vol}_{\text{hyp}}(\Gamma_\sigma\backslash D_\sigma)}{\text{vol}(B_{\text{hyp}}(\mathbf{x}, r))} \\
\times \frac{1}{n-1} \cdot \left((\cosh(u+r))^{n-1} - (\cosh(T_0 - r))^{n-1}\right). \qquad (22)
\end{aligned}
$$

By the definition of the set, we also have

$$
\begin{aligned}
&\sharp\{\eta \in \Gamma_\sigma\backslash\Gamma \mid \eta\mathbf{x} \in V(u) \setminus V(T_0)\} \\
&= \sharp\{\eta \in \Gamma_\sigma\backslash\Gamma \mid \eta\mathbf{x} \in V(u)\} - \sharp\{\eta \in \Gamma_\sigma\backslash\Gamma \mid \eta\mathbf{x} \in V(T_0)\}. \qquad (23)
\end{aligned}
$$

Therefore, from (22) and (23), we have the following inequality

$$
\begin{aligned}
&\sharp\{\eta \in \Gamma_\sigma\backslash\Gamma \mid d_{\text{hyp}}(\eta\mathbf{x}, D_\sigma) < u\ \} \\
&\le \sharp\{\eta \in \Gamma_\sigma\backslash\Gamma \mid d_{\text{hyp}}(\eta\mathbf{x}, D_\sigma) < T_0\ \} + \text{vol}(S^{k-1}) \cdot \frac{\text{vol}_{\text{hyp}}(\Gamma_\sigma\backslash D_\sigma)}{\text{vol}(B_{\text{hyp}}(\mathbf{x}, r))} \cdot \\
&\qquad \times \frac{1}{n-1} \cdot \left((\cosh(u+r))^{n-1} - (\cosh(T_0 - r))^{n-1}\right).
\end{aligned}
$$

From above, the assertion of the lemma holds. □

The following inequality plays an important role to analyze the hyperbolic Eisenstein series.

Lemma 3.5. Let F be a real-valued, smooth, decreasing function defined for $u > 0$ and let g_1, g_2 be real-valued, non decreasing functions defined for $u \ge a > 0$ and satisfying $g_1(u) \le g_2(u)$ for $u \ge a$. Then, the following inequality of Stieltjes integrals holds when both integrals exist.

$$
F(a)g_1(a) + \int_a^\infty F(u)dg_1(u) \le F(a)g_2(a) + \int_a^\infty F(u)dg_2(u). \qquad (24)
$$

Proof. See [3] Section 2.6 or [4] Section 2.7. □

Lemma 3.6. Let $s \in \mathbb{C}$ be a complex number with $\text{Re}(s) > n-1$. Then, for any $\varepsilon > 0$, there exists a sufficient large $T_0 > 0$ such that

$$
\left| \int_{T_0}^\infty (\cosh u)^{-s} dN_\sigma(u; \mathbf{x}, D_\sigma) \right| < \varepsilon. \qquad (25)
$$

Proof. We define real valued functions $F(u)$, $g_1(u)$ and $g_2(u)$ as follows.

$$\begin{aligned}
F(u) &:= (\cosh u)^{-\mathrm{Re}(s)}, \\
g_1(u) &:= N_\sigma(u; \mathbf{x}, D_\sigma), \\
g_2(u) &:= N_\sigma(T_0; \mathbf{x}, D_\sigma) + \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \\
&\qquad \times \frac{1}{n-1} \cdot \left((\cosh(u+r))^{n-1} - (\cosh(T_0 - r))^{n-1} \right).
\end{aligned}$$

Then both g_1 and g_2 are non-decreasing and $g_1(u) \leq g_2(u)$ for $u \geq T_0 > 0$. Elementary calculations imply that

$$dg_2(u) = \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \cdot \left((\cosh(u+r))^{n-2} \sinh(u+r) \right) du.$$

By using Lemma 3.5, we have

$$\begin{aligned}
&\left| \int_{T_0}^{\infty} (\cosh u)^{-s} dN_\sigma(u; \mathbf{x}, D_\sigma) \right| \\
&\quad \leq \int_{T_0}^{\infty} (\cosh u)^{-\mathrm{Re}(s)} dN_\sigma(u; \mathbf{x}, D_\sigma) \\
&\quad \leq \int_{T_0}^{\infty} (\cosh u)^{-\mathrm{Re}(s)} dg_2 + (\cosh T_0)^{-\mathrm{Re}(s)} \left\{ g_2(T_0) - g_1(T_0) \right\}. \qquad (26)
\end{aligned}$$

The second term of (26) has the following estimate

$$\begin{aligned}
&(\cosh T_0)^{-\mathrm{Re}(s)} \left\{ g_2(T_0) - g_1(T_0) \right\} \\
&\quad = (\cosh T_0)^{-\mathrm{Re}(s)} \cdot \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \\
&\qquad\qquad \times \left\{ (\cosh(T_0 + r))^{n-1} - (\cosh(T_0 - r))^{n-1} \right\} \\
&\quad = \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \\
&\qquad\qquad \times (\cosh T_0)^{-\mathrm{Re}(s)} \cdot \left\{ (\cosh(T_0 + r))^{n-1} - (\cosh(T_0 - r))^{n-1} \right\} \\
&\quad \leq \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \cdot \left(\frac{e^{T_0}}{2} \right)^{-\mathrm{Re}(s)} \cdot \left(e^{(T_0 + r)} \right)^{n-1} \\
&\quad = \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \cdot 2^{\mathrm{Re}(s)} \cdot e^{(n-1-\mathrm{Re}(s))T_0 + (n-1)r}. \qquad (27)
\end{aligned}$$

We calculate the first integral as

$$
\begin{aligned}
&\int_{T_0}^{\infty} (\cosh u)^{-\mathrm{Re}(s)} dg_2 \\
&= \int_{T_0}^{\infty} (\cosh u)^{-\mathrm{Re}(s)} \cdot \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \\
&\qquad\qquad\qquad\qquad \times \big((\cosh(u+r))^{n-2} \sinh(u+r)\big)\, du \\
&= \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \\
&\qquad\qquad \times \int_{T_0}^{\infty} (\cosh u)^{-\mathrm{Re}(s)} \cdot (\cosh(u+r))^{n-2} \sinh((u+r)) du \\
&\leq \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \cdot \int_{T_0}^{\infty} \left(\frac{e^u}{2}\right)^{-\mathrm{Re}(s)} \cdot \left(e^{u+r}\right)^{n-2} \cdot \left(\frac{e^{u+r}}{2}\right) du \\
&\leq \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \cdot 2^{\mathrm{Re}(s)-1} \int_{T_0}^{\infty} e^{(n-1-\mathrm{Re}(s))u + (n-1)r} du. \qquad (28)
\end{aligned}
$$

The integral in (28) converges for $\mathrm{Re}(s) > n-1$ and we have

$$
(28) = -2^{\mathrm{Re}(s)-1} \cdot \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \cdot \frac{e^{(n-1-\mathrm{Re}(s))T_0 + (n-1)r}}{n-1-\mathrm{Re}(s)}. \qquad (29)
$$

Summing up (27) and (29), we have

$$
\begin{aligned}
&\left| \int_{T_0}^{\infty} (\cosh u)^{-s} dN_\sigma(u; \mathbf{x}, D_\sigma) \right| \\
&\quad \leq -2^{\mathrm{Re}(s)-1} \cdot \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \cdot \frac{e^{(n-1-\mathrm{Re}(s))T_0 + (n-1)r}}{n-1-\mathrm{Re}(s)} \\
&\qquad\qquad + 2^{\mathrm{Re}(s)} \cdot \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \cdot e^{(n-1-\mathrm{Re}(s))T_0 + (n-1)r} \\
&\quad = 2^{\mathrm{Re}(s)-1} \cdot \mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \\
&\qquad\qquad \times e^{(n-1-\mathrm{Re}(s))T_0 + (n-1)r} \cdot \left(2 - \frac{1}{n-1-\mathrm{Re}(s)}\right). \qquad (30)
\end{aligned}
$$

Therefore, for any $\varepsilon > 0$, by choosing

$$T_0 > \frac{1}{\mathrm{Re}(s) - n + 1}\left\{ -\log\varepsilon + \log\left(\mathrm{vol}(S^{k-1}) \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_\sigma \backslash D_\sigma)}{\mathrm{vol}(B_{\mathrm{hyp}}(\mathbf{x}, r))} \right.\right.$$
$$\left.\left. \times\, 2^{\mathrm{Re}(s)-1} e^{(n-1)r} \left(2 + \frac{1}{\mathrm{Re}(s) - n + 1} \right) \right) \right\},$$

we have that the last term of (27) is less than ε. □

Proof of Proposition 3.2. From Lemma 3.6, the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ converges absolutely for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. This convergence is locally uniformly and absolutely from the definition of $E_\sigma(\mathbf{x}, s)$. Furthermore, direct calculation shows that $E_\sigma(\mathbf{x}, s)$ satisfies the differential equation (17). Hence we complete the proof of Proposition 3.2. □

4 Spectral expansion

4.1 Some lemmas

Lemma 4.1. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$, the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ is bounded as a function of $\mathbf{x} \in \Gamma\backslash U^n$. If Γ is not cocompact and ν is a cusp such that $\nu = A(x_n\infty)$ for some $A \in G$, then we have the estimate

$$|E_\sigma(\mathbf{x}, s)| = O(x_n(A^{-1}\mathbf{x})^{-\mathrm{Re}(s)}) \tag{31}$$

as $P \to \nu$.

Proof. If $\Gamma\backslash U^n$ is compact, the boundedness of the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ is clear. We consider the case that $\Gamma\backslash U^n$ is non compact. Without loss of generality, we may assume that the cusp of $\Gamma\backslash U^n$ is $x_n\infty$. Let $S_0 = \{\mathbf{x} \mid x_n = r_0\}$ be the holosphere such that every point of D_σ has x_n-part less than r_0. We consider $\mathbf{x} \in U^n$ with $x_n > r_0$. Then $d_{\mathrm{hyp}}(\mathbf{x}, S_0) = \log(x_n/r_0)$. We recall the counting function (18)

$$N_\sigma(T; \mathbf{x}, D_\sigma) = \{\eta \in \Gamma_\sigma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta\mathbf{x}, D_\sigma) < T\}$$

and define the new counting function $N'_\sigma(T; S_0, D_\sigma)$ by

$$N'_\sigma(T; S_0, D_\sigma) := \{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta S_0, D_\sigma) < T\}.$$

Every element of the set

$$\{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta\mathbf{x}, D_\sigma) < T\}$$

corresponds to a geodesic path from $\eta\mathbf{x}$ to D_σ of length less than T. It necessarily intersects the holosphere ηS_0. For $\eta \in \{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta\mathbf{x}, D_\sigma) < T\}$, trivially $d_{\mathrm{hyp}}(\eta S_0, D_\sigma) < T - \log(x_n/r_0)$. Thus we have

$$\{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta\mathbf{x}, D_\sigma) < T\} \subset \{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta S_0, D_\sigma) < T - \log(x_n/r_0)\}.$$

We set $d := \log(x_n/r_0)$. Recalling the Stieltjes integral representation of the hyperbolic Eisenstein series (19), we have the following estimate

$$\begin{aligned} |E_\sigma(\mathbf{x}, s)| &\leq \int_d^\infty \cosh(u)^{-\mathrm{Re}(s)} dN_\sigma(u; \mathbf{x}, D_\sigma) \\ &\leq \int_d^\infty \cosh(u)^{-\mathrm{Re}(s)} dN'_\sigma(u-d; S_0, D_\sigma). \end{aligned} \tag{32}$$

Since $e^u/2 \le \cosh(u)$ holds for any $u \in \mathbb{R}$, we have

$$\int_0^\infty \cosh(u)^{-\mathrm{Re}(s)} dN'_\sigma(u-d; S_0, D_\sigma) \le \left(\frac{x_n}{2r_0}\right)^{-\mathrm{Re}(s)} \int_0^\infty e^{-u\mathrm{Re}(s)} dN'_\sigma(u; S_0, D_\sigma). \quad (33)$$

The integral in the right hand side of (33) converges when $\mathrm{Re}(s) > n-1$. This is shown by using the same method with the proof of convergence of the hyperbolic Eisenstein series in the proof of Proposition 3.2. Therefore, from (32) and (33), we obtain the assertion of the lemma. □

Lemma 4.2. Let $\langle \cdot, \cdot \rangle$ be the inner product in $L^2(\Gamma\backslash U^n)$ and ψ be the real-valued, smooth, bounded function on $\mathcal{F}_\Gamma = \Gamma\backslash U^n$. Assume $\varepsilon > 0$ to be the sufficiently small. Then we have the following estimate

$$\langle E_\sigma(\mathbf{x}, s), \psi \rangle = \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \left(\int_{\Gamma_\sigma\backslash D_\sigma} \psi(\mathbf{x}) dv + O(\varepsilon) \right) \cdot \frac{\Gamma\left((s-n+1)/2\right)\Gamma\left(k/2\right)}{\Gamma\left((s-n+k+1)/2\right)}$$

as $s \to \infty$.

Proof. Since Γ is the cofinite discrete subgroup of G_0, the quotient $\Gamma\backslash U^n$ has finite volume. Therefore, $\psi \in L^2(\Gamma\backslash U^n)$. From Lemma 4.1, we have $E_\sigma(\mathbf{x}, s) \in L^2(\Gamma\backslash U^n)$. Then we have

$$\begin{aligned}
&\langle E_\sigma(\mathbf{x}, s), \psi \rangle \\
&= \int_{\mathcal{F}_\Gamma} E_\sigma(\mathbf{x}, s)\psi(\mathbf{x}) dv \\
&= \int_{\mathcal{F}_\Gamma} \left(\sum_{\eta \in \Gamma_\sigma\backslash\Gamma} (\sin\varphi_0(\eta\mathbf{x}))^s \right) \psi(\mathbf{x}) dv \\
&= \int_{\mathcal{F}_{\Gamma_\sigma}} (\sin\varphi_0(\mathbf{x}))^s \psi(\mathbf{x}) dv \\
&= \int_{\Gamma_\sigma\backslash D_\sigma} \int_{S^{k-1}} \int_{\varphi_0=0}^{\frac{\pi}{2}} (\sin\varphi_0(\mathbf{x}))^s \psi(\mathbf{x}) dv \\
&= \int_{\Gamma_\sigma\backslash D_\sigma} \int_{S^{k-1}} \int_{\varphi_0=0}^{\frac{\pi}{2}} (\sin\varphi_0(\mathbf{x}))^s \psi(\mathbf{x}) \cdot \frac{(\cos\varphi_0)^{k-1}}{(\sin\varphi_0)^n} \cdot d\varphi_0 dv_1 dv_2, \qquad (34)
\end{aligned}$$

where $S^{k-1} = (\varphi_1, ..., \varphi_{k-1})$. For $0 < \varepsilon < \pi/2$ such that $\sin(\pi/2 - \varepsilon) < a_\varepsilon < 1$, we have

$$(34) = \int_{\Gamma_\sigma \backslash D_\sigma} \int_{S^{k-1}} \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}} (\sin \varphi_0(\mathbf{x}))^s \psi(\mathbf{x}) \cdot \frac{(\cos \varphi_0)^{k-1}}{(\sin \varphi_0)^n} \cdot d\varphi_0 dv_1 dv_2 + \mathcal{O}(a_\varepsilon^{s-n}) \quad (35)$$

as $s \to \infty$. Furthermore, for $\frac{\pi}{2} - \varepsilon < \varphi_0 < \frac{\pi}{2}$ and $\Theta \in S^{n-1}$, we have

$$\int_{\Gamma_\sigma \backslash U^{n-k}} \psi(\mathbf{x}) dv_2 = \int_{\Gamma_\sigma \backslash D_\sigma} \psi(\mathbf{x}) dv_2 + O(\varepsilon) \quad (36)$$

as $s \to \infty$. From (34), (35) and (37), for any sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
&\langle E_\sigma(\mathbf{x}, s), \psi \rangle \\
&= \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}} \int_{S^{k-1}} \left(\int_{\Gamma_\sigma \backslash D_\sigma} \psi(\mathbf{x}) dv_2 + O(\varepsilon) \right) \\
&\qquad \times (\cos \varphi_0)^{k-1} (\sin \varphi_0)^{s-n} dv_1 d\varphi_0 + O(a_\varepsilon^{s-n}) \\
&= \left(\int_{\Gamma_\sigma \backslash D_\sigma} \psi(\mathbf{x}) dv_2 + O(\varepsilon) \right) \\
&\qquad \times \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}} \int_{S^{k-1}} (\cos \varphi_0)^{k-1} (\sin \varphi_0)^{s-n} dv_1 d\varphi_0 + O(a_\varepsilon^{s-n}) \\
&= \mathrm{vol}(S^{k-1}) \cdot \left(\int_{\Gamma_\sigma \backslash D_\sigma} \psi(\mathbf{x}) dv_2 + O(\varepsilon) \right) \\
&\qquad \times \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}} (\cos \varphi_0)^{k-1} (\sin \varphi_0)^{s-n} d\varphi_0 + O(a_\varepsilon^{s-n}) \\
&= \mathrm{vol}(S^{k-1}) \cdot \left(\int_{\Gamma_\sigma \backslash D_\sigma} \psi(\mathbf{x}) dv_2 + O(\varepsilon) \right) \\
&\qquad \times \int_0^{\frac{\pi}{2}} (\cos \varphi_0)^{k-1} (\sin \varphi_0)^{s-n} d\varphi_0 \qquad (37)
\end{aligned}$$

as $s \to \infty$. For complex numbers s_1, s_2 with $\mathrm{Re}(s_1) > 0$ and $\mathrm{Re}(s_2) > 0$, we have the following formula

$$\int_0^1 x^{s_1 - 1} (1-x)^{s_2 - 1} dx = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1 + s_2)}.$$

Applying this formula to the integral in (37), we have

$$\langle E_\sigma(\mathbf{x}, s), \psi \rangle = \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \left(\int_{\Gamma_\sigma \backslash D_\sigma} \psi(\mathbf{x}) dv_2 + O(\varepsilon) \right) \cdot \frac{\Gamma\left((s-n+1)/2\right)\Gamma\left(k/2\right)}{\Gamma\left((s-n+k+1)/2\right)}$$

as $s \to \infty$. We obtain the assertion of the lemma. □

4.2 Spectral expansion

Theorem 4.3. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$, the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ admits the following spectral expansion.

$$E_\sigma(\mathbf{x}, s) = \sum_{m \in \mathfrak{D}} a_{m,\sigma}(s) e_m(\mathbf{x}) + \frac{1}{4\pi} \sum_{\nu:\text{cusps}} \int_{-\infty}^{\infty} a_{\frac{n-1}{2}+i\mu,\sigma}(s) E_\nu\left(\mathbf{x}, \frac{n-1}{2} + i\mu\right) d\mu. \quad (38)$$

Then the series in right hand side converges absolutely and locally uniformly. The coefficients $a_{m,\sigma}(s)$ and $a_{\frac{n-1}{2}+i\mu,\sigma}(s)$ are given by

$$a_{m,\sigma} = \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \Gamma\left(\frac{k}{2}\right) \times \frac{\Gamma\left(\left(s - \frac{n-1}{2} + \mu_m\right)/2\right)\Gamma\left(\left(s - \frac{n-1}{2} - \mu_m\right)/2\right)}{\Gamma\left(s/2\right)\Gamma\left(\left(s-n+k+1\right)/2\right)} \times \int_{\Gamma_\sigma \backslash D_\sigma} e_m dv_2 \quad (39)$$

and

$$a_{\frac{n-1}{2}+i\mu,\sigma} = \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \Gamma\left(\frac{k}{2}\right) \times \frac{\Gamma\left(\left(s - \frac{n-1}{2} + i\mu\right)/2\right)\Gamma\left(\left(s - \frac{n-1}{2} - i\mu\right)/2\right)}{\Gamma\left(s/2\right)\Gamma\left(\left(s-n+k+1\right)/2\right)} \times \int_{\Gamma_\sigma \backslash D_\sigma} E_\nu\left(\mathbf{x}, \frac{n-1}{2} + i\mu\right) dv_2, \quad (40)$$

where $\mu_m^2 = \left(\frac{n-1}{2}\right)^2 - \lambda_m$ and λ_m is the eigenvalue of the eigenfunction e_m.

Proof. The hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ is a bounded and smooth function on $\Gamma\backslash U^n$ by Definition 3.1 and Lemma 4.1. Since the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ satisfies the differential sift equation (17), $-\Delta^l f$ are bounded and smooth on $\Gamma\backslash U^n$ for any $0 \leq l \leq \lfloor \frac{n+1}{4} \rfloor + 1$. Hence, from Proposition 2.5, the hyperbolic Eisenstein series has the spectral expansion (38) and it converges absolutely and locally uniformly.

In order to give the coefficients $a_{m,\gamma}(s)$ and $a_{\frac{n-1}{2}+i\mu,\gamma}(s)$, we calculate the inner product $\langle E_\sigma, e_m\rangle$, which converges by asymptotic bound proved in Lemma 4.2. Furthermore, we know the asymptotic bounds for eigenfunctions of the Laplace-Beltrami operator Δ by Lemma 4.2. We have the following relation by using the differential equation (17)

$$\begin{aligned}\lambda_m a_{m,\sigma}(s) &= \lambda_m \langle E_\sigma, e_m\rangle = \langle E_\sigma, \lambda_m e_m\rangle = \langle -\Delta E_\sigma, e_m\rangle \\ &= -s\,(s-n+1)\,a_{m,\sigma}(s) + s(s-n+k+1)a_{m,\sigma}(s+2),\end{aligned}$$

which implies the relation

$$a_{m,\sigma}(s+2) = \frac{s(s-n+1)+\lambda_m}{s(s-n+k+1)} a_{m,\sigma}(s).$$

For μ_m with $\mu_m^2 = \left(\frac{n-1}{2}\right)^2 - \lambda_m$ we set the function $g(s)$ as

$$g(s) = \frac{\Gamma((s-\frac{n-1}{2}+\mu_m)/2)\Gamma((s-\frac{n-1}{2}-\mu_m)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)}.$$

Then $g(s)$ satisfies the relation

$$g(s+2) = \frac{s(s-n+1)+\lambda_m}{s(s-n+k+1)} g(s).$$

From this, we conclude that the quotient $a_{m,\gamma}(s)/g(s)$ is invariant under $s \mapsto s+2$. Furthermore, it is bounded in a vertical strip. Therefore, the quotient $a_{m,\sigma}(s)/g(s)$ is constant, so we have

$$a_{m,\sigma}(s) = b_{m,\sigma} \cdot \frac{\Gamma((s-\frac{n-1}{2}+\mu_m)/2)\Gamma((s-\frac{n-1}{2}-\mu_m)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)} \tag{41}$$

for some constant $b_{m,\sigma}$. From Lemma 4.2, for sufficiently small $\varepsilon > 0$, we have following estimate

$$\begin{aligned}a_{m,\sigma} &= \langle E_\sigma(\mathbf{x}, s), e_m\rangle \\ &= \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \left(\int_{\Gamma_\sigma\backslash D_\sigma} e(\mathbf{x}) dv_2 + O(\varepsilon)\right) \cdot \frac{\Gamma\left((s-n+1)/2\right)\Gamma\left(k/2\right)}{\Gamma\left((s-n+k+1)/2\right)}\end{aligned} \tag{42}$$

as $s \to \infty$. In order to determine the constant $b_{m,\sigma}$, we use the Stirling's asymptotic formula for the gamma function, which states that for a complex number $s \in \mathbb{C}$ such that $|\arg s| < \pi - \varepsilon$, the gamma function $\Gamma(s)$ satisfies the following asymptotic formula

$$\log \Gamma(s) = \frac{1}{2}\log(2\pi) - \frac{1}{2}\log(s) + s\log(s) - s + o(1) \tag{43}$$

as $s \to \infty$. From (41), (42) and (43), by comparing the order as $s \to \infty$, we obtain

$$b_{m,\sigma} = \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \Gamma\left(\frac{k}{2}\right) \cdot \int_{\Gamma_\sigma \backslash D_\sigma} e_m(\mathbf{x})\, dv_2$$

as claimed. We complete the proof of Theorem 4.3. □

4.3 Meromorphic continuation

From the spectral expansion of the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$, we can prove the meromorphic continuation of $E_\sigma(\mathbf{x}, s)$ and the location of the possible poles with the residues. Namely, we obtain the following theorem.

Theorem 4.4. The hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ has a meromorphic continuation to all $s \in \mathbb{C}$. The possible poles of the continued function are located at the following points.

(a) $s = \frac{n-1}{2} \pm \mu_m - 2n'$, where $n' \in \mathbb{N}$ and $\mu_m^2 = \left(\frac{n-1}{2}\right)^2 - \lambda_m$ for the eigenvalue λ_m, with residues

$$\begin{aligned}\mathrm{Res}_{s=\frac{n-1}{2}\pm\mu_m-2n'}&\Big[E_\sigma(\mathbf{x}, s)\Big] \\ &= \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \frac{(-1)^{n'}\Gamma(k/2)\Gamma(\pm\mu_m - n')}{n'! \cdot \Gamma((\frac{n-1}{2} \pm \mu_m - 2n')/2)^2} \\ &\qquad \times \int_{\Gamma_\sigma \backslash D_\sigma} e_m(\mathbf{x}) dv_2 \cdot e_m(\mathbf{x}).\end{aligned}$$

(b) $s = \rho_\nu - 2n'$, where $n' \in \mathbb{N}$ and $\omega = \rho_\nu$ is a pole of the Eisenstein series

$E_\nu(\mathbf{x},\omega)$ with $\mathrm{Re}(\rho_\nu) < \frac{n-1}{2}$, with residues

$$\begin{aligned}
&\mathrm{Res}_{s=\rho_\nu-2n'}\Big[E_\sigma(\mathbf{x},s)\Big] \\
&= \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \sum_{j=0}^{m} \frac{(-1)^j\Gamma(k/2)\Gamma(\rho_\nu-2n'+j-(n-1)/2)}{j!\cdot\Gamma((\rho_\nu-2n')/2)\Gamma((\rho_\nu-2n'+k+1)/2)} \\
&\times \sum_{\nu:\text{cusps}} \Big[\mathrm{CT}_{\omega=\rho_\nu-2n'+2j}E_\nu(\mathbf{x},\omega)\cdot\int_{\Gamma_\sigma\backslash D_\sigma}\mathrm{Res}_{\omega=\rho_\nu-2n'+2j}E_\nu(\mathbf{x},\omega)dv_2 \\
&\qquad + \mathrm{Res}_{\omega=\rho_\nu-2n'+2j}E_\nu(\mathbf{x},\omega)\cdot\int_{\Gamma_\sigma\backslash D_\sigma}\mathrm{CT}_{\omega=\rho_\nu-2n'+2j}E_\nu(\mathbf{x},\omega)dv_2\Big],
\end{aligned}$$

where $\mathrm{CT}_\omega E_\nu(\mathbf{x},\omega)$ denotes the constant term of the Laurent expansion of the Eisenstein series E_ν at ω and $m \in \mathbb{N}$ is the real number such that $\frac{n-1}{2} - 2 - 2m + 2n' < \mathrm{Re}(\rho_\nu) \leq \frac{n-1}{2} - 2m + 2n'$.

(c) $s = n-1-\rho_\nu-2n'$, where $n' \in \mathbb{N}$ and $\omega = \rho_\nu$ is a pole of the Eisenstein series $E_\nu(\mathbf{x},\omega)$ with $\mathrm{Re}(\rho_\nu) \in (\frac{n-1}{2}, n-1]$, with residues

$$\begin{aligned}
&\mathrm{Res}_{s=n-1-\rho_\nu-2n'}\Big[E_\sigma(\mathbf{x},s)\Big] = \frac{1}{2}\mathrm{vol}(S^{k-1}) \\
&\times \sum_{j=m-\lfloor\frac{n-1}{4}\rfloor}^{m} \frac{(-1)^j\Gamma(k/2)\Gamma((n-1)/2-\rho_\nu-2n'+j)}{j!\cdot\Gamma((n-1-\rho_\nu-2n')/2)\Gamma((-\rho_\nu-2n'+k)/2)} \\
&\times \sum_{\nu=1}^{h} \Big[\mathrm{CT}_{\omega=\rho_\nu+2n'-2j}E_\nu(\mathbf{x},\omega)\cdot\int_{\Gamma_\sigma\backslash D_\sigma}\mathrm{Res}_{\omega=\rho_\nu+2n'-2j}E_\nu(\mathbf{x},\omega)dv_2 \\
&\qquad + \mathrm{Res}_{\omega=\rho_\nu+2n'-2j}E_\nu(\mathbf{x},\omega)\cdot\int_{\Gamma_\sigma\backslash D_\sigma}\mathrm{CT}_{\omega=\rho_\nu+2n'-2j}E_\nu(\mathbf{x},\omega)dv_2\Big],
\end{aligned}$$

where $\mathrm{CT}_\omega E_\nu(\mathbf{x},\omega)$ denotes the constant term of the Laurent expansion of the Eisenstein series E_ν at ω and $m \in \mathbb{N}$ is the real number such that $\frac{n-1}{2} + 2m - 2n' < \mathrm{Re}(\rho_\nu) \leq \frac{n-1}{2} + 2m - 2n' + 2$.

Proof. First, we give the meromorphic continuation of the hyperbolic Eisenstein series $E_\sigma(\mathbf{x},s)$. The explicit formula (39) shows that the series in (38) arising from the discrete spectrum have the meromorphic continuation to all $s \in \mathbb{C}$. We give the meromorphic continuation of the continuous spectrum in (38). We consider the meromorphic continuation of the following integral

$$I_{\frac{n-1}{2}}(s) = \int_{\mathrm{Re}(z)=\frac{n-1}{2}} \frac{\Gamma((s-n+1+z)/2)\Gamma((s-z)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)} \times \left(\int_{\Gamma_\sigma \backslash D_\sigma} E_\nu(\mathbf{x}, z) dv_2 \right) \cdot E_\nu(\mathbf{x}, z) dt. \quad (44)$$

The function $I_{\frac{n-1}{2}}(s)$ is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{n-1}{2}$ except possibly finitely many points in the segment $(\frac{n-1}{2}, n-1]$ on the real line. Let $\varepsilon > 0$ be sufficiently small such that $E_\nu(\mathbf{x}, s)$ has no poles in the strip $\frac{n-1}{2} < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$. For $s \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$, we have by the residue theorem

$$I_{\frac{n-1}{2}}(s) = I_{\frac{n-1}{2}+\varepsilon}(s) - 4\pi i \frac{\Gamma(s-(n-1)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)} \times \left(\int_{\Gamma_\sigma \backslash D_\sigma} E_\nu(\mathbf{x}, s) dv_2 \right) \cdot E_\nu(\mathbf{x}, s). \quad (45)$$

The right-hand side of (45) is a meromorphic function for $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$ and it gives the meromorphic continuation of the integral $I_{\frac{n-1}{2}}(s)$ to the stripe $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$. Now, for $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2}$ using the residue theorem once again, we have

$$\begin{aligned}(45) = I_{\frac{n-1}{2}}(s) &+ 4\pi i \frac{\Gamma(s-(n-1)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)} \\ &\times \left(\int_{\Gamma_\sigma \backslash D_\sigma} E_\nu(\mathbf{x}, n-1-s) dv_2 \right) \cdot E_\nu(\mathbf{x}, n-1-s) \\ &- 4\pi i \frac{\Gamma(s-(n-1)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)} \\ &\times \left(\int_{\Gamma_\sigma \backslash D_\sigma} E_\nu(\mathbf{x}, s) dv_2 \right) \cdot E_\nu(\mathbf{x}, s). \quad (46)\end{aligned}$$

The right-hand side of (46) is a meromorphic function for $\frac{n-1}{2} - 2 < \mathrm{Re}(s) < \frac{n-1}{2}$ and it gives the meromorphic continuation of (45) to the stripe $\frac{n-1}{2} - 2 < \mathrm{Re}(s) < \frac{n-1}{2}$. Summing up, from (45) and (46), we have the meromorphic continuation of the integral $I_{\frac{n-1}{2}}(s)$ to the stripe $\frac{n-1}{2} - 2 < \mathrm{Re}(s) \leq \frac{n-1}{2}$.

Continuing this process, the meromorphic continuation of the integral $I_{\frac{n-1}{2}}(s)$ to the stripe $\frac{n-1}{2} - 2 - 2m < \mathrm{Re}(s) \le \frac{n-1}{2} - 2m$

$$
\begin{aligned}
I_{\frac{n-1}{2}}(s) + \sum_{l=0}^{m} \frac{4\pi i \cdot (-1)^l}{l!} \cdot \frac{\Gamma(s+l-(n-1)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)} \\
\times \left(\int_{\Gamma_\sigma \backslash D_\sigma} E_\nu(\mathbf{x}, n-1-2l-s) dv_2 \right) \cdot E_\nu(\mathbf{x}, n-1-2l-s) \\
- \sum_{l=0}^{m} \frac{4\pi i \cdot (-1)^l}{l!} \cdot \frac{\Gamma(s+l-(n-1)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)} \\
\times \left(\int_{\Gamma_\sigma \backslash D_\sigma} E_\nu(\mathbf{x}, s+2l) dv_2 \right) \cdot E_\nu(\mathbf{x}, s+2l). \quad (47)
\end{aligned}
$$

In order to determine the poles, we consider the spectral expansion (38). First, we consider the discrete term of the spectral expansion. The poles arising from discrete term of (38) are all derived from Γ-functions. Hence the location of poles are given $s = \frac{n-1}{2} \pm \mu_m - 2l$, where $l \in \mathbb{N}$ and $\mu_m^2 = \left(\frac{n-1}{2}\right)^2 - \lambda_m$ for the eigenvalue λ_m. We can also calculate their residues from the residues of Γ-function. Therefore we obtain (a). Next, we consider the continuous spectrum of (38). We work from their meromorphic continuation (47). Then only poles can arise from the summands $E_\nu(\mathbf{x}, s+2l)$ and $E_\nu(\mathbf{x}, n-1-2l-s)$, where $l = 0, ..., m$. The poles arising from $E_\nu(\mathbf{x}, s+2l)$ are located at $s_{\nu,l} = \rho_\nu - 2l$, where $l = 0, ..., m$ and ρ_ν is a pole of $E_\nu(\mathbf{x}, s)$ with $\frac{n-1}{2} - 2 - 2m + 2l < \mathrm{Re}(\rho_\nu) \le \frac{n-1}{2} - 2m + 2l$. The poles arising from $E_\nu(\mathbf{x}, n-1-2l-s)$ are located at $s'_{\nu,l} = n-1-\rho_\nu-2l$, where $l = 0, ..., m$ and ρ_ν is a pole of $E_\nu(\mathbf{x}, s)$ with $\frac{n-1}{2} + 2m - 2l \le \mathrm{Re}(\rho_\nu) < \frac{n-1}{2} + 2m - 2l + 2$; in this case there are only poles for $m - \lfloor \frac{n-1}{4} \rfloor \le l \le m$, because the parabolic Eisenstein series $E_\nu(\mathbf{x}, s)$ has no poles in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \frac{n-1}{2}\}$ except possibly finitely many simple poles in the segment $(\frac{n-1}{2}, n-1]$ on the real line. The residues can be easily derived from (47). Thereby we complete the proof of Theorem 4.4. □

Remark 4.5. The poles given in (a), (b), and (c) might coincide in parts. If it is in the case, the corresponding residues have to be the sum added the each residue.

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