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Modal and Total Power Spectra of Nikolaevskii Turbulence

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We numerically investigate the *modal* and *total* time correlation functions and the corresponding power spectra for the Nikolaevskii equation. The modal power spectrum exhibits Lorentzian for low-frequency ranges, while the total power spectrum diverges in the low-frequency limit. The dynamic exponent is equal to 2 and 3/2 for smaller and larger system sizes, respectively. The relationship between the Nikolaevskii turbulence and the soft-mode turbulence is also discussed.

KEYWORDS: power spectra, Nikolaevskii turbulence, dynamic exponent, time correlation function

1. Introduction

The universality of statistical properties of turbulence and/or spatiotemporal chaos is a challenging subject.[1–3] Turbulence and spatiotemporal chaos occur from periodic dissipative structures in several systems. However, a rare class of turbulence is found to occur supercritically from uniform static states.

As a model system of this class of turbulence, two types of turbulence, soft-mode turbulence (SMT)[4] and Nikolaevskii turbulence (NT),[5] have been actively studied, and their similarities have been discussed. These two types of turbulence exhibit a similar onset, as follows. The uniform static state is unstable with respect to finite-wavelength perturbations when a small parameter is positive. However, this instability does not lead to spatially periodic static states, because the systems possess a neutral uniform mode due to Galilean invariance or spontaneous symmetry breaking, and the corresponding weakly stable long-wavelength modes interact with the unstable short-wavelength modes. Consequently, the spatially periodic static states do not occur, and instead spatiotemporal chaos occurs supercritically from the uniform static state. Because of this similarity of the onset, we expect that these two types of turbulence may exhibit turbulent fluctuations that belong in the same universality class. However, this has never been confirmed.

The power spectrum, the time correlation function, and the dynamic exponent are important keywords used to investigate the universality class. It was reported that the SMT exhibits a Lorentzian

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power spectrum. [6] In contrast, Matthews-Cox equations,[7] which are small-amplitude equations that approximately describe the NT in one-dimensional space,[8] exhibit a non-Lorentzian power spectrum in an intermediate frequency range.[9] Note that, for a number of reasons, a direct comparison of these two spectra is not possible. The first reason is the difference in spatial dimension. Soft-mode turbulence involves a complex electrohydrodynamic convection in homeotropically aligned nematic liquid crystals in real three-dimensional space. In contrast, the Matthews-Cox equations are derived for the NT in one-dimensional space. Unfortunately, amplitude equations for the NT in higher-dimensional space have never been successfully derived.[9–11] The second reason is the existence of higher harmonics that are ignored when the Matthews-Cox equations are derived. These harmonics affect the stability of localized structures in the NT.[12] For these reasons, we cannot simply compare the above two power spectra. In the present paper, as a partial step to study the second reason, we focus on the power spectra of the NT in one-dimensional space. In particular, we note a difference between modal and total power spectra, and we point out the ambiguity of spectral fitting in a narrow frequency range .

2. Basic Equation

The NT is exhibited by

$$\partial_t u(x,t) = -\partial_x^2[\epsilon - (1+\partial_x^2)^2]u - u\partial_x u, \tag{2.1}$$

which was derived for the propagation of longitudinal seismic waves in viscoelastic media[13] and for phase dynamics in the neighborhood of a codimension-two Turing–Benjamin-Feir point in oscillatory reaction diffusion systems.[14–16] When the small parameter ϵ is positive, the uniform steady state of eq. (2.1), $u = 0$, evolves supercritically into a state characterized by spatiotemporal chaos, called the NT. Recently, the robustness of this turbulence has been actively studied.[16–18] Equation (2.1) exhibits characteristic spatiotemporal chaos when the value of ϵ is no greater than $O(10^{-1})$.[8] Otherwise, eq. (2.1) exhibits Kuramoto-Sivashinsky (KS)-like chaos.[3,8] Thus, we adopt $\epsilon = 0.02$ in the following.

We numerically integrate the N-truncated Fourier transform of eq. (2.1) under the periodic boundary condition,

$$u(x,t) = u(x+L,t),$$

with a pseudo-spectral method. Here, L is the system size. The Fourier coefficient $\hat{u}_n(t)$ of $u(x,t)$ is defined as

$$\hat{u}_n(t) \equiv \int_0^L u(x,t)e^{-ik_n x}dx$$

with $k_n \equiv 2n\pi/L$ ($n = 1, \ldots, N$). Figure 1 shows the energy spectrum,

$$E(k_n) \equiv \frac{\langle \hat{u}_n(t)\hat{u}_n^*(t)\rangle}{2\pi L},$$

which exhibits the characteristic profile of the NT.[8] Here, $*$ represents a complex conjugation. The modal time correlation function $U_n(t)$ of $\hat{u}_n$ is defined as

$$U_n(t) \equiv \langle \hat{u}_n(t)\hat{u}_n^*(0)\rangle = \frac{1}{M}\sum_{j=0}^{M-1} \hat{u}_n(t+T_j)\hat{u}_n^*(T_j),$$

where $T_j = T_0 + T_m j$, the starting time is $T_0 = 1000$, the maximum correlation time is T_m, the ensemble number is $M = (T_f - T_0)/T_m$, and the final time is T_f. The details of this numerical treatment are given in a paper.[19]

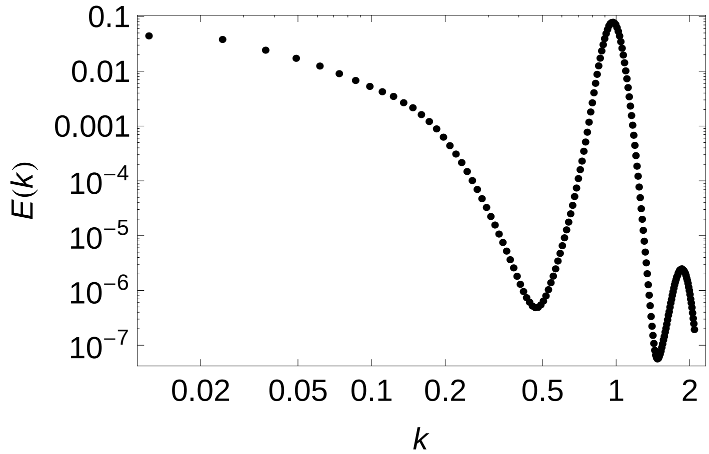

Fig. 1. Log-log plot of the energy spectrum $E(k)$ for $\epsilon = 0.02$, $L = 512$, and $N = 2^8$.

3. Modal Power Spectrum $I_n(\omega)$

Figure 2 shows the modal power spectrum

$$I_n(\omega) \equiv \frac{1}{\pi}\int_0^\infty U_n(t)\cos(\omega t)dt, \tag{3.1}$$

which is obtained from the numerical simulation for $L = 512$, $N = 2^8$, $T_m = 100$, and $T_f = 3 \times 10^8$. As shown in Fig. 2, the asymptotic modal power spectrum is consistent with the exponential decay,

$$I_n(\omega) \propto e^{-\omega\tau_n} \quad \text{for} \quad \omega \to \infty, \tag{3.2}$$

where τ_n is the characteristic time of $U_n(t)$. The exponential dependence of the modal power spectrum on ω in eq. (3.2) allows us to conclude that the corresponding modal time correlation function $U_n(t)$ exhibits the Lorentzian form in the limit $t \to 0$, which is consistent with both the numerical result for the KS equation[20] and the analytical result in one-dimensional turbulence in general.[21] In the opposite limit $t \to \infty$, i.e., $\omega \to 0$, $I_n(\omega)$ is the Lorentzian spectrum, as shown in Fig. 2, which is also consistent with the previous results.[9,20,21]

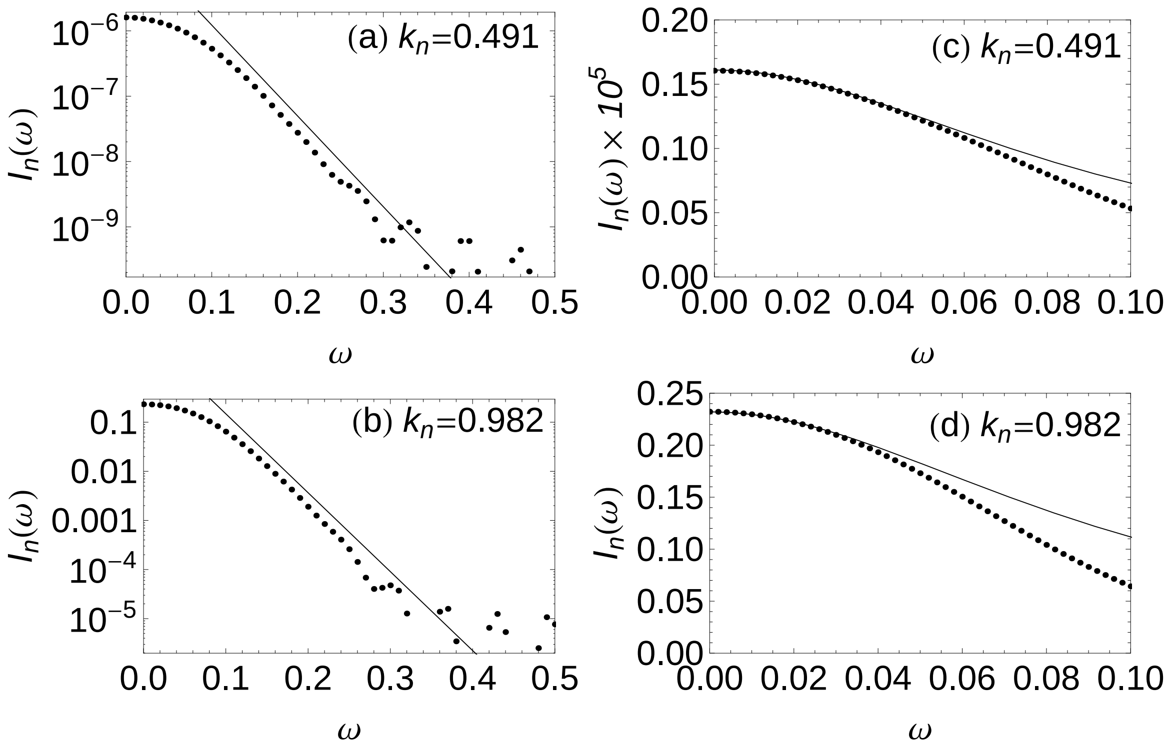

Fig. 2. Semilog plot of the modal power spectra (3.1) with the straight line denoting the exponential function (3.2) for (a) $k_n = 0.491$ and $\tau_n = 32$, and (b) $k_n = 0.982$ and $\tau_n = 37$. Normal plot of the same spectra with the curve denoting the Lorentzian function $I_n(\omega) = I_n(0)/[1 + (\omega\tau_n^{(L)})^2]$ for (c) $k_n = 0.491$, $I_n(0) = 1.61 \times 10^{-6}$, and $\tau_n^{(L)} = 11.0$, and (d) $k_n = 0.982$, $I_n(0) = 0.232$, and $\tau_n^{(L)} = 10.4$.

4. Total Power Spectrum $I(\omega)$

4.1 Numerical evaluation

We now consider the total power spectrum $I(\omega)$ for $\omega \to 0$. Instead of evaluating the total power spectrum directly, we first evaluate the total time correlation functions

$$Q(t) \equiv \langle u(x,t)u(x,0)\rangle = \frac{4\pi}{L}\sum_{n=1}^{N} E(k_n)\frac{U_n(t)}{U_n(0)}, \tag{4.1}$$

which are shown in the modified forms $\sqrt{t}\tilde{Q}(t)$ in Fig. 3 and $t^{2/3}\tilde{Q}(t)$ in Fig. 4, in the three cases $(L, N) = (256, 2^7)$, $(512, 2^8)$, and $(1024, 2^9)$ for $T_m = 2000$ and $T_f = 3 \times 10^9$. Here,

$$\tilde{Q} \equiv \frac{Q(t)}{Q(0)}.$$

Figure 3 indicates that the numerical result is consistent with the power-law decay

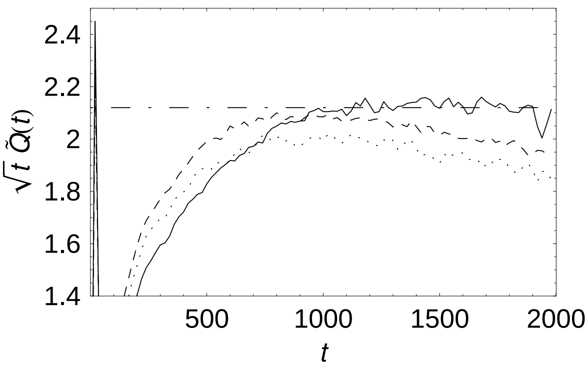

Fig. 3. Modified time correlation function $\sqrt{t}\tilde{Q}(t)$ for three cases: ——, $(L, N) = (256, 2^7)$; $\cdots$, $(512, 2^8)$; – – –, $(1024, 2^9)$, with a straight line, $\tilde{Q}(t) = 2.12t^{-1/2}$.

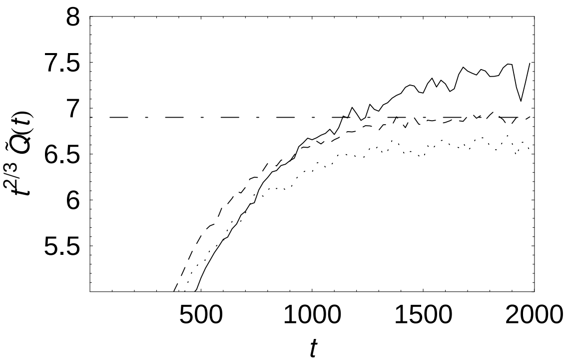

Fig. 4. Modified time correlation function $t^{2/3}\tilde{Q}(t)$ for three cases: ——, $(L, N) = (256, 2^7)$; $\cdots$, $(512, 2^8)$; – – –, $(1024, 2^9)$, with a straight line, $\tilde{Q}(t) = 6.9t^{-2/3}$.

$$2.12t^{-1/2} \quad \text{for} \quad t \gtrsim 1000$$

in the case $(L, N) = (256, 2^7)$, while Fig. 4 indicates that the numerical results approach the power-law decays

$$6.6t^{-2/3} \quad \text{and} \quad 6.9t^{-2/3} \quad \text{as} \quad t \gtrsim 1300$$

in the cases $(L, N) = (512, 2^8)$ and $(1024, 2^9)$, respectively. This means that the asymptotic time correlation function becomes

$$Q(t) \propto t^{-1/z},$$

where z, called the dynamic exponent,[22] is determined to be 2 for $L < L_c$ and 3/2 for $L > L_c$. Here, L_c is the critical system size and satisfies $256 < L_c < 512$. This dependence of the dynamic exponent on the system size was pointed out by Sneppen *et al.*[23] for the KS equation. In order to evaluate the total power spectrum, we supplement the total time correlation function $Q_{\text{num}}(t)$, obtained from the numerical simulation, by giving the power-law decay for larger values of t:

$$Q(t) = \begin{cases} Q_{\rm num}(t) & 0 \le t < T_m, \\ C_\infty t^{-1/z} & T_m \le t, \end{cases} \tag{4.2}$$

where C_∞ is a constant. Finally, substituting eq. (4.2) into

$$I(\omega) \equiv \frac{1}{\pi}\int_0^\infty Q(t)\cos(\omega t)dt, \tag{4.3}$$

we obtain the total power spectrum $I(\omega)$. Figure 5 shows $I(\omega)$ in the case $(L, N) = (256, 2^7)$ for $z = 2$ and $C_\infty = 0.06$. The total power spectrum $I(\omega)$ is graphically indistinguishable from the Lorentzian spectrum for smaller values of ω, except in the case of $\omega \lesssim 0.01$, and $I(\omega)$ diverges as $1/\sqrt{\omega}$ for $\omega \to 0$ owing to the power-law decay of $Q(t)$. Note that this divergence does not bring about the divergence of the total energy $Q(0)/2$.

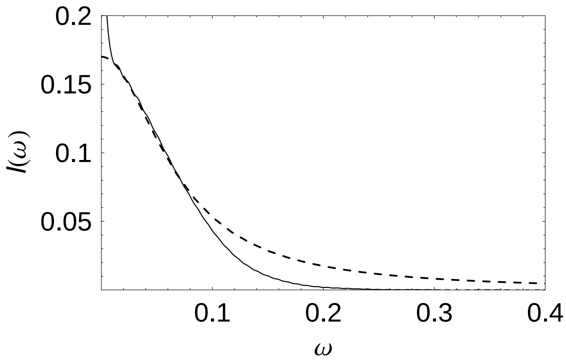

Fig. 5. Total power spectrum $I(\omega)$ with a broken curve showing the Lorentzian spectrum $0.17/[1 + (14.8\omega)^2]$ for comparison. Lorentzian-like spectrum can be observed only in the narrow range.

4.2 Semianalytic evaluation

Next, we again find the power-law decay of the total time correlation function, $Q(t) \propto t^{-1/z}$, by using a semianalytic method. As in eq. (4.1), $Q(t)$ is a summation of $U_n(t)$, and, in the case of $t \to \infty$, $U_n(t)$ for smaller values of k_n has a major contribution to the summation because $U_n(t)$ for smaller values of k_n decays more slowly than that for larger values of k_n. Since the Markov approximation is valid for smaller values of k_n,[21,24] $U_n(t)$ satisfies the exponential decay form

$$U_n(t) \propto e^{-\gamma_n t}, \tag{4.4}$$

where γ_n is the inverse of the characteristic time of $U_n(t)$. Furthermore, in the case $k_n \to 0$, we can expect the dynamic scaling law[25] expressed by

$$\gamma_n = C_\gamma k_n^z, \tag{4.5}$$

where C_γ is a constant. Figure 6 shows the dependence of γ_n on the wavenumber k_n in a log-log plot, indicating the dynamic exponents $z = 2$ for $(L, N) = (256, 2^7)$ and $z = 3/2$ for $(L, N) = (1024, 2^9)$. These exponents are consistent with those in Figs. 3 and 4, and $z = 2$ is consistent with a previous experimental result.[26)] To our knowledge, this is the first time that the dynamic exponent is clearly shown to be $z = 3/2$ in one-dimensional turbulence (see Fig. 2 in Sneppen *et al.*[23)]). Of course, numerical simulations in the case of larger values of L and N bring about better results, but the present simulations are as much as we can do. Figure 1 indicates the energy spectrum

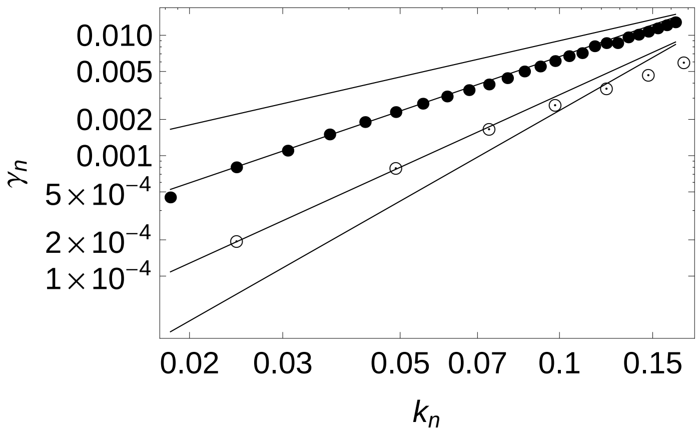

Fig. 6. Log-log plot of the dependence of γ_n on k_n for two cases: •, $(L, N) = (1024, 2^9)$; ⊙, $(256, 2^7)$, with four straight lines showing $\gamma_n = 0.09k_n$, $0.21k_n^{3/2}$, $0.32k_n^2$, and $0.75k_n^{5/2}$.

$$E(k) = 0.07e^{-27k} \tag{4.6}$$

for smaller values of k. Therefore, we can evaluate the asymptotic normalized time correlation function $\tilde{Q}(t)$ for larger values of t by using eqs. (4.4)–(4.6) as follows:

$$\tilde{Q}(t) \sim \begin{cases} 3.9t^{-1/2} & \text{for} \quad (L, N) = (256, 2^7), \\ 6.4t^{-2/3} & \text{for} \quad (L, N) = (1024, 2^9), \end{cases} \tag{4.7}$$

where coefficients 3.9 and 6.4 differ from those obtained from the asymptotic forms in Figs. 3 and 4 by 46 and 8%, respectively.

Next, substituting eq. (4.7) into eq. (4.3), we obtain the asymptotic total power spectrum

$$I(\omega) \propto \omega^{-1/z} \qquad \text{for} \quad \omega \to 0, \tag{4.8}$$

which diverges in this limit and is quite far from the Lorentzian spectrum, while Fig. 2 shows that the modal power spectrum $I_n(\omega)$ is Lorentzian for smaller values of ω. This is likely to be inconsistent. However, if γ_n has a wavenumber dependence of the form (4.5) with the dynamic exponent $z > 1/2$, there is no contradiction because

$$I(\omega) \propto \int_0^\infty \frac{1}{1+(k^z\omega)^2} dk \propto \omega^{-1/z} \quad \text{for} \quad z > 1/2.$$

Therefore, even if $I_n(\omega)$ is Lorentzian for smaller values of ω, the total power spectrum $I(\omega)$ can be non-Lorentzian. It is important to note that the total power spectrum $I(\omega)$ and the modal power spectrum $I_n(\omega)$ should not be confused in the case of a discussion on the spectral form.

We now comment on the dynamic exponent z. Figure 6 indicates the dynamic exponent $z = 2$ for $L = 256$ and $z = 3/2$ for $L = 1024$, while the dynamic renormalization-group method predicts $z = 3/2$ for the noisy Burgers equation, called the KPZ equation.[22,25] Yakhot[27] suggested that the large-scale properties of the KS equation can be described by the KPZ equation with the dynamic renormalization-group method. It is reasonable that the large-scale properties of the Nikolaevskii equation can also be described by the KPZ equation because the dynamic renormalization-group method ignores the higher-order spatial derivatives.

5. Discussion

Finally, we compare our results with the two power spectra exhibited by the Matthews-Cox equations[9] and the SMT.[6] First, Fig. 6 in Tanaka *et al.*[9] shows that the spectrum of the mode represented by A (which corresponds to mode $k_n \sim 1$ in the present paper) is Lorentzian for smaller frequencies, which is consistent with Fig. 2. Note that although the modal power spectra are Lorentzian (Fig. 2), the total power spectrum is not Lorentzian (Fig. 5). This is due to the fact that the characteristic time of each mode depends on the wavenumber of the mode (Fig. 6). In addition, the higher harmonics of mode $k_n \sim 1$, i.e., modes $k_n \sim 2, 3, \ldots$, which are ignored in the Matthews-Cox equations, affect the long-time correlation of turbulent fluctuations exhibited by the Nikolaevskii equation.[12] Second, the spectrum of the NT in the neighborhood of $\omega \sim 0.05$ is graphically indistinguishable from the Lorentzian spectrum (Fig. 5). The spectrum in this frequency range may correspond to the Lorentzian spectrum observed in the SMT.[6] However, the spectrum of the NT does not exhibit a single Lorentzian regime (Fig. 5), and, in particular, the spectrum diverges in the low-frequency limit and is quite far from the Lorentzian spectrum, as shown semianalytically in eq. (4.8). Thus, at present, there is no persuasive evidence with respect to the universality between the SMT and the NT.

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Dr. Dan Tanaka, a promising young scientist in nonequilibrium statistical physics, died suddenly

on 28 November, 2009, during the preparation of this paper. May he rest in peace.

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