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Decay Forms of the Time Correlation Functions for Turbulence and Chaos

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Taking the Rubin model for the one-dimensional Brownian motion and the chaotic Kuramoto-Sivashinsky equation for the one-dimensional turbulence, we derive a generalized Langevin equation in terms of the projection operator formalism, and then investigate the decay forms of the time correlation function $U_k(t)$ and its memory function $\varGamma_k(t)$ for a normal mode $u_k(t)$ of the system with a wavenumber k. Let $\tau_k^{(u)}$ and $\tau_k^{(\gamma)}$ be the decay times of $U_k(t)$ and $\varGamma_k(t)$, respectively, with $\tau_k^{(u)} \geq \tau_k^{(\gamma)}$. Here, $\tau_k^{(u)}$ is a macroscopic time scale if $k \ll 1$, but a microscopic time scale if $k \gtrsim 1$, whereas $\tau_k^{(\gamma)}$ is always a microscopic time scale. Changing the length scale k^{-1} and the time scales $\tau_k^{(u)}$, $\tau_k^{(\gamma)}$, we can obtain various aspects of the systems as follows. If $\tau_k^{(u)} \gg \tau_k^{(\gamma)}$, then the time correlation function $U_k(t)$ exhibits the decay of macroscopic fluctuations, leading to an exponential decay $U_k(t) \propto \exp(-t/\tau_k^{(u)})$. At the singular point where $\tau_k^{(u)} = \tau_k^{(\gamma)}$, however, both $U_k(t)$ and $\varGamma_k(t)$ exhibit anomalous microscopic fluctuations, leading to the power-law decay $U_k(t) \propto t^{-3/2} \cos[(2t/\tau_k^{(u)}) - (3\pi/4)]$ for $t \to \infty$. The above decay forms give us important information on the macroscopic and microscopic fluctuations in the systems and their dissipations.

Subject Index: 051, 056

§1. Introduction

The time correlation functions or the power spectra are the most important quantities for characterizing chaos and turbulence. In particular, the time correlation function $U_k(t)$ and its memory function $\varGamma_k(t)$ for a normal mode $u_k(t)$ with a wavenumber k clearly exhibit structures of chaos and turbulence,[1)–3)] where $\varGamma_k(t)$ is the time correlation function of the fluctuating force $r_k(t)$ of the k mode, representing the second fluctuation-dissipation theorem.[4),5)]

Chaos and turbulence consist of unstable nonperiodic orbits with positive Liapunov exponents, and their sensitive dependence on the initial points brings about an exponential increase of the initial errors, so that the nonperiodic orbits become stochastic and random after an initial regime.[6),7)]

Therefore, the nonperiodic orbits in the time correlation function $U_k(t)$ of a normal mode $u_k(t)$ consist of deterministic short orbits in an initial regime $t < \tau_k^{(\gamma)}$, where $\tau_k^{(\gamma)}$ is a decay time of the memory function $\varGamma_k(t)$. This leads to a dual structure of the randomization of the nonperiodic orbits in $U_k(t)$, which is governed by a non-Markovian evolution equation with the memory function $\varGamma_k(t)$ as the kernel

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of convolution.[1)–3)] Let $\tau_k^{(u)}$ be a decay time of $U_k(t)$ with $\tau_k^{(u)} \geq \tau_k^{(\gamma)}$. If $\tau_k^{(u)} \gg \tau_k^{(\gamma)}$, then $U_k(t)$ exhibits the exponential decay $\exp(-t/\tau_k^{(u)})$ in the final regime $t \gtrsim \tau_k^{(u)}$.[1)] It has been shown in the previous papers[1),2)] that, at the singular point where $\tau_k^{(u)} = \tau_k^{(\gamma)}$, both of $U_k(t)$ and $\Gamma_k(t)$ exhibit the power-law decay $t^{-3/2}$ for $t \to \infty$ and undergo anomalous microscopic fluctuations.

The present paper is organized as follows. In §2, we summarize the statistical mechanical theory of the Brownian motions to formulate the decay form of the time correlation function $U_k(t)$ and its memory function $\Gamma_k(t)$ in terms of the projection operator formalism.[1),4)] In §3, we treat the Rubin model for the 1D Brownian motions and find a singular point where $\tau_k^{(u)} = \tau_k^{(\gamma)}$ theoretically. Then we clarify the decay forms of its $U_k(t)$ and $\Gamma_k(t)$ for turbulence and chaos. In §4, we treat the Kuramoto-Sivashinsky (KS) equation by finding a singular point where $\tau_k^{(u)} = \tau_k^{(\gamma)}$, and clarify the decay forms of its $U_k(t)$ and $\Gamma_k(t)$ for turbulence and chaos. Section 5 is devoted to the summary.

§2. Statistical mechanical theory of Brownian motion

Let us consider the Brownian motion of an impurity of mass M floating in a statistically homogeneous and steady turbulent water. If the mass M is sufficiently larger than the mass m of the water molecule, then the time evolution of the x component of the momentum, $p_0(t)$, of the impurity is given by the Langevin equation

$$\frac{dp_0(t)}{dt} = -\gamma_0 p_0(t) + r_0(t), \tag{2·1}$$

where γ_0 is the friction constant due to water and $r_0(t)$ is the microscopic fluctuating force at the time t, satisfying

$$\langle r_0(t) p_0(0) \rangle = 0 \tag{2·2}$$

and

$$\langle r_0(t) r_0(t') \rangle = 2\gamma_0 \left\langle |p_0(0)|^2 \right\rangle \delta(t-t') \tag{2·3}$$

with the long-time average

$$\langle F(t,t') \rangle \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau F(t+s, t'+s) ds. \tag{2·4}$$

Let us define the two basic time scales

$$\tau_0^{(u)} \equiv \int_0^\infty U_0(t) dt \tag{2·5}$$

and

$$\tau_0^{(\gamma)} \equiv \frac{1}{\Gamma_0(0)} \int_0^\infty \Gamma_0(t) dt, \tag{2·6}$$

which denote the decay time of the time correlation function

$$U_0(t) \equiv \frac{\langle p_0(t) p_0(0) \rangle}{\langle |p_0(0)|^2 \rangle} \tag{2·7}$$

of the momentum $p_0(t)$ and that of the time correlation function

$$\mathit{\Gamma}_0(t) \equiv \frac{\langle r_0(t) r_0(0)\rangle}{\langle |p_0(0)|^2\rangle} \tag{2·8}$$

of the fluctuating force $r_0(t)$, respectively. Then (2·1) and (2·2) lead to

$$\frac{dU_0(t)}{dt} = -\gamma_0 U_0(t), \tag{2·9}$$

which is integrated to give the exponential decay $U_0(t) \propto \exp(-t/\tau_0^{(u)})$. Here, we have used the relation $\tau_0^{(u)} = 1/\gamma_0$ derived from (2·5). The main feature of this decay is irreversible with the decay time $\tau_0^{(u)}$.

Integrating (2·3) with respect to time leads to

$$\gamma_0 = \frac{1}{\langle |p_0(0)|^2\rangle} \int_0^\infty \langle r_0(t) r_0(0)\rangle dt. \tag{2·10}$$

This gives the friction constant γ_0 in terms of the time correlation function of the microscopic fluctuating force $r_0(t)$, and is called the second fluctuation-dissipation theorem.[4),5)]

Here, it should be noted that the time correlation (2·7) represents a macroscopic fluctuation of the Brownian motion only if $M \gg m$, whereas the time correlation (2·8) always represents the microscopic fluctuations at the molecular level. Indeed, the Langevin equation (2·1) is valid only if $M \gg m$, i.e., $\tau_0^{(u)} \gg \tau_0^{(\gamma)}$. Therefore, in the present paper, we take a generalized Langevin equation.

As will be shown in §3, if we take $M = 2m$ as an interesting case of the Rubin model, then we have $\tau_0^{(u)} = \tau_0^{(\gamma)}$ for the decay times of the time correlation functions (2·7) and (2·8). Then the Brownian motion of the impurity exhibits anomalous fluctuations with the decay form $t^{-3/2}$. In this case, the Langevin equation (2·1) must be generalized into a non-Markov evolution equation

$$\frac{d\mathsf{A}(t)}{dt} = \mathsf{L}\mathsf{A}(t) - \int_0^t \mathsf{\Gamma}(t-s)\mathsf{A}(s)ds + \mathsf{r}(t), \tag{2·11a}$$

$$\mathsf{A}(t) = \begin{pmatrix} x_0(t) \\ p_0(t) \end{pmatrix}, \quad \mathsf{L} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{\Gamma}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathit{\Gamma}_0(t) \end{pmatrix}, \quad \mathsf{r}(t) = \begin{pmatrix} 0 \\ r_0(t) \end{pmatrix}, \tag{2·11b}$$

where $x_0(t)$ is the position of the Brownian particle at the time t and the memory function $\mathit{\Gamma}_0(t)$ is given by the time correlation (2·8). As shown by the projection operator formalism,[4)] the fluctuating force $r_0(t)$ is given by

$$r_0(t) = \exp[t(1-\mathcal{P})i\mathcal{L}](1-\mathcal{P})i\mathcal{L}p_0(0) \tag{2·12}$$

in terms of the Liouville operator $\mathcal{L}$ and the projection operator $\mathcal{P}$:

$$\mathcal{P}g = \frac{\langle g x_0(0)\rangle}{\langle |x_0(0)|^2\rangle} x_0(0) + \frac{\langle g p_0(0)\rangle}{\langle |p_0(0)|^2\rangle} p_0(0). \tag{2·13}$$

Therefore, the fluctuations due to the fluctuating force (2·12) are the microscopic fluctuations at the molecular level, because the microscopic projection operator $(1 - \mathcal{P})$ in (2·12) excludes the macroscopic motion brought by $i\mathcal{L}$.

Multiplying the second component of (2·11a) by $p_0(0)$ and taking the long-time average, we obtain

$$\frac{dU_0(t)}{dt} = -\int_0^t \Gamma_0(t-s)U_0(s)ds. \tag{2·14}$$

This is the basic equation that gives the relation between $U_0(t)$ and $\Gamma_0(t)$. Their Fourier-Laplace transform leads to

$$\hat{U}_0(\omega) \equiv \int_0^\infty U_0(t)e^{-i\omega t}dt = \frac{1}{i\omega + \hat{\Gamma}_0(\omega)}, \tag{2·15}$$

where we have defined the Fourier-Laplace transform of (2·8)

$$\hat{\Gamma}_0(\omega) \equiv \int_0^\infty \Gamma_0(t)e^{-i\omega t}dt. \tag{2·16}$$

These quantities give a generalization of the classical theory of the Brownian motion for the macroscopic and microscopic fluctuations of the turbulent and chaotic systems. We shall calculate them explicitly in the following.

§3. Rubin model for 1D Brownian motions

The Hamiltonian of the Rubin model shown in Fig. 1 is given by

$$\mathcal{H} = \frac{P_0^2}{2M} + \sum_{j=1}^N \frac{P_j^2}{2m} + \frac{K}{2}\sum_{j=0}^N (X_{j+1} - X_j)^2 - KX_0^2, \tag{3·1}$$

which consists of N molecules at $x = X_j$ $(j = 1, 2, \ldots, N)$ and a Brownian impurity at $x = X_0$ under the periodic boundary condition,

$$X_{N+1} = X_0. \tag{3·2}$$

The last term of (3·1) is introduced to eliminate the harmonic motion of the Brownian impurity. This model is a one-dimensional lattice system of frequency

$$\nu = 2\sqrt{\frac{K}{m}}, \tag{3·3}$$

coupled by harmonic springs of strength K, with a buffer.[8)]

The first statistical mechanical treatment of the fluctuating force (2·12) and the memory function (2·8) was given by Sakurai[9)] in 1965 and by Fick and Sauermann[10)] in 1990. Here, we briefly describe their treatment from a new point of view.

We obtain[9), 10)]

$$\Gamma_0(t) = \frac{m}{M}\nu^2\frac{J_1(\nu t)}{\nu t}, \tag{3·4}$$

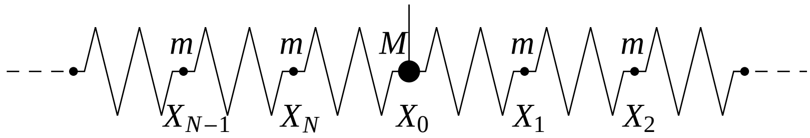

Fig. 1. Heavy impurity of mass M in chain of harmonic oscillators.

where $J_1(z)$ is the Bessel function of the first kind and we have $\Gamma_0(0) = m\nu^2/(2M)$. The derivation of (3·4) is described in detail in Appendix A. The Fourier-Laplace transformation of (3·4) gives

$$\hat{\Gamma}_0(\omega) = \frac{m}{M}\left(-i\omega + \sqrt{\nu^2 - \omega^2}\right), \tag{3·5}$$

and the substitution of (3·5) into (2·15) leads to

$$\hat{U}_0(\omega) = \frac{M/m}{[(M/m)-1]i\omega + \sqrt{\nu^2 - \omega^2}}. \tag{3·6}$$

The two basic time scales $\tau_0^{(u)}$ and $\tau_0^{(\gamma)}$ are expressed by

$$\tau_0^{(u)} = \hat{U}_0(0) = \frac{M}{m\nu} \tag{3·7}$$

and

$$\tau_0^{(\gamma)} = \frac{\hat{\Gamma}_0(0)}{\Gamma_0(0)} = \frac{2}{\nu}, \tag{3·8}$$

where we have $\tau_0^{(u)}/\tau_0^{(\gamma)} = M/(2m)$. In Fig. 2, the broken and solid lines represent the m/M dependence of $\tau_0^{(u)}$ and $\tau_0^{(\gamma)}$, respectively.

First, we consider the case of $\tau_0^{(u)} = \tau_0^{(\gamma)}$ $(M/m = 2)$. Then, (3·6) leads to

$$\hat{U}_0(\omega) = \frac{2}{i\omega + \sqrt{\nu^2 - \omega^2}} = \frac{4}{\nu^2}\hat{\Gamma}_0(\omega). \tag{3·9}$$

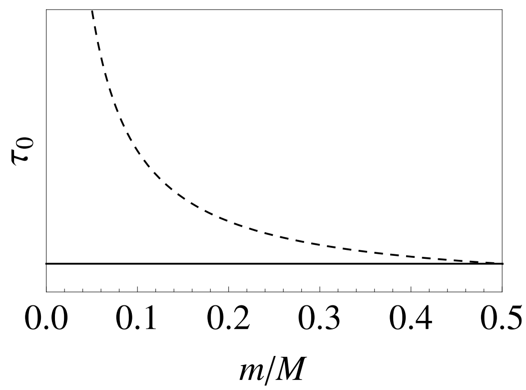

Fig. 2. Broken line: $\tau_0^{(u)}$ in (3·7); solid line: $\tau_0^{(\gamma)}$ in (3·8).

Using the inverse transform of (2·15), we obtain from (3·9)

$$U_0(t) = \frac{4}{\nu^2}\Gamma_0(t) \tag{3·10}$$

$$= 2\frac{J_1(\nu t)}{\nu t}, \tag{3·11}$$

which is equivalent to Rubin's result.[8] Thus, it turns out that, at the singular point $\tau_0^{(u)} = \tau_0^{(\gamma)}$, the decay forms of $U_0(t)$ and $\Gamma_0(t)$ become

$$U_0(t) \approx \frac{1}{\sqrt{\pi}}\left(\frac{t}{\tau_0^{(u)}}\right)^{-3/2} \cos\left(\frac{2t}{\tau_0^{(u)}} - \frac{3\pi}{4}\right) \quad \text{for} \quad t > \tau_0^{(u)} \tag{3·12a}$$

and

$$\Gamma_0(t) \approx \frac{\nu^2}{4\sqrt{\pi}}\left(\frac{t}{\tau_0^{(\gamma)}}\right)^{-3/2} \cos\left(\frac{2t}{\tau_0^{(\gamma)}} - \frac{3\pi}{4}\right) \quad \text{for} \quad t > \tau_0^{(u)}, \tag{3·12b}$$

both of which exhibit $t^{-3/2}$ anomalous microscopic fluctuations. Here, we use the relation

$$J_1(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{3}{4\pi}\right) \quad \text{for} \quad z \gtrsim 2. \tag{3·13}$$

Clearly, (3·11) leads to the algebraic forms

$$U_0(t) \approx \frac{1}{1+\left[t/(\sqrt{2}\tau_0^{(u)})\right]^2} = \frac{1}{1+\left[t\Big/\sqrt{2\tau_0^{(u)}\tau_0^{(\gamma)}}\right]^2} \quad \text{for} \quad t < \tau_0^{(\gamma)} \tag{3·14}$$

and

$$\Gamma_0(t) \approx \frac{\nu^2}{4}\frac{1}{1+\left[t/(\sqrt{2}\tau_0^{(\gamma)})\right]^2} \quad \text{for} \quad t < \tau_0^{(\gamma)}, \tag{3·15}$$

where we use the relation

$$\frac{J_1(z)}{z} \approx \frac{1}{2+(z/2)^2} \quad \text{for} \quad z \lesssim 2. \tag{3·16}$$

Second, we consider the case of $\tau_0^{(u)} \gg \tau_0^{(\gamma)}$ ($M \gg m$). The decay form of $\Gamma_0(t)$ becomes

$$\Gamma_0(t) = 2\int_0^\infty \Gamma_0(s) ds\, \delta(t) = \frac{2\nu m}{M}\int_0^\infty \frac{J_1(s)}{s} ds\, \delta(t) = \frac{2\delta(t)}{\tau_0^{(u)}} \quad \text{for} \quad t > \tau_0^{(u)}, \tag{3·17}$$

because

$$\frac{J_1(\nu t)}{\nu t} = \frac{J_1(2t/\tau_0^{(\gamma)})}{2t/\tau_0^{(\gamma)}} \approx 0 \quad \text{for} \quad t > \tau_0^{(u)} \gg \tau_0^{(\gamma)}. \tag{3·18}$$

Equations (2·14) and (3·17) lead to the exponential form

$$U_0(t) \propto e^{-t/\tau_0^{(u)}} \quad \text{for} \quad t > \tau_0^{(u)}. \tag{3·19}$$

Using (3·16), we obtain

$$\Gamma_0(t) \approx \frac{2M}{m\nu^2}\frac{1}{1+\left[t/(\sqrt{2}\tau_0^{(\gamma)})\right]^2} \quad \text{for} \quad t < \tau_0^{(\gamma)} \tag{3·20}$$

from (3·4). The substitution of (3·20) into (2·14) leads to

$$\frac{dU_0(t)}{dt} \approx -\frac{2M}{m\nu^2}\frac{tU_0(t)}{1+\left[t/(\sqrt{2}\tau_0^{(\gamma)})\right]^2}, \tag{3·21}$$

which is integrated to give the algebraic form

$$U_0(t) \approx \frac{1}{\left\{1+\left[t/(\sqrt{2}\tau_0^{(\gamma)})\right]^2\right\}^{2m/M}} \approx \frac{1}{1+\left[t\Big/\sqrt{2\tau_0^{(u)}\tau_0^{(\gamma)}}\right]^2} \quad \text{for} \quad t < \tau_0^{(\gamma)}. \tag{3·22}$$

We summarize the various decay forms of $U_0(t)$ and $\Gamma_0(t)$ in the initial regime $t < \tau_0^{(\gamma)}$ and in the final regime $t > \tau_0^{(u)}$ in Table I. The form and structure in the initial regime are built by deterministic coherent short orbits, while those in the final regime are built by stochastic and random long orbits. Here, it should be noted that the power-law form $t^{-3/2}$ is created by the microscopic fluctuations at a singular point.

Finally, we consider the case of $2\tau_0^{(u)} = \tau_0^{(\gamma)}$ $(M = m)$, incidentally. The memory function $\Gamma_0(t)$ is described by

$$\Gamma_0(t) = \nu^2\frac{J_1(\nu t)}{\nu t}, \tag{3·23}$$

which is easily obtained from (3·4), while the time correlation function $U_0(t)$ is described by

$$U_0(t) = J_0(\nu t), \tag{3·24}$$

Table I. Rubin model's time correlations characterized by $\tau_0^{(u)}$ and $\tau_0^{(\gamma)}$.

Time scale	Initial regime $t < \tau_0^{(\gamma)}$	Final regime $t > \tau_0^{(u)}$
$\tau_0^{(u)} \gg \tau_0^{(\gamma)}$ $(M \gg 2m)$	$U_0(t) = \dfrac{1}{1+\left[t\Big/\sqrt{2\tau_0^{(u)}\tau_0^{(\gamma)}}\right]^2}$	$U_0(t) \propto e^{-t/\tau_0^{(u)}}$
	: algebraic form	: exponential form
	$\Gamma_0(t) \propto \dfrac{1}{1+\left[t/(\sqrt{2}\tau_0^{(\gamma)})\right]^2}$	$\Gamma_0(t) = \dfrac{2}{\tau_0^{(u)}}\delta(t)$
$\tau_0^{(u)} = \tau_0^{(\gamma)}$ $(M = 2m)$	$U_0(t) = \dfrac{1}{1+\left[t\Big/\sqrt{2\tau_0^{(u)}\tau_0^{(\gamma)}}\right]^2}$	$U_0(t) \propto t^{-3/2}\cos\left(\dfrac{2t}{\tau_0^{(u)}} - \dfrac{3\pi}{4}\right)$
	: algebraic form	: power-law form
	$\Gamma_0(t) \propto \dfrac{1}{1+\left[t/(\sqrt{2}\tau_0^{(\gamma)})\right]^2}$	$\Gamma_0(t) \propto t^{-3/2}\cos\left(\dfrac{2t}{\tau_0^{(\gamma)}} - \dfrac{3\pi}{4}\right)$

which is obtained from (3·6) using the inverse transform of (2·15). Equation (3·24) is equivalent to Rubin's result.[8)]

§4. Two basic decay times of KS equation for turbulence

To clarify the statistical mechanical properties of turbulence, let us consider the one-dimensional KS equation[1),2),11)] from the viewpoint of Brownian motion. The equation of motion for $u(x,t)$ at the position x and the time t is given by

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0. \tag{4·1}$$

Now, we assume that the system size L is sufficiently large so that the system becomes a chaotic and statistically uniform steady state. Taking a mode $u_k(t)$ with the wavenumber k expressed as

$$u_k(t) \equiv \int_0^L u(x,t)e^{-ikx}dx, \tag{4·2a}$$

$$k \equiv \{k_n\}, \qquad k_n \equiv \frac{2\pi n}{L}, \quad (n = \pm 1, \pm 2, \ldots, \pm N), \tag{4·2b}$$

we can write (4·1) as

$$\frac{du_k(t)}{dt} = (k^2 - k^4)u_k(t) + N_k(t), \tag{4·3a}$$

$$N_{k_n}(t) \equiv -\frac{i}{L}\sum_{m=-N}^{N} k_m u_{k_n-k_m}(t)u_{k_m}(t). \tag{4·3b}$$

To clarify the structure of turbulence and chaos due to the nonlinear force $N_k(t)$, the evolution equation (4·3) has been solved numerically for $L = 500$, $N = 2^8$.[1)–3)]

Corresponding to the time correlation (2·7), the time correlation of the mode $u_k(t)$ with wavenumber k is considered

$$U_k(t) \equiv \frac{\langle u_k(t)u_k^*(0)\rangle}{\langle |u_k(0)|^2\rangle}. \tag{4·4}$$

According to the projection operator method,[1),4)] we obtain

$$N_{k_n}(t) = i\sum_m \Omega_{k_n k_m} u_{k_m}(t) - \sum_m \int_0^t \Gamma_{k_n k_m}(t-s)u_{k_m}(s)ds + r_{k_n}(t), \tag{4·5}$$

where we define the frequency matrix[4)]

$$i\Omega_{k_n k_m} \equiv \sum_l \langle N_{k_n}(0)u_{k_l}^\dagger(0)\rangle \left[\langle u_k(0)u_k^\dagger(0)\rangle^{-1}\right]_{lm}, \tag{4·6}$$

the fluctuating force

$$r_k(t) \equiv \exp[t(1-\mathcal{P})i\mathcal{L}](1-\mathcal{P})N_k(0) \tag{4·7}$$

and the memory function

$$\Gamma_{k_n k_m}(t) \equiv \sum_l \langle r_{k_n}(t) r^\dagger_{k_l}(0)\rangle \left[\langle u_k(0) u^\dagger_k(0)\rangle^{-1}\right]_{lm}. \tag{4·8}$$

Since $\langle r_k(t) u^\dagger_k(0)\rangle = 0$, the evolution equation for the time correlation (4·4) takes the form

$$\frac{dU_k(t)}{dt} = (k^2 - k^4)U_k(t) + i\Omega_{kk}U_k(t) - \int_0^t \Gamma_{kk}(t-s)U_k(s)ds. \tag{4·9}$$

Here, we assume that the system is in a statistically uniform steady state. Equation (4·9) is a generalization of (2·14). Using the parity invariance,[12)] we obtain from (4·9)

$$\frac{dU_k(t)}{dt} = -\int_0^t \Gamma_k(t-s)U_k(s)ds, \tag{4·10}$$

where k denotes the k_n of (4·2b), and $\Gamma_k(t)$ is defined as

$$\Gamma_k(t) \equiv \frac{\langle r_k(t) r^\dagger_k(0)\rangle}{\langle |u_k(0)|^2\rangle}. \tag{4·11}$$

Therefore, as the framework of the Brownian motion theory summarized in §2, the evolution equation (4·10) and the memory function (4·11) are obtained. This means that the fluctuations in the KS equation (4·1) can be treated similarly to the Rubin model of §2.

Here, note that if the wavenumber k is small, then the correlation (4·4) exhibits macroscopic fluctuations, but if the wavenumber k is large, then the correlation (4·4) exhibits microscopic fluctuations. The memory function (4·11), however, always exhibits microscopic fluctuations.

Taking the Fourier-Laplace transform of (4·10), we obtain an important relation between $\hat{U}_k(\omega)$ and $\hat{\Gamma}_k(\omega)$:

$$\hat{U}_k(\omega) \equiv \int_0^\infty U_k(t)e^{-i\omega t}dt = \frac{1}{i\omega + \hat{\Gamma}_k(\omega)}, \tag{4·12}$$

where $\hat{U}_k(\omega)$ and $\hat{\Gamma}_k(\omega)$ are the generalizations of (2·15) and (2·16), respectively.

In correspondence with (3·7) and (3·8), the two basic time scales $\tau^{(u)}_k$ and $\tau^{(\gamma)}_k$ are introduced by

$$\tau^{(u)}_k \equiv \int_0^\infty U_k(t)dt = \hat{U}_k(0) = \frac{1}{\hat{\Gamma}_k(0)} \tag{4·13}$$

and

$$\tau^{(\gamma)}_k \equiv \frac{1}{\Gamma_k(0)}\int_0^\infty \Gamma_k(t)dt = \frac{\hat{\Gamma}_k(0)}{\Gamma_k(0)} = \frac{1}{\tau^{(u)}_k \Gamma_k(0)}. \tag{4·14}$$

In Fig. 3, corresponding to Fig. 2, the broken and solid lines represent the k depen-

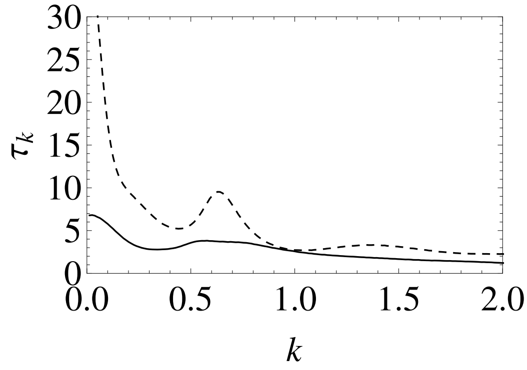

Fig. 3. $\tau_k^{(u)}$ (broken line) and $\tau_k^{(\gamma)}$ (solid line) as functions of k.

dences of $\tau_k^{(u)}$ and $\tau_k^{(\gamma)}$, respectively, thus leading to

$$\tau_k^{(u)} \propto \frac{1}{k^2} \quad \text{for} \quad k \ll 1, \tag{4·15a}$$

$$\tau_k^{(u)} \gg \tau_k^{(\gamma)} \quad \text{for} \quad k \lesssim 0.15, \tag{4·15b}$$

$$\tau_k^{(u)} \approx \tau_k^{(\gamma)} \quad \text{for} \quad k \gtrsim 1. \tag{4·15c}$$

Here, $\tau_k^{(u)}$ with $k \ll 1$ is the decay time of the macroscopic correlation $U_k(t)$, whereas $\tau_k^{(u)}$ with $k \gtrsim 1$ is the decay time of the microscopic correlation $U_k(t)$, and $\tau_k^{(\gamma)}$ is the decay time of the microscopic correlation (4·11) of the fluctuating force $r_k(t)$. At the singular point $\tau_k^{(u)} = \tau_k^{(\gamma)}$, both $U_k(t)$ and $\Gamma_k(t)$ are expected to exhibit microscopic correlations and develop the same power-law decay $t^{-3/2}$, as shown in (3·12). We have not, however, obtained the power-law decay in the present numerical simulation because of the lack of the ensemble number. Thus, we use the semi-analytical result[2] for the power-law decay.

The basic time scales $\tau_k^{(u)}$ and $\tau_k^{(\gamma)}$ respectively defined by (4·13) and (4·14) are just generalizations of the time scales (3·7) and (3·8). Table II shows a summary of the decay form of $U_k(t)$ characterized by $\tau_k^{(u)}/\tau_k^{(\gamma)}$. For the derivation of the time correlation functions in Table II, see Appendix B. Comparing Tables I and II, we found that $U_k(t)$ has the decay form of similar structure to $U_0(t)$.

To end this section, we discuss why the decay form of the time correlation function in the Rubin model is identical to that in the KS equation under the affinity assumption at the singular point $\tau_k^{(u)} = \tau_k^{(\gamma)}$. The time evolution equation (2·14) in the Rubin model is the same as (4·10) in the KS equation. Furthermore, another relation between the time correlation function and the memory function is necessary to solve the time evolution equation, and the relation in the Rubin model is generally different from that in the KS equation. Hence, the time correlation function in the Rubin model is different from that in the KS equation. At the singular point $\tau_k^{(u)} = \tau_k^{(\gamma)}$, however, we see that both of the relations coincide by comparing (3·10) with (B·1). Therefore the time correlation function (3·11) coincides with (B·10) in

Table II. KS equation's time correlations characterized by $\tau_k^{(u)}$ and $\tau_k^{(\gamma)}$.

Time scale	Initial regime $t < \tau_k^{(\gamma)}$	Final regime $t > \tau_k^{(u)}$
$\tau_k^{(u)} \gg \tau_k^{(\gamma)}$ for $k \lesssim 0.15$	$U_k(t) = \dfrac{1}{1 + \left[t \Big/ \sqrt{2\tau_k^{(u)}\tau_k^{(\gamma)}}\right]^2}$: algebraic form	$U_k(t) \propto e^{-t/\tau_k^{(u)}}$: exponential form
$\tau_k^{(u)} \approx \tau_k^{(\gamma)}$ for $k \gtrsim 1$	$U_k(t) = \dfrac{1}{1 + \left[t \Big/ \sqrt{2\tau_k^{(u)}\tau_k^{(\gamma)}}\right]^2}$: algebraic form	$U_k(t) \propto t^{-3/2} \cos\left(\dfrac{2t}{\tau_k^{(u)}} - \dfrac{3\pi}{4}\right)$: power-law form

the entire region at the singular point $\tau_k^{(u)} = \tau_k^{(\gamma)}$, which means that both of the decay forms are also consistent.

§5. Short summary

We have clarified the statistical mechanical structures of turbulence and chaos by taking the one-dimensional Rubin system (3·1) and the one-dimensional KS equation (4·3). Their results are summarized in Tables I and II, which show that the KS equation's time correlation, $U_k(t)$, has the decay form of similar structure to the Rubin model's time correlation $U_0(t)$.

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Dr. Hazime Mori, a prominent scientist in the field of statistical physics, died of cirrhosis of the liver at the age of eighty-five on 28 December, 2011, during the preparation of this paper. May he rest in peace.

Appendix A
—— *Derivation of (3·4)* ——

The Hamiltonian (3·1) yields the equations of motion

$$\frac{d^2x_0}{dt^2} + \sum_{s=1}^{N} \chi_s x_s = 0, \qquad \frac{d^2x_s}{dt^2} + \omega_s^2 x_s + \chi_s x_0 = 0 \quad (s = 1, 2, \ldots, N) \tag{A·1}$$

under a change of the variables[13)]

$$x_0 = \sqrt{M} X_0, \qquad x_s = \sqrt{\frac{m}{N}} \sum_{j=1}^{N} X_j \sin\frac{\pi j s}{N+1}, \quad (s = 1, 2, \ldots, N) \tag{A·2}$$

where

$$\chi_s \equiv -\frac{K}{\sqrt{MmN}}[1 - (-1)^s] \sin\frac{\pi s}{N+1}, \tag{A·3}$$

$$\omega_s \equiv \nu \sin \frac{\pi s}{2(N+1)}. \tag{A·4}$$

It is important to note that x_i and x_j $(i \neq j)$ become independent as N approaches infinity, because χ_s becomes zero as N approaches infinity. Thus,

$$\langle x_i x_j \rangle \to 0 \quad (i \neq j) \quad \text{for} \quad N \to \infty, \tag{A·5}$$

whereas $\langle X_i X_j \rangle \neq 0$ even for $i \neq j$ and $N \to \infty$. In the following, x_i is denoted by $x_i(t)$.

Expanding the fluctuating force (2·12) into the Taylor series in time, we obtain

$$r_0(t) = r_0(0) + \sum_{n=1}^{\infty} \frac{t^n}{n!} r_0^{(n)}, \tag{A·6}$$

where, using (2·13), we have

$$\begin{aligned} r_0(0) &= (1-\mathcal{P}) i\mathcal{L} p_0(0) \\ &= i\mathcal{L} p_0(0) - \frac{\langle [i\mathcal{L} p_0(0)] x_0(0) \rangle}{\langle |x_0(0)|^2 \rangle} x_0(0) - \frac{\langle [i\mathcal{L} p_0(0)] p_0(0) \rangle}{\langle |p_0(0)|^2 \rangle} p_0(0), \end{aligned} \tag{A·7}$$

so that we have[9)]

$$\begin{aligned} r_0^{(n)} &= (1-\mathcal{P}) i\mathcal{L} r_0^{(n-1)} \\ &= i\mathcal{L} r_0^{(n-1)} - \frac{\left\langle \left[i\mathcal{L} r_0^{(n-1)} \right] x_0(0) \right\rangle}{\langle |x_0(0)|^2 \rangle} x_0(0) - \frac{\left\langle \left[i\mathcal{L} r_0^{(n-1)} \right] p_0(0) \right\rangle}{\langle |p_0(0)|^2 \rangle} p_0(0). \end{aligned} \tag{A·8}$$

Using (A·5) and $\langle x_i(t) p_0(t) \rangle = 0$, we obtain

$$r_0 = -\sum_{s=1}^{\infty} \chi_s x_s(0) \tag{A·9}$$

from (A·7). Similarly, we obtain

$$r_0^{(n)} = \begin{cases} -\sum_s \chi_s \omega_s^n x_s(0), & (n : \text{even}) \\ -\sum_s \chi_s \omega_s^{n-1} p_s(0). & (n : \text{odd}) \end{cases} \tag{A·10}$$

It is important to note that the fluctuating force $r_0(t)$ does not include the macroscopic motion related to $x_0(0)$ or $p_0(0)$: in other words, the projection operator $\mathcal{P}$ successfully excludes the macroscopic motion from $dp_0(t)/dt$.

Using (A·10) and $\langle x_s(t) p_s(t) \rangle = 0$, we obtain

$$\left\langle r_0^{(n)} r_0 \right\rangle = \begin{cases} (-1)^{n/2} \langle |p_0(0)|^2 \rangle \sum_{s=1}^{\infty} \chi_s^2 \omega_s^{n-2}, & (n : \text{even}) \\ 0. & (n : \text{odd}) \end{cases} \tag{A·11}$$

Here, we use the equipartition law for $N \to \infty$:

$$\langle |p_0(t)|^2 \rangle = \omega_s^2 \langle |x_s(t)|^2 \rangle, \tag{A·12}$$

which is derived from the Hamiltonian of (A·1),

$$\mathcal{H} = \frac{p_0^2}{2} + \sum_{s=1}^{N} \left(\frac{p_s^2}{2} + \frac{\omega_s^2 x_s^2}{2} \right) + x_0 \sum_{s=1}^{N} \chi_s x_s. \tag{A·13}$$

Equations (A·11) and (A·6) lead to

$$\langle r_0(t) r_0 \rangle = \langle |p_0(0)|^2 \rangle \sum_{s=1}^{\infty} \left(\frac{\chi_s}{\omega_s} \right)^2 \cos(\omega_s t). \tag{A·14}$$

Therefore, substituting (A·14) into the memory function (2·8), we obtain

$$\begin{aligned} \Gamma_0(t) &= \lim_{N\to\infty} \sum_{s=0}^{N} \left(\frac{\chi_s}{\omega_s} \right)^2 \cos(\omega_s t) = \lim_{N\to\infty} \sum_{s=0}^{(N-1)/2} \left(\frac{\chi_{2s+1}}{\omega_{2s+1}} \right)^2 \cos(\omega_{2s+1} t) \\ &= \lim_{N\to\infty} \frac{2}{\pi} \frac{m}{M} \frac{N+1}{N} \int_0^{\nu} \sqrt{\nu^2 - \omega^2} \cos(\omega t) d\omega = \frac{m}{M} \nu^2 \frac{J_1(\nu t)}{\nu t}, \end{aligned} \tag{A·15}$$

where $J_1(x)$ is the Bessel function of the first kind. We use the relations

$$ds = \frac{2(N+1)}{\pi} \frac{d\omega}{\sqrt{\nu^2 - \omega^2}}, \qquad \left(\frac{\chi_{2s+1}}{\omega_{2s+1}} \right)^2 = \frac{m}{M} \frac{\nu^2 - \omega_{2s+1}^2}{N}, \tag{A·16}$$

obtained from (A·3) and (A·4).

Appendix B
—— *Time Correlation Functions under the Affinity between the Time Correlation Function and the Memory Function* ——

We briefly state the derivation of the time correlation functions in the initial and final regimes,[2] which are given in Table II. Let us assume the affinity between the time correlation function $U_k(t)$ and the memory function $\Gamma_k(t)$ as follows:

$$Q_k(T) \equiv U_k\left(T \tau_k^{(u)}\right) = \frac{\Gamma_k\left(T \tau_k^{(\gamma)}\right)}{\Gamma_k(0)}. \tag{B·1}$$

Substituting (B·1) into (4·10), we obtain the closure equation

$$\frac{dQ_k(T)}{dT} = -\int_0^{T/\tilde{\tau}_k} Q_k(S) Q_k(T - \tilde{\tau}_k S) dS, \tag{B·2}$$

where we use (4·13) and define $\tilde{\tau}_k \equiv \tau_k^{(\gamma)} / \tau_k^{(u)}$.

B.1. *Case of* $t < \tau_k^{(\gamma)}$

In the case of $T \ll \tilde{\tau}_k$, (B·2) is reduced to

$$\frac{dQ_k(T)}{dT} \approx -\frac{[Q_k(T)]^2 T}{\tilde{\tau}_k}, \tag{B·3}$$

and its solution is

$$Q_k(T) = \frac{2\tilde{\tau}_k}{T^2 + 2\tilde{\tau}_k} \quad \text{for} \quad T \ll \tilde{\tau}_k, \tag{B·4}$$

which is called an algebraic form and is identical to

$$U_k(t) = \frac{1}{1 + \left[t \middle/ \sqrt{2\tau_k^{(u)}\tau_k^{(\gamma)}}\right]^2} \quad \text{for} \quad t \ll \tau_k^{(\gamma)}. \tag{B·5}$$

In Table II, $t \ll \tau_k^{(\gamma)}$ is replaced with $t < \tau_k^{(\gamma)}$ because the numerical results support the latter condition.[2)]

B.2. *Case of* $\tau_k^{(u)} \gg \tau_k^{(\gamma)}$ *and* $t > \tau_k^{(u)}$

In the case of $\tilde{\tau}_k \ll T$, (B·2) yields the solution

$$Q_k(T) \propto e^{-T} \quad \text{for} \quad T \gtrsim 1, \tag{B·6}$$

which is called an exponential form and is identical to

$$U_k(t) \propto e^{-t/\tau_k^{(u)}} \quad \text{for} \quad t > \tau_k^{(u)}. \tag{B·7}$$

B.3. *Case of* $\tau_k^{(u)} = \tau_k^{(\gamma)}$ *and* $t > \tau_k^{(u)}$

The Fourier-Laplace transform of (B·2) yields

$$\hat{Q}_k(\Omega)\hat{Q}_k(\Omega\tilde{\tau}_k) + i\Omega\hat{Q}_k(\Omega) - 1 = 0. \tag{B·8}$$

In the case of $\tilde{\tau}_k = 1$, we obtain the exact solution

$$\hat{Q}_k(\Omega) = -i\frac{\Omega}{2} + \sqrt{1 - \left(\frac{\Omega}{2}\right)^2} \tag{B·9}$$

to (B·8). The inverse transform of (B·9) yields the time correlation function

$$Q_k(T) = \frac{1}{T}J_1(2T) \approx \frac{T^{-3/2}}{\sqrt{\pi}} \cos\left(2T - \frac{3\pi}{4}\right) \quad \text{for} \quad T \to \infty, \tag{B·10}$$

which becomes

$$U_k(t) \propto t^{-3/2} \cos\left(\frac{2t}{\tau_k^{(u)}} - \frac{3\pi}{4}\right) \quad \text{for} \quad t > \tau_k^{(u)}. \tag{B·11}$$

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