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Modal time correlation functions for homogeneous isotropic turbulence in a projection operator method

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Applying a projection operator method to homogeneous isotropic turbulence in the Lagrangian and Eulerian specifications of the flow field, we obtain a closure equation, which is called the SA equation, for the time correlation function. We compare its solutions with the time correlation function obtained from direct numerical simulation in the Eulerian specification and with that obtained from direct interaction approximation in the Lagrangian specification. Results of the comparison show that the SA equation can express time correlation for homogeneous isotropic turbulence in the Lagrangian and Eulerian specifications.

I. INTRODUCTION

Time correlation can describe flow dynamics and is therefore an important physical quantity in turbulence. However, it has not been actively treated in turbulence studies except in direct interaction approximation (DIA). Some papers[1–4] treat the time correlation function in connection with the characteristic time τ dependence on wave number k. The Eulerian DIA[1] yields $\tau \sim k^{-1}$, while the Lagrangian DIA[2] gives $\tau \sim k^{-2/3}$. Numerical simulations support these results,[3,4] although Eulerian DIA gives an unsuitable energy spectrum in the inertial range. Even in DIA, the central topic is the energy spectrum, which corresponds to the modal time correlation function $\langle \hat{u}_\alpha(\boldsymbol{k};t)\hat{u}^*_\beta(\boldsymbol{k};0)\rangle$ at $t = 0$, where $\hat{u}_\alpha(\boldsymbol{k};t)$ is the Fourier coefficient of the fluid velocity $u_\alpha(\boldsymbol{x},t)$, and the angular brackets $\langle\cdot\rangle$ and $*$ denote the ensemble average and complex conjugation, respectively. Only the modal time correlation function is treated in the present paper, and thus "modal" is dropped from this point forward.

In statistical mechanics, the time correlation function is a central topic of research, including the fluctuation-dissipation theorem which expresses the relationship between impulse response and time correlation.[5] Using a projection operator method developed in statistical mechanics,[6] the present author has treated time correlation functions for one-dimensional Brownian motion[7] and one-dimensional turbulence, such as the Kuramoto-Sivashinsky equation.[8–10] These results are satisfactory but may not interest fluid dynamicists because these treated equations are not entirely related to fluid dynamics. In this paper, we apply the projection operator method to homogeneous isotropic turbulence in the Navier-Stokes equations.

The present paper is organized as follows. In Sec. II, we review the projection operator method by applying it to a set of general evolution equations and by deriving the generalized Langevin equation. In Sec. III, we apply the projection operator method to homogeneous isotropic turbulence in the Navier-Stokes equations with two representations, the Eulerian specification and the Lagrangian specification. We obtain the evolution equation of the time correlation function, which is not closed. In Sec. IV, we propose a closure assumption to obtain a closed equation for the time correlation function and compare its solutions with the time correlation function obtained from direct numerical simulation under the Eulerian specification and with that obtained from DIA under the Lagrangian specification. Conclusions are drawn in Sec. V.

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II. REVIEW OF PROJECTION OPERATOR METHOD

We briefly review the projection operator method[6,11] by using a set of M evolution equations for $U_n(t)$

$$\frac{dU_n(t)}{dt} = H_n(\boldsymbol{U}(t)), \qquad n = 1, 2, \ldots, M, \tag{1}$$

$$\boldsymbol{U}(t) = (U_1(t), U_2(t), \ldots, U_M(t))^T, \tag{2}$$

where T denotes the transpose operation and $H_n(\boldsymbol{U}(t))$ is a given arbitrary function of $\boldsymbol{U}(t)$.

We begin by introducing three important operators $\mathcal{P}$, $\mathcal{P}'$ and Λ. The first operator $\mathcal{P}$ projects a function $f(\boldsymbol{U}(t))$ onto an M-dimensional vector space $\boldsymbol{U}(0)$ and is defined as

$$\mathcal{P} f(\boldsymbol{U}(t)) = \sum_{i=1}^{M} \sum_{j=1}^{M} \langle f(\boldsymbol{U}(t)) U_i^*(0)\rangle \left[\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle^{-1}\right]_{ij} U_j(0), \tag{3}$$

where $\dagger$ denotes the Hermite conjugation and thus $\boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)$ becomes an $M \times M$ matrix. The operator $\mathcal{P}$ is called a projection operator because $\mathcal{P}^2 = \mathcal{P}$. The second projection operator $\mathcal{P}'$ is defined as

$$\mathcal{P}' = 1 - \mathcal{P}, \tag{4}$$

which gives the rest of the projection by $\mathcal{P}$. It is easy to derive the relation

$$\mathcal{P}\mathcal{P}' = 0, \tag{5}$$

which shows that the two projections by $\mathcal{P}$ and $\mathcal{P}'$ are orthogonal. The third operator Λ is defined as

$$\Lambda = \sum_{n=1}^{M} \frac{dU_n(0)}{dt} \frac{\partial}{\partial U_n(0)}, \tag{6}$$

where we have used the abridged notation $dU_n(0)/dt$ defined as $dU_n(0)/dt = dU_n(t)/dt|_{t=0}$. Because the operator Λ satisfies two relations

$$U_n(t) = e^{\Lambda t} U_n(0) \quad \text{and} \quad H_n(\boldsymbol{U}(t)) = e^{\Lambda t} H_n(\boldsymbol{U}(0)), \tag{7}$$

$e^{\Lambda t}$ becomes a time evolution operator.

By using (4) and (7), the left-hand side of (1) is transformed into

$$\frac{dU_n(t)}{dt} = e^{\Lambda t}\frac{dU_n(0)}{dt} = e^{\Lambda t}\mathcal{P}\frac{dU_n(0)}{dt} + e^{\Lambda t}\mathcal{P}'\frac{dU_n(0)}{dt}. \tag{8}$$

Substituting (3) into the first term of the right-hand side of (8), we obtain

$$e^{\Lambda t}\mathcal{P}\frac{dU_n(0)}{dt} = \sum_{j=1}^{M} \Omega_{nj} U_j(t), \tag{9}$$

where Ω_{nj} is defined as

$$\Omega_{nj} = \sum_{i=1}^{M} \left\langle \frac{dU_n(0)}{dt} U_i^*(0) \right\rangle \left[\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle^{-1}\right]_{ij}. \tag{10}$$

Substituting the identity

$$e^{\Lambda t} = \int_0^t e^{\Lambda(t-s)} \mathcal{P} \Lambda e^{\mathcal{P}' \Lambda s} ds + e^{\mathcal{P}' \Lambda t} \tag{11}$$

into the second term of the right-hand side of (8) and using (3), we obtain

$$e^{\Lambda t} \mathcal{P}' \frac{dU_n(0)}{dt} = -\sum_{j=1}^{M} \int_0^t \Gamma_{nj}(s) U_j(t-s) ds + r_n(t), \tag{12}$$

where $\Gamma_{nj}(t)$ and $r_n(t)$ are defined as

$$\begin{aligned} \Gamma_{nj}(t) &= -\sum_{i=1}^{M} \langle [\Lambda r_n(t)] U_i^*(0) \rangle \left[\langle \boldsymbol{U}(0) \boldsymbol{U}^\dagger(0) \rangle^{-1} \right]_{ij} \\ &= \sum_{i=1}^{M} \langle r_n(t) r_i^*(0) \rangle \left[\langle \boldsymbol{U}(0) \boldsymbol{U}^\dagger(0) \rangle^{-1} \right]_{ij} \end{aligned} \tag{13}$$

and

$$r_n(t) = e^{\mathcal{P}' \Lambda t} \mathcal{P}' \frac{dU_n(0)}{dt} = e^{\mathcal{P}' \Lambda t} r_n(0), \tag{14}$$

where the two definitions are transformed into convenient forms. The derivation of (13) is given in Appendix A. It is important to note that statistical steadiness is used to derive (13).

From (8), (9) and (12), we obtain the generalized Langevin equation[5]

$$\frac{dU_n(t)}{dt} = \sum_{j=1}^{M} \Omega_{nj} U_j(t) - \sum_{j=1}^{M} \int_0^t \Gamma_{nj}(s) U_j(t-s) ds + r_n(t), \tag{15}$$

which is exactly derived under the assumption of statistical steadiness.

Now we consider the roles of $\mathcal{P}$ and $\mathcal{P}'$. Assume that $U_n(t)$ has large slowly varying modes and small rapidly varying modes, and hence $dU_n(t)/dt$ has small slowly varying modes and large rapidly varying modes because the characteristic time scale of the rapidly varying mode is much shorter than that of the slowly varying mode. Of course, $U_n(t)$ and $dU_n(t)/dt$ exhibit randomness. The projection operator $\mathcal{P}$ defined in (3) is expressed by $U_n(0)$, and hence $\mathcal{P} f(\boldsymbol{U}(t))$ exhibits slowly varying motion, while $\mathcal{P}'$ defined in (4) is the remainder of the projection from $\mathcal{P}$, and hence $\mathcal{P}' f(\boldsymbol{U}(t))$ exhibits rapidly varying motion. Equation (9) shows that the first term of the right-hand side in (15) is the time evolution of $\mathcal{P}(dU_n(0)/dt)$, which is the projection by $\mathcal{P}$. Hence, this term exhibits slowly varying motion at $t \geq 0$. Equation (12) shows that the second and third terms in (15) are the time evolution of $\mathcal{P}'(dU_n(0)/dt)$, which is the projection by $\mathcal{P}'$ and is thus related to the rapidly varying motion. Because $e^{\Lambda t} \mathcal{P}'(dU_n(0)/dt)$ produces a new slowly varying motion for $t > 0$, these terms exhibit rapidly varying motion only at $t = 0$ and not for $t > 0$. This slowly varying motion corresponds to the second term's convolution integral, which is expressed by $U_j(t)$. The second term depends on the entire evolution history of $U_n(t)$ and is related to friction, including the memory effect. For this reason, the function $\Gamma_{nj}(t)$ is called the memory function. This function is related to a type of dissipation due to chaotic mixing such as eddy viscosity in turbulent flows. The last term $r_n(t)$ is considered to be a fluctuating force because the slowly varying motion, which is produced as $\mathcal{P}'(dU_n(0)/dt)$ evolves with time, is removed at every time step. We can state from (14) that $e^{\mathcal{P}' \Lambda t}$ is a time evolution operator for $r_n(t)$ and its time evolution is extraordinary because of the newly produced slowly varying motion removal at every time step.

The above-mentioned assumption means that chaotic motion is successfully divided into slowly varying motion and rapidly varying motion by the projection operator. Equation

(15) is exact but we do not know whether this division is reasonable. We ascertain the validity of this assumption only for the Kuramoto-Sivashinsky equation.[12]

Multiplying (15) by $U_m^*(0)$ and taking the average, we obtain a set of evolution equations for $Q_{nm}(t)$,

$$\frac{dQ_{nm}(t)}{dt} = \sum_{j=1}^{M} \Omega_{nj} Q_{jm}(t) - \sum_{j=1}^{M} \int_0^t \Gamma_{nj}(s) Q_{jm}(t-s) ds, \tag{16}$$

where $Q_{nm}(t)$ is the time correlation function, defined as

$$Q_{nm}(t) = \langle U_n(t) U_m^*(0) \rangle. \tag{17}$$

Here, we have used the relation

$$\langle r_n(t) U_m^*(0) \rangle = 0, \tag{18}$$

which is derived from the orthogonality condition (5). The derivation of (18) is given in Appendix B. It is important to note that (16) is exact under the assumption of statistical steadiness.

III. APPLICATION TO HOMOGENEOUS ISOTROPIC TURBULENCE

Consider an incompressible fluid with constant density and viscosity ν occupying a periodic cube V with a scale of L. We apply the projection operator method, reviewed in the previous section, to homogeneous isotropic turbulence in the Navier-Stokes equations with two representations, the Eulerian specification and the Lagrangian specification. We use Fourier space because it is more convenient than real space for treating isotropic turbulence.

Now we comment on homogeneous isotropic turbulence. The Navier-Stokes equations are invariant under space transformation, time transformation, isotropy and parity, and hence the statistical quantities for the Navier-Stokes equations are also expected to be invariant under the four transformations in fully developed turbulence.[14] This expectation is reasonable and therefore stationary homogeneous isotropic turbulence can be realized.

A. Eulerian specification of the flow field

The truncated Fourier transform of the fluid velocity $u_\alpha(\boldsymbol{x}, t)$ is defined as

$$\hat{u}_\alpha(\boldsymbol{k}; t) = \left(\frac{1}{L}\right)^3 \int_V u_\alpha(\boldsymbol{x}, t) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x}, \qquad \alpha = 1, 2, 3, \tag{19}$$

where

$$\boldsymbol{k} = \frac{2\pi}{L}(n_1, n_2, n_3), \qquad n_\alpha = \pm 1, \pm 2, \ldots, \pm N. \tag{20}$$

The incompressibility condition in Fourier space becomes

$$\sum_{\alpha=1}^{3} k_\alpha \hat{u}_\alpha(\boldsymbol{k}; t) = 0. \tag{21}$$

Because $u_\alpha(\boldsymbol{x}, t)$ is real, (19) yields the property

$$\hat{u}_\alpha(\boldsymbol{k}; t) = \hat{u}_\alpha^*(-\boldsymbol{k}; t). \tag{22}$$

The incompressible Navier-Stokes equations with an external force in Fourier space is written as[13]

$$\frac{d\hat{u}_\alpha(\boldsymbol{k};t)}{dt} = -\nu k^2 \hat{u}_\alpha(\boldsymbol{k};t) + \sum_{\beta=1}^{3}\sum_{\gamma=1}^{3} M_{\alpha\beta\gamma}(\boldsymbol{k}) \sum_{\boldsymbol{j}} \hat{u}_\beta(\boldsymbol{j};t)\hat{u}_\gamma(\boldsymbol{k}-\boldsymbol{j};t) + F_\alpha(\boldsymbol{k}), \tag{23}$$

where

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = \frac{1}{2i}\left[k_\beta D_{\alpha\gamma}(\boldsymbol{k}) + k_\gamma D_{\alpha\beta}(\boldsymbol{k})\right] \tag{24}$$

and

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|\boldsymbol{k}|^2}. \tag{25}$$

Here $\delta_{\alpha\beta}$ denotes the Kronecker delta and the external force $F_\alpha(\boldsymbol{k})$ satisfies $\sum_\alpha k_\alpha F_\alpha(\boldsymbol{k}) = 0$ due to the incompressibility and $F_\alpha(\boldsymbol{k}) = -F_\alpha(-\boldsymbol{k})$ due to parity invariance. With the projection operator method, the concrete form of the Navier-Stokes equations (23) is not of interest, but the four symmetries of it, space transformation, time transformation, rotation and parity, are of interest.[14] Thus we do not use (23) from this point forward.

The incompressibility condition

$$\sum_{\alpha=1}^{3} k_\alpha D_{\alpha\beta}(\boldsymbol{k}) = \sum_{\beta=1}^{3} k_\beta D_{\alpha\beta}(\boldsymbol{k}) = 0 \tag{26}$$

is derived from (21) and (25). The time correlation function in Fourier space is expressed by

$$\langle \hat{u}_\alpha(\boldsymbol{k};t)\hat{u}^*_\beta(\boldsymbol{k}';0)\rangle = \left(\frac{2\pi}{L}\right)^3 \delta_{\boldsymbol{k}\boldsymbol{k}'} Q_{\alpha\beta}(\boldsymbol{k};t), \tag{27}$$

where

$$Q_{\alpha\beta}(\boldsymbol{k};t) = D_{\alpha\beta}(\boldsymbol{k})Q(k;t), \tag{28}$$

under the assumptions of statistical homogeneity and isotropy.[13] Relation (28) is a general form for the isotropic second-rank tensor in an incompressible fluid.

Each $\boldsymbol{k}$ defined in (20) is numbered consecutively from 1 to $N' = (2N+1)^3 - 1$ for matrix representation convenience. Here, the -1 term represents the exclusion of $(0,0,0)$.

$$\boldsymbol{k}^{[1]} = \frac{2\pi}{L}(-N,-N,-N), \quad \boldsymbol{k}^{[2]} = \frac{2\pi}{L}(-N,-N,-N+1), \quad \ldots, \quad \boldsymbol{k}^{[N']} = \frac{2\pi}{L}(N,N,N). \tag{29}$$

1. Projection onto all modes

The Navier-Stokes equations (23) are expressed by (1) when an M-dimensional vector $\boldsymbol{U}(t)$ is set as

$$\boldsymbol{U}(t) = \Big(\hat{u}_1(\boldsymbol{k}^{[1]};t),\ldots,\hat{u}_1(\boldsymbol{k}^{[N']};t),\hat{u}_2(\boldsymbol{k}^{[1]};t),\ldots,\hat{u}_2(\boldsymbol{k}^{[N']};t),\hat{u}_3(\boldsymbol{k}^{[1]};t),\ldots,\hat{u}_3(\boldsymbol{k}^{[N']};t)\Big)^T, \tag{30}$$

where $N' = M/3$. We begin by examining whether the square matrix $\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle$ is regular because the projection operator $\mathcal{P}$ defined in (3) includes the inverse of $\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle$. The square matrix A is written as

$$\mathsf{A} = \langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle = \begin{pmatrix} \mathsf{K}_{11} & \mathsf{K}_{12} & \mathsf{K}_{13} \\ \mathsf{K}_{21} & \mathsf{K}_{22} & \mathsf{K}_{23} \\ \mathsf{K}_{31} & \mathsf{K}_{32} & \mathsf{K}_{33} \end{pmatrix}, \tag{31}$$

where $\mathsf{K}_{\alpha\beta}$ is a diagonal matrix defined as

$$\mathsf{K}_{\alpha\beta} = \left(\frac{2\pi}{L}\right)^3 \begin{pmatrix} Q_{\alpha\beta}(\boldsymbol{k}^{[1]};0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_{\alpha\beta}(\boldsymbol{k}^{[N']};0) \end{pmatrix}. \tag{32}$$

Introducing an M-dimensional vector

$$\boldsymbol{K} = \left(k_1^{[1]}, \ldots, k_1^{[N']}, k_2^{[1]}, \ldots, k_2^{[N']}, k_3^{[1]}, \ldots, k_3^{[N']}\right)^T, \tag{33}$$

we obtain

$$\mathsf{A}\boldsymbol{K} = \left(\frac{2\pi}{L}\right)^3 \begin{pmatrix} \sum_\beta Q_{1\beta}(\boldsymbol{k}^{[1]};0)k_\beta^{[1]} \\ \vdots \\ \sum_\beta Q_{1\beta}(\boldsymbol{k}^{[N']};0)k_\beta^{[N']} \\ \sum_\beta Q_{2\beta}(\boldsymbol{k}^{[1]};0)k_\beta^{[1]} \\ \vdots \\ \sum_\beta Q_{2\beta}(\boldsymbol{k}^{[N']};0)k_\beta^{[N']} \\ \sum_\beta Q_{3\beta}(\boldsymbol{k}^{[1]};0)k_\beta^{[1]} \\ \vdots \\ \sum_\beta Q_{3\beta}(\boldsymbol{k}^{[N']};0)k_\beta^{[N']} \end{pmatrix} = \left(\frac{2\pi}{L}\right)^3 Q(k;0) \begin{pmatrix} \sum_\beta D_{1\beta}(\boldsymbol{k}^{[1]})k_\beta^{[1]} \\ \vdots \\ \sum_\beta D_{1\beta}(\boldsymbol{k}^{[N']})k_\beta^{[N']} \\ \sum_\beta D_{2\beta}(\boldsymbol{k}^{[1]})k_\beta^{[1]} \\ \vdots \\ \sum_\beta D_{2\beta}(\boldsymbol{k}^{[N']})k_\beta^{[N']} \\ \sum_\beta D_{3\beta}(\boldsymbol{k}^{[1]})k_\beta^{[1]} \\ \vdots \\ \sum_\beta D_{3\beta}(\boldsymbol{k}^{[N']})k_\beta^{[N']} \end{pmatrix} = 0, \tag{34}$$

where we have used (26) and (28). Because $\boldsymbol{K} \neq 0$, we obtain

$$|\mathsf{A}| = |\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle| = 0, \tag{35}$$

which means that the square matrix $\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle$ is not regular due to the incompressibility. Hence, projection onto all modes is not appropriate.

2. Projection onto mode parts

Because projection onto all modes does not work well, we choose another projection operator $\mathcal{P}_\alpha$ which is defined as

$$\mathcal{P}_\alpha f(\boldsymbol{U}(t)) = \sum_{i=1}^{N'}\sum_{j=1}^{N'} \langle f(\boldsymbol{U}(t))\hat{u}_\alpha^*(\boldsymbol{k}^{[i]};0)\rangle \left[\langle \tilde{\boldsymbol{U}}_\alpha(0)\tilde{\boldsymbol{U}}_\alpha^\dagger(0)\rangle^{-1}\right]_{ij} \hat{u}_\alpha(\boldsymbol{k}^{[j]};0), \tag{36}$$

where

$$\boldsymbol{U}(t) = \begin{pmatrix} \tilde{\boldsymbol{U}}_1(t) \\ \tilde{\boldsymbol{U}}_2(t) \\ \tilde{\boldsymbol{U}}_3(t) \end{pmatrix}, \qquad \tilde{\boldsymbol{U}}_\alpha(t) = \begin{pmatrix} \hat{u}_\alpha(\boldsymbol{k}^{[1]};t) \\ \vdots \\ \hat{u}_\alpha(\boldsymbol{k}^{[N']};t) \end{pmatrix}. \tag{37}$$

The M-dimensional vector $\boldsymbol{U}(t)$ in (37) is the same as that in (30), but the projected vector space dimension in (36) is different from that in (3). Note that we do not employ the summation convention in which repeated indices are summed. Using homogeneity, (36) is simplified to

$$\mathcal{P}_\alpha f(\boldsymbol{U}(t)) = \sum_{j=1}^{N'} \frac{\langle f(\boldsymbol{U}(t))\hat{u}_\alpha^*(\boldsymbol{k}^{[j]};0)\rangle}{\langle \hat{u}_\alpha(\boldsymbol{k}^{[j]};0)\hat{u}_\alpha^*(\boldsymbol{k}^{[j]};0)\rangle} \hat{u}_\alpha(\boldsymbol{k}^{[j]};0). \tag{38}$$

It is simple to show that $\mathcal{P}_\alpha^2 = \mathcal{P}_\alpha$. We define $\mathcal{P}'_\alpha$ as

$$\mathcal{P}'_\alpha = 1 - \mathcal{P}_\alpha \tag{39}$$

like we did in (4). It is trivial that the square matrix $\langle \tilde{\boldsymbol{U}}_\alpha(0)\tilde{\boldsymbol{U}}_\alpha^\dagger(0)\rangle$ is regular.

Applying the projection operator (38) to the Navier-Stokes equations (23) and using homogeneity, we obtain the following by the manner explained in the previous subsection,

$$\frac{d\hat{u}_\alpha(\boldsymbol{k};t)}{dt} = \Omega_\alpha(\boldsymbol{k})\hat{u}_\alpha(\boldsymbol{k};t) - \int_0^t \Gamma_\alpha(\boldsymbol{k};s)\hat{u}_\alpha(\boldsymbol{k};t-s)ds + r_\alpha(\boldsymbol{k};t), \tag{40}$$

where

$$\Omega_\alpha(\boldsymbol{k}) = \frac{\left\langle \dfrac{d\hat{u}_\alpha(\boldsymbol{k};0)}{dt}\hat{u}_\alpha^*(\boldsymbol{k};0)\right\rangle}{\langle \hat{u}_\alpha(\boldsymbol{k};0)\hat{u}_\alpha^*(\boldsymbol{k};0)\rangle}, \tag{41}$$

$$\Gamma_\alpha(\boldsymbol{k};t) = \frac{\langle r_\alpha(\boldsymbol{k};t)r_\alpha^*(\boldsymbol{k};0)\rangle}{\langle \hat{u}_\alpha(\boldsymbol{k};0)\hat{u}_\alpha^*(\boldsymbol{k};0)\rangle}, \qquad r_\alpha(\boldsymbol{k};t) = e^{\mathcal{P}'_\alpha \Lambda t}\mathcal{P}'_\alpha \frac{d\hat{u}_\alpha(\boldsymbol{k};0)}{dt}. \tag{42}$$

We obtain from (C8)

$$\Omega_\alpha(\boldsymbol{k}) = 0. \tag{43}$$

Using the relations

$$\langle r_\alpha(\boldsymbol{k};t)r_\alpha^*(\boldsymbol{k};0)\rangle = \left(\frac{2\pi}{L}\right)^3 D_{\alpha\alpha}(\boldsymbol{k})R(k;t) \tag{44}$$

and

$$\langle \hat{u}_\alpha(\boldsymbol{k};t)\hat{u}_\alpha^*(\boldsymbol{k};0)\rangle = \left(\frac{2\pi}{L}\right)^3 D_{\alpha\alpha}(\boldsymbol{k})Q(k;t) \tag{45}$$

obtained from (27) and (28), we obtain

$$\Gamma_\alpha(\boldsymbol{k};t) = \frac{R(k;t)}{Q(k;0)} = \Gamma(k;t), \tag{46}$$

which shows that $\Gamma_\alpha(\boldsymbol{k};t)$ is independent of α and hence is equivalent to $\Gamma(k;t)$.

Multiplying (40) by $\hat{u}_\beta^*(\boldsymbol{k};0)$, taking the average, using three relations (18), (43) and (46), and then summing over β after setting $\alpha = \beta$, we obtain the evolution equation of $Q(k;t)$,

$$\frac{dQ(k;t)}{dt} = -\int_0^t \Gamma(k;s)Q(k;t-s)ds, \tag{47}$$

which is the same as that in one-dimensional turbulence with statistical homogeneity, steadiness and parity invariance, such as the Kuramoto-Sivashinsky equation.[9]

B. Lagrangian specification of the flow field

We now treat Lagrangian fluid velocity $v_\alpha(t|\boldsymbol{a},t')$, which is fluid particle velocity at time t that was at the point $\boldsymbol{a}$ at time t'.[15] The Eulerian velocity is simply

$$u_\alpha(\boldsymbol{x},t) = v_\alpha(t|\boldsymbol{x},t). \tag{48}$$

The Lagrangian fluid velocity $v_\alpha(t|\boldsymbol{a},t')$ satisfies the incompressibility condition (21) for $t=t'$, while it does not for $t\neq t'$. The truncated Fourier transform of the Lagrangian fluid velocity $v_\alpha(t|\boldsymbol{a},t')$ is defined as

$$\hat{v}_\alpha(\boldsymbol{k};t)=\left(\frac{1}{L}\right)^3\int_V v_\alpha(t|\boldsymbol{a},t')e^{-i\boldsymbol{k}\cdot\boldsymbol{a}}d\boldsymbol{a}, \tag{49}$$

where t' is omitted for simplification. The time correlation function in Fourier space is expressed by

$$\langle\hat{v}_\alpha(\boldsymbol{k};t)\hat{v}^*_\beta(\boldsymbol{k}';0)\rangle=\left(\frac{2\pi}{L}\right)^3\delta_{\boldsymbol{k}\boldsymbol{k}'}Q_{\alpha\beta}(\boldsymbol{k};t), \tag{50}$$

under the assumption of statistical homogeneity. Note that the functional form of $Q_{\alpha\beta}(\boldsymbol{k};t)$ in (50) is different from that in (27), although we use the same notation. The Navier-Stokes equations with the incompressibility condition are expressed by

$$\frac{d\hat{v}_\alpha(\boldsymbol{k};t)}{dt}=H_\alpha(\boldsymbol{k};t), \tag{51}$$

where $H_\alpha(\boldsymbol{k};t)$ is a given function but it is not necessary to have knowledge of its concrete form. Obviously, (51) holds the four symmetries, space transformation, time transformation, rotation and parity.

First, we consider the same case as in Sec. III A 1. Setting $t'=0$ and

$$\boldsymbol{U}(t)=\Big(\hat{v}_1(\boldsymbol{k}^{[1]};t),\ldots,\hat{v}_1(\boldsymbol{k}^{[N']};t),\hat{v}_2(\boldsymbol{k}^{[1]};t),\ldots,\hat{v}_2(\boldsymbol{k}^{[N']};t),\hat{v}_3(\boldsymbol{k}^{[1]};t),\ldots,\hat{v}_3(\boldsymbol{k}^{[N']};t)\Big)^T, \tag{52}$$

we obtain

$$|\langle\boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle|=0, \tag{53}$$

which means that the square matrix $\langle\boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle$ is not regular because $\hat{v}_\alpha(\boldsymbol{k};0)$ satisfies the incompressibility condition (21) for $t'=0$. It is important to note that the relation (28) is still correct even though $\hat{v}_\alpha(\boldsymbol{k};t)$ does not satisfy the incompressibility condition (21) for $t\neq 0$. Hence, projection to all modes is not appropriate.

Second, we consider the same case as in Sec. III A 2. Setting $t'=0$ and

$$\tilde{\boldsymbol{U}}_\alpha(t)=\Big(\hat{v}_\alpha(\boldsymbol{k}^{[1]};t),\ldots,\hat{v}_\alpha(\boldsymbol{k}^{[N']};t)\Big)^T, \tag{54}$$

we obtain the evolution equation of $Q(k;t)$,

$$\frac{dQ(k;t)}{dt}=-\int_0^t\Gamma(k;s)Q(k;t-s)ds, \tag{55}$$

which is the same as (47). However, the functional form of $Q(k;t)$ in (55) is different from that in (47), although we use the same notation $Q(k;t)$.

Finally, we consider the same case as in Sec. III A 1 again. This time we set $t'=-T<0$ and use $\boldsymbol{U}(t)$ in (52), and hence $\hat{v}_\alpha(\boldsymbol{k};-T)$ satisfies the incompressibility condition (21) while $\hat{v}_\alpha(\boldsymbol{k};0)$ does not. Therefore $\langle\boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle$ is regular. The projection operator defined in (3) is transformed into

$$\begin{aligned}\mathcal{P}f(\boldsymbol{U}(t))&=\sum_{\alpha,\beta=1}^{3}\sum_{i,j=1}^{M}\Big\langle f(\boldsymbol{U}(t))\hat{v}^*_\alpha(\boldsymbol{k}^{[i]};0)\Big\rangle\left[\langle\boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle^{-1}\right]_{\alpha,i;\beta,j}\hat{v}_\beta(\boldsymbol{k}^{[j]};0)\\&=\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\sum_{j=1}^{M}\Big\langle f(\boldsymbol{U}(t))\hat{v}^*_\alpha(\boldsymbol{k}^{[j]};0)\Big\rangle\Big[\Big\langle\hat{v}_\alpha(\boldsymbol{k}^{[j]};0)\hat{v}^*_\beta(\boldsymbol{k}^{[j]};0)\Big\rangle\Big]^{-1}\hat{v}_\beta(\boldsymbol{k}^{[j]};0),\end{aligned} \tag{56}$$

where we have used the statistical homogeneity in (50). Applying the projection operator (56) to the Navier-Stokes equations (51), we obtain the following by the manner explained in Sec II,

$$\frac{d\hat{v}_\alpha(\boldsymbol{k};t)}{dt} = \sum_{\beta=1}^{3} \Omega_{\alpha\beta}(\boldsymbol{k})\hat{v}_\beta(\boldsymbol{k};t) - \sum_{\beta=1}^{3} \int_0^t \Gamma_{\alpha\beta}(\boldsymbol{k};s)\hat{v}_\beta(\boldsymbol{k};t-s)ds + r_\alpha(\boldsymbol{k};t), \tag{57}$$

where

$$\Omega_{\alpha\beta}(\boldsymbol{k}) = \sum_{\gamma=1}^{3} \left\langle \frac{d\hat{v}_\alpha(\boldsymbol{k};0)}{dt}\hat{v}_\gamma^*(\boldsymbol{k};0) \right\rangle \left[\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle^{-1}\right]_{\gamma\beta} = 0, \tag{58}$$

obtained from (C7),

$$\Gamma_{\alpha\beta}(\boldsymbol{k};t) = \sum_{\gamma=1}^{3} \left\langle r_\alpha(\boldsymbol{k};t) r_\gamma^*(\boldsymbol{k};0)\right\rangle \left[\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle^{-1}\right]_{\gamma\beta} \tag{59}$$

and

$$r_\alpha(\boldsymbol{k};t) = e^{\mathcal{P}'\Lambda t}\mathcal{P}'\frac{d\hat{v}_\alpha(\boldsymbol{k};0)}{dt}. \tag{60}$$

Multiplying (57) by $\hat{v}_\gamma^*(\boldsymbol{k};0)$, taking the average, and using (18), we obtain the evolution equation of $Q_{\alpha\gamma}(\boldsymbol{k};t)$, defined in (50),

$$\frac{dQ_{\alpha\gamma}(\boldsymbol{k};t)}{dt} = -\sum_{\beta=1}^{3} \int_0^t \Gamma_{\alpha\beta}(\boldsymbol{k};s) Q_{\beta\gamma}(\boldsymbol{k};t-s)ds. \tag{61}$$

The time correlation function $Q_{\alpha\beta}(\boldsymbol{k};t)$ is invariant under time transformation and $\hat{v}_\alpha(\boldsymbol{k};t)$ satisfies the incompressibility condition (21) only at $t = t' = -T$. Therefore, after transforming t into $t+T$, $\hat{v}_\alpha(\boldsymbol{k};0)$ can satisfy the incompressibility condition without changing the form of (61). The fluctuating force $r_\alpha(\boldsymbol{k};0)$ also satisfies the incompressibility condition because

$$\sum_\alpha k_\alpha r_\alpha(\boldsymbol{k};0) = \sum_\alpha k_\alpha \mathcal{P}'\frac{d\hat{v}_\alpha(\boldsymbol{k};0)}{dt} = \sum_\alpha k_\alpha \mathcal{P}'\frac{d\hat{u}_\alpha(\boldsymbol{k};0)}{dt} = \mathcal{P}'\frac{d}{dt}\sum_\alpha k_\alpha \hat{u}_\alpha(\boldsymbol{k};0) = 0. \tag{62}$$

We can therefore obtain the relation (28) for $Q_{\alpha\beta}(\boldsymbol{k};t)$ and a similar relation

$$\Gamma_{\alpha\beta}(\boldsymbol{k};t) = D_{\alpha\beta}(\boldsymbol{k})\Gamma(k;t) \tag{63}$$

for $\Gamma_{\alpha\beta}(\boldsymbol{k};t)$. Note that (28) and (63) hold true even when $\hat{v}_\alpha(\boldsymbol{k};t)$ does not satisfy the incompressibility condition (21) for $t \neq 0$. Substituting (28) and (63) into (61), and then summing over α after setting $\gamma = \alpha$, we obtain

$$\frac{dQ(k;t)}{dt} = -\int_0^t \Gamma(k;s)Q(k;t-s)ds, \tag{64}$$

which is the same as (47) for the Eulerian specification. However, the functional form of $Q(k;t)$ in (64) is different from that in (47).

IV. TIME CORRELATION FUNCTIONS

We have now obtained the evolution equation of $Q(k;t)$

$$\frac{dQ(k;t)}{dt} = -\int_0^t \Gamma(k;s)Q(k;t-s)ds \tag{65}$$

under the assumptions of statistical homogeneity, steadiness, isotropy and parity invariance for both the Lagrangian and Eulerian specifications. Setting $t = 0$ in (65), we obtain

$$\frac{dQ(k;0)}{dt} = 0, \tag{66}$$

which is valid for both the Lagrangian and Eulerian specifications. It is important to note that we do not use the closure assumption (67) to derive (66).

A. Closure

Because (65) has two unknowns, $Q(k;t)$ and $\Gamma(k;t)$, it is not closed. As shown in (27), (28), (59) and (63), $\Gamma(k;t)$ and $Q(k;t)$ are the scalar components of the time correlation functions with respect to the fluctuating force $r_\alpha(\boldsymbol{k};t)$ and fluid velocity $\hat{u}_\alpha(\boldsymbol{k};t)$, respectively. In turbulence, all time correlation functions decay to zero for $t \to \infty$. We can expect that the differences between their decay forms are not important except for the characteristic time. Therefore, it is reasonable to assume that when the time correlation functions $Q(k;t)$ and $\Gamma(k;t)$ are normalized to one at $t = 0$ and time t is scaled using characteristic time $\tau^{(u)}(k)$ or $\tau^{(\gamma)}(k)$, their normalized functions can be approximated by a function $G(k;t)$[9]

$$G(k;t) = \frac{Q(k;\tau^{(u)}(k)t)}{Q(k;0)} = \frac{\Gamma(k;\tau^{(\gamma)}(k)t)}{\Gamma(k;0)}, \tag{67}$$

where

$$\tau^{(u)}(k) = \frac{1}{Q(k;0)}\int_0^\infty Q(k;t)dt, \qquad \tau^{(\gamma)}(k) = \frac{1}{\Gamma(k;0)}\int_0^\infty \Gamma(k;t)dt. \tag{68}$$

Integrating (65) with respect to t from 0 to ∞ and using (68), we obtain

$$\Gamma(k;0)\tau^{(\gamma)}(k)\tau^{(u)}(k) = 1. \tag{69}$$

Equations (67) and (68) yield

$$\int_0^\infty G(k;t)dt = 1. \tag{70}$$

Substituting (67) into (65) and using (69), we obtain a closed equation of $G(k;t)$, called the SA equation[9]

$$\frac{d}{dt}G(k;t) = -\int_0^{t/\tilde{\tau}(k)} G(k;s)G(k;t-\tilde{\tau}(k)s)ds, \tag{71}$$

where

$$\tilde{\tau}(k) = \frac{\tau^{(\gamma)}(k)}{\tau^{(u)}(k)}. \tag{72}$$

It is reasonable that

$$0 \le \tilde{\tau}(k) \le 1 \tag{73}$$

because $Q(k,t)$ and $\Gamma(k,t)$ are related to the slowly varying motion and rapidly varying motion, respectively, and hence $\tau^{(u)}(k) > \tau^{(\gamma)}(k)$. It is important to note that $\Gamma(k;t)$ is the scalar component of the time correlation function with respect to the fluctuating force $r_\alpha(\boldsymbol{k};t)$ as shown in (59) and (63).

Differentiating (71) with respect to t and then setting $t = 0$, we obtain

$$\tilde{\tau}(k) = -\frac{1}{d^2G(k;0)/dt^2}. \tag{74}$$

We cannot evaluate $d^2G(k;0)/dt^2$ in the projection operator method. Thus we determine it with $G(t)$ from direct numerical simulations or other closure models, and otherwise it is treated as an arbitrary parameter. The SA equation (71) is the same form for both Lagrangian and Eulerian specifications, and hence all information with their difference is simplified to $\tilde{\tau}(k)$.

The time correlation function for homogeneous isotropic turbulence in Fourier space is independent of wave number k using the characteristic time $\tau \sim k^{-1}$ in the dissipation range for the Eulerian specification[4] and using the characteristic time $\tau \sim k^{-2/3}$ in the inertial range for the Lagrangian specification.[2] Hence, $\tilde{\tau}(k)$ is independent of k; in other words, the k-dependence of $\tau^{(\gamma)}(k)$ is the same as that of $\tau^{(u)}(k)$. In this case, (71) and (74) are simplified to

$$\frac{d}{dt}G(t) = -\int_0^{t/\tilde{\tau}} G(s)G(t-\tilde{\tau}s)ds \tag{75}$$

and

$$\tilde{\tau} = -\frac{1}{d^2G(0)/dt^2}, \tag{76}$$

respectively. Now we can obtain a solution to (75) with two initial conditions, $G(0) = 1$ and $dG(0)/dt = 0$, if there is some information on the value of $d^2G(0)/dt^2$.

Figure 1 shows four solutions to (75) for $\tilde{\tau} = j/3$, $(j = 0, 1, \ldots, 3)$. Two of them are exact solutions

$$G(t) = \begin{cases} e^{-t} & \text{for } \tilde{\tau} = 0, \\ \dfrac{1}{t}J_1(2t) & \text{for } \tilde{\tau} = 1, \end{cases} \tag{77}$$

where $J_1(t)$ is the Bessel function of the first kind. Furthermore, we know that the time correlation function $G(t)$ decays exponentially for $\tilde{\tau} < \tilde{\tau}_c$ and exhibits an oscillatory exponential decay for $\tilde{\tau} > \tilde{\tau}_c$,[9,10] where $\tilde{\tau}_c$ is the critical value

$$\tilde{\tau}_c = 0.38. \tag{78}$$

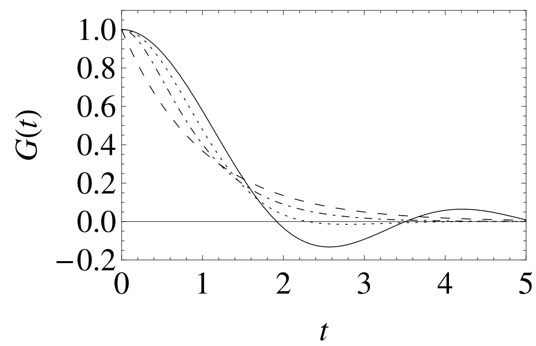

FIG. 1. Time correlation functions obtained from the SA equation (75) for four values of $\tilde{\tau}$: dashed line, $\tilde{\tau} = 0$; dash-dotted line, $\tilde{\tau} = 1/3$; dotted line, $\tilde{\tau} = 2/3$; solid line, $\tilde{\tau} = 1$.

B. Comparison

1. Eulerian specification

For the Eulerian specification, we compare the time correlation functions obtained from (75) with those obtained from direct numerical simulations of the Navier-Stokes equations with 256^3 grid points and adopted feedback-acceleration forcing[16] by Carini and Quadrio.[4] We use the raw data from Fig. 8 in their paper. This figure shows that the normalized time correlation functions with convective scaling collapse into a single curve in the dissipation range $0.61\kappa_d \leq k \leq 1.1\kappa_d$ at the Taylor-scale Reynolds number $Re_\lambda = 94$, where κ_d is the Kolmogorov wave number.

It is impossible to evaluate $d^2G(0)/dt^2$ accurately, at least to three significant figures, from the numerical simulation data because the time difference used in these data is not fine enough. Instead, minimizing the difference between the solution to (75) and numerical simulation data for $0 < t < 2$, we obtain the optimum value of

$$\tilde{\tau} = 0.643. \tag{79}$$

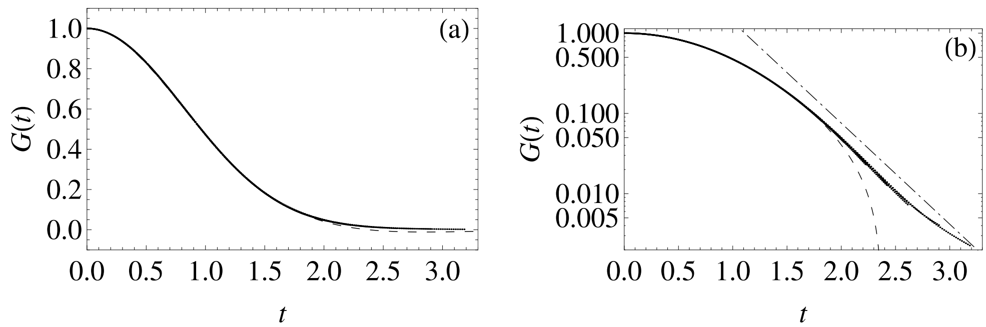

FIG. 2. (a) Linear-linear and (b) semi-logarithmic plots of the time correlation functions: dashed line, solution to (75) for $\tilde{\tau} = 0.643$; dotted line (cannot be distinguished from the solid line), numerical simulation data; dash-dotted line, $G(t) \propto e^{-2.9t}$.

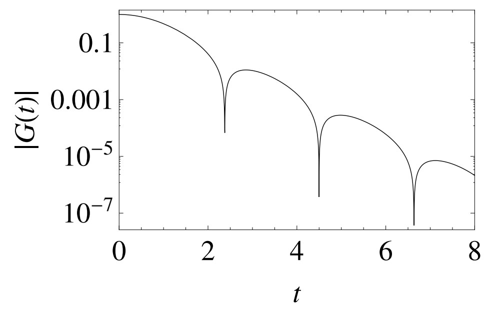

FIG. 3. Semi-logarithmic plot of the absolute value of the time correlation function obtained from (75) for $\tilde{\tau} = 0.643$.

Figure 2 shows a comparison between the solution to (75) for $\tilde{\tau} = 0.643$ and numerical data. Their agreement is quite good, especially for $0 \leq t < 2$, while these two are noticeably different for $t > 2$. Figure 2 (b) suggests that the simulation data exhibit exponential decay but they deviate from this for $t > 2.5$ because the periodicity may disturb results for large times such as $t > 2.5$. Figure 3 shows the absolute value of the time correlation function obtained from (75) for $\tilde{\tau} = 0.643$ and $0 \leq t \leq 8$. This figure shows that the time correlation function obtained from (75) for $\tilde{\tau} = 0.643$ exhibits oscillatory exponential decay. At present, we cannot determine which decay form is correct, but we provide an example of oscillatory exponential decay, which was obtained from direct numerical simulation of the Kuramoto-Sivashinsky equation. See Fig. 5 in the reference.[9] It is necessary to carry out further detailed numerical simulation of the Navier-Stokes equations to obtain the time correlation function for large times.

Now we discuss the time derivative at $t = 0$ for the time correlation function $G(k;t)$ and the response function $\mathcal{G}(k;t)$. Here, for statistical isotropy and stationarity, the response function is expressed by

$$D_{\alpha\beta}(\boldsymbol{k})\mathcal{G}(k;t-t')\delta(\boldsymbol{k}-\boldsymbol{k}') = \left\langle \frac{\delta u_\alpha(\boldsymbol{k};t)}{\delta f_\beta(\boldsymbol{k}';t')} \right\rangle \qquad \text{for} \quad t > t',$$

where $\delta(\boldsymbol{k})$ and $\delta u_\alpha(\boldsymbol{k};t)/\delta f_\beta(\boldsymbol{k}';t')$ denote the Dirac delta function and the functional derivative of $u_\alpha(\boldsymbol{k};t)$ with respect to $f_\beta(\boldsymbol{k}';t')$, respectively. Figure 2 clearly shows that the numerical data satisfy (66). Carini and Quadrio[4] indicate that, as shown in Fig. 5 in their paper, the response function $\mathcal{G}(k;t)$ obtained from numerical simulation is consistent with the viscous Gaussian convective response

$$\mathcal{G}_{\mathrm{GC}}(k;t) = \exp(-\nu k^2 t - \tfrac{1}{2}u_0^2 k^2 t^2), \tag{80}$$

where u_0 denotes the root mean square of turbulent fluctuations. This means that the response function $\mathcal{G}(k;t)$ does not satisfy (66) for $\nu \neq 0$, while Fig. 2 shows that the velocity time correlation function $G(k;t)$ does even for $\nu \neq 0$. Note again that we do not use the closure assumption (67) to derive (66).

2. Lagrangian specification

For the Lagrangian specification, we compare the time correlation function obtained from (75) with that obtained from DIA,[2,17] which is fairly reliable in the inertial range because it reproduces the Kolmogorov energy spectrum as well as the Kolmogorov constant.

In the inertial range, the evolution equation of the time correlation function $G_{\mathrm{DIA}}(t)$ for DIA is

$$\frac{dG_{\mathrm{DIA}}(t)}{dt} = -G_{\mathrm{DIA}}(t)\int_0^\infty dp\, J(p)\int_0^t dt' G_{\mathrm{DIA}}(pt'), \qquad G_{\mathrm{DIA}}(0) = 1, \tag{81}$$

where

$$J(p) = \frac{3}{32p^5}\left[\frac{(1-p^3)^4}{2p^{3/2}}\log\frac{1+p^{3/2}}{|1-p^{3/2}|} - \frac{1}{3}(1+p^3)(3p^6 - 14p^3 + 3)\right]. \tag{82}$$

Note here that $\int_0^\infty G_{\mathrm{DIA}}(t)dt \neq 1$. Differentiating (81) with respect to t and then setting $t = 0$, we obtain

$$\frac{d^2 G_{\mathrm{DIA}}(0)}{dt^2} = -\int_0^\infty J(p)dp = -\frac{972\sqrt{3}}{5005}\pi. \tag{83}$$

Using the numerical solution to (81), we obtain

$$\int_0^\infty G_{\mathrm{DIA}}(t)dt = 1.5576. \tag{84}$$

Equations (76), (83) and (84) yield

$$\tilde{\tau} = 0.390, \tag{85}$$

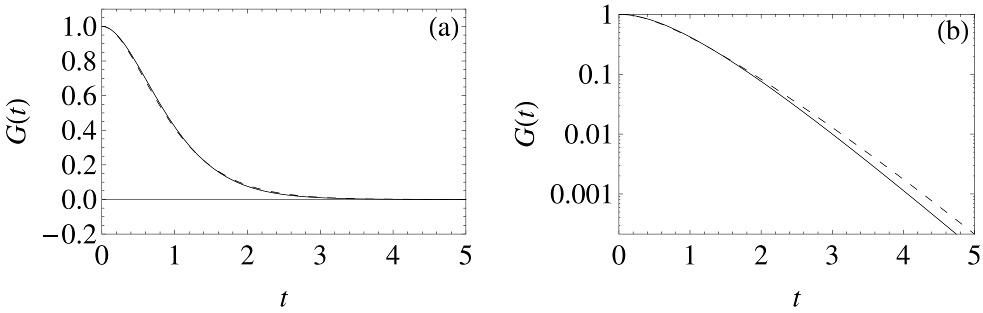

FIG. 4. (a) Linear-linear and (b) semi-logarithmic plots of the time correlation functions: dashed line, solution to the SA equation (75) for $\tilde{\tau} = 0.390$; solid line, solution to the DIA equation (81).

which is close to the critical value in (78).[10] Thus we can solve the SA equation (75).

Figure 4 shows the comparison between the solution to the SA equation (75) for $\tilde{\tau} = 0.390$ and that from the DIA equation (81) in the inertial range. Their solutions are consistent throughout the entire domain in the linear-linear plot, but a small difference can be noticed between them for $t > 2$ in the semi-logarithmic plot. This agreement is quite surprising because these two evolution equations are fairly different.

We end this section by discussing the time correlation function in the dissipation range. In the dissipation range, the equations (not shown here) obtained from DIA[2,17] indicate that neither the time correlation function nor the response function satisfies (66), while the time correlation function obtained from (75) does even for $\nu \neq 0$. Hence, the time correlation function obtained from DIA may be inconsistent with that obtained from (75), especially for smaller values of t. At present, we cannot determine which result is correct because we have a lack of data on the time correlation function obtained from direct numerical simulation in the Lagrangian specification.

V. CONCLUSIONS

The SA equation (71) or (75) was derived under the assumptions of statistical homogeneity, steadiness, isotropy and parity invariance, and by using the closure assumption (67). Its solutions for $\tilde{\tau} = 0.643$ and $\tilde{\tau} = 0.390$ are consistent with the time correlation function obtained from direct numerical simulation in the dissipation range for the Eulerian specification and that obtained from the DIA equation in the inertial range for the Lagrangian specification, respectively. The previous paper[9] showed that the solutions to the SA equation (71) for some $\tilde{\tau}(k)$ are consistent with those obtained from the Kuramoto-Sivashinsky equation for wave number k corresponding to $\tilde{\tau}(k)$. Additionally, the solution to (75) for $\tilde{\tau} = 1$ is the same as that from the Eulerian DIA equation for $\nu \to 0$,[1] which gives an unsuitable energy spectrum in the inertial range. Therefore, the SA equation (71) or (75) is expected to express the time correlation function for homogeneous turbulence in general.

ACKNOWLEDGMENTS

I am grateful to Maurizio Quadrio and Marco Carini for providing the raw data from Fig. 8 in their paper.[4]

Appendix A: Derivation of (13)

Introducing the time correlation function

$$F_{nj}(t,s) = \langle r_n(t+s)U_j^*(s)\rangle \tag{A1}$$

and considering statistical steadiness $\partial F_{nj}/\partial s = \Lambda F_{nj} = 0$, we obtain

$$0 = \Lambda\langle r_n(t)U_j^*(0)\rangle = \langle[\Lambda r_n(t)]U_j^*(0)\rangle + \langle r_n(t)\Lambda U_j^*(0)\rangle. \tag{A2}$$

Equations (7) and (15) yield

$$\Lambda U_j^*(0) = \frac{dU_j^*(0)}{dt} = \sum_{i=1}^{M} \Omega_{ji}^* U_i^*(0) + r_j^*(0). \tag{A3}$$

Using (A3) and (18), we obtain

$$\langle r_n(t)\Lambda U_j^*(0)\rangle = \langle r_n(t) r_j^*(0)\rangle. \tag{A4}$$

Equations (A2) and (A4) yield

$$\langle[\Lambda r_n(t)]U_j^*(0)\rangle + \langle r_n(t) r_j^*(0)\rangle = 0, \tag{A5}$$

from which we obtain (13).

We end this appendix by noting that the statistical steadiness $\Lambda\langle r_n(t)U_j^*(0)\rangle = 0$ in (A2) cannot yield

$$\left\langle \frac{dr_n(t)}{dt} U_j^*(0)\right\rangle + \left\langle r_n(t)\frac{dU_j^*(0)}{dt}\right\rangle = 0, \tag{A6}$$

because $e^{\Lambda t}$ is the time evolution operator for $U_j(t)$, while $e^{\mathcal{P}'\Lambda t}$ is the time evolution operator for $r_n(t)$, as shown in (7) and (14).

Appendix B: Derivation of (18)

Using the equation $\mathcal{P} r_n(t) = 0$ obtained from (5) and (14), and the definition of $\mathcal{P}$ in (3), we obtain

$$\sum_{i=1}^{M}\sum_{j=1}^{M} \langle r_n(t)U_i^*(0)\rangle \left[\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle^{-1}\right]_{ij} U_j(0) = 0. \tag{B1}$$

The independence of $U_j(0)$ reduces (B1) to

$$\sum_{i=1}^{M} \langle r_n(t)U_i^*(0)\rangle \left[\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle^{-1}\right]_{ij} = 0, \tag{B2}$$

and using the regularity of the matrix $\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle$, we obtain $\langle r_n(t)U_i^*(0)\rangle = 0$ from (B2).

Appendix C: Derivations of (43) and (58)

Statistical parity invariance means that a statistical quantity is invariant under the transformation

$$\hat{v}_\alpha(\boldsymbol{k};t) \to -\hat{v}_\alpha(-\boldsymbol{k};t) \tag{C1}$$

in Fourier space. Note that $\hat{v}_\alpha(\boldsymbol{k};t)$ denotes either Lagrangian or Eulerian velocity in this appendix. For example, using parity invariance and (22), we obtain

$$\left\langle \frac{d\hat{v}^*_\alpha(\boldsymbol{k};0)}{dt}\hat{v}_\beta(\boldsymbol{k};0)\right\rangle = \left\langle \frac{d\hat{v}^*_\alpha(-\boldsymbol{k};0)}{dt}\hat{v}_\beta(-\boldsymbol{k};0)\right\rangle = \left\langle \frac{d\hat{v}_\alpha(\boldsymbol{k};0)}{dt}\hat{v}^*_\beta(\boldsymbol{k};0)\right\rangle, \tag{C2}$$

which means that $\langle (d\hat{v}^*_\alpha(\boldsymbol{k};0)/dt)\hat{v}_\beta(\boldsymbol{k};0)\rangle$ is real. Because the isotropic second-rank tensor $Q_{\alpha\beta}$ is invariant under the exchange between α and β,[18] we obtain

$$\left\langle \frac{d\hat{v}^*_\alpha(\boldsymbol{k};0)}{dt}\hat{v}_\beta(\boldsymbol{k};0)\right\rangle = \left\langle \hat{v}_\alpha(\boldsymbol{k};0)\frac{d\hat{v}^*_\beta(\boldsymbol{k};0)}{dt}\right\rangle. \tag{C3}$$

Using (C2) and (C3), we obtain

$$\left\langle \hat{v}_\alpha(\boldsymbol{k};0)\frac{d\hat{v}^*_\beta(\boldsymbol{k};0)}{dt}\right\rangle = \left\langle \frac{d\hat{v}^*_\alpha(\boldsymbol{k};0)}{dt}\hat{v}_\beta(\boldsymbol{k};0)\right\rangle = \left\langle \frac{d\hat{v}_\alpha(\boldsymbol{k};0)}{dt}\hat{v}^*_\beta(\boldsymbol{k};0)\right\rangle. \tag{C4}$$

Statistical steadiness yields

$$\frac{d}{dt}\langle \hat{v}_\alpha(\boldsymbol{k};t)\hat{v}^*_\beta(\boldsymbol{k};t)\rangle = \left\langle \frac{d\hat{v}_\alpha(\boldsymbol{k};t)}{dt}\hat{v}^*_\beta(\boldsymbol{k};t)\right\rangle + \left\langle \hat{v}_\alpha(\boldsymbol{k};t)\frac{d\hat{v}^*_\beta(\boldsymbol{k};t)}{dt}\right\rangle = 0. \tag{C5}$$

Using (C4) and (C5), we obtain

$$\left\langle \frac{d\hat{v}_\alpha(\boldsymbol{k};t)}{dt}\hat{v}^*_\beta(\boldsymbol{k};t)\right\rangle = 0, \tag{C6}$$

which yields

$$\Omega_{\alpha\beta}(\boldsymbol{k}) = \sum_{\gamma=1}^{3}\left\langle \frac{d\hat{v}_\alpha(\boldsymbol{k};0)}{dt}\hat{v}^*_\gamma(\boldsymbol{k};0)\right\rangle \left[\langle \boldsymbol{U}(0)\boldsymbol{U}^\dagger(0)\rangle^{-1}\right]_{\gamma\beta} = 0, \tag{C7}$$

and then

$$\Omega_\alpha(\boldsymbol{k}) = \Omega_{\alpha\alpha}(\boldsymbol{k}) = 0. \tag{C8}$$

[1] R. H. Kraichnan, "The structure of isotropic turbulence at very high Reynolds numbers," J. Fluid Mech. **5**, 497–543 (1959).
[2] Y. Kaneda, "Inertial range structure of turbulent velocity and scalar fields in a Lagrangian renormalized approximation," Phys. Fluids **29**, 701–708 (1986).
[3] Y. Kaneda, T. Ishihara, and K. Gotoh, "Taylor expansions in powers of time of Lagrangian and Eulerian two-point two-time velocity correlations in turbulence," Phys. Fluids **11**, 2154–2166 (1999).
[4] M. Carini and M. Quadrio, "Direct-numerical-simulation-based measurement of the mean impulse response of homogeneous isotropic turbulence," Phys. Rev. E **82**, 066301 (2010).
[5] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II, Nonequilibrium Statistical Mechanics*, (Springer-Verlag, Berlin, 1991).
[6] H. Mori, "Transport, collective motion, and Brownian motion," Prog. Theor. Phys. **33**, 423–455 (1965).
[7] H. Mori and M. Okamura, "Decay forms of the time correlation functions for turbulence and chaos," Prog. Theor. Phys. **127**, 615–629 (2012).
[8] H. Mori and M. Okamura, "Dynamic structures of the time correlation functions of chaotic nonequilibrium fluctuations," Phys. Rev. E **76**, 061104 (2007).
[9] M. Okamura and H. Mori, "Time correlation functions in a similarity approximation for one-dimensional turbulence," Phys. Rev. E **79**, 056312 (2009).
[10] M. Okamura, "Universality of modal time correlation functions in medium scale," J. Phys. Soc. Jpn. **83**, 074004 (2014).
[11] H. Mori and H. Fujisaka, "Transport and entropy production due to chaos or turbulence," Phys. Rev. E **63**, 026302 (2001).
[12] M. Okamura, "Validity of the essential assumption in a projection operator method," Phys. Rev. E **74**, 046210 (2006).
[13] W. D. McComb, *The Physics of Fluid Turbulence* (Oxford University Press, New York, 1990).

[14]U. Frisch, *Turbulence: The legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[15]R. H. Kraichnan, "Lagrangian-history closure approximation for turbulence," Phys. Fluids **8**, 575–598 (1965).
[16]A. G. Lamorgese, D. A. Caughey, and S. B. Pope, "Direct numerical simulation of homogeneous turbulence with hyperviscosity," Phys. Fluids **17**, 015106 (2005).
[17]S. Kida and S. Goto, "A Lagrangian direct-interaction approximation for homogeneous isotropic turbulence," J. Fluid Mech. **345**, 307–345 (1997).
[18]G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, 1953).