IMPLICATION AND FUNCTIONAL DEPENDENCY IN INTENSIONAL CONTEXTS

Ishida, Toshikazu
Department of Informatics, Kyushu University

Honda, Kazumasa
Department of Informatics, Kyushu University

Kawahara, Yasuo
Department of Informatics, Kyushu University

https://doi.org/10.5109/18997

出版情報：Bulletin of informatics and cybernetics. 40, pp.101-111, 2008-12. Research Association of Statistical Sciences

IMPLICATION AND FUNCTIONAL DEPENDENCY IN INTENSIONAL CONTEXTS

by

Toshikazu Ishida, **Kazumasa Honda** and **Yasuo Kawahara**

Reprinted from the Bulletin of Informatics and Cybernetics
Research Association of Statistical Sciences, Vol.40

FUKUOKA, JAPAN
2008

Bulletin of Informatics and Cybernetics, Vol. 40, 2008

IMPLICATION AND FUNCTIONAL DEPENDENCY IN INTENSIONAL CONTEXTS

By

Toshikazu Ishida[*], **Kazumasa Honda**[†] and **Yasuo Kawahara**[‡]

Abstract

Formal concept analysis is a mathematical field applied to data mining. Usually, a formal concept is defined as a pair of sets, called extents and intents, for a given formal context in binary relation. In this paper we review the idea that Armstrong's inference rules are complete and sound for functional dependencies. Then, we prove that Armstrong's inference rules are complete and sound for implications of formal contexts. Still, we give an example which shows the difference between implication and functional dependency. Besides, we show that functional dependency can be reduced to implication. Finally, we give the condition on which a set of implications and a set of functional dependencies for the intensional context are equivalent.

Key Words and Phrases: Formal concept, Functional dependency, Implication, Armstrong inference rules

1. Introduction

In the philosophical theory, a concept is defined as a pair of an extent and an intent. The extent is a subset of all objects that belong to the concept, and the intent is a subset of all attributes whose object are common. In the mathematical theory, formal concepts mean formal models of concepts as defined above. Our subjects are the correlations between formal concepts. The method to analyze them is called formal concept analysis. Based on the lattice theory, it was proposed by Wille in 1970's. The standard textbook is Ganter and Wille (1999).

What we call a formal context is really a database which consists of a set of objects, a set of attributes and the binary relation between them. The formal context defines formal concepts as suitable pairs of an extent and an intent. The idea is an extension of Dedekind cuts of real numbers. The set of all formal concepts forms a complete lattice by inclusion between extents. The complete lattice implies the features of a formal context. Therefore, formal concept analysis is applicable to data mining.

To find a correlation of attributes is important for analyzing a relational database. Codd (1970) introduced a notion of functional dependency, which is a constraint between two sets of attributes. Armstrong (1974) proposed so-called Armstrong inference rules. Beeri, Fagin and Howard (1977) proposed that those inference rules are complete and sound for functional dependencies. On the other hand, Ganter and Wille defined implication, or another type of dependency in a formal context.

[*] Department of Informatics, Kyushu University, Fukuoka, 819-0395, Japan. t-ishida@i.kyushu-u.ac.jp
[†] Department of Informatics, Kyushu University
[‡] Department of Informatics, Kyushu University

In this paper, we review a simple proof that Armstrong inference rules are sound and complete for functional dependencies in formal contexts. Also we demonstrate that Armstrong inference rules are sound and complete for implications for formal contexts. Further, to distinguish semantics and syntax, we give a common proof of implication and functional dependency. Still, we give an example which shows the difference between implication and functional dependency.

The paper is organized as follows. In Section 2 we introduce an intensional context which could be discussed more clearly than a formal context. In Section 3 we review Armstrong inference rules for the dependency of attributes. In section 4 we examine the functional dependency in intensional context, and show that Armstrong inference rules are sound for functional dependencies. In Section 5 we explain a notion of implication in intensional contexts, and then prove some properties of implication to show that Armstrong inference rules are sound for it. In Section 6 we demonstrate that Armstrong inference rules are complete for implications and functional dependencies. Still, in Section 7, we notice an example which shows the difference between implication and functional dependency.

2. Intensional Contexts

A formal context is a binary relation between objects and attributes. The following shows a simple example of formal contexts.

	a	b	c
x_0	×	×	
x_1		×	×
x_2			×
x_3		×	×

Where a, b and c are attributes, x_0, x_1, x_2 and x_3 objects. The formal concept lattice C is:

$$\begin{tikzcd}
 & (\{x_0,x_1,x_2,x_3\},\emptyset) & \\
(\{x_0,x_1,x_3\},\{b\}) \arrow[ur] & & (\{x_1,x_2,x_3\},\{c\}) \arrow[ul] \\
(\{x_0\},\{a,b\}) \arrow[u] & & (\{x_1,x_3\},\{b,c\}) \arrow[u] \arrow[ull] \\
 & (\emptyset,\{a,b,c\}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Construct the family $\mathcal{T} = \{\{a,b\},\{b,c\},\{c\}\}$ of the subsets of attributes related with each object, and the closure system

$$\mathcal{T}^* = \{\{a,b,c\},\{a,b\},\{b,c\},\{b\},\{c\},\emptyset\}$$

generated by $\mathcal{T}$. It is easy to see that $\mathcal{T}^*$ considers with the set of all intents (second components) of C. For constructing concept lattices, it is enough to treat with a family of subsets of attributes instead of a formal context. In this paper, we argue by using such framework.

Definition 2.1. Let Y be a set of attributes and $\wp(Y)$ the power set of Y. A subset $\mathcal{T}$ of $\wp(Y)$ is called an *intensional context* on Y. □

The basic idea on Formal Concept Analysis (FCA) due to Ganter and Wille (1999), uses closure operations defined for formal contexts. For intensional contexts, the closure operation is modified as follows: $B^+ = \bigcap\{T \in \mathcal{T} \mid B \subseteq T\}$ for a subset B of Y. In this sense the formal concept lattice $\mathcal{T}^*$ for $\mathcal{T}$ is a subset $\mathcal{T}^* = \{\cap\mathcal{A} \mid \mathcal{A} \subseteq \mathcal{T}\}$ of $\wp(Y)$. The following proposition is fundamental.

PROPOSITION 2.2. *Let B be a subset of attributes Y and $\mathcal{T}$ an intentional context on Y. Then the following holds.*

(a) $B \subseteq B^+ = B^{++}$,

(b) $Y \in \mathcal{T}^*$,

(c) $\mathcal{T}^*$ *is a complete lattice.*

3. Armstrong Inference Rules

Armstrong (1974) introduced so-called Armstrong inference rules as a basic framework to treat the logical structure on dependencies on attribute sets.

DEFINITION 3.1. Let A, B, C and D be subsets of an attribute set Y.

(a) A formal expression $A \rhd B$, namely, an ordered pair of subsets A and B of Y, is called a *dependency* on Y.

(b) *Armstrong inference rules* consist of the following three inference rules:

$$[\mathrm{A0}]\ \frac{}{A \rhd A} \qquad [\mathrm{A1}]\ \frac{A \rhd B}{A \cup C \rhd B} \qquad [\mathrm{A2}]\ \frac{A \rhd B \quad B \cup C \rhd D}{A \cup C \rhd D}$$

(c) A dependency $A \rhd B$ is *provable* from a set $\mathcal{L}$ of dependencies on Y (written $\mathcal{L} \vdash A \rhd B$) if $A \rhd B$ is obtained by using assumptions $\mathcal{L}$ and Armstrong inference rules [A0], [A1] and [A2]. □

PROPOSITION 3.2. *The following system of inference rules* [A0'], [A1'] *and* [A2'] *is equivalent to Armstrong inference rules.*

$$[\mathrm{A0'}]\ \frac{A \supseteq B}{A \rhd B} \qquad [\mathrm{A1'}]\ \frac{A \rhd B \quad C \supseteq D}{A \cup C \rhd B \cup D} \qquad [\mathrm{A2'}]\ \frac{A \rhd B \quad B \rhd C}{A \rhd C}$$

PROOF. [A0'], [A1'] and [A2'] imply [A0], [A1] and [A2]:

$$[\mathrm{A0'}]\ \frac{\overline{A \supseteq A}}{A \rhd A} \quad \cdots\ [\mathrm{A0}]$$

$$[\mathrm{A1'}]\ \frac{A \rhd B \quad \overline{C \supseteq \emptyset}}{A \cup C \rhd B} \quad \cdots\ [\mathrm{A1}]$$

$$[\mathrm{A2'}]\ \frac{[\mathrm{A1'}]\ \dfrac{A \rhd B \quad \overline{C \supseteq C}}{A \cup C \rhd B \cup C} \quad B \cup C \rhd D}{A \cup C \rhd D} \quad \cdots\ [\mathrm{A2}]$$

[A0], [A1] and [A2] imply [A0'], [A1'] and [A2']:

$$\dfrac{[\text{A1}]\,\dfrac{[\text{A0}]\,\dfrac{}{B \rhd B}}{B \cup A \rhd B} \quad A \supseteq B}{A \rhd B} \quad \cdots\ [\text{A0'}]$$

$$[\text{A2}]\,\dfrac{A \rhd B \quad [\text{A0'}]\,\dfrac{\dfrac{C \supseteq D}{B \cup C \supseteq B \cup D}}{B \cup C \rhd B \cup D}}{A \cup C \rhd B \cup D} \quad \cdots\ [\text{A1'}]$$

$$\dfrac{[\text{A2}]\,\dfrac{A \rhd B \quad [\text{A1}]\,\dfrac{B \rhd C}{B \cup A \rhd C}}{A \cup A \rhd C}}{A \rhd C} \quad \cdots\ [\text{A2'}]$$

□

Also the union rule

$$[\text{A3}] \quad \dfrac{A \rhd B \quad A \rhd C}{A \rhd B \cup C}$$

is proved from [A1'] and [A2'] as follows:

Proof.

$$[\text{A2'}]\,\dfrac{[\text{A1'}]\,\dfrac{A \rhd B \quad A \supseteq A}{A \rhd A \cup B} \qquad [\text{A1'}]\,\dfrac{A \rhd C \quad B \supseteq B}{A \cup B \rhd B \cup C}}{A \rhd B \cup C}$$

□

For a set $\mathcal{L}$ of dependencies we define a subset $A_{\mathcal{L}}$ of Y by $A_{\mathcal{L}} = \{y \in Y \mid \mathcal{L} \vdash A \rhd \{y\}\}$.

Lemma 3.3. *If B is a finite subset of Y, then $\mathcal{L} \vdash A \rhd B$ if and only if $B \subseteq A_{\mathcal{L}}$.*

Proof. ($\rightarrow$) Assume that $\mathcal{L} \vdash A \rhd B$ and let $y \in B$. Then $\vdash B \rhd \{y\}$ by [A0'] and so $\mathcal{L} \vdash A \rhd \{y\}$ by [A2']. Hence $y \in A_{\mathcal{L}}$.
($\leftarrow$) Assume that $B \subseteq A_{\mathcal{L}}$. Then, for all $y \in B$, we have $\mathcal{L} \vdash A \rhd \{y\}$ by the definition of $A_{\mathcal{L}}$ and hence $\mathcal{L} \vdash A \rhd B$ by the union rule [A3], because B is finite. □

4. Functional Dependency

In this section, we review a functional dependency introduced in Codd (1970). In definition of a functional dependency, we use an equivalence relation that called indiscernibility relation in Düntsch and Günther (2000).

Definition 4.1. For each subset A of Y we define an equivalence relation $\theta[A]$ on $\wp(Y)$ by $(S, T) \in \theta[A] \leftrightarrow S \cap A = T \cap A$. □

The following proposition is trivial.

Proposition 4.2.

(a) *$(S, T) \in \theta[\emptyset]$ for all subsets S and T of Y,*

(b) $(S, T) \in \theta[Y]$ *if and only if* $S = T$. $\square$

We define a functional dependency as follows.

Definition 4.3. Let $\mathcal{T}$ be an intensional context on Y, $A \rhd B$ a dependency on Y, and $\mathcal{L}$ a set of dependencies.

$$\begin{aligned}\mathcal{T} \models_F A \rhd B &\leftrightarrow \forall S, T \in \mathcal{T}.\ (S \cap A = T \cap A \rightarrow S \cap B = T \cap B)\\ &\leftrightarrow \forall S, T \in \mathcal{T}.\ (S, T) \in \theta[A] \rightarrow (S, T) \in \theta[B].\\ \mathcal{T} \models_F \mathcal{L} &\leftrightarrow \forall A \rhd B \in \mathcal{L}.\ \mathcal{T} \models_F A \rhd B.\end{aligned}$$

$\square$

If $\mathcal{T} \models_F A \rhd B$ then $A \rhd B$ is called a *functional dependency* on $\mathcal{T}$ and a dependency $A \rhd B$ is *valid* (as functional dependency) for an intensional context $\mathcal{T}$ on Y. A set of functional dependencies $\mathcal{L}$ is valid for $\mathcal{T}$ on Y if $\mathcal{T} \models_F \mathcal{L}$. Then the following four propositions hold.

Proposition 4.4. *Let $\mathcal{T} = \{S, T\}$ be an intensional context on Y. Then*

$$\mathcal{T} \models_F A \rhd B \text{ if and only if } ((S, T) \in \theta[A] \rightarrow (S, T) \in \theta[B]).$$

Proof is omitted. $\square$

The next proposition means that Armstrong inference rules are sound for functional dependencies.

Proposition 4.5. *Let $\mathcal{T}$ be an intensional context and $A \rhd B$ a dependency on Y. Then*
(A0') *If $A \supseteq B$ then $\mathcal{T} \models_F A \rhd B$,*
(A1') *If $\mathcal{T} \models_F A \rhd B$ and $C \supseteq D$ then $\mathcal{T} \models_F A \cup C \rhd B \cup D$,*
(A2') *If $\mathcal{T} \models_F A \rhd B$ and $\mathcal{T} \models_F B \rhd C$ then $\mathcal{T} \models_F A \rhd C$.*

Proof. (A0') For all $S, T \in \mathcal{T}$ it holds that

$$\begin{aligned}S \cap A = T \cap A &\rightarrow S \cap A \cap B = T \cap A \cap B\\ &\rightarrow S \cap B = T \cap B. \qquad \{\ A \supseteq B\ \}\end{aligned}$$

(A1') For all $S, T \in \mathcal{T}$ we have

$$\begin{aligned}& S \cap (A \cup C) = T \cap (A \cup C)\\ \leftrightarrow\ & S \cap A = T \cap A,\ S \cap C = T \cap C\\ \rightarrow\ & S \cap B = T \cap B,\ S \cap D = T \cap D \quad \{\ \mathcal{T} \models_F A \rhd B,\ C \supseteq D\ \}\\ \leftrightarrow\ & S \cap (B \cup D) = T \cap (B \cup D).\end{aligned}$$

(A2') For all $S, T \in \mathcal{T}$ it holds that

$$\begin{aligned}S \cap A = T \cap A &\rightarrow S \cap B = T \cap B \quad \{\ \mathcal{T} \models_F A \rhd B\ \}\\ &\rightarrow S \cap C = T \cap C. \quad \{\ \mathcal{T} \models_F B \rhd C\ \}\end{aligned}$$

$\square$

(A0'), (A1') and (A2') are called reflexive law, augmentation law and transitive law, respectively.

The following proposition will be used in our proof of the completeness of Armstrong inference rules for functional dependencies.

Proposition 4.6. *Let A be a proper subset of Y. There exists a set $\mathcal{T}_0$ such that*

(a) *$C \subseteq A$ if and only if $\mathcal{T}_0 \models_F \emptyset \rhd C$,*

(b) *$C \not\subseteq A$ if and only if $\mathcal{T}_0 \models_F C \rhd Y$.*

Proof. (a)

$$\begin{array}{rcll} \mathcal{T}_0 \models_F \emptyset \rhd C & \leftrightarrow & (A,Y) \in \theta[\emptyset] \rightarrow (A,Y) \in \theta[C] & \{ \text{ 4.4 } \} \\ & \leftrightarrow & (A,Y) \in \theta[C] & \{ \text{ 4.2(a) } \} \\ & \leftrightarrow & A \cap C = Y \cap C = C & \{ \ C \subseteq Y \ \} \\ & \leftrightarrow & C \subseteq A. & \end{array}$$

(b)

$$\begin{array}{rcll} \mathcal{T}_0 \models_F C \rhd Y & \leftrightarrow & (A,Y) \in \theta[C] \rightarrow (A,Y) \in \theta[Y] & \{ \text{ 4.4 } \} \\ & \leftrightarrow & (A,Y) \notin \theta[C] \vee A = Y & \{ \text{ 4.2(b) } \} \\ & \leftrightarrow & (A,Y) \notin \theta[C] & \{ \ A \neq Y \ \} \\ & \leftrightarrow & C \not\subseteq A. & \{ \text{ (a) } \} \end{array}$$

□

Proposition 4.7. *Let $\mathcal{T}$ be an intensional context on Y and $a, b \in Y$. If $\mathcal{T} \models_F \{a\} \rhd \{b\}$ and $\mathcal{T} \not\models_F \{b\} \rhd \{a\}$, then $\mathcal{T} \models_F \emptyset \rhd \{b\}$.*

Proof. Since $\mathcal{T} \not\models_F \{b\} \rhd \{a\}$, there exist $S_0, T_0 \in \mathcal{T}$ such that $a \in S_0$, $a \notin T_0$ and $\{b\} \cap S_0 = \{b\} \cap T_0$. Consider $\mathcal{T}_a = \{S \in \mathcal{T} | a \in S\}$ and $\mathcal{T}_{\neg a} = \{S \in \mathcal{T} | a \notin S\}$. Then $\{b\} \cap S = \{b\} \cap S_0$ and $\{b\} \cap T = \{b\} \cap T_0$ for all $S \in \mathcal{T}_a$ and all $T \in \mathcal{T}_{\neg a}$, respectively, because $\mathcal{T} \models_F \{a\} \rhd \{b\}$. Hence, for all $S \in \mathcal{T}_a$ and $T \in \mathcal{T}_{\neg a}$, $\{b\} \cap S = \{b\} \cap S_0 = \{b\} \cap T_0 = \{b\} \cap T$. And then $\{b\} \cap S = \{b\} \cap T$ for all $S, T \in \mathcal{T}$. Therefore we have $\mathcal{T} \models_F \emptyset \rhd \{b\}$. □

5. Implication

In this section, we review implication introduced in Ganter and Wille (1999).

Definition 5.1. Let $\mathcal{T}$ be an intensional context on Y, $A \rhd B$ a dependency on Y and $\mathcal{L}$ a set of dependencies on Y. We define two notations $\mathcal{T} \models_I A \rhd B$ and $\mathcal{T} \models_I \mathcal{L}$ as follows:

(a) $\mathcal{T} \models_I A \rhd B$ if $A \subseteq T$ implies $B \subseteq T$ for all $T \in \mathcal{T}$,

(b) $\mathcal{T} \models_I \mathcal{L}$ if $\mathcal{T} \models_I A \rhd B$ for all $A \rhd B \in \mathcal{L}$. □

If $\mathcal{T} \models_I A \rhd B$ then $A \rhd B$ is *valid* (as implication) for an intensional context $\mathcal{T}$ on Y or is called an *implication* on $\mathcal{T}$. A set of implication $\mathcal{L}$ is valid for $\mathcal{T}$ on Y if $\mathcal{T} \models_I \mathcal{L}$.

The next proposition means that Armstrong inference rules are sound for implications.

PROPOSITION 5.2. *Let $\mathcal{T}$ be an intensional context on Y, $A \rhd B$ a dependency on Y, and L a set of dependencies.*

(A0) $\mathcal{T} \models_I A \rhd A$,

(A1) *If* $\mathcal{T} \models_I A \rhd B$ *then* $\mathcal{T} \models_I A \cup C \rhd B$,

(A2) *If* $\mathcal{T} \models_I A \rhd B$ *and* $\mathcal{T} \models_I B \cup C \rhd D$ *then* $\mathcal{T} \models_I A \cup C \rhd D$.

PROOF. (A0) It is trivial.
(A1) For all $T \in \mathcal{T}$ we have

$$\begin{array}{rcl} A \cup C \subseteq T & \rightarrow & A \subseteq T \\ & \rightarrow & B \subseteq T. \quad \{\mathcal{T} \models_I A \rhd B\} \end{array}$$

(A2) For all $T \in \mathcal{T}$ we have

$$\begin{array}{rcl} (A \subseteq T \rightarrow B \subseteq T) & \rightarrow & (A \cup C \subseteq T \rightarrow B \cup C \subseteq T) \\ & \rightarrow & (A \cup C \subseteq T \rightarrow D \subseteq T). \qquad \{\mathcal{T} \models_I (B \subseteq C \rhd D)\} \end{array}$$

□

The following proposition will be used in our proof of the completeness of Armstrong inference rules for implications.

PROPOSITION 5.3. *Let A be a proper subset of Y. There exists a set $\mathcal{T}_0$ such that*

(a) $C \subseteq A$ *if and only if* $\mathcal{T}_0 \models_I \emptyset \rhd C$,

(b) $C \not\subseteq A$ *if and only if* $\mathcal{T}_0 \models_I C \rhd Y$.

PROOF. (a) $\mathcal{T}_0 \models_I \emptyset \rhd C \;\leftrightarrow\; (\emptyset \subseteq A \rightarrow C \subseteq A) \;\leftrightarrow\; C \subseteq A$.
(b) $T \models_I C \rhd Y \;\leftrightarrow\; (C \subseteq A \rightarrow Y \subseteq A) \;\leftrightarrow\; C \not\subseteq A$. □

PROPOSITION 5.4. *Let $\mathcal{T}$ be an intensional context and $A \rhd B$ a dependency. Then*

(a) $\mathcal{T} \models_I A \rhd B$ *if and only if* $\mathcal{T}^* \models_I A \rhd B$,

(b) $\mathcal{T} \models_I A \rhd B$ *if and only if* $B \subseteq A^+$.

PROOF. (a)

$$\begin{array}{rcl} \mathcal{T} \models_I A \rhd B & \leftrightarrow & \forall T \in \mathcal{T}.\ (A \subseteq T \rightarrow B \subseteq T) \\ & \leftrightarrow & \forall \mathcal{S} \subseteq \mathcal{T}.\ \forall T \in \mathcal{S}.\ (A \subseteq T \rightarrow B \subseteq T) \\ & \leftrightarrow & \forall \mathcal{S} \subseteq \mathcal{T}.\ (A \subseteq \cap\mathcal{S} \rightarrow B \subseteq \cap\mathcal{S}) \\ & \leftrightarrow & \mathcal{T}^* \models_I A \rhd B. \end{array}$$

(b)

$$\begin{array}{rcl} \mathcal{T} \models_I A \rhd B & \leftrightarrow & \forall T \in \mathcal{T}.\ (A \subseteq T \rightarrow B \subseteq T) \\ & \leftrightarrow & B \subseteq \cap\{T \in \mathcal{T} | A \subseteq T\} \\ & \leftrightarrow & B \subseteq A^+. \end{array}$$

□

6. Soundness and Completeness

Now we will state the soundness and the completeness theorems of implication and functional dependency for intensional contexts. In the proof, the symbol $[A2']$ by square bracket will be used in semantics and the symbol $(A2')$ by round bracket will be used in syntax.

Theorem 6.1. *Let $A \rhd B$ be a dependency and $\mathcal{L}$ a set of dependencies on a finite set Y. Then the following equivalence holds:*

$$\mathcal{L} \vdash A \rhd B \ \text{ if and only if } \ \forall \mathcal{T} \subseteq \wp(Y).\ (\mathcal{T} \models_\bullet \mathcal{L} \to \mathcal{T} \models_\bullet A \rhd B),$$

where $\bullet = F$ *or* I.

Proof. (soundness) We have already seen the soundness in Proposition 4.5 and 5.2
(completeness) Assume $\forall \mathcal{T}.\ (\mathcal{T} \models_\bullet \mathcal{L} \to \mathcal{T} \models_\bullet A \rhd B)$.
(I) In the case of $A_\mathcal{L} = Y$.

$$\begin{array}{rcll} A_\mathcal{L} = Y & \to & B \subseteq A_\mathcal{L} & \{\ B \subseteq Y\ \} \\ & \leftrightarrow & \mathcal{L} \vdash A \rhd B. & \{\ 3.3\ \} \end{array}$$

(II) In the case of $A_\mathcal{L} \neq Y$. Choose an intensional context $\mathcal{T}_0$ satisfying the condition (4.6 and 5.3). Then we can see $\mathcal{T}_0 \models_\bullet \mathcal{L}$, that is, $\mathcal{T}_0 \models_\bullet C \rhd D$ for all $C \rhd D \in \mathcal{L}$:

(II-i) In the case of $C \subseteq A_\mathcal{L}$.

$$\begin{array}{rcll} C \subseteq A_\mathcal{L} & \leftrightarrow & \mathcal{L} \vdash A \rhd C & \{\ 3.3\ \} \\ & \to & \mathcal{L} \vdash A \rhd D & \{\ C \rhd D \in \mathcal{L},\ [\text{A2'}]\ \} \\ & \leftrightarrow & D \subseteq A_\mathcal{L} & \{\ 3.3\ \} \\ & \leftrightarrow & \mathcal{T}_0 \models_\bullet \emptyset \rhd D & \{\ 4.6(\text{a}),\ 5.3(\text{a})\ \} \\ & \to & \mathcal{T}_0 \models_\bullet C \rhd D. & \{\ (\text{A0'})\ \mathcal{T}_0 \models_\bullet C \rhd \emptyset,\ (\text{A2'})\ \} \end{array}$$

(II-ii) In the case of $C \not\subseteq A_\mathcal{L}$.

$$\begin{array}{rcll} C \not\subseteq A_\mathcal{L} & \leftrightarrow & \mathcal{T}_0 \models_\bullet C \rhd Y & \{\ 4.6(\text{b}),\ 5.3(\text{b})\ \} \\ & \to & \mathcal{T}_0 \models_\bullet C \rhd D. & \{\ (\text{A0'})\ \mathcal{T}_0 \models_\bullet Y \rhd D,\ (\text{A2'})\ \} \end{array}$$

Therefore $\mathcal{T}_0 \models_\bullet A \rhd B$ holds by the assumption and so we have

$$\begin{array}{rcll} \mathcal{L} \vdash A \rhd A & \leftrightarrow & A \subseteq A_\mathcal{L} & \{\ 3.3\ \} \\ & \leftrightarrow & \mathcal{T}_0 \models_\bullet \emptyset \rhd A & \{\ 4.6(\text{a}),\ 5.3(\text{a})\ \} \\ & \to & \mathcal{T}_0 \models_\bullet \emptyset \rhd B & \{\ \mathcal{T}_0 \models_\bullet A \rhd B,\ (\text{A2'})\ \} \\ & \leftrightarrow & B \subseteq A_\mathcal{L} & \{\ 4.6(\text{b}),\ 5.3(\text{a})\ \} \\ & \leftrightarrow & \mathcal{L} \vdash A \rhd B. & \{\ 3.3\ \} \end{array}$$

□

Definition 6.2. A set $\mathcal{L}$ of dependencies is closed, if

$$\forall A \rhd B. \mathcal{L} \vdash A \rhd B \to A \rhd B \in \mathcal{L}.$$

□

Then, the following theorem holds.

THEOREM 6.3 MAIER. *A set $\mathcal{L}$ of dependencies is closed if and only if $\mathcal{L}$ satisfies the following.*

(a) $A \rhd A \in \mathcal{L}$,

(b) *if* $A \rhd B \in \mathcal{L}$ *then* $A \cup C \rhd B \in \mathcal{L}$,

(c) *if* $A \rhd B, B \cup C \rhd D \in \mathcal{L}$ *then* $A \cup C \rhd D \in \mathcal{L}$.

PROOF. $(\leftarrow)$ is trivial by structural induction.
$(\rightarrow)$
(a)

$$\begin{array}{rcll} true & \rightarrow & \mathcal{L} \vdash A \rhd A & \{ \text{[A0]} \} \\ & \rightarrow & A \rhd A \in \mathcal{L}. & \{ \mathcal{L} : \text{closed} \} \end{array}$$

(b)

$$\begin{array}{rcll} A \rhd B \in \mathcal{L} & \rightarrow & \mathcal{L} \vdash A \rhd B & \\ & \rightarrow & \mathcal{L} \vdash A \cup C \rhd B & \{ \text{[A1]} \} \\ & \rightarrow & A \cup C \rhd B \in \mathcal{L}. & \{ \mathcal{L} : \text{closed} \} \end{array}$$

(c)

$$\begin{array}{rcll} A \rhd B,\ B \cup C \rhd D \in \mathcal{L} & \rightarrow & \mathcal{L} \vdash A \rhd B \wedge \mathcal{L} \vdash B \cup C \rhd D & \\ & \rightarrow & \mathcal{L} \vdash A \cup C \rhd D & \{ \text{[A2]} \} \\ & \rightarrow & A \cup C \rhd D \in \mathcal{L}. & \{ \mathcal{L} : \text{closed} \} \end{array}$$

Conversely assume that $\mathcal{L}$ satisfies the conditions (a), (b) and (c). Recall the completeness theorem

$$\mathcal{L} \vdash A \rhd B \ \text{ if and only if } \ \forall T.\ (T \models_I \mathcal{L} \rightarrow T \models_I A \rhd B).$$

If $\forall T.\ (T \models_I \mathcal{L} \rightarrow T \models_I A \rhd B)$, then $\mathcal{L} \vdash A \rhd B$ by the completeness theorem and so $A \rhd B \in \mathcal{L}$ by the conditions (a), (b) and (c). □

Therefore, Armstrong Inference rules are sound and complete for functional dependencies and implications.

7. Difference between Implication and Functional Dependency

In this section, we show the difference between an implication and a functional dependency, by using examples.

Let an intensional context $\mathcal{T}$ be $\{\{a, b\}, \{b, c\}, \{c\}\}$.

	a	b	c
x	×	×	
y		×	×
z			×

We consider about the dependency $\{a\} \rhd \{b\}$. The set $\{a\}^+$ is $\{a, b\}$. Therefore $\mathcal{T} \models_I \{a\} \rhd \{b\}$. But, for $\{b, c\}, \{c\} \in \mathcal{T}$, $\{b, c\} \cap \{a\} = \{c\} \cap \{a\}$, and $\{b, c\} \cap \{b\} \neq$

$\{c\} \cap \{b\}$. Therefore $\{a\} \triangleright \{b\}$ is not a functional dependency on $\mathcal{T}$. On the other hand, we consider about the dependency $\{a\} \triangleright \{c\}$. It is a functional dependency on $\mathcal{T}$, but it is not an implication on $\mathcal{T}$. Hence, an implication and a functional dependency are different.

PROPOSITION 7.1. *For an intensional context $\mathcal{T}$, there is not always $\mathcal{U} \subseteq \wp(Y)$ such that $\mathcal{T} \models_I A \triangleright B$ if and only if $\mathcal{U} \models_F A \triangleright B$.*

PROOF. Let an intensional context $\mathcal{T}$ be $\{\{a, b\}, \{b, c\}, \{c\}\}$. Then $\mathcal{T} \models_I \{a\} \triangleright \{b\}$, $\mathcal{T} \not\models_I \{b\} \triangleright \{a\}$ and $\mathcal{T} \not\models_I \emptyset \triangleright \{b\}$.

By assumption of an intensional context $\mathcal{U}$, it satisfy (1) $\mathcal{U} \models_F \{a\} \triangleright \{b\}$, (2) $\mathcal{U} \not\models_F \{b\} \triangleright \{a\}$ and (3) $\mathcal{U} \not\models_F \emptyset \triangleright \{b\}$.

Since (1) and (2), we get $\mathcal{U} \models_F \emptyset \triangleright \{b\}$. However it is inconsistent with (3). □

We consider the condition on which the functional dependency and the implication of an intensional context are equivalent. And we find the following condition.

PROPOSITION 7.2. *Let $\mathcal{T}$ be an intensional context. Define a set $\mathcal{T}'$ of subsets of Y by $\mathcal{T}' = \{(S^- \cup T) \cap (T^- \cup S) \mid S, T \in \mathcal{T}\}$.*

(a) *$\mathcal{T}' \models_I A \triangleright B$ if and only if $\mathcal{T} \models_F A \triangleright B$,*

(b) *If $\mathcal{T} \subseteq \mathcal{T}' \subseteq \mathcal{T}^*$, then $\mathcal{T} \models_F A \triangleright B$ if and only if $\mathcal{T} \models_I A \triangleright B$.*

PROOF. (a) First remark that

$$\begin{aligned} S \cap A = T \cap A &\leftrightarrow (S \cap A \subseteq T) \wedge (T \cap A \subseteq S) \\ &\leftrightarrow (A \subseteq S^- \cup T) \wedge (A \subseteq T^- \cup S) \\ &\leftrightarrow A \subseteq (S^- \cup T) \cap (T^- \cup S). \end{aligned}$$

($\rightarrow$) Let $S, T \in \mathcal{T}$ and set $U = (S^- \cup T) \cap (T^- \cup S)$. Then $U \in \mathcal{T}'$ and so we have

$$\begin{aligned} S \cap A = T \cap A &\leftrightarrow A \subseteq U \\ &\rightarrow B \subseteq U \qquad \{\ \mathcal{T} \models_I A \triangleright B\ \} \\ &\leftrightarrow S \cap B = T \cap B. \end{aligned}$$

($\leftarrow$) For all $U \in \mathcal{T}'$ there exists a pair of subsets $S, T \in \mathcal{T}$ such that $U = (S^- \cup T) \cap (T^- \cup S)$ and so we have

$$\begin{aligned} A \subseteq U &\leftrightarrow S \cap A = T \cap A \\ &\rightarrow S \cap B = T \cap B \quad \{\ \mathcal{T} \models_F A \triangleright B\ \} \\ &\leftrightarrow B \subseteq U. \end{aligned}$$

(b) ($\rightarrow$)

$$\begin{aligned} \mathcal{T} \models_F A \triangleright B &\leftrightarrow \mathcal{T}' \models_I A \triangleright B \\ &\rightarrow \mathcal{T} \models_I A \triangleright B. \quad \{\ \mathcal{T} \subseteq \mathcal{T}'\} \end{aligned}$$

($\leftarrow$)

$$\begin{aligned} \mathcal{T} \models_I A \triangleright B &\leftrightarrow \mathcal{T}^* \models_I A \triangleright B \\ &\rightarrow \mathcal{T}' \models_I A \triangleright B \quad \{\ \mathcal{T}' \subseteq \mathcal{T}^*\ \} \\ &\leftrightarrow \mathcal{T} \models_F A \triangleright B. \end{aligned}$$

□

8. Summary and Outlook

In this paper, we review the idea that Armstrong inference rules are sound and complete for functional dependencies. Then, we prove that Armstrong inference rules are sound and complete for implications of formal contexts. Further, to distinguish the semantic and syntax, we give a proof which is different from that of Ganter and Wille (1999). In the proof, we ordered the thing of semantic or syntax theory. Still, we give an example which shows the difference between implication and functional dependency. Besides, we show that functional dependency can be reduced to implication. Finally, we give the condition that a set of implications and a set of functional dependencies for a intensional context are equivalent.

In the future, we will consider other conditions on which implication and functional dependency are equivalent.

References

Ganter, B., Wille, R. (1999). Formal Concept Analysis, *Springer-Verlag.*

Codd, E. (1970). A relational model of data for large shared data banks, *Communications of the ACM **13**. 377–387*

Armstrong, W. W. (1974). Dependency structures in data base relationships, *IFIP Congress, Geneva, Switzerland. 580–583*

Beeri, C., Fagin, R., Howard, J.H. (1977). A complete axiomatization for functional and multivalued dependencies in database relations, *Proceedings of the 1977 ACM SIGMOD International Conference on Management of Data, Toronto, Canada. 47–61*

Düntsch, I., Günther, G. (2000). Rough set data analysis. A road to non-invasive knowledge discovery, *Methoδos Publishers*

Received June 24, 2008
Revised October 6, 2008