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Abstract

We give sets of fairly easy conditions under which a multidimensional diffusion with compound-Poisson jumps possesses several global-stability properties: (exponential) ergodicity, (exponential) β-mixing property, and also boundedness of moments. These are important to statistical inference under long-time asymptotics. The proof in this article is based on Masuda (2007), but we here demonstrate an explicit construction of a "T-chain kernel", which enables us to deal with a broad class of finite-jump parts under smoothness of the coefficients plus pointwise nondegeneracy of the diffusion-coefficient matrix.

Key Words and Phrases: Boundedness of moments, diffusion with compound-Poisson jumps, (exponential) ergodicity, (exponential) β-mixing property.

1. Introduction and statement of results

When attempting statistical inference for a continuous-time stochastic process $X = (X_t)_{t\in[0,T]}$ based on long-time asymptotics, namely for $T \to \infty$, most often (but not always!) required are a law of large numbers, typically referred to as *ergodicity.* Moreover, in case of higher-order inference *fast decay of a mixing coefficient* (see Section 1.3.) is often indispensable; of course, this is also the case for discrete-time time series, see Liebscher (2005) and the references therein. Previously, for multidimensional diffusions with possibly infinitely many jumps on compact intervals, Masuda (2007), henceforth referred to as [M] (with the corrections to as [M-Corrections]), derived sufficient conditions for such global stabilities. Although the results in [M] and [M-Corrections] are general to cover a wide range of diffusions with jumps, the conditions include a kind of topological continuity of the transition semigroup (see [M, Assumption 2] and [M-Corrections, Assumption 2(a)′]), for which one may be forced to consult some advanced results on existence and smoothness of a transition density for general diffusions with jumps: this might cause some inconvenience to readers unfamiliar with such results.

The purpose of this article is to provide fairly easy conditions for the above-mentioned global stabilities of X when drift and diffusion coefficients are smooth with the latter being pointwise elliptic, and the jump intensity is finite. Our emphasize here is put on ease of verification of the conditions. As a matter of fact, once the coefficients and the Lévy measure are given, all the assumptions employed in this article can be verified only by elementary calculus. The scenario of the proofs we will take here is in parallel with that in [M], except that we will utilize the "T-chain property" of a skeleton

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chain of X in an explicit way: actually, this enables us to pick out a nice property of the diffusion part with leaving the finite-jump part almost arbitrary.

In the rest of this section we describe our objective and results, part of which are applicable to much more general diffusions with jumps than our main objective (1) below. The proofs are given in Section 2.

1.1. Objective

Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a d-dimensional càdlàg [1] Markov process given by the time-homogeneous Itô's stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dw_t + \int_0^t \int \zeta(X_{t-}, z)\mu(dt, dz), \tag{1}$$

which is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$. Here the ingredients are given as follows.

- The coefficients $b = (b^i) : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma = (\sigma^{ij}) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^k$, and $\zeta = (\zeta^i) : \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}^d$ are measurable functions.
- w is a k-dimensional standard Wiener process.
- μ is a time-homogeneous Poisson random measure on $\mathbb{R} \times \mathbb{R}^r \backslash \{0\}$ with Lévy measure $\nu(dz)$.
- The initial variable X_0 is $\mathcal{F}_0$-measurable and independent of (w, μ).

We will suppose $\nu(\mathbb{R}^r) < \infty$, which implies that number of X's jumps is a.s. finite over any compact time interval and that the stochastic integral with respect to μ in the right-hand side of (1) is well defined. The process X is a diffusion with compound-Poisson jumps, which extends the diffusion process (where $\zeta \equiv 0$) and forms a broad class of Markov processes accommodating accidental large state change as well as diffusive small fluctuation; consult the references in [M] for a comprehensive account for theory of general diffusion with jumps.

We will write $\mathbb{E}$ for the expectation operator and η for the law of initial variable X_0. The symbol $\mathbb{P}_\eta$ (resp. $\mathbb{E}_\eta$) will be used instead of $\mathbb{P}$ (resp. $\mathbb{E}$), when we emphasize the dependence on η; $\mathbb{P}_x$ corresponds to the case of $\eta = \delta_x$ for some $x \in \mathbb{R}^d$, where δ_x stands for the Dirac delta measure at x. We will denote by $(P_t)_{t \in \mathbb{R}_+}$ the transition semigroup of X, namely, $P_t(x, dy) = \mathbb{P}_x[X_t \in dy]$.

The following basic notation will be used in the sequel. For a matrix $M = (M^{ij})$ let $|M| := \{\sum_{i,j} (M^{ij})^2\}^{1/2}$ and $M^{\otimes 2} := MM^\top$, where $\top$ denotes the trasposition. Write $a \lesssim a'$ if $a \le ca'$ for some generic constant $c > 0$.

1.2. Assumptions

[C1] *For every $x_1, x_2 \in \mathbb{R}^d$ and $z_1, z_2 \in \mathbb{R}^r$, we have $\zeta(x_1, 0) = 0$ and*

$$\begin{aligned}
&|b(x_1) - b(x_2)| + |\sigma(x_1) - \sigma(x_2)| \lesssim |x_1 - x_2|, \\
&|\zeta(x_1, z_1) - \zeta(x_2, z_1)| \lesssim |z_1||x_1 - x_2|, \\
&|\zeta(x_1, z_1) - \zeta(x_1, z_2)| \lesssim \rho(x_1)|z_1 - z_2|,
\end{aligned}$$

[1] A function $t \mapsto x_t$ on $\mathbb{R}_+$ is called càdlg̀ if it is right-continuous and if $\lim_{s \uparrow t, s<t} x_s$ exists for each $t > 0$.

where $\rho : \mathbb{R}^d \to \mathbb{R}_+$ *is a locally bounded function such that* $|\zeta(x,z)| \le \rho(x)|z|$ *for every* (x,z) *and that* $\lim_{|x|\to\infty} \rho(x)/|x| = 0$.

[C2] $\nu(\mathbb{R}^r) < \infty$.

[C3] *For every* $i \le d$ *and* $j \le k$ *the functions* b^i *and* σ^{ij} *are of class* $\mathcal{C}^\infty$ *and have bounded derivatives of any positive order. Moreover,* $\sigma^{\otimes 2}(x)$ *is positive-definite for every* $x \in \mathbb{R}^d$.

Under **[C1]** the stochastic differential equation (1) admits a unique solution, which is $(\mathcal{F}_t)$-adapted, non-explosive, càdlàg, strong-Markov, and weak-Feller. **[C2]** is essential in this article, while we put **[C3]** for simplicity of the description (see the third remark in page 65). Nevertheless, **[C3]** allows us to deal with possibly unbounded b and σ having linear growth, i.e. $|b(x)| + |\sigma(x)| \lesssim 1 + |x|$, with σ being *not* uniformly elliptic.

We need to prepare two more conditions. For $q > 0$ and $x = (x_i)_{i=1}^d \in \mathbb{R}^d \backslash \{0\}$, we define

$$
\begin{aligned}
B_q(x) &= q|x|^{q-2} x^\top b(x), \\
D_q(x) &= \frac{1}{2} q|x|^{q-2} \mathrm{trace}\left\{ \left((q-2)[x_i x_j]_{i,j=1}^d |x|^{-2} + I_d \right) \sigma(x)^{\otimes 2} \right\}, \\
G_q(x) &= B_q(x) + D_q(x), \\
J_q(x) &= \{\rho(x)\}^2 |x|^{q-2} + \{\rho(x)\}^q + |x|^{q-1} \rho(x) \mathbf{1}_{(1,\infty)}(q),
\end{aligned}
$$

where $\mathbf{1}_{(1,\infty)}(q)$ is defined to be 0 or 1 according as $q \in (0,1]$ or $q \in (1,\infty)$. Note that $x \mapsto G_q(x)$ is formally the diffusion part of the generator of X applied to $x \mapsto |x|^q$.

[E] *At least one of the following two holds true.*

- *There exists a constant* $q > 0$ *such that* $\int_{|z|>1} |z|^q \nu(dz) < \infty$ *and that:*
  - (i) *there exists a constant* $c > 0$ *such that*
  $$G_q(x) \le -c$$
  for every $|x|$ *large enough; and*
  - (ii) $\lim_{|x|\to\infty} J_q(x) = 0$.
- *There exists a constant* $q > 0$ *such that* $\int_{|z|>1} |z|^q \nu(dz) < \infty$ *and that:*
  - (i') $B_q(x) \to -\infty$ *as* $|x| \to \infty$*; and*
  - (ii') $\lim_{|x|\to\infty} \{D_q(x) \vee J_q(x)\}/B_q(x) = 0$.

[EE] *There exist constants* $q > 0$ *and* $c' > 0$ *such that* $\int_{|z|>1} |z|^q \nu(dz) < \infty$ *and that*

$$G_q(x) \le -c'|x|^q$$

for every $|x|$ *large enough.*

We can simplify the conditions if (σ,ρ) does not become so large for $|x|\to\infty$. Clearly, we can replace "$G_q(x)$" with "$B_q(x)$" in **[E]**(i) if $\sigma(x)=o(|x|^{1-q/2})$. Also, as

$$\left|\frac{D_q(x)\vee J_q(x)}{B_q(x)}\right|\lesssim\frac{\{|\sigma(x)|\vee\rho(x)\}^2}{|x^\top b(x)|}+\left\{\left(\frac{\rho(x)}{|x|}\right)^q+\mathbf{1}_{(1,\infty)}(q)\frac{\rho(x)}{|x|}\right\}\frac{|x|^2}{|x^\top b(x)|},$$

the condition **[E]**(ii') is fulfilled as soon as $\{|\sigma(x)|\vee\rho(x)\}^2/|x^\top b(x)|\to 0$ and $|x|^2/|x^\top b(x)|$ stay bounded for $|x|\to\infty$. Moreover, we can replace "$G_q(x)\le -c'|x|^q$" with "$x^\top b(x)\le -c'|x|^2$" in **[EE]** if $\sigma(x)=o(|x|)$.

1.3. Main results

Let $\|m\|:=\sup_{|g|\le 1}|\int g(x)m(dx)|$ stand for the total variation norm of a signed measure m. The β-mixing (absolute-regular) coefficient of X is given by

$$\beta_X(t)=\sup_{s\in\mathbb{R}_+}\int\|P_t(x,\cdot)-\eta P_{s+t}(\cdot)\|\eta P_s(dx),$$

where ηP_t stands for the marginal law of X_t. Then X is called:

- *β-mixing* if $\beta_X(t)=o(1)$ for $t\to\infty$;
- *exponentially β-mixing* if there exists a constant $\gamma>0$ such that $\beta_X(t)=O(e^{-\gamma t})$ for $t\to\infty$.

Now we can state our main result.

Theorem 1.1. *Suppose* **[C1]**, **[C2]**, *and* **[C3]** *hold true. Then:*

(a) *under* **[E]**, (P_t) *admits a unique invariant law* π *for which*

$$\|P_t(x,\cdot)-\pi(\cdot)\|\to 0 \tag{2}$$

as $t\to\infty$ *for every* $x\in\mathbb{R}^d$, *and* X *is* β*-mixing for any* η*;*

(b) *under* **[EE]**, (P_t) *admits a unique invariant law* π *fulfilling*

$$\int|x|^q\pi(dx)<\infty, \tag{3}$$

for which there exist positive constants a *and* c *such that*

$$\|P_t(x,\cdot)-\pi(\cdot)\|\le c(1+|x|^q)e^{-at} \tag{4}$$

for every $x\in\mathbb{R}^d$ *and* $t\in\mathbb{R}_+$. *If moreover* $\int|x|^q\eta(dx)<\infty$, *then there exist constants* $a'>0$ *and* $c'>0$ *such that* $\beta_X(t)\le c'e^{-a't}$ *for each* $t\in\mathbb{R}_+$, *hence* X *is exponentially* β*-mixing.*

In both cases we have the ergodic theorem: for every π*-integrable* F

$$\frac{1}{T}\int_0^T F(X_t)dt\to\int F(x)\pi(dx) \tag{5}$$

as $T\to\infty$ *in* $\mathbb{P}_\eta$*-probability whatever* η *is.*

Remark. We can consult Kulik (2007) for an exponential β-mixing result for $\sigma \equiv 0$; in this case, we inevitably need some nondegeneracy conditions on the jump part.

Remark. Even in the first-order inference (such as M-estimation) concerning X, the *boundedness of moments* in the sense that, e.g.,

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}_\eta[|X_t|^k] < \infty \tag{6}$$

for sufficiently large $k > 0$, may be also crucial in order to deduce suitable limit theorems for estimating functions. We note that (6) can be readily verified by applying [M, Theorem 2.2 (i)] without any topological continuity condition of (P_t). Specifically:

"under **[C1]** and **[EE]** we have (6) for any $k \leq q$".

Especially, (6) holds true for every $k > 0$ if q in **[EE]** can be any positive real.

Remark. In this article we have introduced **[C3]** because of its simplicity. However, we must note that **[C3]** can be relaxed by means of well-known "Hörmander's condition" (e.g., Watanabe (1984)) together with "positivity criteria for transition density" (cf. Skorokhod (1989, Section I.2.2)). Especially, we should note that it suffices that the law of X_t admits a smooth and Lebesgue-a.e. positive density, and this is true if for every $x \in \mathbb{R}^d$ the subspace spanned by $\{\sigma_j(x) : j \leq k\}$ coincides with $\mathbb{R}^d$, where $\sigma_j(x)$ denotes the jth column vector of $\sigma(x)$.

It is possible to deduce the (exponential) β-mixing property and variants of the uniform boundedness (6) under different sets of conditions. Among others, we here focus on the case where the drift function b is bounded and ν admits an exponential moments outside a neighborhood of the origin. In this instance we can derive the same conclusion as in Theorem 1.1(b) and additionally an exponential-moment version of (6) as was done by a part of Gobet (2002) for diffusions. Before stating the result, let us introduce new sets of conditions.

[C1b] *In addition to* **[C1]**, *the functions b, σ, and ρ are bounded.*

Under **[C1b]**, we write $\rho^* := \sup_x |\rho(x)| \in [0, \infty)$.

[EEb] *There exist constants $r > 0$ and $c_0 > \rho^* \int |z|\nu(dz)$ such that $\int_{|z|>1} \exp(r|z|)\nu(dz) < \infty$ and that*

$$x^\top b(x) \leq -c_0|x|$$

for every $|x|$ large enough.

Theorem 1.2. *Suppose* **[C1b]**, **[C2]**, **[C3]**, *and* **[EEb]**. *Then the same statement as in Theorem 1.1*(b) *with* "**[EEb]**" *instead of* "**[EE]**" *holds true. Moreover, there exists a constant $r_0 \in (0, r/\rho^*)$* [2] *such that:*

- *for any $r_1 \in [0, r_0)$ we have $\int \exp(r_1|x|)\pi(dx) < \infty$; and that*

[2] Regard $r_0 \in (0, r/\rho^*)$ as $r_0 > 0$ if $\rho^* = 0$.

- *for any* $r_2 \in [0, r_0)$ *meeting* $\int \exp(r_2|x|)\eta(dx) < \infty$, *we have*

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}_\eta[\exp(r_2|X_t|)] < \infty. \tag{7}$$

Prior to the proofs of Theorems 1.1 and 1.2, we proceed to some lemmas.

1.4. Some lemmas applicable to more general setup

Here we prepare some lemmas, part of which will be used in the proof of Theorem 1.1. The following Lemmas 1.3, 1.4, and 1.5 are slight refinements of Lemmas 2.4, 2.5(i), and 3.9 of [M], respectively. The lemmas presented in this section can work on much more general diffusions with jumps than (1).

In Lemmas 1.3 and 1.4 below, we forget the objective (1), and instead deal with the general diffusion with jumps given by

$$\begin{aligned} dX'_t &= b(X'_t)dt + \sigma(X'_t)dw_t \\ &\quad + \int_0^t \int_{|z|\le 1} \zeta(X'_{t-}, z)\tilde{\mu}(dt, dz) + \int_0^t \int_{|z|>1} \zeta(X'_{t-}, z)\mu(dt, dz), \end{aligned} \tag{8}$$

where $\tilde{\mu}(dt, dz) = \mu(dt, dz) - \nu(dz)dt$ denotes the compensated Poisson random measure; note that X' may have infinitely many small jumps over each compact time interval. We used the same notation as in (1) for the coefficient of the stochastic differential equation (8) just for the convenience; for X', we will consistently put the descriptions of **[C1]**, **[E]**, and **[EE]** to use.

To state the lemmas we need some more notation. As in [M], let $\mathcal{Q}$ denote the set of all $\mathcal{C}^2$ functions $f : \mathbb{R}^d \to \mathbb{R}_+$ such that there exists a locally bounded measurable function $\bar{f}$ for which

$$\int_{|z|>1} f(x + \zeta(x, z))\nu(dz) \le \bar{f}(x)$$

for every $x \in \mathbb{R}^d$, and put $\mathcal{Q}^* = \mathcal{Q} \cap \{f : \mathbb{R}^d \to \mathbb{R}_+ |\ f(x) \to \infty \text{ as } |x| \to \infty\}$. Define the extended generator $\mathcal{A}$ of X' by

$$\mathcal{A}f = \mathcal{G}f + \mathcal{J}_*f + \mathcal{J}^*f \tag{9}$$

for $f \in \mathcal{Q}$, where

$$\begin{aligned} \mathcal{G}f(x) &= \nabla f(x)b(x) + \frac{1}{2}\mathrm{trace}\{\nabla^2 f(x)\sigma(x)\sigma(x)^\top\}, \\ \mathcal{J}_*f(x) &= \int_{|z|\le 1} \Big(f(x + \zeta(x, z)) - f(x) - \nabla f(x)\zeta(x, z)\Big)\nu(dz), \\ \mathcal{J}^*f(x) &= \int_{|z|>1} \Big(f(x + \zeta(x, z)) - f(x)\Big)\nu(dz). \end{aligned}$$

The function $x \mapsto \mathcal{A}f(x)$ is actually well defined and locally bounded as soon as $f \in \mathcal{Q}$ (see [M, Section 3.1.2] for details). Now let us recall the drift conditions used in [M] (the conditions **[D]** and **[D*]** below are termed Assumption 3 and Assumption 3* in [M], respectively):

[D] *There exist $f \in \mathcal{Q}$ and a constant $c > 0$ such that $\mathcal{A}f(x) \le -c$ for every $|x|$ large enough.*

[D*] *There exist $f \in \mathcal{Q}^*$ and a constant $c' > 0$ such that $\mathcal{A}f(x) \le -c' f(x)$ for every $|x|$ large enough.*

In [M], we have seen that the β-mixing property (resp. the exponential β-mixing property) can be derived under the following three kinds of conditions (see Section 2.1. for more detail): **[C1]**, a kind of irreducibility and continuity of the transition semigroup (cf. Assumption 2 of [M]), and drift conditions **[D]** (resp. **[D*]**). The next lemma serves as a tool for verification of the last one.

Lemma 1.3. *Suppose* **[C1]** *holds true. Then,* **[D]** *(resp.* **[D*]***) is implied by* **[E]** *(resp.* **[EE]***).*

The scenario of the proof is equal to Kulik (2007, Proposition 4.1), which previously obtained **[D*]** in case of $\sigma \equiv 0$. However, in Section 2.3. we will give a full proof in order to clarify how to derive **[D]**.

The next one is a refinement of [M, Lemma 2.5(i)] dealing with a very heavy-tailed ν, but we do not use it in this article.

Lemma 1.4. *Suppose* **[C1]** *and*

$$\int_{|z|>1} \log(1+|z|)\nu(dz) < \infty, \tag{10}$$

and that $|\sigma(x)| = o(|x|)$ for $|x| \to \infty$. Furthermore, suppose that

$$\limsup_{|x|\to\infty} \frac{x^\top b(x)}{|x|(1+|x|)} < 0. \tag{11}$$

Then there exists an $f \in \mathcal{Q}^*$ for which **[D]** *holds true.*

We end with the following lemma, which can apply to general continuous-time Markov processes.

Lemma 1.5. *Let $Y = (Y_t)_{t\in\mathbb{R}_+}$ be a Markov process taking its values in a locally compact separable metric space $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$, $\mathcal{B}(\mathbb{Y})$ denoting the Borel field on $\mathbb{Y}$. Let η, $(P_t)_{t\in\mathbb{R}_+}$, and $\beta_Y(t)$ respectively denote initial distribution, transition semigroup, and β-mixing coefficient of Y. Suppose that there exists a probability measure π on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$ for which*

$$V_t(y) := \|P_t(y,\cdot) - \pi(\cdot)\| \to 0$$

as $t \to \infty$ for every $y \in \mathbb{Y}$. Then, for each $t \in \mathbb{R}_+$ and $u \in (0,t)$ we have

$$\beta_Y(t) \le \eta(V_t) + 2\eta(V_u) + \pi(V_{t-u}). \tag{12}$$

Especially:

(a) *Y is β-mixing for any η;*

(b) *$\beta_Y(t) \lesssim \delta(t/2)$ for each $t \in \mathbb{R}_+$ if $V_t(y) \le h(y)\delta(t)$ for a finite measurable function $h : \mathbb{Y} \to \mathbb{R}_+$ and a nonincreasing function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$, and if $\pi(h) \vee \eta(h) < \infty$.*

2. Proofs

2.1. Proof of Theorem 1.1

We will proceed as in [M, Theorems 2.1 and 2.2] for the most part. However, differently from [M] we will here utilize an explicit T-chain kernel given by (16) below. Specifically, we will achieve the proof through "weak-Feller property", "open-set irreducibility and T-chain property of some skeleton chain", and "Foster-Lyapunov drift conditions".

First, let us mention that the proof of Theorem 1.1 reduces to the verification of the condition (13) below. We will apply Meyn and Tweedie (1993b, Theorems 5.1 and 6.1) for the ergodic properties (2) and (4). There one of the crucial steps is to prove that

$$\text{every compact sets are petite for some skeleton chain } (X_{\Delta m})_{m\in\mathbb{Z}_+}, \tag{13}$$

where $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$, and $\Delta > 0$ is some constant: see Meyn and Tweedie (1993a) for a detailed account for the notion of *petite sets*. X is a non-explosive right process under **[C1]**, so that, in order to prove Theorem 1.1 it is sufficient to show:

- "(13) and **[D]**" for (2);
- "(13) and **[D*]**" for (4).

This sufficiency follows from the argument in [M, Section 3.1.1]. The drift conditions **[D]** and **[D*]** are directly verified by means of Lemma 1.3. Also, under **[EE]** we immediately get (3); see [M, the last paragraph in p.50]. Once (2) (resp. (4)) is verified, then the β-mixing property (resp. the β-mixing bound) readily follows on applying Lemma 1.5; Lemma 1.5(b) can be applied as we know that (3) holds true. Furthermore, the ergodic theorem (5) is a direct consequence of (2); see [M, Theorem 2.1 and Section 3.1.4]. Therefore, in order to achieve the proof of Theorem 1.1 it remains to prove (13).

Unlike the drift criteria, (13) is not straightforward to verify as such. Here we will make use of the fact that (13) is implied by the following conditions (at least for *one* $\Delta > 0$):

[T1] (Open-set irreducibility) *For every open set $O \subset \mathbb{R}^d$ and every $x \in \mathbb{R}^d$, there exists a constant $m = m(x,O) \in \mathbb{N}$ for which $\mathbb{P}_x[X_{m\Delta} \in O] > 0$;*

[T2] (T-chain property with δ_Δ as the sampling distribution) *there exists a kernel $T_\Delta : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \to [0,1]$ such that*

(i) *$x \mapsto T_\Delta(x,A)$ is lower semicontinuous for every $A \in \mathcal{B}(\mathbb{R}^d)$, that*

(ii) *$P_\Delta(x,A) \geq T_\Delta(x,A)$ for every $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$, and that*

(iii) *$T_\Delta(x,\mathbb{R}^d) > 0$ for every $x \in \mathbb{R}^d$.*

As a matter of fact, if **[T1]** and **[T2]** are fulfilled for some $\Delta > 0$, then (13) follows from Meyn and Tweedie (1993a, the last half of Theorem 6.2.5(ii)).

Building on the observations made above, we see that it suffices to prove **[T1]** and **[T2]**. In the rest of this proof we will suppose that $\nu(\mathbb{R}^r) > 0$, as the case of null ν (i.e. the case of diffusion processes), which is implicitly contained in our framework, is easier to handle.

Proof of [**T1**]. Take any $x \in \mathbb{R}^d$, and define a diffusion $Y = (Y_t)_{t\in\mathbb{R}_+}$ by

$$Y_t = x + \int^t b(Y_s)ds + \int_0^t \sigma(Y_s)dw_s. \tag{14}$$

Fix any $\Delta > 0$. On the event

$$E_\Delta := \{\omega \in \Omega : \mu((0,\Delta], \mathbb{R}^r\backslash\{0\}) = 0\},$$

the original X agrees with Y over the time interval $[0,\Delta]$ ($\mathbb{P}_x$-a.s.); we know that $\mathbb{P}[E_\Delta] > 0$ under [**C2**]. As in [M, the proof of Claim 1 under Assumption 2(a) in Proposition 3.1] (see also [M-Corrections]), by restricting the situation to E_Δ it suffices to show that $\mathbb{P}_x[Y_\Delta \in O] > 0$ for any open $O \subset \mathbb{R}^d$: specifically, by means of the independence between w and μ we see that

$$\mathbb{P}_x[X_\Delta \in O] \geq \mathbb{P}_x[X_\Delta \in O\} \cap E_\Delta] = \mathbb{P}_x[Y_\Delta \in O\} \cap E_\Delta] = \mathbb{P}[E_\Delta]\mathbb{P}_x[Y_\Delta \in O]. \tag{15}$$

Under [**C3**], Y_Δ admits an smooth density $p^Y_\Delta(x,y)$ which is (Lebesgue-)a.e. positive; see Watanabe (1984, Theorem 2.7) and Skorokhod (1989, Theorem I.13). Hence (15) becomes

$$\mathbb{P}_x[X_\Delta \in O] \geq \mathbb{P}[E_\Delta]\int_O p^Y_\Delta(x,y)dy > 0,$$

since $\mathbb{P}[E_\Delta] > 0$ and O is open. Accordingly, [**T1**] holds true for *any* $\Delta > 0$ under [**C1**], [**C2**], and [**C3**].

Proof of [**T2**]. Again fix any $x \in \mathbb{R}^d$ and $\Delta > 0$. In [M], the existence of a bounded transition density of the "original X with jumps" was supposed. We can do away this assumption by reducing the situation to Y given by (14) through an explicit T-chain kernel.

Set $T_\Delta(x,A) = \mathbb{P}_x[\{X_\Delta \in A\} \cap E_\Delta]$, so that

$$T_\Delta(x,A) = \mathbb{P}_x[Y_\Delta \in A]\mathbb{P}[E_\Delta] \tag{16}$$

for every $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$. Obviously we then have [**T2**](ii). Reminding that $\mathbb{P}[E_\Delta] > 0$, we also get [**T2**](iii). So it remains to prove [**T2**](i). To this end we will utilize Cline and Pu (1998, Lemma 3.1) as in [M].

Fix any $\epsilon > 0$ and nonempty compact $K_1, K_2 \subset \mathbb{R}^d$. Then we know that

$$c(\Delta; K_1, K_2) := \sup_{y\in K_2}\sup_{x\in K_1} p^Y_\Delta(x,y) \in (0,\infty)$$

under [**C3**]. Take any $\delta > 0$ such that

$$\delta < \epsilon\bigg\{\mathbb{P}[E_\Delta]c(\Delta; K_1, K_2)\bigg\}^{-1}, \tag{17}$$

and then fix any $A \subset K_2$ such that $\ell(A) < \delta$, where ℓ stands for the Lebesgue measure on $\mathbb{R}^d$. According to (16) and (17) we see that

$$\begin{aligned}\sup_{x\in K_1} T_\Delta(x,A) &= \mathbb{P}[E_\Delta]\sup_{x\in K_1}\int_A p_\Delta^Y(x,y)dy\\ &\le \ell(A)\Big\{\mathbb{P}[E_\Delta]c(\Delta;K_1,K_2)\Big\}\\ &< \delta\Big\{\mathbb{P}[E_\Delta]c(\Delta;K_1,K_2)\Big\}\\ &< \epsilon,\end{aligned}$$

verifying the condition (i) of Cline and Pu (1998, Lemma 3.1). On the other hand, since the diffusion Y is weak-Feller under **[C1]**, the lower semicontinuity of $x \mapsto T_\Delta(x, O')$ for every open $O' \subset \mathbb{R}^d$ follows on account of (16), cf. Meyn and Tweedie (1993a, Proposition 6.1.1(i)):

$$\liminf_{y\to x} T_\Delta(y,O') = \Big(\liminf_{y\to x}\mathbb{P}_y[Y_\Delta\in O']\Big)\mathbb{P}[E_\Delta] \ge \mathbb{P}_x[Y_\Delta\in O']\mathbb{P}[E_\Delta] = T_\Delta(x,O').$$

This verifies the remaining condition (ii) of Cline and Pu (1998, Lemma 3.1), thereby yielding the lower semicontinuity of $x \mapsto T_\Delta(x, A)$ for any $A \in \mathcal{B}(\mathbb{R}^d)$. Thus the proof of **[T2]**(i) is complete.

2.2. Proof of Theorem 1.2

Theorem 1.2 can be achieved in much the same way as in the proof of Theorem 1.1, so we will only mention the points. Actually, (13) can be derived as in the proof of Theorem 1.1, hence it remains to look at the drift condition **[D*]** and (7); as in Theorem 1.1, that "for any $r_1 \in [0, r_0)$ we have $\int \exp(r_1|x|)\pi(dx) < \infty$" in the statement follows from **[D*]**.

Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be a $\mathcal{C}^2$ function such that $f(x) = \exp(\alpha|x|)$ for $|x| \ge 1$, where $\alpha > 0$ is a constant, and that $f(x) \le \exp(\alpha|x|)$ for every $x \in \mathbb{R}^d$. For any $\alpha \in (0, r/\rho^*)$ (for any $\alpha > 0$ if $\rho^* = 0$; recall that $\rho^* := \sup_x |\rho(x)|$) we have

$$\int_{|z|>1} f(x+\zeta(x,z))\nu(dz) \le \exp(\alpha|x|)\int_{|z|>1}\exp(\alpha\rho^*|z|)\nu(dz) \lesssim \exp(\alpha|x|),$$

so that $f \in \mathcal{Q}^*$. Below we will control $\alpha \in (0, r/\rho^*)$ in order to verify **[D*]** and (7).

Let us recall (9), which here reads (X is given by (1))

$$\mathcal{A}f(x) = \mathcal{G}f(x) + \int \Big(f(x+\zeta(x,z)) - f(x)\Big)\nu(dz), \tag{18}$$

where $\mathcal{G}f$ is given in (9). Let $|x| \ge 1$ in the sequel. Simple algebra leads to

$$\nabla f(x) = \frac{\alpha}{|x|}f(x)x, \tag{19}$$

$$\nabla^2 f(x) = \alpha f(x)\left\{\frac{\alpha}{|x|^2}[x_ix_j]_{i,j=1}^d + \left(\frac{1}{|x|}I_d - \frac{1}{|x|^3}[x_ix_j]_{i,j=1}^d\right)\right\}. \tag{20}$$

First, it follows from **[C1b]**, (19), and (20) that $\mathcal{G}f(x)$ takes the form of

$$\mathcal{G}f(x) = \alpha f(x)\left\{\frac{x^\top b(x)}{|x|} + D_\alpha(x)\right\}, \tag{21}$$

where $|D_\alpha(x)| \lesssim \alpha + o(1)$ for $|x| \to \infty$. Next, for $|x| \geq 1$

$$\begin{aligned}\int \Big(f(x+\zeta(x,z)) - f(x)\Big)\nu(dz) &= \int f(x+\zeta(x,z))\nu(dz) - f(x)\int \nu(dz) \\ &\leq \alpha f(x) \int \frac{\exp(\alpha\rho^*|z|) - 1}{\alpha}\nu(dz) \\ &= \alpha f(x)\int \kappa_\alpha(z)\nu(dz), \quad \text{say.} \end{aligned}\tag{22}$$

Combining (18), (21), (22), and **[EEb]**, we see that there exists a constant $C > 0$ such that ($\int \kappa_\alpha(z)\nu(dz) = 0$ if $\rho^* = 0$)

$$\mathcal{A}f(x) \leq \alpha f(x)\left\{-\left(c_0 - C\alpha - \int \kappa_\alpha(z)\nu(dz)\right) + o(1)\right\} \tag{23}$$

for $|x| \to \infty$. Now take any $r' \in (\alpha\rho^*, r)$, so that $\int |z|\exp(r'|z|)\nu(dz) < \infty$. Then for any $\alpha \in (0, r/\rho^*)$

$$\kappa_\alpha(z) = \int_0^1 \rho^*|z|\exp(u\alpha\rho^*|z|)du \leq \rho^*|z|\exp(\alpha\rho^*|z|) \leq \rho^*|z|\exp(r'|z|) \in L^1(\nu).$$

Moreover, for any $|z| \neq 0$ it follows from L'Hôpital's rule that $\kappa_\alpha(z) \to \rho^*|z|$ as $\alpha \to 0$. Thus the dominated convergence theorem yields that $\int \kappa_\alpha(z)\nu(dz) \to \rho^* \int |z|\nu(dz)$ as $\alpha \to 0$. Therefore, by (23) and the assumption $\rho^* \int |z|\nu(dz) < c_0$, it is easy to see that **[D*]** follows on letting $\alpha > 0$ be sufficiently small. Once **[D*]** is verified we can readily derive the moment bound (7) as in [M, pp.50–51] and [M-Corrections, Remark 3], completing the proof of Theorem 1.2.

2.3. Proof of Lemma 1.3

Let $q > 0$ be the constant given in **[E]** or **[EE]**. In analogy with [M] and [M-Corrections], we will target at a $\mathcal{C}^2$ function $f : \mathbb{R}^d \to \mathbb{R}_+$ such that $f(x) = |x|^q$ for $|x| \geq 1$, and that $f(x) \leq |x|^q$ for every $x \in \mathbb{R}^d$: we know that $f \in \mathcal{Q}^*$, see [M, Lemma 2.3]. We are going to show that this kind of f serves as the required function.

First we look at **[D]** under **[E]**. For every $|x| \geq 1$ we have

$$\mathcal{G}f(x) = G_q(x). \tag{24}$$

By means of **[C1]** we can find a constant $K' \geq 1$ such that for $|x| \geq K'$

$$\frac{1}{2}|x| \leq \inf_{|z|\leq 1, u\in[0,1]} |x + u\zeta(x,z)| \leq \sup_{|z|\leq 1, u\in[0,1]} |x + u\zeta(x,z)| \leq \frac{3}{2}|x|, \tag{25}$$

since we have

$$1 - \frac{\rho(x)}{|x|} \leq \frac{|x + u\zeta(x,z)|}{|x|} \leq 1 + \frac{\rho(x)}{|x|}$$

for every $x \in \mathbb{R}^d$, $z \in \mathbb{R}^r$ such that $|z| \le 1$, and $u \in [0,1]$. Therefore Taylor's formula yields that

$$\mathcal{J}_* f(x) \lesssim |x|^{q-2}\{\rho(x)\}^2 \int_{|z|\le 1} |z|^2 \nu(dz) \lesssim |x|^{q-2}\{\rho(x)\}^2 \tag{26}$$

for every $|x| \ge 2K'$. As for $\mathcal{J}^* f$, first we consider $q \in (0,1]$. we can apply the inequality $|A+B|^q \le |A|^q + |B|^q$ valid for $q \in (0,1]$ to get

$$\mathcal{J}^* f(x) \le \int_{|z|>1} (|x+\zeta(x,z)|^q - |x|^q)\nu(dz) \le \{\rho(x)\}^q \int_{|z|>1} |z|^q \nu(dz) \lesssim \{\rho(x)\}^q, \tag{27}$$

using the presupposed property $f(x) \le |x|^q$ for every $x \in \mathbb{R}^d$. In case of $q > 1$, without loss of generality we additionally suppose that $|\nabla f(x)| \lesssim |x|^{q-1}$ for every $x \in \mathbb{R}^d$. Then, by means of Taylor's expansion we obtain

$$\begin{aligned} \mathcal{J}^* f(x) &\lesssim |x|^{q-1}\rho(x) \int_{|z|>1} |z|\nu(dz) + \{\rho(x)\}^q \int_{|z|>1} |z|^q \nu(dz) \\ &\lesssim |x|^{q-1}\rho(x) + \{\rho(x)\}^q \end{aligned} \tag{28}$$

Putting (26), (27) and (28) together, we have

$$\mathcal{J}_* f(x) + \mathcal{J}^* f(x) \lesssim J_q(x) \tag{29}$$

for every $|x| \ge 2K'$, where J_q is defined in Section 1.2. It follows from (24) and (29) that there exists a constant $c_0 > 0$ such that

$$\begin{aligned} \mathcal{A} f(x) &\le G_q(x) + c_0 J_q(x) \\ &\lesssim B_q(x)\left(1 + \frac{D_q(x) \vee J_q(x)}{B_q(x)}\right) \end{aligned} \tag{30}$$

for every $|x|$ large enough, from which **[D]** readily follows on **[E]**.

Now suppose **[EE]**. In view of (30) and **[C1]** we can bound $\mathcal{A}f$ as

$$\begin{aligned} \mathcal{A} f(x) &\le |x|^q \left[\frac{G_q(x)}{|x|^q} + c_0 \left\{\left(\frac{\rho(x)}{|x|}\right)^2 + \left(\frac{\rho(x)}{|x|}\right)^q + \mathbf{1}_{(1,\infty)}(q)\frac{\rho(x)}{|x|}\right\}\right] \\ &\lesssim |x|^q\{-c' + o(1)\} \end{aligned}$$

for $|x| \to \infty$, yielding **[D*]**.

2.4. Proof of Lemma 1.4

The proof is analogous to [M-Corrections], so we only give a sketch.

Let $f : \mathbb{R}^d \to \mathbb{R}_+$ fulfil that $f(x) = \log(1+|x|)$ for $|x| \ge 1$, and that $f(x) \le \log(1+|x|)$ for every $x \in \mathbb{R}^d$. In this case we have $f \in \mathcal{Q}^*$ (cf. [M, Lemma 2.3]), and

$$\begin{aligned} \nabla f(x) &= \frac{1}{|x|(1+|x|)} x^\top, \quad |x| \ge 1, \\ |\nabla^2 f(x)| &= O(|x|^{-2}), \quad |x| \to \infty. \end{aligned}$$

Therefore Taylor's formula together with (25) and **[C1]** implies that

$$\mathcal{J}_* f(x) \lesssim (\rho(x)/|x|)^2 = o(1).$$

Further, in view of the choice of f made above we get

$$\mathcal{J}^* f(x) \le \int_{|z|>1} \log\left(1 + \frac{\rho(x)}{1+|x|}|z|\right)\nu(dz)$$

for $|x|$ large enough, the upper bound tending to 0 as $|x| \to \infty$ by means of the condition (10), the dominated convergence theorem, and **[C1]**. Thus, taking the condition on σ into account we arrive at

$$\mathcal{A}f(x) \le \frac{x^\top b(x)}{|x|(1+|x|)} + o(1),$$

so that Lemma 1.4 follows on (11).

2.5. Proof of Lemma 1.5

The proof consists of a modification of Liebscher (2005, Proposition 3). By triangular inequality we see that $\beta_Y(t) \le \beta_{Y,1}(t) + \beta_{Y,2}(t)$, where

$$\beta_{Y,1}(t) := \sup_{s\in\mathbb{R}_+} \|\eta P_{t+s}(y,\cdot) - \pi(\cdot)\|,$$

$$\beta_{Y,2}(t) := \sup_{s\in\mathbb{R}_+} \int \|P_t(y,\cdot) - \pi(\cdot)\| \eta P_s(dy).$$

Since $t \mapsto V_t(y)$ is nonincreasing for each $y \in \mathbb{Y}$, we have

$$\beta_{Y,1}(t) \le \int \sup_{s\in\mathbb{R}_+} \|P_{t+s}(y,\cdot) - \pi(\cdot)\|\eta(dy) \le \int \|P_t(y,\cdot) - \pi(\cdot)\|\eta(dy) = \eta(V_t). \tag{31}$$

Now fix any $u \in (0,t)$. Applying the Chapman-Kolmogorov relation and using the fact $\sup_{t\in\mathbb{R}_+, y\in\mathbb{Y}} |V_t(y)| \le 2$, we get

$$\begin{aligned}
\beta_{Y,2}(t) &= \sup_{s\in\mathbb{R}_+} \int \left\{ \sup_{|g|\le 1} \left| \int \left(\int g(z) P_{t-u}(x,dz) - \int g(z)\pi(dz) \right) P_u(y,dx) \right| \right\} \eta P_s(dy) \\
&\le \sup_{s\in\mathbb{R}_+} \iint V_{t-u}(x) P_u(y,dx) \eta P_s(dy) \\
&= \sup_{s\in\mathbb{R}_+} \iint V_{t-u}(x) P_{s+u}(z,dx) \eta(dz) \\
&\le 2 \sup_{s\in\mathbb{R}_+} \int V_{s+u}(z)\eta(dz) + \pi(V_{t-u}) \\
&\le 2\eta(V_u) + \pi(V_{t-u}).
\end{aligned} \tag{32}$$

Hence (31) and (32) yield (12). Now (a) is obvious by taking $u = t/2$ in (12) and then applying the dominated convergence theorem. Finally, under the assumptions it directly follows from (12) that $\beta_Y(t) \lesssim \delta(u \wedge (t-u))$, leading to (b) again by taking $u = t/2$.

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