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PERIODIC BEHAVIORS OF QUANTUM CELLULAR AUTOMATA

By

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Abstract

In this paper we investigate quantum cellular automata whose global transitions are defined using a global transition function of classical cellular automata. And we prove the periodicity of behaviors of some quantum cellular automata.

Key words: cellular automata, quantization, reversibility, periodicity

1. Introduction

Since classical (discrete) cellular automata (CA, for short) were introduced by J. von Neumann about 60 years ago, CA have been applied to various fields and much research is still reported.

Feynman (1982) proposed first the notion of cellular automata on principle of quantum mechanics, and Watrous (1995) introduced the first formal model of quantum CA as a kind of quantum computer. He proved that there exists a partitioned quantum CA which can simulate any quantum Turing machine efficiently with constant slowdown. After introduction of the formal model some researches on properties of quantum CA were reported. Dürr and Santha (2002) considered the properties between the local function of quantum CA and the unitarity of the global transition function, and proposed an algorithm to decide if a linear quantum CA is unitary. Van Dam (1996) investigated quantum CA with circular bounded configurations and proved the existence of a universal quantum CA. Although most quantum CA investigated till now have infinite cell space, Inokuchi and Mizoguchi (2005) dealt with quantum CA with finite cell array. They introduced a notion of quantum CA with cyclic finite cell array and showed a sufficient condition for local transition functions to form quantum CA. And they introduced some examples of quantization method of reversible classical CA. And several types of construction method of quantum cellular automata were proposed by Schumacher and Werner (2004) and they showed that any quantum cellular automaton is structurally reversible. Nevertheless much research on quantum CA has been published, few research on dynamical behaviors of quantum CA has been reported very much. Inui et al. (2005) studied statistical dynamical behaviors of a quantum CA. They

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calculated the time averaged probability of finding a configuration for cell size 4 exactly in finite quantum CA defined by quantization of classical CA with Wolfram's rule 150, in addition they proved that the time averaged mean density of cells with the state 1 is 0.5 for arbitrary cell size.

Because quantum computation operates under the unitary law, the reversibility of both quantum and classical systems has received much attention. A lot of papers concerned with the reversibility of CA have been published and particularly the research by Morita and Harao (1989) is widely noticed. They introduced partitioned CA with which we can easily construct reversible CA, and proved that reversible CA are capable of universal computation. Wolfram (2002) investigated the reversibility of several models of CA of the elementary CA with infinite cell array and showed that only six CA, whose transition functions are identity function, right-shift function, left-shift function and these complement functions, are reversible. In addition, Inokuchi et al. (2005) investigated the reversibility of elementary CA with one dimensional finite cell array and proved that some CA including non-trivial CA are reversible and the other CA are not reversible.

A quantization of classical CA by rotation of classical cells is introduced in Inokuchi and Mizoguchi (2005) and we get quantizable classical CA, i.e. reversible CA, from the results of Inokuchi et al. (2005). In this paper we investigate quantum CA with finite cell array which can be determined by reversible CA and the quantization, and we focus on the periodic behaviors of quantum CA. Bertoni and Carpentieri (2001) proved that for any unitary matrix Δ and any $\varepsilon > 0$ there exists $t \in \mathbb{N}$ such that $||\Delta^t - id|| \leq \varepsilon$. Hence any quantum CA behave almost periodically. Reversible classical CA with finite cell array behave periodically because CA with finite cell array are finite transition systems. It can be conjectured intuitively that quantum CA which are determined by reversible CA and rotation of the product of π and a rational number behave periodically. For example, the quantum CA, which is defined by a classical CA with behaviors of period 5 and rotation of $\frac{2\pi}{3}$, have periodic behaviors of period 15. In the following discussion we will prove for some quantum CA to behave periodically.

2. Preliminaries

In this section we define quantum CA and we mention the reversibility of classical CA with one dimensional finite cell array. This definition of quantum CA was introduced by Inokuchi and Mizoguchi (2005). The global transition function of quantum CA is defined by the global transition function of classical CA and a rotation matrix.

Let $\mathbb{C}$ be the set of all complex numbers and I a singleton set $\{*\}$.

Definition 2.1. Let X and Y be finite sets. A (transition) matrix of size $|Y| \times |X|$ is a mapping

$$\alpha : X \times Y \to \mathbb{C}$$

Definition 2.2. For each element $x \in X$ we define the matrix $\varepsilon_x : I \times X \to \mathbb{C}$ of size $|X| \times |I|$ by

$$\varepsilon_x(*, a) = \begin{cases} 1 & a = x \\ 0 & a \neq x \end{cases}$$

Example 2.3. We let $Q = \{0,1\}$. Then $\varepsilon_{00} : I \times Q^2 \to \mathbb{C}$ is

$$\varepsilon_{00} = \begin{pmatrix} \varepsilon_{00}(*,(0,0)) \\ \varepsilon_{00}(*,(0,1)) \\ \varepsilon_{00}(*,(1,0)) \\ \varepsilon_{00}(*,(1,1)) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Lemma 2.4. *Every matrix $\rho : I \times X \to \mathbb{C}$ can be uniquely represented as a linear combination*

$$\rho = \sum_{x \in X} k_x \varepsilon_x$$

where k_x is a complex number.

Definition 2.5. A matrix $\alpha : X \times Y \to \mathbb{C}$ is called a quantum matrix (q-matrix) if it satisfies the unitary law

$$\alpha^T \alpha = Id_{|X|}$$

where α^T is the transposed matrix of α and $Id_{|X|}$ is the identity matrix of size $|X|$.

Lemma 2.6.

- *Every composite of q-matrices is also a q-matrix.*
- *A matrix $M : X \times Y \to \mathbb{C}$ is a q-matrix if and only if $\mu^T \rho = (M\mu)^T (M\rho)$ for all matrices $\rho, \mu : I \times X \to \mathbb{C}$.*

Let Q denote the set $\{0,1\}$ of binary digits and n a positive integer and $f : Q^3 \to Q$ be a local function with rule number R ($0 \leq R \leq 255$), where

$$R = \sum_{abc \in Q^3} 2^{4a+2b+c} f(abc).$$

A classical CA $CA - R_c(n)$ with cyclic boundary condition has a global transition function $\delta_{R,c} : Q^n \to Q^n$ defined by

$$\delta_{R,c}(x_1 x_2 \cdots x_n) = f(x_n x_1 x_2) f(x_1 x_2 x_3) \cdots f(x_{n-1} x_n x_1),$$

and a classical CA $CA - R_{a-b}(n)$ with fixed boundary condition $a - b$ ($a, b \in Q$) has a global transition function $\delta_{R,a-b} : Q^n \to Q^n$ defined by

$$\delta_{R,a-b}(x_1 x_2 \cdots x_n) = f(a x_1 x_2) f(x_1 x_2 x_3) \cdots f(x_{n-1} x_n b)$$

for $x_1 x_2 \cdots x_n \in Q^n$.

Definition 2.7. A (rotation) matrix $\lambda_\theta : Q \times Q \to \mathbb{C}$ is defined by

$$\lambda_\theta = \begin{pmatrix} \lambda_\theta(0,0) & \lambda_\theta(0,1) \\ \lambda_\theta(1,0) & \lambda_\theta(1,1) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : Q \times Q \to \mathbb{C}$$

Trivially λ_θ satisfies the unitary law, that is, λ_θ is a q-matrix. We define the tensor product of λ_θs by

$$\begin{aligned}\lambda_\theta \otimes \lambda_\theta &= \begin{pmatrix} \cos\theta\lambda_\theta & -\sin\theta\lambda_\theta \\ \sin\theta\lambda_\theta & \cos\theta\lambda_\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta & -\cos\theta\sin\theta & -\cos\theta\sin\theta & \sin^2\theta \\ \cos\theta\sin\theta & \cos^2\theta & -\sin^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ \sin^2\theta & \cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\lambda_\theta \otimes \lambda_\theta \otimes \cdots \otimes \lambda_\theta &= \lambda_\theta \otimes (\lambda_\theta \otimes \cdots \otimes \lambda_\theta) \\ &= \begin{pmatrix} \cos\theta(\lambda_\theta \otimes \cdots \otimes \lambda_\theta) & -\sin\theta(\lambda_\theta \otimes \cdots \otimes \lambda_\theta) \\ \sin\theta(\lambda_\theta \otimes \cdots \otimes \lambda_\theta) & \cos\theta(\lambda_\theta \otimes \cdots \otimes \lambda_\theta) \end{pmatrix}\end{aligned}$$

And for $m, n \in \mathbb{C}$

$$m \otimes n = m \times n.$$

Definition 2.8. A quantum CA $QCA-[\delta,\theta](n)$ with n cells is a system consisting of a bijection $\delta : Q^n \to Q^n$ and a q-matrix $\lambda_\theta : Q \times Q \to \mathbb{C}$.

$$QCA-[\delta,\theta](n) = (\delta : Q^n \to Q^n, \lambda_\theta : Q \times Q \to \mathbb{C})$$

For a function $\delta : Q^n \to Q^n$ a matrix $\Gamma : Q^n \times Q^n \to \mathbb{C}$ is defined by

$$\Gamma(y,x) = \begin{cases} 1 & (y = \delta(x)) \\ 0 & (y \neq \delta(x)). \end{cases}$$

If δ is a bijection it is trivial that Γ is a q-matrix. For a quantum CA $QCA-[\delta,\theta](n)$ its global transition q-matrix $\Delta : Q^n \times Q^n \to \mathbb{C}$ is defined by

$$\Delta = (\lambda_\theta \otimes \lambda_\theta \otimes \cdots \otimes \lambda_\theta)\Gamma : Q^n \times Q^n \to \mathbb{C},$$

$$\begin{aligned}\Delta(y,x) &= \lambda_\theta(y_1, \delta(x)_1) \otimes \lambda_\theta(y_2, \delta(x)_2) \otimes \cdots \otimes \lambda_\theta(y_n, \delta(x)_n) \qquad (x, y \in Q^n) \\ &= \lambda_\theta(y_1, \delta(x)_1)\lambda_\theta(y_2, \delta(x)_2) \cdots \lambda_\theta(y_n, \delta(x)_n)\end{aligned}$$

And we say that $QCA-[\delta,\theta](n)$ and $QCA-[\delta',\theta](n)$ are symmetric if $\delta : Q^n \to Q^n$ and $\delta' : Q^n \to Q^n$ are symmetric (cf. Inokuchi et al. (2005)).

Example 2.9. QCA-$[\delta_{90,0-0},\theta](2)$ is constructed by $\delta_{90,0-0} : Q^2 \to Q^2$ and $\lambda_\theta : Q \times Q \to \mathbb{C}$ where $\delta_{90,0-0}$ is the global transition function of CA-$90_{0-0}(2)$. The global transition q-matrix Δ is

$$\begin{aligned}\Delta &= \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \otimes \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta & \cos\theta\sin\theta & \sin^2\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta & -\sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & -\cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}\end{aligned}$$

Hence the q-matrix $\Delta\varepsilon_{01}$ after one step transition from initial q-matrix ε_{01} is

$$\Delta\varepsilon_{01} = \Delta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta\sin\theta \\ -\sin^2\theta \\ \cos^2\theta \\ -\cos\theta\sin\theta \end{pmatrix}$$

Usually we say that a classical CA(or its global transition function δ) is reversible if δ is injective. Since the cell size of classical CA treated with in this paper is finite, the injectivity and the bijectivity of δ are equivalent.
Generally the global transition function $\delta : Q^n \to Q^n$ of a classical CA is not always a bijection. Since quantum computations have to satisfy the unitarity law, we can construct a quantum CA $QCA - [\delta, \theta](n)$ if and only if the transition function δ of a classical CA is reversible. The following table is an extract from the results of Inokuchi et al. (2005) and shows the reversibility of 1D finite classical CA $CA - R(n)$ with fixed and cyclic boundary conditions.

Rule numbers	Fixed boundary	Cyclic boundary
204, 51	Reversible	Reversible
240,15, 170, 85	Not reversible	Reversible
90, 165	Reversible iff $n = 0 \pmod 2$	Not reversible
60,195,102,153	Reversible	Not reversible
150,105	Reversible iff $n \neq 2 \pmod 3$	Reversible iff $n \neq 0 \pmod 3$
166,180,154,210, 89,75,101,45	Not reversible	Reversible iff $n = 1 \pmod 2$
others	Not reversible	Not reversible

We introduce the following term concerned with periodic behaviors of quantum CA.

DEFINITION 2.10. A quantum CA $QCA - [\delta, \theta](n)$ is called "globally periodic" if there exists a positive integer p such that

$$\Delta^p = Id_{|Q^n|}.$$

Now we will use new notations of $\kappa_0 = \cos\theta$, $\kappa_1 = \sin\theta$ and $\oplus$ defined as follows:

$$(x \oplus y)_i = x_i + y_i \ (\text{mod}2) \qquad (x, y \in Q^n)$$

Then using those notations, the (a, b)-th component of λ_θ is represented by

$$\lambda_\theta(a, b) = (-1)^{(1-a)b}\kappa_{a\oplus b}(\theta).$$

And we let $u = 11\cdots1 \in Q^n$. Then for $x, y \in Q^n$ we define

$$< x, y >= \sum_{k=1}^{n} x_k y_k,$$

which is the absolute sum but not the sum modulo 2, and

$$m(x) =< x, u > .$$

It is trivial that $m(x \oplus u) = n - m(x)$ and $< x, y > + < x, z > = < x, y \oplus z >$ (mod2).

At the end of this section we prove the following lemma which shows the elements of the global transition q-matrix Δ of quantum CA.

Lemma 2.11. *For all configurations $x, y \in Q^n$ the (y, x)-th component of the q-matrix $\Delta : Q^n \times Q^n \to \mathbb{C}$ of $QCA - [\delta, \theta](n)$ is given by*

$$\Delta(y, x) = (-1)^{<\delta(x), y \oplus u>} \cos^{n-m(\delta(x)\oplus y)} \theta \sin^{m(\delta(x)\oplus y)} \theta.$$

Proof. Let $x, y, z \in Q^n$ and $x' = \delta(x)$. Then we have

$$\begin{aligned}
& \Delta(y, x) \\
= \; & \lambda_\theta(y_1, x'_1)\lambda_\theta(y_2, x'_2) \cdots \lambda_\theta(y_n, x'_n) \\
= \; & (-1)^{(1-y_1)x'_1} \kappa_{y_1 \oplus x'_1}(\theta)(-1)^{(1-y_2)x'_2} \kappa_{y_2 \oplus x'_2}(\theta) \cdots (-1)^{(1-y_n)x'_n} \kappa_{y_n \oplus x'_n}(\theta) \\
= \; & (-1)^{<y \oplus u, x'>} \kappa_{y_1 \oplus x'_1}(\theta)\kappa_{y_2 \oplus x'_2}(\theta) \cdots \kappa_{y_n \oplus x'_n}(\theta) \\
= \; & (-1)^{<x', y \oplus u>} \cos^{n-m(x' \oplus y)} \theta \sin^{m(x' \oplus y)} \theta
\end{aligned}$$

3. Analysis of the Behaviors of Quantum Cellular Automata

All reversible CA with 2 states, 1D finite cell array were listed in the previous section. We can consider that the behaviors of a quantum CA and its symmetric quantum CA are isomorphic. Therefore in this section we deal with either a reversible CA or its symmetric CA.

We will show the results of computer simulations of quantum CA QCA-$[\delta_{60,0-0}, \frac{\pi}{3}](5)$ (Figure 1) and QCA-$[\delta_{60,0-0}, \frac{\pi}{4}](5)$ (Figure 2). First we calculated the probability of

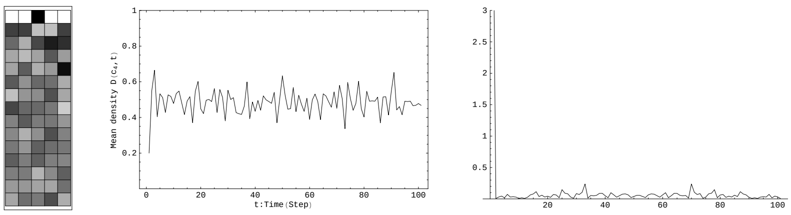

Figure 1: QCA-$[\delta_{60,0-0}, \frac{\pi}{3}](5)$

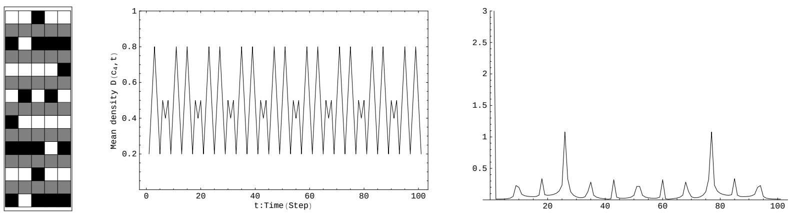

Figure 2: QCA-$[\delta_{60,0-0}, \frac{\pi}{4}](5)$

finding the state 1 at each cells where the initial q-matrix is ε_{00100}, and the results are presented by darkness in the left-hand side figures. If quantum CA is globally periodic we can see it from the left-hand side figures. In order to examine in detail we calculated the change of the mean density of cells with the state 1 (the middle graphs in Figure 1 and 2) and the discrete Fourier transformation of the middle graphs (the right-hand side graphs in Figure 1 and 2). From Figure 1 we cannot obtain the result that QCA-$[\delta_{60,0-0}, \frac{\pi}{3}](5)$ is globally periodic. But from Figure 2 we can guess that QCA-$[\delta_{60,0-0}, \frac{\pi}{4}](5)$ is globally periodic. From our computer simulation we can guess the following theorem which show the globally periodicity of the behaviors of some quantum CA.

THEOREM 3.1. *Let n be a positive integer. Then for QCA-$[\delta_{R,b.c.}, \theta](n)$ the following table hold where r is a rational number, g.p. shows that QCA-$[\delta_{R,b.c.}, \theta](n)$ is globally periodic for any n, and g.p.$_1$, g.p.$_2$, g.p.$_3$ and g.p.$_4$ show that QCA-$[\delta_{R,b.c.}, \theta](n)$ is globally periodic if n is even, if $n \neq 0$ (mod3), if $n \neq 2$ (mod3) and if n is odd, respectively.*

θ	$\frac{\pi}{2}$		$\frac{\pi}{4}$		$r\pi$		*others*	
b.c.	*0-0*	*c*	*0-0*	*c*	*0-0*	*c*	*0-0*	*c*
R=204	*g.p.*	*g.p.*	*g.p.*	*g.p.*	*g.p.*	*g.p.*	*?*	*?*
R=51	*g.p.*	*g.p.*	*g.p.*	*g.p.*	*g.p.*	*g.p.*	*g.p.*	*g.p.*
R=240,15, 170,85	*?*	*g.p.*	*?*	*g.p.*	*?*	*g.p.*	*?*	*?*
R=90,165	*g.p.*$_1$	*?*	*g.p.*$_1$	*?*	*?*	*?*	*?*	*?*
R=60,195 102,153	*g.p*	*?*	*g.p.*	*?*	*?*	*?*	*?*	*?*
R=150,105	*g.p.*$_2$	*g.p.*$_3$	*g.p.*$_2$	*g.p.*$_3$	*?*	*?*	*?*	*?*
R=166,180, 154,210, 89,75, 101,45	*?*	*g.p.*$_4$	*?*	*?*	*?*	*?*	*?*	*?*

The following lemma is very useful for proving the theorem in the following discussion.

LEMMA 3.2. *For any $x \in Q^n$ the equation*

$$\sum_{y \in Q^n} (-1)^{<x,y>} = \begin{cases} 2^n & if \quad x = 0^n \\ 0 & otherwise \end{cases}$$

holds.

PROOF.

$$\begin{aligned} \sum_{y \in Q^n} (-1)^{<x,y>} &= \sum_{y \in Q^n} (-1)^{<x,y>} \\ &= \sum_{y \in Q^n} (-1)^{x_1 y_1 + \cdots + x_n y_n} \\ &= \prod_{i=1}^{n} (1 + (-1)^{x_i}) \\ &= \begin{cases} 2^n & if \quad x = 0^n \\ 0 & otherwise \end{cases} \end{aligned}$$

3.1. Quantum CA with Rotation Angle $\theta = \frac{\pi}{2}$

In this subsection we discuss the behaviors of quantum CA decided by reversible CA and the $\frac{\pi}{2}$ rotation.

The rotation q-matrix λ_θ of $\theta = \frac{\pi}{2}$ is presented by

$$\lambda_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

And the negation function $\neg : Q \to Q$ is easily represented by a unitary matrix

$$\neg = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : Q \times Q \to \mathbb{C}$$

The negation function $\neg : Q \to Q$ is extended into a function $\neg_n : Q^n \to Q^n$ by

$$\neg_n = \neg \otimes \cdots \otimes \neg.$$

Now we let $\delta_R : Q^n \to Q^n$ be a global transition function of a reversible classical CA and $\delta_{R'} : Q^n \to Q^n$ the compliment function of δ_R, that is,

$$\delta_{R'}(x) = \neg_n(\delta_R(x))$$

and let $\Delta : Q^n \times Q^n \to \mathbb{C}$ and $\Delta' : Q^n \times Q^n \to \mathbb{C}$ be global transition q-matrices of $QCA - [\delta_R, \frac{\pi}{2}](n)$ and $QCA - [\delta_{R'}, 0](n)$, respectively. Note that $\delta_{R'}$ is reversible and $R' = 255 - R$.
Then we have

$$\begin{aligned} \Delta(z,x) &= (\lambda_{\frac{\pi}{2}} \otimes \cdots \otimes \lambda_{\frac{\pi}{2}})(z, \delta_R(x)) \\ &= \begin{cases} (-1)^{m(\delta(x))} & if\ z = \delta_R(x) \oplus u \\ 0 & otherwise \end{cases} \end{aligned}$$

and $\delta'(x) = \delta(x) \oplus u$ for $x, z \in Q^n$. Hence the equation

$$|\Delta(z,x)| = |\Delta'(z,x)|$$

holds for any $x, z \in Q^n$. The quantum CA $QCA - [\delta_{R'}, 0](n)$ is essentially the classical CA $CA - R'_{b.c.}(n)$ and $CA - R'_{b.c.}(n)$ is reversible. Therefore we have the following proposition.

Proposition 3.3. *For any quantum cellular automata $QCA - [\delta_R, \frac{\pi}{2}](n)$ defined by the global transition function δ_R of reversible classical CA and the $\frac{\pi}{2}$ rotation q-matrix $\lambda_{\frac{\pi}{2}}$ there exists a positive integer p such that*

$$\Delta^p = Id_{|Q^n|}.$$

3.2. Quantum CA with Rule 204

Trivially the global transition function δ_{204} of $CA - 204(n)$ is the identity function Id_{Q^n} on Q^n. We let $\Delta = \lambda_\theta \otimes \lambda_\theta \otimes \cdots \otimes \lambda_\theta : Q^n \times Q^n \to \mathbb{C}$ be the global transition q-matrix of the quantum CA $QCA - [\delta_{204}, \theta](n)$. First we will prove the following lemma.

LEMMA 3.4. *For any $\theta, \theta' \in \mathbb{R}$ and any $x \in Q^n$ the followings hold:*

1. $(\lambda_\theta \otimes \cdots \otimes \lambda_\theta)(\lambda_{\theta'} \otimes \cdots \otimes \lambda_{\theta'}) = (\lambda_{\theta+\theta'} \otimes \cdots \otimes \lambda_{\theta+\theta'})$
2. $\Delta(x, x) = \cos^n \theta$,
3. $\Delta(x \oplus u, x) = (-1)^{n-m(x)} \sin^n \theta$.

PROOF. 1. For any $\theta, \theta' \in \mathbb{R}$ we can easily check $\lambda_\theta \lambda_{\theta'} = \lambda_{\theta+\theta'}$. Hence we have

$$\begin{aligned} & (\lambda_\theta \otimes \cdots \otimes \lambda_\theta)(\lambda_{\theta'} \otimes \cdots \otimes \lambda_{\theta'}) \\ = & ((\lambda_\theta \lambda_{\theta'}) \otimes \cdots \otimes (\lambda_\theta \lambda_{\theta'})) \\ = & (\lambda_{\theta+\theta'} \otimes \cdots \otimes \lambda_{\theta+\theta'}). \end{aligned}$$

2.

$$\begin{aligned} \Delta(x, x) &= (\lambda_\theta \otimes \cdots \otimes \lambda_\theta)(x, x) \\ &= \lambda_\theta(x_1, x_1) \times \cdots \times \lambda_\theta(x_n, x_n) \\ &= \cos^n \theta. \end{aligned}$$

3.

$$\begin{aligned} \Delta(x \oplus u, x) &= (\lambda_\theta \otimes \cdots \otimes \lambda_\theta)(x \oplus u, x) \\ &= \lambda_\theta(1 - x_1, x_1) \times \cdots \times \lambda_\theta(1 - x_n, x_n) \\ &= \sin^{m(x)} \theta (-\sin \theta)^{n-m(x)} \\ &= (-1)^{n-m(x)} \sin^n \theta. \end{aligned}$$

The global transition q-matrix Δ is unitary and using the above two lemmas we have

$$\Delta^m(x, x) = \cos^n m\theta.$$

Hence we can get the following proposition:

PROPOSITION 3.5. *In $QCA - [\delta_{204}, \theta](n)$ the following equation*

$$\Delta^m = Id_{Q^n}$$

holds if there exists m satisfying

$$m = \begin{cases} \min\{\ m' \in \mathbb{N} \mid m'\theta \in \pi\mathbb{Z}\ \} & \textit{if } n \textit{ is even}, \\ \min\{\ m' \in \mathbb{N} \mid m'\theta \in 2\pi\mathbb{Z}\ \} & \textit{otherwise} \end{cases}$$

3.3. Quantum CA with Rule 51

The global transition function δ_{51} of the CA $CA - 51(n)$ is the negation function $\neg_n$. The global transition q-matrix of the quantum CA $QCA - [\delta_{51}, \theta](n)$ is given by

$$\Delta = (\lambda_\theta \neg) \otimes \cdots \otimes (\lambda_\theta \neg) : Q^n \times Q^n \to \mathbb{C}.$$

And we have

$$\begin{aligned}(\lambda_\theta\neg)^2 &= \left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)^2\\ &= \begin{pmatrix}\sin\theta & \cos\theta\\ \cos\theta & -\sin\theta\end{pmatrix}^2\\ &= Id_{|Q|}.\end{aligned}$$

Therefore we can prove the following proposition:

Proposition 3.6. *In $QCA-[\delta_{51},\theta](n)$ the following equation*

$$\Delta^2 = Id_{|Q^n|}$$

holds.

Proof.

$$\begin{aligned}\Delta^2 &= ((\lambda_\theta\neg)\otimes\cdots\otimes(\lambda_\theta\neg))^2\\ &= (\lambda_\theta\neg)^2\otimes\cdots\otimes(\lambda_\theta\neg)^2\\ &= Id_{|Q|}\otimes\cdots\otimes Id_{|Q|}\\ &= Id_{|Q^n|}\end{aligned}$$

3.4. Quantum CA with Rule 240 and 15

Classical CA with the local functions of rule number 240 and 15 are reversible only in the case of cyclic boundary condition. We can prove the following lemmas on quantum CA $QCA-[\delta_{240,c},\theta](n)$ and $QCA-[\delta_{15,c},\theta](n)$.

Lemma 3.7. *Let $\Gamma : Q^n\times Q^n \to \mathbb{C}$ and $\Gamma' : Q^n\times Q^n\to\mathbb{C}$ be matrices determined by $\delta_{240,c} : Q^n\to Q^n$ and $\delta_{15,c} : Q^n\to Q^n$ respectively. For any θ the following holds:*

1. $\Gamma(\lambda_\theta\otimes\cdots\otimes\lambda_\theta) = (\lambda_\theta\otimes\cdots\otimes\lambda_\theta)\Gamma$,
2. $((\lambda_\theta\otimes\cdots\otimes\lambda_\theta)\Gamma')^2 = \Gamma^2$

Proof. 1. For any $x,\ y\in Q^n$ we have

$$\begin{aligned}&((\lambda_\theta\otimes\cdots\otimes\lambda_\theta)\Gamma)(y,x)\\ =\ & (\lambda_\theta\otimes\cdots\otimes\lambda_\theta)(y,x_nx_1x_2\cdots x_{n-1})\\ =\ & \lambda_\theta(y_1,x_n)\times\lambda_\theta(y_2,x_1)\times\lambda_\theta(y_3,x_2)\times\cdots\times\lambda_\theta(y_n,x_{n-1})\\ =\ & \lambda_\theta(y_2,x_1)\times\lambda_\theta(y_3,x_2)\times\cdots\times\lambda_\theta(y_n,x_{n-1})\times\lambda_\theta(y_1,x_n)\\ =\ & ((\lambda_\theta\otimes\cdots\otimes\lambda_\theta))(y_2y_3\cdots y_ny_1,x)\\ =\ & (\Gamma(\lambda_\theta\otimes\cdots\otimes\lambda_\theta))(y,x).\end{aligned}$$

2. The equations $\neg_n\Gamma = \Gamma\neg_n$ and $\Gamma' = \neg_n\Gamma$ can be checked easily. Hence we have

$$\begin{aligned}((\lambda_\theta\otimes\cdots\otimes\lambda_\theta)\Gamma')^2 &= ((\lambda_\theta\otimes\cdots\otimes\lambda_\theta)\neg_n\Gamma)^2\\ &= (\lambda_\theta\otimes\cdots\otimes\lambda_\theta)\neg_n(\lambda_\theta\otimes\cdots\otimes\lambda_\theta)\Gamma\neg_n\Gamma\\ &= ((\lambda_\theta\otimes\cdots\otimes\lambda_\theta)\neg_n)^2\Gamma^2\\ &= \Gamma^2\end{aligned}$$

Using the above two lemmas we can get the configuration after 1 or 2 transition steps in $QCA-[\delta_{240,c},\theta](n)$ and $QCA-[\delta_{15,c},\theta](n)$ from $QCA-[\delta_{204},\theta](n)$. Hence we get the following proposition:

PROPOSITION 3.8. *Let Δ, Δ' and Δ'' be the global transition q-matrices of $QCA-[\delta_{204},\theta](n)$, $QCA-[\delta_{240,c},\theta](n)$ and $QCA-[\delta_{15,c},\theta](n)$, respectively. If there exists a positive integer p such that $\Delta^p = Id_{Q^n}$ then the equations*

$$(\Delta')^{p'} = Id_{|Q^n|},$$

and

$$(\Delta'')^{p''} = Id_{|Q^n|}$$

hold where $p' = lcm(p,n)$ and $p'' = 2p'$.

3.5. Quantum CA with Rule 90

Classical CA with the local function of rule number 90 are reversible only in the case of $n = 0 \pmod 2$ and fixed boundary condition. In this subsection we discuss the behaviors of quantum CA decided by $\delta_{90,0-0}$, $\theta = \frac{\pi}{4}$ and cell size $n = 0$ (mod2).

When n is even, in classical CA $CA-90(n)$ of $0-0$ boundary condition it can be checked that $\delta_{90,0-0}(x) = 0^n$ if and only if $x = 0^n$ for $x \in Q^n$. By using it and lemma 3.2 we can prove the following lemma.

LEMMA 3.9. *In $QCA-[\delta_{90,0-0},\frac{\pi}{4}](n)$ for any $x \in Q^n$ the equation*

$$\Delta^2(x \oplus u, x) = (-1)^{x_1+x_n}$$

holds where n is even.

PROOF. We let x, y, z be arbitrary configurations in Q^n. Then we can check that the following equation holds;

$$\begin{aligned}
& < \delta_{90,0-0}(y), z \oplus u > + < \delta_{90,0-0}(x), y \oplus u > \\
= \;& < \delta_{90,0-0}(y), z > + < \delta_{90,0-0}(y), u > + < \delta_{90,0-0}(x), y > + < \delta_{90,0-0}(x), u > \quad (\text{mod}2) \\
= \;& < \delta_{90,0-0}(x), u > + < (y_2, y_1+y_3, y_2+y_4, \cdots, y_{n-2}+y_n, y_{n-1}), z > \\
& + < (y_2, y_1+y_3, y_2+y_4, \cdots, y_{n-2}+y_n, y_{n-1}), u > + < \delta_{90,0-0}(x), y > \quad (\text{mod}2) \\
= \;& < \delta_{90,0-0}(x), u > + < \delta_{90,0-0}(x), y > \\
& +(z_2+1)y_1 + (z_1+1+z_3+1)y_2 + (z_2+1)+z_4+1)y_3 \\
& + \cdots + (z_{n-2}+1+z_n+1)y_{n-1} + (z_{n-1}+1)y_n \quad (\text{mod}2) \\
= \;& < \delta_{90,0-0}(x), u > + < \delta_{90,0-0}(x), y > + < \delta_{90,0-0}(z \oplus u), y > \quad (\text{mod}2) \\
= \;& < \delta_{90,0-0}(x), u > + < \delta_{90,0-0}(x \oplus z \oplus u), y > \quad (\text{mod}2)
\end{aligned}$$

Therefore by lemma 3.2 we have

$$\begin{aligned}
\Delta^2(z,x) &= \sum_{y \in Q^n} \Delta(z,y)\Delta(y,x) \\
&= \sum_{y \in Q^n} \left\{ \frac{(-1)^{<\delta_{90,0-0}(y), z \oplus u>}}{(\sqrt{2})^n} \times \frac{(-1)^{<\delta_{90,0-0}(x), y \oplus u>}}{(\sqrt{2})^n} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{<\delta_{90,0-0}(x),u>}}{2^n} \sum_{y \in Q^n} (-1)^{<\delta_{90,0-0}(x \oplus z \oplus u),y>} \\
&= \begin{cases} (-1)^{<\delta_{90,0-0}(x),u>} & if\ \delta_{90,0-0}(x \oplus z \oplus u) = 0^n \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{<\delta_{90,0-0}(x),u>} & if\ x \oplus z \oplus u = 0^n \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{x_1+x_n} & if\ z = x \oplus u \\ 0 & otherwise \end{cases}
\end{aligned}
$$

Proposition 3.10. *In $QCA - [\delta_{90,0-0}, \frac{\pi}{4}](n)$ the equation*

$$\Delta^4 = Id_{|Q^n|}$$

holds where n is even.

Proof. For any $x, z \in Q^n$ we have

$$
\begin{aligned}
\Delta^4(z,x) &= \sum_{y \in Q^n} \Delta^2(z,y)\Delta^2(y,x) \\
&= \begin{cases} \Delta^2(z,y)\Delta^2(y,x) & if\ y = x \oplus u\ and\ z = y \oplus u \\ 0 & otherwise \end{cases} \\
&= \begin{cases} 1 & if\ z = x \\ 0 & otherwise \end{cases}
\end{aligned}
$$

3.6. Quantum CA with Rule 165

Classical CA with the local function of rule number 165 are reversible only in the case of $n = 0 \pmod 2$ and fixed boundary condition. In this subsection we discuss the behaviors of quantum CA decided by $\delta_{165,0-0}$, $\theta = \frac{\pi}{4}$ and cell size $n = 0$ (mod2).

In classical CA $CA - 165(n)$ of $0 - 0$ boundary condition we can check easily that $\delta_{165,0-0}(w) = 0^n$ if and only if

$$w = \begin{cases} (0110)^k & if\ n = 4k \\ (1100)^k 11 & if\ n = 4k+2 \end{cases}$$

for $w \in Q^n$ where n is even and $k = [\frac{n}{4}]$. By lemma 3.2 and it we can get the following lemma:

Lemma 3.11. *Let $w \in Q^n$ be a configuration such that $\delta_{165,0-0}(w) = 0^n$. Then in $QCA - [\delta_{165,0-0}, \frac{\pi}{4}](n)$*

$$\Delta^2(z,x) = \begin{cases} (-1)^{x_2+x_3+\cdots+x_{n-1}} & if\ z = w \oplus x \oplus u \\ 0 & otherwise \end{cases}$$

where n is even.

PROOF. For any $x, y, z \in Q^n$ we can get the following equation by the same way as the proof of lemma 3.9;

$$
\begin{aligned}
& < \delta_{165,0-0}(y), z \oplus u > + < \delta_{165,0-0}(x), y \oplus u > \\
= \; & < \delta_{165,0-0}(x) \oplus z \oplus u, u > + < \delta_{165,0-0}(x \oplus z \oplus u), y > \qquad (\mathrm{mod} 2)
\end{aligned}
$$

Then by lemma 3.2 we have

$$
\begin{aligned}
\Delta^2(z,x) &= \sum_{y \in Q^n} \Delta(z,y)\Delta(y,x) \\
&= \sum_{y \in Q^n} \left\{ \frac{(-1)^{<\delta_{165,0-0}(y), z \oplus u>}}{(\sqrt{2})^n} \times \frac{(-1)^{<\delta_{165,0-0}(x), y \oplus u>}}{(\sqrt{2})^n} \right\} \\
&= \frac{(-1)^{<\delta_{165,0-0}(x) \oplus z \oplus u, u>}}{2^n} \sum_{y \in Q^n} (-1)^{<\delta_{165,0-0}(x \oplus z \oplus u), y>} \\
&= \begin{cases} (-1)^{<\delta_{165,0-0}(x) \oplus z \oplus u, u>} & if\ \delta_{165,0-0}(x \oplus z \oplus u) = 0^n \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{<\delta_{165,0-0}(x) \oplus z \oplus u, u>} & if\ x \oplus z \oplus u = w \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{x_2 + x_3 + \cdots + x_{n-1}} & if\ z = w \oplus x \oplus u \\ 0 & otherwise \end{cases}
\end{aligned}
$$

PROPOSITION 3.12. *In* $QCA - [\delta_{165,0-0}, \frac{\pi}{4}](n)$ *the equation*

$$\Delta^4 = Id_{|Q^n|}$$

holds where n *is even.*

PROOF. Let $w \in Q^n$ be a configuration such that $\delta_{165,0-0}(w) = 0^n$. Then we have

$$
\begin{aligned}
\Delta^4(z,x) &= \sum_{y \in Q^n} \Delta^2(z,y)\Delta^2(y,x) \\
&= \begin{cases} \Delta^2(z,y)\Delta^2(y,x) & if\ y = w \oplus x \oplus u\ and\ z = w \oplus y \oplus u \\ 0 & otherwise \end{cases} \\
&= \begin{cases} 1 & if\ z = x \\ 0 & otherwise \end{cases}
\end{aligned}
$$

3.7. Quantum CA with Rule 60

Classical CA with the local function of rule number 60 are reversible in the case of any cell size n and fixed boundary condition. In this subsection we discuss the behaviors of quantum CA decided by $\delta_{60,0-0}$ and $\theta = \frac{\pi}{4}$.

For $x \in Q^n$ and an integer i $(0 \le i \le n)$ we define $x(i) \in Q^n$ as follows:

- $x(0) = x$
- $x(i)_i = x_{n-i+1} + 1 \pmod 2$,

- $x(i)_{i+j} = x_j + x_{n-i+1} + 1 \pmod 2$ for $1 \le j \le n-i$,
- $x(i)_{i-j} = x_{n-i+1} + x_{n-j+1} \pmod 2$ for $1 \le j \le i-1$.

We can easily check the followings:

- $x(1) = (0, x_1, x_2, \ldots, x_{n-1}) \oplus (x_n, x_n, \ldots, x_n) \oplus u$
- $x(i+1) = x(i)(1)$
- $x(n)(1) = x$

In addition, we can easily check that for any $x \in Q^n$ the equation $\delta_{60,0-0}(x) \oplus \delta_{102,0-0}(z \oplus u) = 0^n$ holds if $z = x(1)$ Then the following lemma can be proved.

Lemma 3.13. *In* $QCA - [\delta_{60,0-0}, \frac{\pi}{4}](n)$ *the equation*

$$\Delta^{2i}(z,x) = \begin{cases} (-1)^{\sum_{j=1}^{i} x(j-1)_n} & if\ z = x(i) \\ 0 & otherwise \end{cases}$$

holds for any configuration $x \in Q^n$ *and* $1 \le i \le n$.

Proof. We prove it by induction on i. First, for $x, y, z \in Q^n$ we have

$$\begin{aligned}
& < \delta_{60,0-0}(x), y \oplus u > + < \delta_{60,0-0}(y), z \oplus u > \\
= \;& < \delta_{60,0-0}(x), y > + < \delta_{60,0-0}(x), u > + < \delta_{60,0-0}(y), z > + < \delta_{60,0-0}(y), u > \pmod 2 \\
= \;& < \delta_{60,0-0}(x), u > + < \delta_{60,0-0}(x), y > \\
& + < (y_1, y_1 + y_2, y_2 + y_3, , y_{n-2} + y_{n-1}, y_{n-1} + y_n), z > \\
& + < (y_1, y_1 + y_2, y_2 + y_3, \cdots, y_{n-2} + y_{n-1}, y_{n-1} + y_n), u > \pmod 2 \\
= \;& < \delta_{60,0-0}(x), u > + < \delta_{60,0-0}(x), y > \\
& + (z_1 + 1 + z_2 + 1) y_1 + (z_2 + 1 + z_3 + 1) y_2 + (z_3 + 1 + z_4 + 1) y_3 \\
& + \cdots + (z_{n-1} + 1 + z_n + 1) y_{n-1} + (z_n + 1) y_n \pmod 2 \\
= \;& < \delta_{60,0-0}(x), u > + < \delta_{60,0-0}(x), y > + < \delta_{102,0-0}(z \oplus u), y > \pmod 2 \\
= \;& < \delta_{60,0-0}(x), u > + < \delta_{60,0-0}(x) \oplus \delta_{102,0-0}(z \oplus u), y > \pmod 2
\end{aligned}$$

since the local function $f_{102} : Q^3 \to Q$ with rule number 102 is represented by $f_{102}(a,b,c) = b + c \pmod 2$ for $a, b, c \in Q$. Therefore by lemma 3.2 we have

$$\begin{aligned}
\Delta^2(x,z) &= \sum_{y \in Q^n} \Delta(x,y)\Delta(y,z) \\
&= \sum_{y \in Q^n} \left\{ \frac{(-1)^{<\delta_{60,0-0}(x), y \oplus u>}}{(\sqrt{2})^n} \times \frac{(-1)^{<\delta_{60,0-0}(y), z \oplus u>}}{(\sqrt{2})^n} \right\} \\
&= \frac{(-1)^{<\delta_{60,0-0}(x), u>}}{2^n} \sum_{y \in Q^n} (-1)^{<\delta_{60,0-0}(x) \oplus \delta_{102,0-0}(z \oplus u), y>} \\
&= \begin{cases} (-1)^{<\delta_{60,0-0}(x), u>} & if\ \delta_{60,0-0}(x) \oplus \delta_{102,0-0}(z \oplus u) = 0^n \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{<\delta_{60,0-0}(x), u>} & if\ z = x(1) \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{x(0)_n} & if\ z = x(1) \\ 0 & otherwise \end{cases} .
\end{aligned}$$

And we assume that the equation holds in the case of $i \leq k$. Then we prove that the equation of $i = k+1$ holds.

$$\begin{aligned}
\Delta^{2k+2}(x,z) &= \sum_{y \in Q^n} \Delta^{2k}(x,y)\Delta^2(y,z) \\
&= \begin{cases} (-1)^{\sum_{j=1}^{k} x(j-1)_n}(-1)^{y_n} & if\ y = x(k)\ and\ z = y(1) \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{\sum_{j=1}^{k+1} x(j-1)_n} & if\ z = x(k+1) \\ 0 & otherwise \end{cases}
\end{aligned}$$

For any $x \in Q^n$ we have

$$\sum_{j=0}^{n} x(j)_n = 2\sum_{j=1}^{n} x_j + n.$$

Therefore the following proposition holds:

PROPOSITION 3.14. *In* $QCA - [\delta_{60,0-0}, \frac{\pi}{4}](n)$ *the following*

$$\Delta^{2n+2} = (-1)^n Id_{|Q^n|}$$

holds.

PROOF. For any $x, z \in Q^n$ we have

$$\begin{aligned}
\Delta^{2n+2}(z,x) &= \sum_{y \in Q^n} \Delta^2(z,y)\Delta^{2n}(y,x) \\
&= \begin{cases} (-1)^{\sum_{j=1}^{n} x(j-1)_n}(-1)^{y_n} & if\ \ y = x(n)\ \ and\ \ z = y(1) \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{\sum_{j=0}^{n} x(j)_n} & if\ z = x \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^n & if\ z = x \\ 0 & otherwise. \end{cases}
\end{aligned}$$

3.8. Quantum CA with Rule 195

Classical CA with the local function of rule number 195 are reversible in the case of any cell size n and fixed boundary condition. In this subsection we discuss the behaviors of quantum CA decided by $\delta_{195,0-0}$ and $\theta = \frac{\pi}{4}$.

For $x \in Q^n$ and an integer i $(0 \leq i \leq n)$ we define $x(i) \in Q^n$ as follows:

- $x(0) = x$
- $x(i)_i = x_{n-i+1} + ni + 1 \pmod 2$,
- $x(i)_{i+j} = x_j + x_{n-i+1} + i(n+j) + 1 \pmod 2$
  for $1 \leq j \leq n-i$,

- $x(i)_{i-j} = x_{n-i+1} + x_{n-j+1} + i(n+j) + j(n+1) \pmod 2$ for $1 \leq j \leq i-1$.

We can easily check the following equations:

- $x(1) = (x_n, x_n, \cdots, x_n) \oplus (0, x_1, x_2, \cdots, x_{n-1}) \oplus (n-1, n-2, \cdots, 0)$
- $x(i+1) = x(i)(1)$
- $x(n)(1) = x$

And we can easily check that for $x \in Q^n$ the equation $\delta_{195,0-0}(x) \oplus \delta_{170,0-0}(z \oplus u) \oplus z \oplus u = 0^n$ holds if $z = x(1)$. Then the following lemma holds by using the above fact and lemma 3.2:

LEMMA 3.15. *In* $QCA - [\delta_{195,0-0}, \frac{\pi}{4}](n)$ *the equation*

$$\Delta^{2i}(z,x) = \begin{cases} (-1)^{\sum_{j=1}^{i}\{<x(j-1),u>+nx(j-1)_n+\frac{n(n+1)}{2}\}} & if\ z = x(i) \\ 0 & otherwise \end{cases}$$

holds for any configuration $x \in Q^n$ *and* $1 \leq i \leq n$.

PROOF. We prove it by induction on i. First, since the local function $f_{170} : Q^3 \to Q$ with rule number 170 is represented by $f_{170}(a,b,c) = c \pmod 2$ for $a, b, c \in Q$ we have the following equation modulo 2 by the same way as the proof of lemma 3.13:

$$\begin{aligned} & < \delta_{195,0-0}(x), y \oplus u > + < \delta_{195,0-0}(y), z \oplus u > \\ = \ & < \delta_{195,0-0}(x) \oplus z \oplus u, u > + < \delta_{195,0-0}(x) \oplus \delta_{170,0-0}(z \oplus u) \oplus z \oplus u, y > \pmod 2 \end{aligned}$$

Therefore by lemma 3.2 we have

$$\begin{aligned} \Delta^2(x,z) &= \sum_{y \in Q^n} \Delta(x,y)\Delta(y,z) \\ &= \sum_{y \in Q^n} \left\{ \frac{(-1)^{<\delta_{195,0-0}(x), y \oplus u>}}{(\sqrt{2})^n} \times \frac{(-1)^{<\delta_{195,0-0}(y), z \oplus u>}}{(\sqrt{2})^n} \right\} \\ &= \frac{(-1)^{<\delta_{195,0-0}(x) \oplus z \oplus u, u>}}{2^n} \sum_{y \in Q^n} (-1)^{<\delta_{195,0-0}(x) \oplus \delta_{170,0-0}(z \oplus u) \oplus z \oplus u, y>} \\ &= \begin{cases} (-1)^{<\delta_{195,0-0}(x) \oplus z \oplus u, u>} & if\ \delta_{195,0-0}(x) \oplus \delta_{170,0-0}(z \oplus u) \oplus z \oplus u = 0^n \\ 0 & otherwise \end{cases} \\ &= \begin{cases} (-1)^{<\delta_{195,0-0}(x) \oplus z \oplus u, u>} & if\ z = x(1) \\ 0 & otherwise \end{cases} \\ &= \begin{cases} (-1)^{<x,u>+nx_n+\frac{n(n+1)}{2}} & if\ z = x(1) \\ 0 & otherwise \end{cases} \end{aligned}$$

And we assume that the equation holds in the case of $i \leq k$. Then we have

$$\Delta^{2k+2}(x,z) = \sum_{y \in Q^n} \Delta^{2k}(x,y)\Delta^2(y,z)$$

$$= \begin{cases} (-1)^{\sum_{j=1}^{k}\{<x(j-1),u>+nx(j-1)_n+\frac{n(n+1)}{2}\}+<y(0),u>+ny(0)_n+\frac{n(n+1)}{2}} \\ \qquad\qquad\qquad\qquad if\ y = x(k)\ and\ z = y(1) \\ 0 \\ \qquad\qquad\qquad\qquad otherwise \end{cases}$$

$$= \begin{cases} (-1)^{\sum_{j=1}^{k+1}\{<x(j-1),u>+nx(j-1)_n+\frac{n(n+1)}{2}\}} & if\ z = x(k+1) \\ 0 & otherwise. \end{cases}$$

PROPOSITION 3.16. *In* $QCA - [\delta_{195,0-0}, \frac{\pi}{4}](n)$ *the following*

$$\Delta^{2n+2} = \begin{cases} -Id_{|Q^n|} & n = 4k+3 \\ Id_{|Q^n|} & otherwise \end{cases}$$

holds.

PROOF. For any $x, z \in Q^n$ we have

$$\begin{aligned} & \Delta^{2n+2}(z,x) \\ = & \sum_{y\in Q^n} \Delta^2(z,y)\Delta^{2n}(y,x) \\ = & \begin{cases} (-1)^{\sum_{j=1}^{n}\{<x(j-1),u>+nx(j-1)_n+\frac{n(n+1)}{2}\}}(-1)^{\{<y,u>+ny_n+\frac{n(n+1)}{2}\}} \\ \qquad\qquad\qquad if\ \ y = x(n)\ \ and\ \ z = y(1) \\ 0 \\ \qquad\qquad\qquad otherwise \end{cases} \\ = & \begin{cases} (-1)^{\sum_{j=1}^{n+1}\{<x(j-1),u>+nx(j-1)_n+\frac{n(n+1)}{2}\}} & if\ z = x \\ 0 & otherwise \end{cases} \\ = & \begin{cases} (-1)^{\{\frac{n^3+2n^2+n}{2}+\frac{4n^4+3n^3+5n^2}{6}+\frac{7n^4+8n^3+29n^2+4n}{12}\}} & if\ z = x \\ 0 & otherwise \end{cases} \end{aligned}$$

and we can check that

- $\frac{n^3+2n^2+n}{2} + \frac{4n^4+3n^3+5n^2}{6} + \frac{7n^4+8n^3+29n^2+4n}{12}$ is odd if $n = 4k+3$
- $\frac{n^3+2n^2+n}{2} + \frac{4n^4+3n^3+5n^2}{6} + \frac{7n^4+8n^3+29n^2+4n}{12}$ is even otherwise.

Therefore we have

$$\Delta^{2n+2} = \begin{cases} -Id_{|Q^n|} & if\ n = 4k+3 \\ Id_{|Q^n|} & otherwise. \end{cases}$$

3.9. Quantum CA with Rule 150

Classical CA with the local function of rule number 150 are reversible in the following cases:

- $n \neq 2 \pmod 3$ and fixed boundary condition
- $n \neq 0 \pmod 3$ and cyclic boundary condition

where n is cell size. In this subsection we discuss the behaviors of each quantum CA decided as follows:

- $\delta_{150,c}$, any θ and cell size $n = 4$
- $\delta_{150,c}$, $\theta = \frac{\pi}{4}$ and cell size $n \neq 0 \pmod 3$
- $\delta_{150,0-0}$, $\theta = \frac{\pi}{4}$ and cell size $n \neq 2 \pmod 3$

3.9.1. $QCA - [\delta_{150,c}, \theta](4)$

We set

$$\begin{aligned}
g(n) &= \frac{1}{2}\left(\sum_{i=1}^{n} \cos(4i-2)\theta\right)\sin 2\theta,\\
A1(n) &= \begin{pmatrix} \cos^2(2n\theta) & -g(n) & -g(n) & 0 \\ g(n) & \cos^2(2n\theta) & 0 & -g(n) \\ g(n) & 0 & \cos^2(2n\theta) & -g(n) \\ 0 & g(n) & g(n) & \cos^2(2n\theta) \end{pmatrix},\\
A2(n) &= \begin{pmatrix} g(n) & 0 & 0 & g(n) \\ 0 & g(n) & -g(n) & 0 \\ 0 & -g(n) & g(n) & 0 \\ g(n) & 0 & 0 & g(n) \end{pmatrix},\\
A3(n) &= \begin{pmatrix} 0 & -g(n) & -g(n) & \sin^2(2n\theta) \\ g(n) & 0 & -\sin^2(2n\theta) & -g(n) \\ g(n) & -\sin^2(2n\theta) & 0 & -g(n) \\ \sin^2(2n\theta) & g(n) & g(n) & 0 \end{pmatrix},\\
B1(n) &= \begin{pmatrix} A1(n) & -A2(n) \\ A2(n) & A1(n) \end{pmatrix},\\
B2(n) &= \begin{pmatrix} A2(n) & -A3(n) \\ A3(n) & A2(n) \end{pmatrix}, \textit{and}\\
C(n) &= \begin{pmatrix} B1(n) & -B2(n) \\ B2(n) & B1(n) \end{pmatrix}.
\end{aligned}$$

Then we can show the following lemma:

Lemma 3.17. *In $QCA - [\delta_{150,c}, \theta](4)$ the following equation*

$$\Delta^{2k} = C(k)$$

holds for any positive integer k.

Therefore considering positive integer m such that $\cos^2(2m\theta) = 1$ we can get the following proposition

Proposition 3.18. *In $QCA - [\delta_{150,c}, \theta](4)$ the following equation*

$$\Delta^{2m} = Id_{|Q^n|}$$

holds if there exists m satisfying $m = \min\{\ m' \in \mathbb{N} \mid 2m'\theta \in \pi\mathbb{Z}\ \}$.

3.9.2. $QCA-[\delta_{150,c},\frac{\pi}{4}](n)$

In the classical CA $CA-150$ of cyclic boundary condition we can easily check that $\delta_{150,c}(x)=0^n$ if and only if $x=0^n$ for $x\in Q^n$ where $n\neq 0$ (mod 3). Using the fact and lemma 3.2 we can prove the following lemma:

LEMMA 3.19. *In* $QCA-[\delta_{150,c},\frac{\pi}{4}](n)$ *the equation*

$$\Delta^2(z,x)=\begin{cases}(-1)^{<x,u>} & if\ z=x\oplus u\\ 0 & otherwise\end{cases}$$

holds for all $x,z\in Q^n$ *where* $n\neq 0(\mathrm{mod}3)$.

PROOF. For any $x,z\in Q^n$ we have

$$\begin{aligned}\Delta^2(z,x) &= \sum_{y\in Q^n}\Delta(z,y)\Delta(y,x)\\ &= \sum_{y\in Q^n}\left\{\frac{(-1)^{<\delta_{150,c}(y),z\oplus u>}}{(\sqrt{2})^n}\times\frac{(-1)^{<\delta_{150,c}(x),y\oplus u>}}{(\sqrt{2})^n}\right\}\\ &= \frac{(-1)^{<x,u>}}{2^n}\sum_{y\in Q^n}(-1)^{<\delta_{150,c}(x\oplus z\oplus u),y>}\\ &= \begin{cases}(-1)^{<x,u>} & if\ \delta_{150,c}(x\oplus z\oplus u)=0^n\\ 0 & otherwise\end{cases}\\ &= \begin{cases}(-1)^{<x,u>} & if\ x\oplus z=u\\ 0 & otherwise\end{cases}\\ &= \begin{cases}(-1)^{<x,u>} & if\ z=x\oplus u\\ 0 & otherwise\end{cases}\end{aligned}$$

The following proposition can be got immediately from the above lemma.

PROPOSITION 3.20. *In* $QCA-[\delta_{150,c},\frac{\pi}{4}](n)$ *the equation*

$$\Delta^4=(-1)^n Id_{|Q^n|}$$

holds where $n\neq 0(\mathrm{mod}3)$.

PROOF. For any $x,z\in Q^n$ we have

$$\begin{aligned}\Delta^4(z,x) &= \sum_{y\in Q^n}\Delta^2(z,y)\Delta^2(y,x)\\ &= \begin{cases}\Delta^2(x,x\oplus u)\Delta^2(x\oplus u,x) & if\ x=z\\ 0 & otherwise\end{cases}\\ &= \begin{cases}(-1)^n & if\ x=z\\ 0 & otherwise\end{cases}\end{aligned}$$

3.9.3. $QCA-[\delta_{150,0-0},\frac{\pi}{4}](n)$

In the classical CA $CA-150$ of $0-0$ boundary condition we can easily check that $\delta_{150,0-0}(x)=0^n$ if and only if $x=0^n$ for $x\in Q^n$ where $n\neq 2 \pmod 3$. Then we can prove the following lemma:

Lemma 3.21. *In* $QCA-[\delta_{150,0-0},\frac{\pi}{4}](n)$ *the equation*

$$\Delta^2(z,x)=\begin{cases}(-1)^{x_2+x_3+\cdots+x_{n-1}} & if\ z=x\oplus u\\ 0 & otherwise\end{cases}$$

holds for all $x,z\in Q^n$ *where* $n\neq 2(\mathrm{mod}3)$.

Proof. For any $x,z\in Q^n$ we have

$$\begin{aligned}\Delta^2(z,x) &= \sum_{y\in Q^n}\Delta(z,y)\Delta(y,x)\\ &= \sum_{y\in Q^n}\left\{\frac{(-1)^{<\delta_{150,0-0}(y),z\oplus u>}}{(\sqrt{2})^n}\times\frac{(-1)^{<\delta_{150,0-0}(x),y\oplus u>}}{(\sqrt{2})^n}\right\}\\ &= \frac{(-1)^{x_2+x_3+\cdots+x_{n-1}}}{2^n}\sum_{y\in Q^n}(-1)^{<\delta_{150,0-0}(x\oplus z\oplus u),y>}\\ &= \begin{cases}(-1)^{x_2+x_3+\cdots+x_{n-1}} & if\ \delta_{150,0-0}(x\oplus z\oplus u)=0^n\\ 0 & otherwise\end{cases}\\ &= \begin{cases}(-1)^{x_2+x_3+\cdots+x_{n-1}} & if\ z=x\oplus u\\ 0 & otherwise.\end{cases}\end{aligned}$$

Using this lemma we can get the following proposition:

Proposition 3.22. *In* $QCA-[\delta_{150,0-0},\frac{\pi}{4}](n)$

$$\Delta^4=(-1)^n Id_{|Q^n|}$$

holds where $n\neq 2(\mathrm{mod}3)$.

Proof. For any $x,z\in Q^n$ we have

$$\begin{aligned}\Delta^4(z,x) &= \sum_{y\in Q^n}\Delta^2(z,y)\Delta^2(y,x)\\ &= \begin{cases}\Delta^2(x,x\oplus u)\Delta^2(x\oplus u,x) & if\ x=z\\ 0 & otherwise\end{cases}\\ &= \begin{cases}(-1)^{x_2+x_3+\cdots+x_{n-1}}(-1)^{(x_2+1)+(x_3+1)+\cdots+(x_{n-1}+1)} & if\ x=z\\ 0 & otherwise\end{cases}\\ &= \begin{cases}(-1)^{n-2} & if\ x=z\\ 0 & otherwise.\end{cases}\end{aligned}$$

3.10. Quantum CA with Rule 105

Classical CA with the local function of rule number 105 are reversible in the following cases:

- $n \neq 2 \pmod 3$ and fixed boundary condition
- $n \neq 0 \pmod 3$ and cyclic boundary condition

where n is cell size. In this subsection we discuss the behaviors of each quantum CA decided by as follows:

- $\delta_{105,c}$, $\theta = \frac{\pi}{4}$ and cell size $n \neq 0$ (mod3)
- $\delta_{105,0-0}$, $\theta = \frac{\pi}{4}$ and cell size $n \neq 2$ (mod3)

3.10.1. $QCA - [\delta_{105,c}, \frac{\pi}{4}](n)$

In the classical CA $CA-105$ of cyclic boundary condition we can easily check that $\delta_{105,c}(x) = 0^n$ if and only if $x = 1^n$ for $x \in Q^n$ where $n \neq 0 \pmod 3$. Then we can prove the following proposition:

Proposition 3.23. *In* $QCA - [\delta_{105,c}, \frac{\pi}{4}](n)$ *the equation*

$$\Delta^2 = Id_{|Q^n|}$$

holds where $n \neq 0(\text{mod}3)$.

Proof. For any $x, z \in Q^n$ by lemma 3.2 we have

$$\begin{aligned}
\Delta^2(z,x) &= \sum_{y \in Q^n} \Delta(z,y)\Delta(y,x) \\
&= \sum_{y \in Q^n} \left\{ \frac{(-1)^{<\delta(y), z\oplus u>}}{(\sqrt{2})^n} \times \frac{(-1)^{<\delta(x), y\oplus u>}}{(\sqrt{2})^n} \right\} \\
&= \frac{(-1)^{n+3m(x)+m(z\oplus u)}}{2^n} \sum_{y \in Q^n} (-1)^{<\delta(x\oplus z\oplus u), y>} \\
&= \begin{cases} 1 & if\ \delta(x \oplus z \oplus u) = 0^n \\ 0 & otherwise \end{cases} \\
&= \begin{cases} 1 & if\ x \oplus z \oplus u = u \\ 0 & otherwise \end{cases} \\
&= \begin{cases} 1 & if\ z = x \\ 0 & otherwise \end{cases}
\end{aligned}$$

3.10.2. $QCA - [\delta_{105,0-0}, \frac{\pi}{4}](n)$

In the classical CA $CA-105$ of $0-0$ boundary condition we can easily check that $\delta_{105,0-0}(w) = 0^n$ if and only if

$$w = \begin{cases} (010)^k & if \quad n = 3k \\ (100)^k 1 & if \quad n = 3k+1. \end{cases}$$

for $w \in Q^n$ where $n \neq 2 \pmod 3$. Then the following lemma can be proved by the above fact and lemma 3.2:

Lemma 3.24. *Let $w \in Q^n$ be a configuration such that $\delta_{105,0-0}(w) = 0$. Then in $QCA - [\delta_{105,0-0}, \frac{\pi}{4}](n)$ the equation*

$$\Delta^2(z,x) = \begin{cases} (-1)^{x_1+x_n} & if\ z = w \oplus x \oplus u \\ 0 & otherwise \end{cases}$$

holds for any $x, z \in Q^n$ where $n \neq 2(\mathrm{mod} 3)$.

Proof. For any $x, z \in Q^n$ we have

$$\begin{aligned}
\Delta^2(z,x) &= \sum_{y \in Q^n} \Delta(z,y)\Delta(y,x) \\
&= \sum_{y \in Q^n} \left\{ \frac{(-1)^{<\delta_{105,0-0}(y), z \oplus u>}}{(\sqrt{2})^n} \times \frac{(-1)^{<\delta_{105,0-0}(x), y \oplus u>}}{(\sqrt{2})^n} \right\} \\
&= \frac{(-1)^{<\delta_{105,0-0}(x) \oplus z \oplus u, u>}}{2^n} \sum_{y \in Q^n} (-1)^{<\delta_{105,0-0}(x \oplus z \oplus u), y>} \\
&= \begin{cases} (-1)^{<\delta_{105,0-0}(x) \oplus z \oplus u, u>} & if\ \delta_{105,0-0}(x \oplus z \oplus u) = 0^n \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{<\delta_{105,0-0}(x) \oplus z \oplus u, u>} & if\ x \oplus z \oplus u = w \\ 0 & otherwise \end{cases} \\
&= \begin{cases} (-1)^{x_1+x_n} & if\ z = w \oplus x \oplus u \\ 0 & otherwise \end{cases}
\end{aligned}$$

Proposition 3.25. *In $QCA - [\delta_{105,0-0}, \frac{\pi}{4}](n)$ the equation*

$$\Delta^4 = Id_{|Q^n|}$$

holds where $n \neq 2(\mathrm{mod} 3)$.

Proof. For any $x, z \in Q^n$ we have

$$\begin{aligned}
\Delta^4(z,x) &= \sum_{y \in Q^n} \Delta^2(z,y)\Delta^2(y,x) \\
&= \begin{cases} \Delta^2(z,y)\Delta^2(y,x) & if\ y = w \oplus x \oplus u\ and\ z = w \oplus y \oplus u \\ 0 & otherwise \end{cases} \\
&= \begin{cases} 1 & if\ z = x \\ 0 & otherwise. \end{cases}
\end{aligned}$$

4. Conclusion

In this paper we treated quantum CA whose global transition function is defined by the global transition function of classical CA and rotation of cells. A quantum cell can be represented by any point on the sphere and a transition in quantum CA is considered as a movement from a point to another point on the sphere. While trivially classical CA with finite cell array behave in finite space, we think that generally quantum CA (including quantum CA quantizated classical finite CA) behave infinitely for any initial configuration. In this paper we proved that some quantum CA is globally periodic. Globally periodic quantum CA have infinite variety of configurations and finite behaviors from any initial configuration. That is, in globally periodic quantum CA for any initial configuration we can get the same configuration after some step transitions. By this property we may apply globally periodic quantum CA to cryptography theory or quantum communication theory by using rule number and rotation angle as keys.

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Appendix A Computer Simulations

In this appendix we will show three kinds of results of computer simulations for each $QCA - [\delta_{R,bc}, \lambda_\theta](n)$. We let $c_2 = \varepsilon_{0^{n-2}10}$ and $c_4 = \varepsilon_{0^{n-3}100}$. The first results (the left-hand side figures) shows the probability of finding the state 1 at each cells by darkness. The second results (the middle graphs) shows the change of the mean density of cells with state 1 and the third result (the right-hand side graphs) shows the discrete Fourier transformation of the second results.

Appendix A1. Behaviors of $QCA - [\delta_{60,0-0}, \lambda_\theta](5)$

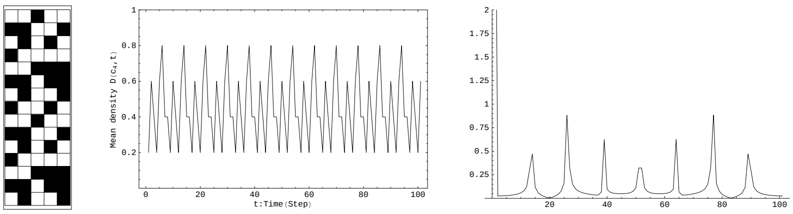

Figure 3: $\theta = \frac{\pi}{2}$

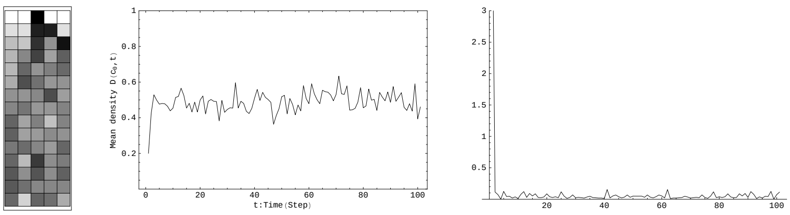

Figure 4: $\theta = 0.35764$

Appendix A2. Behaviors of $QCA - [\delta_{204,c}, \theta](5)$

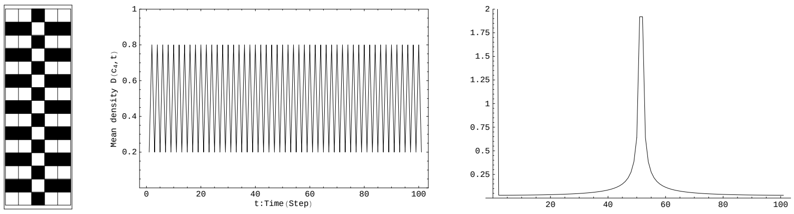

Figure 5: $\theta = \frac{\pi}{2}$

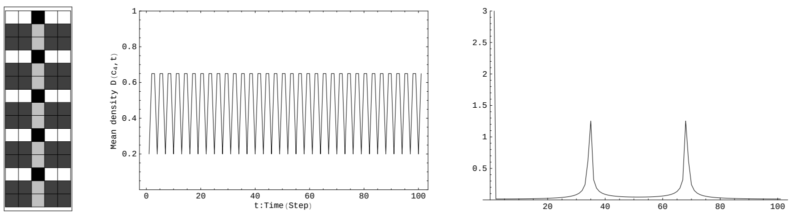

Figure 6: $\theta = \frac{\pi}{3}$

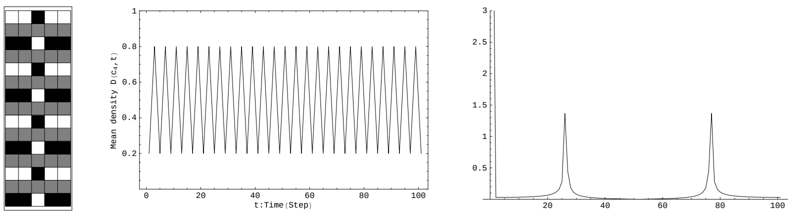

Figure 7: $\theta = \frac{\pi}{4}$

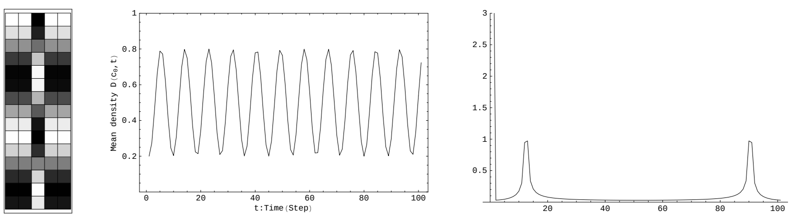

Figure 8: $\theta = 0.35764$

Appendix A3. Behaviors of $QCA - [\delta_{51,c}, \lambda_\theta](5)$

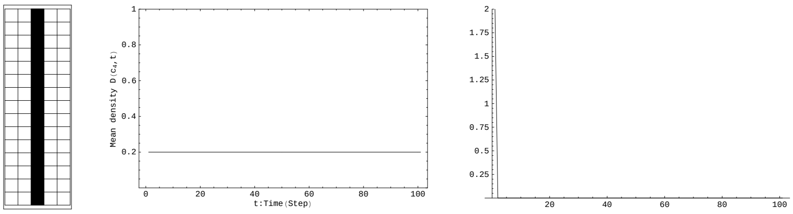

Figure 9: $\theta = \frac{\pi}{2}$

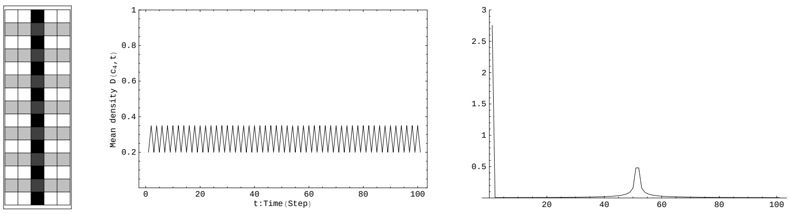

Figure 10: $\theta = \frac{\pi}{3}$

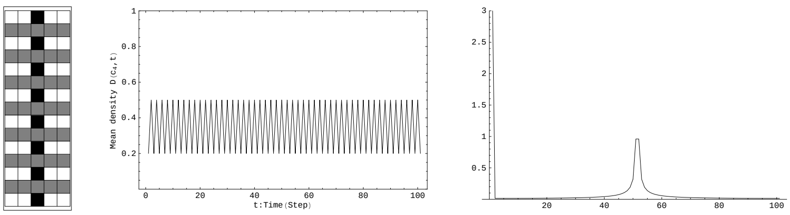

Figure 11: $\theta = \frac{\pi}{4}$

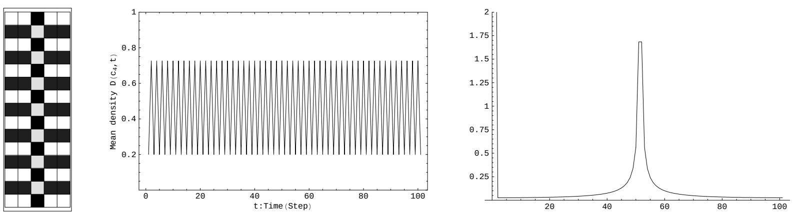

Figure 12: $\theta = 0.35764$

Appendix A4. Behaviors of $QCA - [\delta_{240,c}, \lambda_\theta](5)$

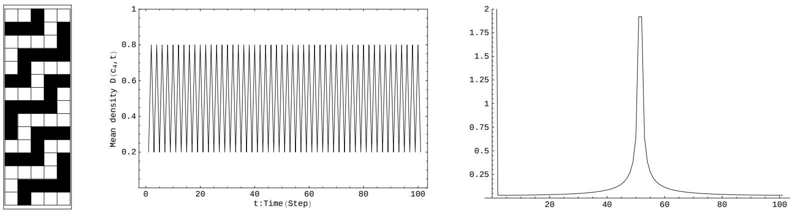

Figure 13: $\theta = \frac{\pi}{2}$

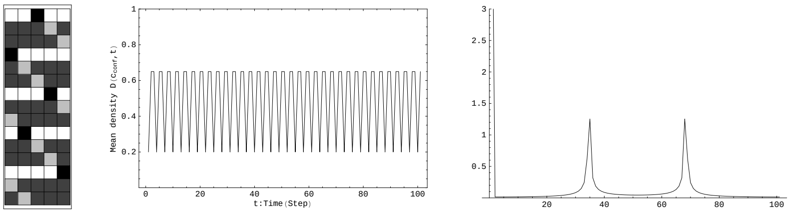

Figure 14: $\theta = \frac{\pi}{3}$

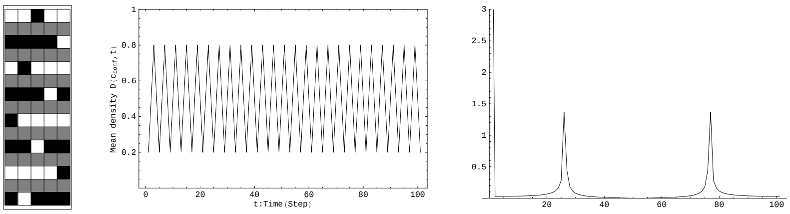

Figure 15: $\theta = \frac{\pi}{4}$

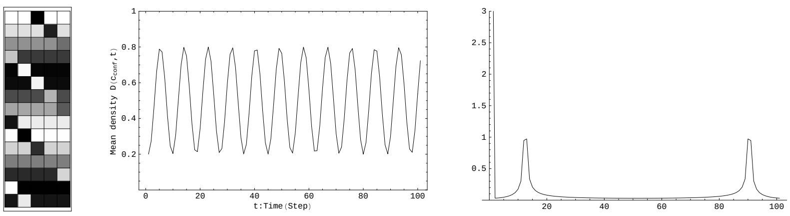

Figure 16: $\theta = 0.35764$

Appendix A5. Behaviors of $QCA - [\delta_{90,0-0}, \lambda_\theta](4)$

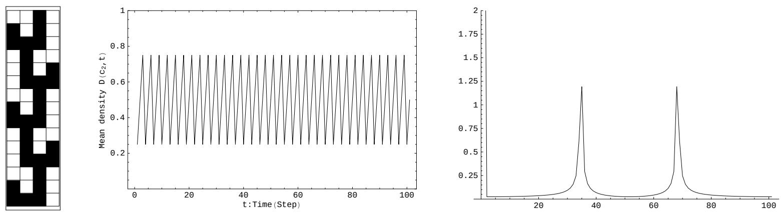

Figure 17: $\theta = \frac{\pi}{2}$

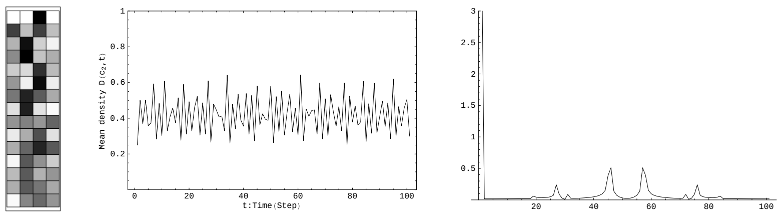

Figure 18: $\theta = \frac{\pi}{3}$

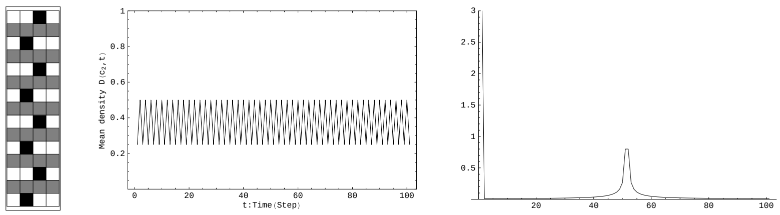

Figure 19: $\theta = \frac{\pi}{4}$

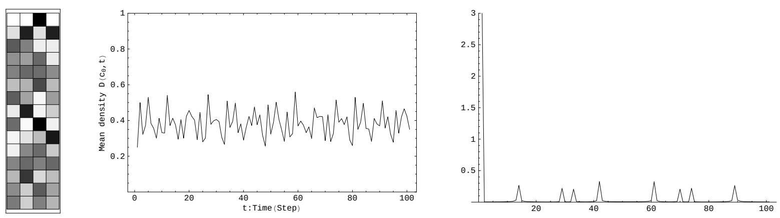

Figure 20: $\theta = 0.35764$

Appendix A6. Behaviors of $QCA - [\delta_{165,0-0}, \lambda_\theta](4)$

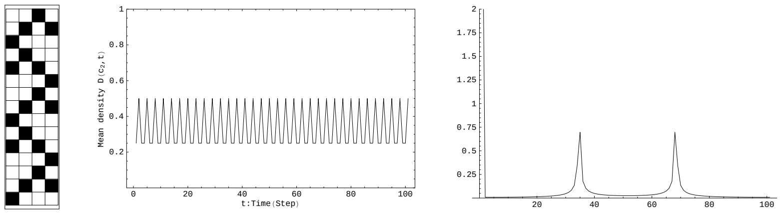

Figure 21: $\theta = \frac{\pi}{2}$

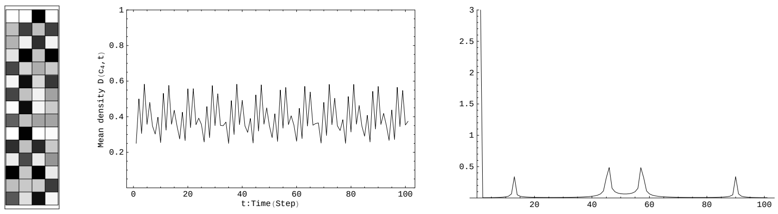

Figure 22: $\theta = \frac{\pi}{3}$

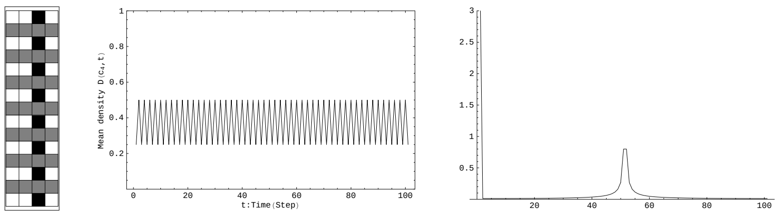

Figure 23: $\theta = \frac{\pi}{4}$

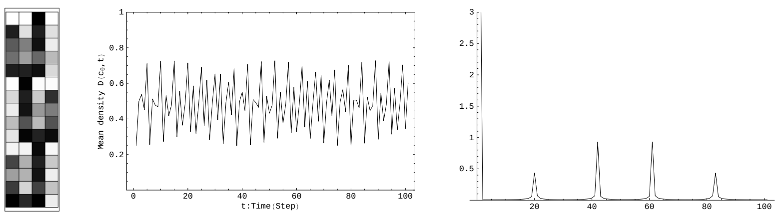

Figure 24: $\theta = 0.35764$

Appendix A7. Behaviors of $QCA - [\delta_{195,0-0}, \lambda_\theta](5)$

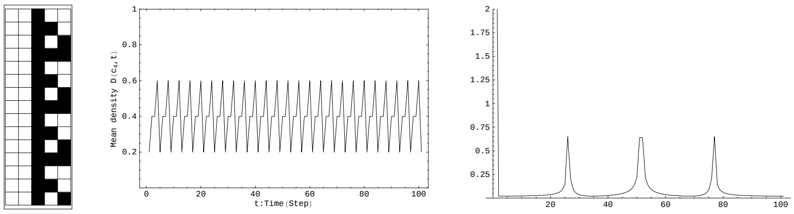

Figure 25: $\theta = \frac{\pi}{2}$

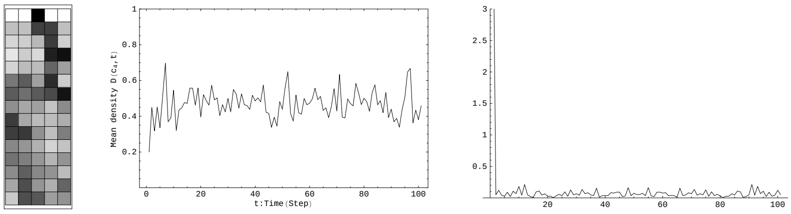

Figure 26: $\theta = \frac{\pi}{3}$

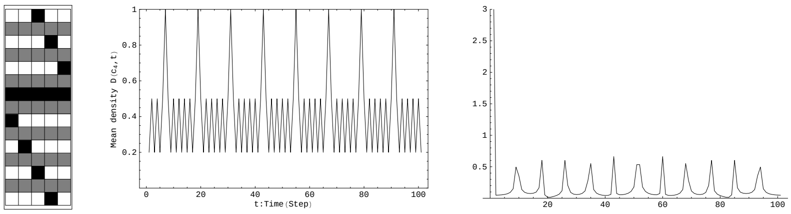

Figure 27: $\theta = \frac{\pi}{4}$

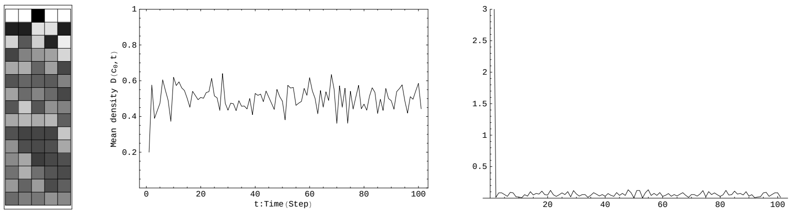

Figure 28: $\theta = 0.35764$

Appendix A8. Behaviors of $QCA-[\delta_{150,0-0}, \lambda_\theta](4)$

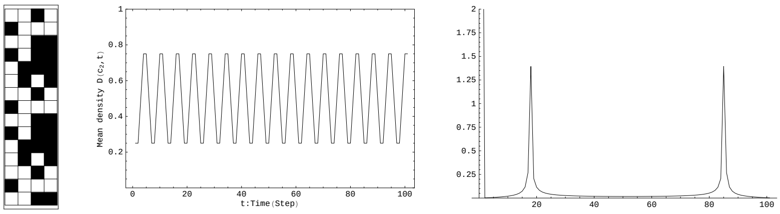

Figure 29: $\theta = \frac{\pi}{2}$

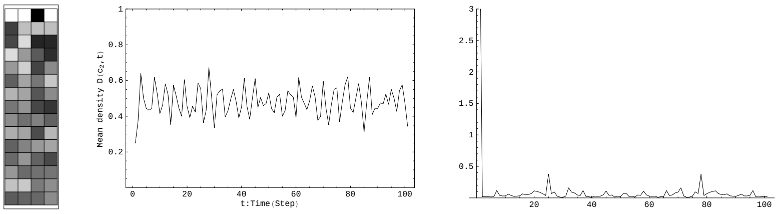

Figure 30: $\theta = \frac{\pi}{3}$

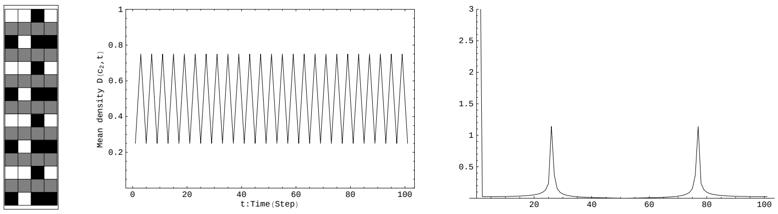

Figure 31: $\theta = \frac{\pi}{4}$

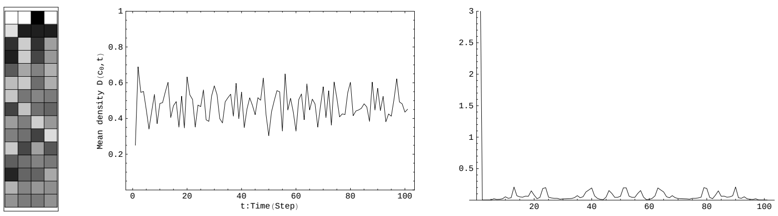

Figure 32: $\theta = 0.35764$

Appendix A9. Behaviors of $QCA - [\delta_{150,c}, \lambda_\theta](5)$

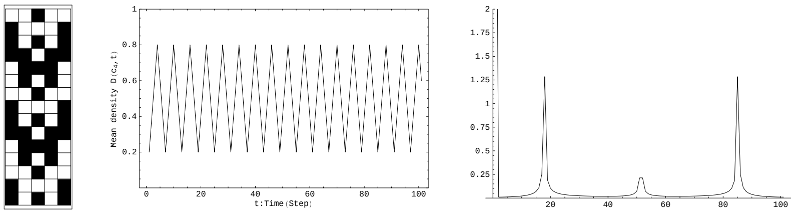

Figure 33: $\theta = \frac{\pi}{2}$

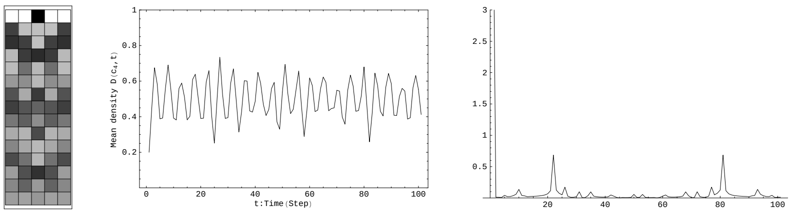

Figure 34: $\theta = \frac{\pi}{3}$

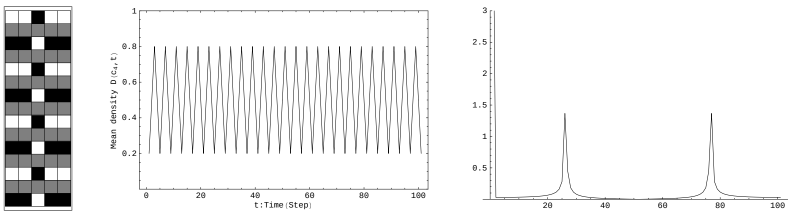

Figure 35: $\theta = \frac{\pi}{4}$

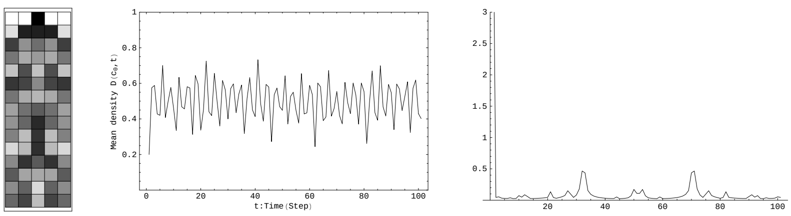

Figure 36: $\theta = 0.35764$

Appendix A10. Behaviors of $QCA - [\delta_{105,0-0}, \lambda_\theta](4)$

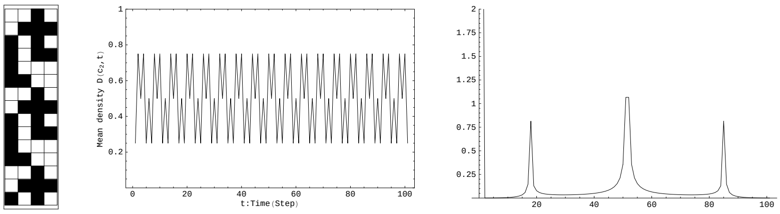

Figure 37: $\theta = \frac{\pi}{2}$

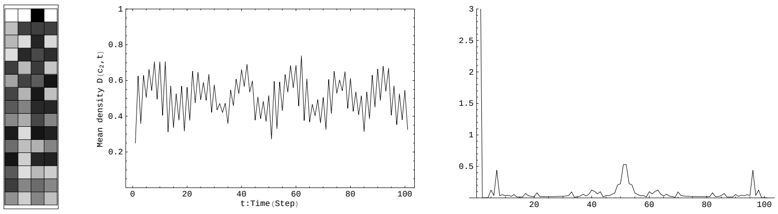

Figure 38: $\theta = \frac{\pi}{3}$

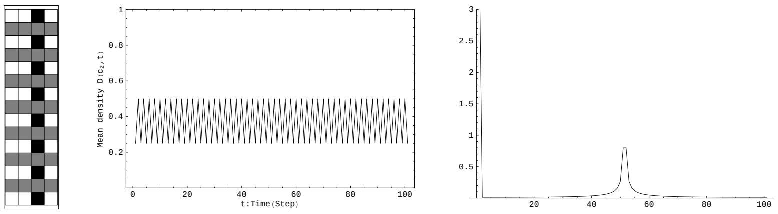

Figure 39: $\theta = \frac{\pi}{4}$

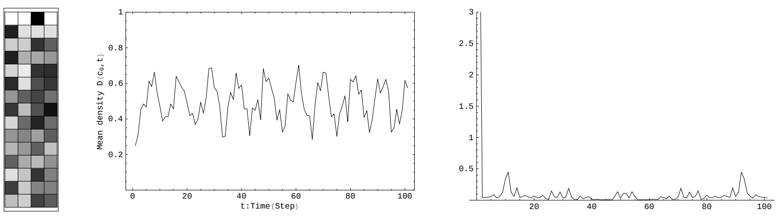

Figure 40: $\theta = 0.35764$

Appendix A11. Behaviors of $QCA-[\delta_{105,c},\lambda_\theta](5)$

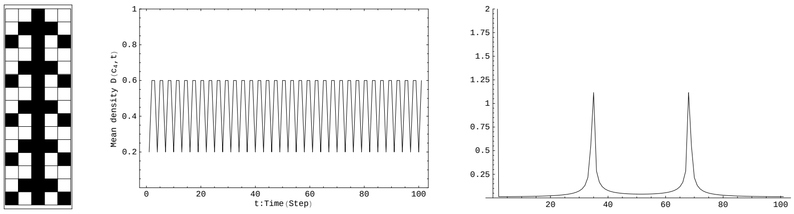

Figure 41: $\theta = \frac{\pi}{2}$

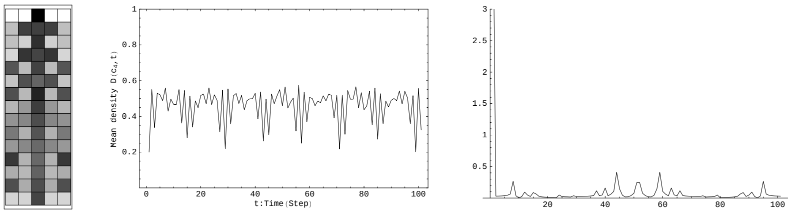

Figure 42: $\theta = \frac{\pi}{3}$

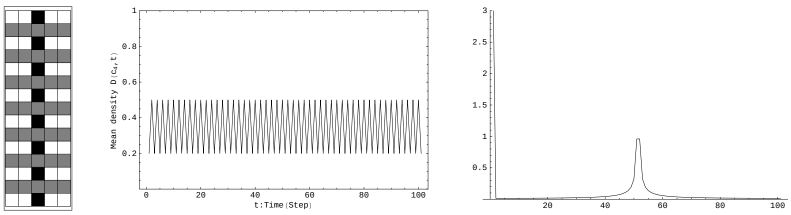

Figure 43: $\theta = \frac{\pi}{4}$

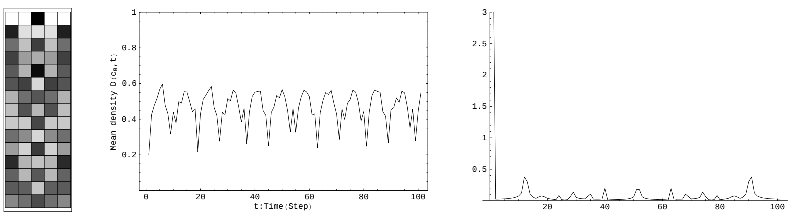

Figure 44: $\theta = 0.35764$