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https://hdl.handle.net/2324/18979

出版情報：MI Preprint Series. 2011-5, 2011-02-21. 九州大学大学院数理学研究院
バージョン：
権利関係：

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MI Preprint Series

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MI 2011-5

(Received February 21, 2011)

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Implications in Compressed Sensing and the Restricted Isometry Property

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Abstract

We give a simple idea for verifying the Restricted Isometry Property(Candès [3]). Although our approach is based on the Candès proofs, the proposed results are more flexible and meaningful than those of Candès. In this note, we establish our new results about the accuracy of the reconstruction from undersampled measurements which are possible to improve estimation depending on the situation.

Key Words and Phrases: Compressed sensing, Restricted isometry property.

1 Introduction

We suppose that we observe

$$\boldsymbol{y} = A\boldsymbol{x}, \quad \boldsymbol{x} \in \boldsymbol{R}^n, \tag{1}$$

where A is a $m \times n$ matrix. Our goal is to reconstract $\boldsymbol{x} \in \boldsymbol{R}^n$ with good accuracy. We are interested in $m < n$ case. It occurs the problem is of course ill-posed, but we know an important results when we suppose $\boldsymbol{x}$ is known to be sparse or nearly sparse and A obeys restricted isometry property(RIP) introduced below. Then we can reconstract $\boldsymbol{x} \in \boldsymbol{R}^n$ with good accuracy. In detail, this premise changes the problem, making the search for solutions feasible. In fact, we show that the solution $\boldsymbol{x}^\star$ to the following optimization problem

$$\min_{\tilde{\boldsymbol{x}} \in \boldsymbol{R}^n} \parallel \tilde{\boldsymbol{x}} \parallel_1 \quad \textit{subject to} \;\; \boldsymbol{y} = A\tilde{\boldsymbol{x}} \tag{2}$$

recovers $\boldsymbol{x}$ exactly, where $\| \cdot \|_1$ is l_1-norm. Furthermore, we extend the results for noiseless recovery to the case of noisy recovery. We observe

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{z}, \tag{3}$$

where $\boldsymbol{z}$ is an unknown noise term. In this context, we consider reconstructing $\boldsymbol{x}$ as the solution $\boldsymbol{x}^\star$ to the optimization problem

$$\min_{\tilde{\boldsymbol{x}} \in \boldsymbol{R}^n} \| \tilde{\boldsymbol{x}} \|_1 \quad \textit{subject to} \quad \| \boldsymbol{y} - A\tilde{\boldsymbol{x}} \|_2 \leq \varepsilon, \tag{4}$$

where ε is an upper bounded on the size of the noisy contribution and $\| \cdot \|_2$ is l_2-norm.

Definition 1.1. A matrix A satisfies the Restricted Isometry Property(RIP) of order s if there exists a constant δ with $0 < \delta < 1$ such that

$$(1 - \delta) \| \boldsymbol{a} \|_2^2 \leq \| A\boldsymbol{a} \|_2^2 \leq (1 + \delta) \| \boldsymbol{a} \|_2^2 \tag{5}$$

for all s-sparse vectors $\boldsymbol{a}$. A vector is said to be s-sparse if it has at most s nonzero entries. The minimum of above constants δ is said to be the isometry constant of A and it is denoted by δ_s.

The condition (5) is equivalent to requiring that the matrix $A_S^T A_S$ has all of its eigenvalues in $[1 - \delta_s, 1 + \delta_s]$, where A_S is the $m \times |S|$ matrix composed of these columns for any subset S of $\{1, 2, \cdots, n\}$.

It is well-known in[1, 2, 4, 5] that RIP is very useful to study the general robustness of CS. In particular, Candès [3] has obtained the following results:

Theorem 1.1 (Noiseless recovery). Assume that $\delta_{2s} < \sqrt{2} - 1$. Then the solution $\boldsymbol{x}^\star$ to (2) obeys

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_1 \leq C_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \tag{6}$$

and

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_2 \leq C_0 \frac{1}{\sqrt{s}} \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \tag{7}$$

for some constant C_0 given explicitly, where $\boldsymbol{x}_s$ is the vector $\boldsymbol{x}$ with all but the largest s components set to zero. In particular, if $\boldsymbol{x}$ is s-sparse, the recovery is exact.

Theorem 1.2 (Noisy recovery). Assume that $\delta_{2s} < \sqrt{2} - 1$ and $\| \boldsymbol{z} \|_2 \leq \varepsilon$. Then the solution $\boldsymbol{x}^\star$ to (4) obeys

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_2 \leq C_0 \frac{1}{\sqrt{s}} \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + C_1 \varepsilon \tag{8}$$

where C_0, C_1 are explicitly given constants.

We shall roughly state the Candès idea. Let $\{\boldsymbol{e}_k\}_{k=1,2,\cdots,n}$ be the basic vectors in $\boldsymbol{R}^n$. Let $\boldsymbol{a} = \sum_{k=1}^n a_k \boldsymbol{e}_k \in \boldsymbol{R}^n$ and $T \subset \{1, 2, \cdots, n\}$. We put $\boldsymbol{a}_T = \sum_{k=1}^n a_k^T \boldsymbol{e}_k$, where $a_k^T = a_k$ if $k \in T$ and $a_k = 0$ if otherwise. Candès obtained the above results by taking first the location T_0 of the s-largest coefficients of $\boldsymbol{x}$, next the location $T_1 \subset T_0^c$ of the s-largest coefficients of $\boldsymbol{h} \equiv \boldsymbol{x} - \boldsymbol{x}^\star$ and repeating this method, and by investigating $\boldsymbol{h}_{T_0 \cup T_1}$ and $\boldsymbol{h}_{(T_0 \cup T_1)^c}$. In this paper we shall improve the Candès results (Theorem 1.1 and Theorem 1.2) by taking the numbers s' that are different from s of above $T_1, T_2, \cdots$. By the difference of using the RIP, we have two main results:

Let $s, s' \in \boldsymbol{N}$ with $s < n$ and $s' < n - s$. We put

$$\alpha = \frac{2\sqrt{1 + \delta_{s+s'}}}{1 - \delta_{s+s'}} \quad and \quad \rho = \begin{cases} \frac{\sqrt{2}\delta_{s+s'}}{1-\delta_{s+s'}} & , s' \leq s \\ \frac{\sqrt{2}\delta_{2s'}}{1-\delta_{s+s'}} & , s' \geq s \end{cases}$$

Suppose that $\max(\delta_{s+s'}, \delta_{2s'}) < \frac{1}{1+\sqrt{\frac{2s}{s'}}}$. Then, since $\delta_{s+s'} \geq \delta_{2s'}$ if $s' \leq s$ and $\delta_{s+s'} \leq \delta_{2s'}$ if $s' \geq s$, we have $\rho < 1$. Under this preparation the following main theorems hold:

Theorem 1.3. Assume that A satisfies the RIP of order $max\,(s + s', 2s')$ and

$$\frac{1}{1 + \sqrt{\frac{2s}{s'}}} > \max(\delta_{s+s'}, \delta_{2s'}) = \begin{cases} \delta_{s+s'} & , s' \leq s \\ \delta_{2s'} & , s' \geq s \end{cases} \tag{9}$$

and put

$$C_0 = \begin{cases} \left(\frac{1+\rho}{1-\rho}\right) \sqrt{\frac{s}{s'}} & , s' \leq s \\ \left(\frac{1+\rho}{1-\rho}\right) \sqrt{\frac{s'}{s}} & , s' \geq s \end{cases} \tag{10}$$

and

$$C_1 = \begin{cases} \frac{\alpha}{1-\rho} 2\sqrt{s} & , s' \leq s \\ \frac{\alpha}{1-\rho} 2\sqrt{s'} & , s' \geq s \end{cases} \tag{11}$$

Then the following hold:

Noiseless recovery. The solution $\boldsymbol{x}^\star$ to (2) obeys

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_1 \leq C_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \tag{12}$$

and

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_2 \leq \frac{2}{\sqrt{s}} C_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \tag{13}$$

In particular, if $\boldsymbol{x}$ is s-sparse, the recovery is exact.

Noisy recovery. The solution $\boldsymbol{x}^\star$ to (4) obeys

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_1 \leq C_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + C_1 \varepsilon \tag{14}$$

and

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_2 \leq \frac{2}{\sqrt{s}} C_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + \frac{1}{\sqrt{s'}} C_1 \varepsilon \tag{15}$$

Theorem 1.4. Assume that A satisfies the RIP of order $(s + 2s')$ and

$$\delta_{s+s'} + \sqrt{\frac{s}{s'}} \delta_{s+2s'} < 1 \tag{16}$$

and put

$$\begin{aligned} \gamma &= \frac{\delta_{s+2s'}}{1 - \delta_{s+s'}}, \\ D_0 &= \left\{ \begin{array}{ll} \left(\frac{1+\gamma}{1-\gamma}\right) \sqrt{\frac{s}{s'}} & , s' \leq s \\ \left(\frac{1+\gamma}{1-\gamma}\right) \sqrt{\frac{s'}{s}} & , s' \geq s \end{array} \right. , \quad D_1 = \left\{ \begin{array}{ll} \frac{\alpha}{1-\gamma} 2\sqrt{s} & , s' \leq s \\ \frac{\alpha}{1-\gamma} 2\sqrt{s'} & , s' \geq s \end{array} \right. \end{aligned} \tag{17}$$

Then the following hold:

Noiseless recovery. The solution $\boldsymbol{x}^\star$ to (2) obeys

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_1 \leq D_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \tag{18}$$

and

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_2 \leq \frac{2}{\sqrt{s}} D_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \tag{19}$$

Noisy recovery. The solution $\boldsymbol{x}^\star$ to (4) obeys

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_1 \leq D_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + D_1 \varepsilon \tag{20}$$

and

$$\| \boldsymbol{x}^\star - \boldsymbol{x} \|_2 \leq \frac{2}{\sqrt{s}} D_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + \frac{1}{\sqrt{s'}} D_1 \varepsilon \tag{21}$$

The RIP requires bounded condition number for all submatrices built by selecting s arbitrary columns and the spectral norm of a matrix is not generally easy to commute. Hence, it is meaningful to improve the assumption: $\delta_{2s} < \sqrt{2} - 1$ of Theorem 1.1 and Theorem 1.2. In Section 3, we shall show that our results are more flexible than those of Candès by taking some appropriate numbers s'.

2 Proofs

In this section we prove Theorem 1.3 and Theorem 1.4. The proofs are based on those of Theorem 1.1 and Theorem 1.2 in [3].

Proof of Theorem 1.3. It suffices to show the case of noisy recovery. By Lemma 2.1 in [3] we have

$$|\langle A\boldsymbol{a},\ A\boldsymbol{b}\rangle| \leq \delta_{s+s'} \parallel \boldsymbol{a} \parallel_2 \parallel \boldsymbol{b} \parallel_2 \tag{22}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^n$ supported on disjoint subsets $T, T' \subset \{1, 2, \cdots, n\}$ with $|T| \leq s$ and $|T'| \leq s'$.

We put

$$\boldsymbol{h} = \boldsymbol{x}^{\star} - \boldsymbol{x} \tag{23}$$

By the linearity of$A$ and the triangle equality we have

$$\parallel A\boldsymbol{h} \parallel_2 \leq 2\varepsilon \tag{24}$$

For the simplity we use the following symbol: For $^\forall \boldsymbol{a} \in \boldsymbol{R}^n$ and $T \subset \{1, 2, \cdots, n\}$ we put

$$\boldsymbol{a}_T = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix},\ a_i^T = \begin{cases} a_i & , i \in T \\ 0 & , i \in T^c \end{cases}$$

Let T_0 be the location of the s-largest coefficients of $\boldsymbol{x}$. Then, $\boldsymbol{x}_{T_0} = \boldsymbol{x}_s$ and for $^\forall k \in T_0$ we have

$$\left|x_k^{T_0}\right| \geq |x_j|,\ \ ^\forall j \geq s+1 \tag{25}$$

Let T_1 be the location of s'-largest coefficients of $\boldsymbol{h}_{T_0^c}$. Then,

$$\left|h_k^{T_1}\right| \geq |h_i|,\ \ ^\forall k \in T_1,\ \ ^\forall i \in (T_0 \cup T_1)^c \tag{26}$$

Repeating this method, we get vectors $\boldsymbol{h}_{T_j}(j \geq 2)$ satisfying the conditions

$$\left|h_k^{T_j}\right| \geq |h_i|, \ \ ^\forall k \in T_j, \ \ ^\forall i \in (T_0 \cup T_1 \cup \cdots \cup T_j)^c \tag{27}$$

Let $\{1, 2, \cdots, n\} = T_0 \cup T_1 \cup \cdots \cup T_{r-1} \cup T_r$, $|T_r| \leq s'$. Let $2 \leq^\forall j \leq r-1$. Clearly,

$$\| \boldsymbol{h}_{T_j} \|_2 = \left(\sum_{k=1}^{s'} \left| h_k^{T_j} \right|^2 \right)^{\frac{1}{2}} \leq \sqrt{s'} \, \| \boldsymbol{h}_{T_j} \|_\infty \equiv \sqrt{s'} \max_{k \in T_j} \left| h_k^{T_j} \right| \tag{28}$$

and it follows from (26) and (28) that

$$\| \boldsymbol{h}_{T_{j-1}} \|_1 = \sum_{k=1}^{s'} \left| h_k^{T_{j-1}} \right| \geq s' \, \| \boldsymbol{h}_{T_j} \|_\infty, \tag{29}$$

which implies that

$$\| \boldsymbol{h}_{T_j} \|_2 \leq \frac{1}{\sqrt{s'}} \| \boldsymbol{h}_{T_{j-1}} \|_1, \ \ 2 \leq^\forall j \leq r-1 \tag{30}$$

Furthermore, it follows that

$$\| \boldsymbol{h}_{T_{r-1}} \|_1 = \sum_{k=1}^{s'} |h_k^{T_{r-1}}| \geq s' \, \| \boldsymbol{h}_{T_r} \|_\infty$$

and

$$\begin{aligned} \| \boldsymbol{h}_{T_r} \|_2 &\leq \sqrt{|T_r|} \, \| \boldsymbol{h}_{T_r} \|_\infty \\ &\leq \sqrt{s'} \, \| \boldsymbol{h}_{T_r} \|_\infty \\ &\leq \frac{1}{\sqrt{s'}} \| \boldsymbol{h}_{T_{r-1}} \|_1 \end{aligned} \tag{31}$$

By (30) and (31) we have

$$\| \boldsymbol{h}_{T_j} \|_2 \leq \frac{1}{\sqrt{s'}} \| \boldsymbol{h}_{T_{j-1}} \|_1, \ \ 2 \leq j \leq r \tag{32}$$

We next show

$$\| \boldsymbol{h}_{(T_0 \cup T_1)^c} \|_2 \leq \frac{1}{\sqrt{s'}} \| \boldsymbol{h}_{T_0^c} \|_1 \tag{33}$$

Indeed, this follows from (32) that

$$\begin{aligned} \| \boldsymbol{h}_{(T_0 \cup T_1)^c} \|_2 &= \| \sum_{j \geq 2} h_{T_j} \|_2 \\ &\leq \sum_{j \geq 2} \| \boldsymbol{h}_{T_j} \|_2 \\ &\leq \frac{1}{\sqrt{s'}} \sum_{j \geq 1} \| \boldsymbol{h}_{T_j} \|_1 \\ &= \frac{1}{\sqrt{s'}} \| \sum_{j \geq 1} \boldsymbol{h}_{T_j} \|_1 \\ &= \frac{1}{\sqrt{s'}} \| \boldsymbol{h}_{T_0^c} \|_1 \qquad (34) \end{aligned}$$

Since

$$\begin{aligned} \| \boldsymbol{x} \|_1 &\geq \| \boldsymbol{x}^\star \|_1 \\ &= \| \boldsymbol{x} + \boldsymbol{h} \|_1 \\ &= \| \boldsymbol{x}_{T_0} + \boldsymbol{h}_{T_0} + \boldsymbol{x}_{T_0^c} + \boldsymbol{h}_{T_0^c} \|_1 \\ &= \| \boldsymbol{x}_{T_0} + \boldsymbol{h}_{T_0} \|_1 + \| \boldsymbol{x}_{T_0^c} + \boldsymbol{h}_{T_0^c} \|_1 \\ &\geq \| \boldsymbol{x}_{T_0} \|_1 - \| \boldsymbol{h}_{T_0} \|_1 + \| \boldsymbol{h}_{T_0^c} \|_1 - \| \boldsymbol{x}_{T_0^c} \|_1, \end{aligned}$$

it follows that

$$\begin{aligned} \| \boldsymbol{h}_{T_0^c} \|_1 &\leq \| \boldsymbol{x} \|_1 - \| \boldsymbol{x}_{T_0} \|_1 + \| \boldsymbol{x}_{T_0^c} \|_1 + \| \boldsymbol{h}_{T_0} \|_1 \\ &= \| \boldsymbol{x}_{T_0^c} \|_1 + \| \boldsymbol{x}_{T_0^c} \|_1 + \| \boldsymbol{h}_{T_0} \|_1 \\ &= 2 \| \boldsymbol{x}_{T_0^c} \|_1 + \| \boldsymbol{h}_{T_0} \|_1 \\ &= 2 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + \| \boldsymbol{h}_{T_0} \|_1, \qquad (35) \end{aligned}$$

which implies by (28) that

$$\begin{aligned} \| \boldsymbol{h}_{(T_0 \cup T_1)^c} \|_2 &\leq \frac{1}{\sqrt{s'}} \| \boldsymbol{h}_{T_0^c} \|_1 \\ &\leq \frac{1}{\sqrt{s'}} (\| \boldsymbol{h}_{T_0} \|_1 + 2 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1) \\ &\leq \sqrt{\frac{s}{s'}} \| \boldsymbol{h}_{T_0} \|_2 + \frac{2}{\sqrt{s'}} \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \quad (\textit{by Schwartz inequality}) \\ &\leq \sqrt{\frac{s}{s'}} \| \boldsymbol{h}_{T_0 \cup T_1} \|_2 + \frac{2}{\sqrt{s'}} \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \qquad (36) \end{aligned}$$

Suppose that $s' \le s$. Then we have $\delta_{2s'} \le \delta_{s+s'}$ by the definition of restricted isometry constants. Hence it follows from (22) that for $^\forall j \ge 2$

$$\begin{aligned}
\left|\langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}_{T_j}\rangle\right| &\le \left|\langle A\boldsymbol{h}_{T_0}, A\boldsymbol{h}_{T_j}\rangle\right| + \left|\langle A\boldsymbol{h}_{T_1}, A\boldsymbol{h}_{T_j}\rangle\right| \\
&\le \delta_{s+s'} \parallel \boldsymbol{h}_{T_0} \parallel_2 \parallel \boldsymbol{h}_{T_j} \parallel_2 + \delta_{2s'} \parallel \boldsymbol{h}_{T_1} \parallel_2 \parallel \boldsymbol{h}_{T_j} \parallel_2 \\
&\le \delta_{s+s'} \parallel \boldsymbol{h}_{T_j} \parallel_2 (\parallel \boldsymbol{h}_{T_0} \parallel_2 + \parallel \boldsymbol{h}_{T_1} \parallel_2) \\
&\le \delta_{s+s'} \parallel \boldsymbol{h}_{T_j} \parallel_2 \sqrt{2} \left(\parallel \boldsymbol{h}_{T_0} \parallel_2^2 + \parallel \boldsymbol{h}_{T_1} \parallel_2^2\right)^{\frac{1}{2}} \quad \textit{by } \boldsymbol{h}_{T_0} \perp \boldsymbol{h}_{T_1} \\
&= \sqrt{2}\delta_{s+s'} \parallel \boldsymbol{h}_{T_j} \parallel_2 \parallel \boldsymbol{h}_{T_0} + \boldsymbol{h}_{T_1} \parallel_2 \\
&= \sqrt{2}\delta_{s+s'} \parallel \boldsymbol{h}_{T_j} \parallel_2 \parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2,
\end{aligned}$$

which implies by (24) and (32) that

$$\begin{aligned}
\parallel A\boldsymbol{h}_{T_0\cup T_1} \parallel_2^2 &= \langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h} - \sum_{j\ge 2} A\boldsymbol{h}_{T_j}\rangle \\
&= \langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}\rangle - \langle A\boldsymbol{h}_{T_0\cup T_1}, \sum_{j\ge 2} A\boldsymbol{h}_{T_j}\rangle \\
&\le \parallel A\boldsymbol{h}_{T_0\cup T_1} \parallel_2 \parallel A\boldsymbol{h} \parallel_2 + \sum_{j\ge 2} \left|\langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}_{T_j}\rangle\right| \qquad (37) \\
&\le \sqrt{1+\delta_{s+s'}} \parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2 2\varepsilon \\
&+ \sqrt{2}\delta_{s+s'} \left(\sum_{j\ge 2} \parallel \boldsymbol{h}_{T_j} \parallel_2\right) \parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2 \\
&= \parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2 \left(2\varepsilon\sqrt{1+\delta_{s+s'}} + \sqrt{2}\delta_{s+s'} \sum_{j\ge 2} \parallel \boldsymbol{h}_{T_j} \parallel_2\right) \\
&\le \parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2 \left(2\varepsilon\sqrt{1+\delta_{s+s'}} + \sqrt{2}\delta_{s+s'} \frac{1}{\sqrt{s'}} \sum_{j\ge 1} \parallel \boldsymbol{h}_{T_j} \parallel_1\right) \\
&= \parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2 \left(2\varepsilon\sqrt{1+\delta_{s+s'}} + \sqrt{2}\delta_{s+s'} \frac{1}{\sqrt{s'}} \parallel \boldsymbol{h}_{T_0^c} \parallel_1\right)
\end{aligned}$$

Hence we have

$$\begin{aligned}
(1-\delta_{s+s'}) \parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2^2 &\le \parallel A\boldsymbol{h}_{T_0\cup T_1} \parallel_2^2 \\
&\le \parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2 \left(2\varepsilon\sqrt{1+\delta_{s+s'}} + \sqrt{2}\delta_{s+s'} \frac{1}{\sqrt{s'}} \parallel \boldsymbol{h}_{T_0^c} \parallel_1\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\parallel \boldsymbol{h}_{T_0\cup T_1} \parallel_2 &\le \frac{2\sqrt{1+\delta_{s+s'}}}{1-\delta_{s+s'}}\varepsilon + \frac{\sqrt{2}\delta_{s+s'}}{1-\delta_{s+s'}} \frac{1}{\sqrt{s'}} \parallel \boldsymbol{h}_{T_0^c} \parallel_1 \\
&= \alpha\varepsilon + \frac{\rho}{\sqrt{s'}} \parallel \boldsymbol{h}_{T_0^c} \parallel_1 \qquad (38)
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
\| \boldsymbol{h}_{T_0^c} \|_1 &\leq \| \boldsymbol{h}_{T_0} \|_1 + 2 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \quad (by\ \ (35)) \\
&\leq \sqrt{s} \| \boldsymbol{h}_{T_0} \|_2 + 2 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \\
&\leq \sqrt{s} \| \boldsymbol{h}_{T_0 \cup T_1} \|_2 + 2 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1,
\end{aligned} \tag{39}$$

it follows from (38) that

$$\left(1 - \sqrt{\frac{s}{s'}}\rho\right) \| \boldsymbol{h}_{T_0 \cup T_1} \|_2 \leq \alpha\varepsilon + \frac{2}{\sqrt{s'}}\rho \| \boldsymbol{x} - \boldsymbol{x}_s \|_1$$

Hence, the assumption $\delta_{s+s'} < \frac{1}{1+\sqrt{\frac{2s}{s'}}}$ (iff $1 - \sqrt{\frac{s}{s'}}\rho > 0$), we have

$$\| \boldsymbol{h}_{T_0 \cup T_1} \|_2 \leq \frac{\alpha}{1 - \sqrt{\frac{s}{s'}}\rho}\varepsilon + \frac{2\rho}{\left(1 - \sqrt{\frac{s}{s'}}\rho\right)} \frac{1}{\sqrt{s'}} \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \tag{40}$$

Thus we have by (36) and (40)

$$\begin{aligned}
\| \boldsymbol{x} - \boldsymbol{x}^\star \|_2 = \| \boldsymbol{h} \|_2 &\leq \| \boldsymbol{h}_{T_0 \cup T_1} \|_2 + \| \boldsymbol{h}_{(T_0 \cup T_1)^c} \|_2 \\
&\leq \| \boldsymbol{h}_{T_0 \cup T_1} \|_2 + \sqrt{\frac{s}{s'}} \| \boldsymbol{h}_{T_0 \cup T_1} \|_2 + \frac{2}{\sqrt{s'}} \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \\
&= \left(1 + \sqrt{\frac{s}{s'}}\right) \| \boldsymbol{h}_{T_0 \cup T_1} \|_2 + \frac{2}{\sqrt{s'}} \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \\
&\leq \frac{1 + \sqrt{\frac{s}{s'}}}{1 - \sqrt{\frac{s}{s'}}\rho}\alpha\varepsilon + \frac{2}{\sqrt{s'}} \left(\frac{1 + \rho}{1 - \sqrt{\frac{s}{s'}}\rho}\right) \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \\
&\leq \left(1 + \sqrt{\frac{s}{s'}}\right) \left(\frac{\alpha}{1 - \rho}\right) \varepsilon + \frac{2}{\sqrt{s'}} \left(\frac{1 + \rho}{1 - \rho}\right) \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \\
&\leq 2\sqrt{\frac{s}{s'}} \left(\frac{\alpha}{1 - \rho}\right) \varepsilon + \frac{2}{\sqrt{s}} \left(\sqrt{\frac{s}{s'}} \frac{1 + \rho}{1 - \rho}\right) \| \boldsymbol{x} - \boldsymbol{x}_s \|_1
\end{aligned}$$

Hence, we have

$$\| \boldsymbol{x} - \boldsymbol{x}^\star \|_2 \leq \frac{2}{\sqrt{s}} C_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + \frac{1}{\sqrt{s'}} C_1 \varepsilon$$

Furthermore, we have by (39) and (40),

$$\begin{aligned}
\| \boldsymbol{x} - \boldsymbol{x}^\star \|_1 &= \| \boldsymbol{h}_{T_0} \|_1 + \| \boldsymbol{h}_{T_0^c} \|_1 \\
&\leq 2\sqrt{s} \| \boldsymbol{h}_{T_0 \cup T_1} \|_2 + 2 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \\
&\leq \frac{2\sqrt{s}\alpha}{1 - \sqrt{\frac{s}{s'}}\rho}\varepsilon + \left(\frac{1 + \sqrt{\frac{s}{s'}}\rho}{1 - \sqrt{\frac{s}{s'}}\rho}\right) \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \\
&\leq 2\sqrt{s} \left(\frac{\alpha}{1 - \rho}\right) \varepsilon + \sqrt{\frac{s}{s'}} \left(\frac{1 + \rho}{1 - \rho}\right) \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 \\
&= C_1\varepsilon + C_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1
\end{aligned}$$

This completes the proof in case of $s' \leq s$.

Suppose that $s' \geq s$ and $\delta_{2s'} < \frac{1}{1+\sqrt{\frac{2s}{s'}}}$. Then $\delta_{s+s'} \leq \delta_{2s'}$ and the assumption $\delta_{2s'} < \frac{1}{1+\sqrt{\frac{2s}{s'}}}$ implies that $\left(1 - \sqrt{\frac{s}{s'}}\rho\right) > 0$. Hence we can prove (11) and (12) at the same way as the case $s' \leq s$. This completes the proof of Theorem 1.3.

Proof of Theorem 1.4. Using (22) on (37) directly, we can obtain the inequality:

$$\begin{aligned} \| A\boldsymbol{h}_{T_0\cup T_1} \|_2^2 &\leq \| A\boldsymbol{h}_{T_0\cup T_1} \|_2 \| A\boldsymbol{h} \|_2 + \sum_{j\geq 2} \left|\langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}_{T_j}\rangle\right| \\ &\leq \sqrt{1+\delta_{s+s'}} \| \boldsymbol{h}_{T_0\cup T_1} \|_2 \, 2\varepsilon + \sum_{j\geq 2} \delta_{s+2s'} \| \boldsymbol{h}_{T_0\cup T_1} \|_2 \| \boldsymbol{h}_{T_j} \|_2 \\ &= \| \boldsymbol{h}_{T_0\cup T_1} \|_2 \left(2\varepsilon\sqrt{1+\delta_{s+s'}} + \delta_{s+2s'} \sum_{j\geq 2} \| \boldsymbol{h}_{T_j} \|_2 \right), \end{aligned}$$

and by (32), (35), (36), (39) and (40)

$$\| \boldsymbol{x} - \boldsymbol{x}^\star \|_2 \leq \frac{\sqrt{1+\frac{s}{s'}}}{1-\sqrt{\frac{s}{s'}}\gamma}\alpha\varepsilon + \frac{2}{\sqrt{s'}}\left(\frac{1+\gamma}{1-\sqrt{\frac{s}{s'}}\gamma}\right) \| \boldsymbol{x} - \boldsymbol{x}_s \|_1,$$

which implies that

$$\| \boldsymbol{x} - \boldsymbol{x}^\star \|_2 \leq \frac{2}{\sqrt{s}} D_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + \frac{1}{\sqrt{s'}} D_1 \varepsilon,$$

and then

$$\| \boldsymbol{x} - \boldsymbol{x}^\star \|_1 \leq D_0 \| \boldsymbol{x} - \boldsymbol{x}_s \|_1 + D_1\varepsilon$$

This completes the proof.

3 Discussions

By taking appropriate numbers s' for s, we shall search good conditions under which Theorem 1.3 and Theorem 1.4 hold and under which the Candès results are improved.

(1) In Theorem 1.3 and Theorem 1.4, taking $s' = s$, the assumption (9) coincides with the assumption

$$\delta_{2s} < \sqrt{2} - 1 \tag{41}$$

in Theorem 1.1 and Theorem 1.2, and the assumption (16) coincides with the assumption

$$\delta_{2s} + \delta_{3s} < 1 \tag{42}$$

in the Candès. [2].

(2) Let $s = 2k,\ k \in \boldsymbol{N}$. Taking $s' = k$, the assumption (9) becomes

$$\delta_{\frac{3}{2}s} < \frac{1}{3} \tag{43}$$

and the assumption (16) becomes

$$\delta_{\frac{3}{2}s} + \sqrt{2}\delta_{2s} < 1 \tag{44}$$

Since $\delta_{\frac{3}{2}s} + \sqrt{2}\delta_{2s} < \left(1 + \sqrt{2}\right)\delta_{2s}$, this condition is better than the condition (41) in Theorem 1.1 and Theorem 1.2.

(3) Let $s = 2k+1,\ k \in \boldsymbol{N}$. Taking $s' = k+1$, Theorem 1.3 holds under the assumption

$$\delta_{\frac{3s+1}{2}} < \frac{1}{3} \tag{45}$$

and Theorem 1.4 holds under the assumption

$$\delta_{\frac{3s+1}{2}} + \sqrt{2}\delta_{2s+1} < 1 \tag{46}$$

(4) We take $s' = 2s$. Then, Theorem 1.3 holds under the assumption

$$\delta_{4s} < \frac{1}{2} \tag{47}$$

and Theorem 1.4 holds under

$$\delta_{3s} + \frac{1}{\sqrt{2}}\delta_{5s} < 1 \tag{48}$$

(5) Let $s = 2k,\ k \in \boldsymbol{N}$. Taking $s' = 3k$, Theorem 1.3 holds under

$$\delta_{3s} < \frac{1}{1 + \sqrt{\frac{2}{3}}} \tag{49}$$

and Theorem 1.4 holds under

$$\delta_{\frac{5}{2}s} + \sqrt{\frac{2}{3}}\delta_{4s} < 1 \tag{50}$$

References

[1] Baraniuk, R., Davenport, M., De Vore, R. and Wakin, M. (2008). A Simple Proof of the Restricted Isometry Property for Random Matrices. *Constructive Approximation* **28**(3),253–263.

[2] Candès, E. (2006). Compressive sampling. *Proceeding of the International Congress of Mathematicians, Madrid, Spain.*

[3] Candès, E. (2008). The Restricted Isometry Property and Its Implication for Compressed Sensing. *Comptes Rendus Mathematique* **346**(9-10),589–592.

[4] Candès, E. and Wakin, M. (2008). An introduction to compressive sampling. *IEEE Signal Processing Magazine* **March 2008**, 21–30.

[5] Donoho, D.L. (2006). Compressed sensing. *IEEE Trans Inform Theory* **52**,1289–1306.

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