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出版情報：MI Preprint Series. 2011-2, 2011-02-15. 九州大学大学院数理学研究院
バージョン：
権利関係：

MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

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MI 2011-2

(Received February 15, 2011)

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Abstract

Copulas have lately attracted much attention as a tool for dealing with multiple risks that cannot be considered independent. The normal copula, widely used in practice, is known to have the same tail dependence parameter as the product copula. The present paper brings into question the common interpretation of this fact as evidence that the normal copula lacks tail dependence, both by providing numerical examples and by mathematically determining the asymptotic behaviour of the tail dependence.

Keywords: copula, normal copula, tail dependence.
JEL: C16.
2010 MSC: 62H20 (primary), 62P05 (secondary).

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1. Introduction

1.1. Copulas

Copulas have gained increasing popularity in risk management as a tool for investigating dependent risks. We begin by reviewing rudimentary definitions and facts on copulas. See Nelsen (2006) for further reference.

Definition 1.1. A *copula* is $C\colon [0,1]^2 \to [0,1]$ with the following properties:

(1) $C(u,0) = C(0,v) = 0$, $C(u,1) = u$, and $C(1,v) = v$ for all $u, v \in [0,1]$;

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(2) if $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$, then $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0$.

Example 1.2. The function $C(u, v) = uv$ is a copula and called the *product copula.*

For a bivariate random variable (X, Y), let F_X and F_Y denote the marginal distribution functions and let $F_{X,Y}$ denote the joint distribution function: $F_X(x) = P(X \leq x)$, $F_Y(y) = P(Y \leq y)$, and $F_{X,Y}(x, y) = P(X \leq x,\ Y \leq y)$ for $x, y \in \mathbb{R}$. We say that (X, Y) is *continuous* if F_X and F_Y are both continuous.

Theorem 1.3 (Sklar). *If (X, Y) is a continuous bivariate random variable, then there exists a unique copula $C_{X,Y}$ such that*

$$F_{X,Y}(x, y) = C_{X,Y}\big(F_X(x), F_Y(y)\big)$$

for all $x, y \in \mathbb{R}$.

Example 1.4. The independence of X and Y is equivalent to $C_{X,Y}$ being the product copula.

Remark 1.5. If we write $F^{-1}(u) = \inf\{x \in \mathbb{R} \mid F(x) \geq u\}$ for univariate distribution functions F, we have $C_{X,Y}(u, v) = F_{X,Y}\big(F_X^{-1}(u), F_Y^{-1}(v)\big)$.

In this paper, the focus will be on the *normal copula*:

Definition 1.6. Let $-1 < \rho < 1$. If (X, Y) is a normally distributed bivariate random variable such that $E[X] = E[Y] = 0$, $V(X) = V(Y) = 1$, and $\mathrm{Cov}(X, Y) = \rho$, then $C_{X,Y}$ is called the *normal copula* (or *Gaussian copula*) with correlation ρ and denoted by C_ρ.

1.2. Tail dependence of copulas

Definition 1.7. Let C be a copula. We define $\lambda_C \colon (0, 1) \to [0, 1]$ by

$$\lambda_C(t) = \frac{1 - 2t + C(t, t)}{1 - t}.$$

We call $\lim_{t \nearrow 1} \lambda_C(t)$ the *upper tail dependence parameter* of C, if it exists.

	Product copula	Normal copula C_ρ with $\rho = 0.5$
$t = 0.8$	0.2000	0.4358
$t = 0.9$	0.1000	0.3240
$t = 0.95$	0.0500	0.2438
$t = 0.99$	0.0100	0.1294
$t = 0.995$	0.0050	0.0993
$t = 0.999$	0.0010	0.0543

Table 1: Upper tail dependence $\lambda_C(t)$ of the product and normal copulas

Remark 1.8. If (X, Y) is a continuous bivariate random variable, then

$$\begin{aligned}\lambda_{C_{X,Y}}(t) &= \frac{1 - P\big(X \le F_X^{-1}(t)\big) - P\big(Y \le F_Y^{-1}(t)\big) + P\big(X \le F_X^{-1}(t),\ Y \le F_Y^{-1}(t)\big)}{1 - P\big(X \le F_X^{-1}(t)\big)} \\ &= \frac{P\big(X > F_X^{-1}(t),\ Y > F_Y^{-1}(t)\big)}{P\big(X > F_X^{-1}(t)\big)} \\ &= P\big(Y > F_Y^{-1}(t) \bigm| X > F_X^{-1}(t)\big).\end{aligned}$$

Example 1.9. If C is the product copula, then $\lambda_C(t) = 1 - t \to 0$ as $t \nearrow 1$.

The normal copula is known to have upper tail dependence parameter 0:

Proposition 1.10. *The normal copula with arbitrary correlation $\rho \in (-1, 1)$ has upper tail dependence parameter* 0*.*

This proposition, with Example 1.9 in mind, is often interpreted to mean that the normal copula exhibits no tail dependence. However, Table 1 suggests that the product and normal copulas have different rates at which $\lambda_C(t)$ converges to 0. The purpose of this paper is to completely describe how $\lambda_{C_\rho}(t)$ converges to 0.

Now we state a particular case of our main theorem, of which the complete statement will be given in Section 2 (Theorem 2.3).

Theorem 1.11. *We have*

$$\lambda_{C_\rho}(t) = \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, e^{-\frac{1-\rho}{2(1+\rho)}s^2} \left(s^{-1} - \frac{1 + 2\rho - \rho^2}{1-\rho} s^{-3} + O(s^{-5}) \right)$$

as $t \nearrow 1$, where $s = \Phi^{-1}(t) \nearrow \infty$, with Φ denoting the distribution function of the standard normal distribution: $t = \Phi(s) = (2\pi)^{-1/2} \int_{-\infty}^{s} \exp(-x^2/2)\, dx$.

Remark 1.12. By using Proposition 3.2, we may infer from Theorem 1.11 that the leading behaviour of $\lambda_{C_\rho}(t)$ is

$$\begin{aligned}\lambda_{C_\rho}(t) &\sim \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, e^{-\frac{1-\rho}{2(1+\rho)}s^2} s^{-1} \\ &\sim (4\pi)^{-\frac{\rho}{1+\rho}} \sqrt{\frac{(1+\rho)^3}{1-\rho}} (1-t)^{\frac{1-\rho}{1+\rho}} \big(-\log(1-t)\big)^{-\frac{\rho}{1+\rho}}.\end{aligned}$$

Note that Heffernan (2000) mentions the order $(1-t)^{(1-\rho)/(1+\rho)}$ in a different language.

2. Precise statement of the main theorem

This section is devoted to giving the precise statement of our main theorem. Henceforth we fix a real number ρ with $-1 < \rho < 1$ and denote $\lambda_{C_\rho}(t)$ simply by $\lambda(t)$.

Definition 2.1. We define sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of real numbers by

$$\begin{aligned} a_n &= (-1)^n n!(1+\rho)^n \sum_{l=0}^{n} \frac{(2l-1)!!}{l!}(1-\rho)^{-l}, \\ b_n &= (-1)^n (2n-1)!! \end{aligned}$$

where $(-1)!! = 1$ by definition. We further define a sequence $(c_n)_{n\geq 0}$ of real numbers by the following equation between formal power series in X:

$$\sum_{n=0}^{\infty} c_n X^n = \frac{\sum_{n=0}^{\infty} a_n X^n}{\sum_{n=0}^{\infty} b_n X^n} \in \mathbb{R}[[X]].$$

In other words, we define $(c_n)_{n\geq 0}$ recursively by setting $c_0 = a_0/b_0$ and

$$c_n = \frac{1}{b_0}\left(a_n - \sum_{k=0}^{n-1} b_{n-k} c_k\right)$$

for $n \geq 1$.

Example 2.2. The first three terms of the sequences are as follows:

$$
\begin{aligned}
&a_0 = 1, \quad a_1 = -(1+\rho)\left(1+\frac{1}{1-\rho}\right), && a_2 = (1+\rho)^2\left(2+\frac{2}{1-\rho}+\frac{3}{(1-\rho)^2}\right),\\
&b_0 = 1, \quad b_1 = -1, && b_2 = 3,\\
&c_0 = 1, \quad c_1 = -\frac{1+2\rho-\rho^2}{1-\rho}, && c_2 = \frac{3+13\rho-3\rho^2-3\rho^3+2\rho^4}{(1-\rho)^2}.
\end{aligned}
$$

Now our main theorem goes as follows:

Theorem 2.3 (Main Theorem). *For every positive integer N, we have*

$$
\lambda(t) = \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, e^{-\frac{1-\rho}{2(1+\rho)}s^2}\left(\sum_{n=0}^{N-1} c_n s^{-2n-1} + O(s^{-2N-1})\right)
$$

as $t \nearrow 1$, where $s = \Phi^{-1}(t) \nearrow \infty$.

Remark 2.4. Theorem 1.11 is the $N = 2$ case of our main theorem.

3. Proof of the main theorem

Let $1/2 < t < 1$ and put $s = \Phi^{-1}(t) > 0$. If we set

$$
\begin{aligned}
A &= \int_s^\infty \int_s^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right) dx\, dy,\\
B &= \int_s^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx,
\end{aligned}
$$

then $\lambda(t) = A/B$ by Remark 1.8. We shall estimate A and B separately.

Let $\mathbb{R}_+$, $\mathbb{N}_0$, and $\mathbb{N}$ denote the sets of positive real numbers, nonnegative integers, and positive integers, respectively.

3.1. Estimate of B

Proposition 3.1. *If $\theta \in \mathbb{R}_+$ and $N \in \mathbb{N}$, then*

$$
(-1)^N \int_\theta^\infty e^{-x^2/2}\, dx > (-1)^N e^{-\theta^2/2} \sum_{n=0}^{N-1} b_n \theta^{-2n-1}.
$$

PROOF. For $n \in \mathbb{N}_0$, set

$$I_n = \int_\theta^\infty x^{-n} e^{-x^2/2}\, dx.$$

Then the left-hand side of the required inequality is $(-1)^N I_0$.

Since integration by parts gives

$$\begin{aligned} I_n &= -\int_\theta^\infty x^{-n-1} (e^{-x^2/2})'\, dx \\ &= -[x^{-n-1} e^{-x^2/2}]_\theta^\infty + \int_\theta^\infty (-n-1) x^{-n-2} e^{-x^2/2}\, dx \\ &= \theta^{-n-1} e^{-\theta^2/2} - (n+1) I_{n+2}, \end{aligned}$$

we have

$$\begin{aligned} (-1)^N e^{-\theta^2/2} \sum_{n=0}^{N-1} b_n \theta^{-2n-1} &= \sum_{n=0}^{N-1} (-1)^{N+n} (2n-1)!! \theta^{-2n-1} e^{-\theta^2/2} \\ &= \sum_{n=0}^{N-1} (-1)^{N+n} (2n-1)!! \big(I_{2n} + (2n+1) I_{2n+2} \big) \\ &= \sum_{n=0}^{N-1} \big((-1)^{N+n} (2n-1)!! I_{2n} - (-1)^{N+n+1} (2n+1)!! I_{2n+2} \big) \\ &= (-1)^N I_0 - (2N-1)!! I_{2N} \\ &< (-1)^N I_0. \end{aligned}$$

Proposition 3.2. *For every $N \in \mathbb{N}$, we have*

$$B = \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \left(\sum_{n=0}^{N-1} b_n s^{-2n-1} + O(s^{-2N-1}) \right)$$

as $s \nearrow \infty$.

PROOF. If N' is an even integer with $N' \geq N$, then Proposition 3.1 shows that

$$B > \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \sum_{n=0}^{N'-1} b_n s^{-2n-1} = \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \left(\sum_{n=0}^{N-1} b_n s^{-2n-1} + O(s^{-2N-1}) \right).$$

By taking N' to be an odd integer with $N' \geq N$, we may similarly obtain

$$B < \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \left(\sum_{n=0}^{N-1} b_n s^{-2n-1} + O(s^{-2N-1}) \right).$$

The proposition follows from these estimates.

3.2. Estimate of A

Definition 3.3. We set $\alpha = \sqrt{(1-\rho)/2}$ and $\beta = \sqrt{(1+\rho)/2}$, so that α and β are positive real numbers with $\alpha^2 + \beta^2 = 1$.

Lemma 3.4. *We have*

$$A = \frac{\beta}{\pi} e^{-s^2/2} \int_{\alpha s/\beta}^{\infty} \left(\int_{\alpha w + \beta s}^{\infty} e^{-z^2/2}\, dz \right) e^{(\alpha w + \beta s)^2/2} e^{-w^2/2}\, dw.$$

Proof. Symmetry gives

$$\begin{aligned} A &= 2 \iint_{x \geq y \geq s} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right) dx\, dy \\ &= \frac{1}{2\pi\alpha\beta} \iint_{x \geq y \geq s} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right) dx\, dy. \end{aligned}$$

We use the change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \beta z + \alpha\beta w - \alpha^2 s \\ \beta z - \alpha\beta w + \alpha^2 s \end{pmatrix} \iff \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} (x+y)/2\beta \\ (x-y)/2\alpha\beta + \alpha s/\beta \end{pmatrix}.$$

Since

$$\begin{aligned} x \geq y \geq s &\iff \beta z + \alpha\beta w - \alpha^2 s \geq \beta z - \alpha\beta w + \alpha^2 s \geq s \\ &\iff w \geq \alpha s/\beta,\ z \geq \alpha w + \beta s \end{aligned}$$

and

$$\begin{aligned} \frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} &= \frac{(x+y)^2}{4(1+\rho)} + \frac{(x-y)^2}{4(1-\rho)} = \frac{(2\beta z)^2}{8\beta^2} + \frac{(2\alpha\beta w - 2\alpha^2 s)^2}{8\alpha^2} \\ &= \frac{z^2}{2} + \frac{(\beta w - \alpha s)^2}{2} = \frac{z^2}{2} - \frac{(\alpha w + \beta s)^2}{2} + \frac{w^2 + s^2}{2}, \end{aligned}$$

we have

$$A = \frac{1}{2\pi\alpha\beta} \int_{\alpha s/\beta}^{\infty} \int_{\alpha w+\beta s}^{\infty} \exp\left(-\frac{z^2}{2} + \frac{(\alpha w + \beta s)^2}{2} - \frac{w^2 + s^2}{2} \right) \left| \det \begin{pmatrix} \beta & \alpha\beta \\ \beta & -\alpha\beta \end{pmatrix} \right| dz\, dw$$
$$= \frac{\beta}{\pi} e^{-s^2/2} \int_{\alpha s/\beta}^{\infty} \left(\int_{\alpha w+\beta s}^{\infty} e^{-z^2/2}\, dz \right) e^{(\alpha w+\beta s)^2/2} e^{-w^2/2}\, dw.$$

Lemma 3.5. *For every $K \in \mathbb{N}$, we have*

$$(-1)^K A > (-1)^K \frac{\beta}{\pi} e^{-s^2/2} \sum_{n=0}^{K-1} b_n \int_{\alpha s/\beta}^{\infty} (\alpha w + \beta s)^{-2n-1} e^{-w^2/2}\, dw.$$

Proof. Lemma 3.4 and Proposition 3.1 show that

$$\begin{aligned} (-1)^K A &= \frac{\beta}{\pi} e^{-s^2/2} \int_{\alpha s/\beta}^{\infty} \left((-1)^K \int_{\alpha w+\beta s}^{\infty} e^{-z^2/2}\, dz \right) e^{(\alpha w+\beta s)^2/2} e^{-w^2/2}\, dw \\ &> \frac{\beta}{\pi} e^{-s^2/2} \int_{\alpha s/\beta}^{\infty} (-1)^K \left(\sum_{n=0}^{K-1} b_n (\alpha w + \beta s)^{-2n-1} \right) e^{-w^2/2}\, dw \\ &= (-1)^K \frac{\beta}{\pi} e^{-s^2/2} \sum_{n=0}^{K-1} b_n \int_{\alpha s/\beta}^{\infty} (\alpha w + \beta s)^{-2n-1} e^{-w^2/2}\, dw. \end{aligned}$$

Definition 3.6. For $n \in \mathbb{N}$ and $j, k \in \mathbb{N}_0$ with $j \leq k$, we define

$$r_{j,k,n} = \frac{(2k-j)!(n+j-1)!}{(2k-2j)!!j!(n-1)!}.$$

Lemma 3.7. *If $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, then we have the following:*

(1) $r_{0,k+1,n} = r_{0,k,n}(2k+1)$.
(2) $r_{k+1,k+1,n} = r_{k,k,n}(n+k)$.
(3) $r_{j,k+1,n} = r_{j,k,n}(2k-j+1) + r_{j-1,k,n}(n+j-1)$ *for* $j = 1, \ldots, k$.

Proof. (1) We have

$$r_{0,k,n}(2k+1) = \frac{(2k)!}{(2k)!!}(2k+1) = \frac{(2k+2)!}{(2k+2)!!} = r_{0,k+1,n}.$$

(2) We have

$$r_{k,k,n}(n+k) = \frac{(n+k-1)!}{(n-1)!}(n+k) = \frac{(n+k)!}{(n-1)!} = r_{k+1,k+1,n}.$$

(3) We have

$$\begin{aligned}
&r_{j,k,n}(2k-j+1) + r_{j-1,k,n}(n+j-1) \\
&\quad = \frac{(2k-j)!(n+j-1)!}{(2k-2j)!!j!(n-1)!}(2k-j+1) \\
&\qquad\quad + \frac{(2k-j+1)!(n+j-2)!}{(2k-2j+2)!!(j-1)!(n-1)!}(n+j-1) \\
&\quad = \frac{(2k-j+1)!(n+j-1)!}{(2k-2j+2)!!j!(n-1)!}\big((2k-2j+2)+j\big) \\
&\quad = \frac{(2k-j+2)!(n+j-1)!}{(2k-2j+2)!!j!(n-1)!} \\
&\quad = r_{j,k+1,n}.
\end{aligned}$$

Lemma 3.8. *If $n, K \in \mathbb{N}$, then*

$$\begin{aligned}
&(-1)^K \int_{\alpha s/\beta}^{\infty} (\alpha w + \beta s)^{-n} e^{-w^2/2}\, dw \\
&\quad > (-1)^K \sum_{0 \le j \le k \le K-1} (-1)^k r_{j,k,n} \alpha^{-2k+2j-1} \beta^{n+2k+1} s^{-n-2k-1} e^{-\alpha^2 s^2/2\beta^2}.
\end{aligned}$$

Proof. Put $u = s/\beta$ for simplicity. For $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, set

$$I_{m,n} = \int_{\alpha s/\beta}^{\infty} w^{-m}(\alpha w + \beta s)^{-n} e^{-w^2/2}\, dw = \int_{\alpha u}^{\infty} w^{-m}(\alpha w + \beta^2 u)^{-n} e^{-w^2/2}\, dw.$$

Then what we need to show is that

$$(-1)^K I_{0,n} > (-1)^K \sum_{k=0}^{K-1} (-1)^k \sum_{j=0}^{k} r_{j,k,n} \alpha^{-2k+2j-1} u^{-n-2k-1} e^{-\alpha^2 u^2/2}.$$

Since integration by parts gives

$$
\begin{aligned}
I_{m,n} &= -\int_{\alpha u}^{\infty} w^{-m-1}(\alpha w+\beta^2 u)^{-n}(e^{-w^2/2})'\,dw \\
&= -\big[w^{-m-1}(\alpha w+\beta^2 u)^{-n}e^{-w^2/2}\big]_{\alpha u}^{\infty} \\
&\quad + \int_{\alpha u}^{\infty}\big((-m-1)w^{-m-2}(\alpha w+\beta^2 u)^{-n} + w^{-m-1}(-\alpha n)(\alpha w+\beta^2 u)^{-n-1}\big)e^{-w^2/2}\,dw \\
&= \alpha^{-m-1}u^{-m-n-1}e^{-\alpha^2u^2/2} - (m+1)I_{m+2,n} - \alpha n I_{m+1,n+1},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\sum_{j=0}^{k} r_{j,k,n}\alpha^j(\alpha^{-2k+j-1}u^{-n-2k-1}e^{-\alpha^2u^2/2} - I_{2k-j,n+j}) \\
&\quad = \sum_{j=0}^{k} r_{j,k,n}\alpha^j\big((2k-j+1)I_{2k-j+2,n+j} + \alpha(n+j)I_{2k-j+1,n+j+1}\big) \\
&\quad = \sum_{j=0}^{k} r_{j,k,n}\alpha^j(2k-j+1)I_{2k-j+2,n+j} + \sum_{j=1}^{k+1} r_{j-1,k,n}\alpha^j(n+j-1)I_{2k-j+2,n+j} \\
&\quad = \sum_{j=0}^{k+1} r_{j,k+1,n}\alpha^j I_{2k-j+2,n+j}
\end{aligned}
$$

by Lemma 3.7. It follows that

$$
\begin{aligned}
&(-1)^K \sum_{k=0}^{K-1} (-1)^k \sum_{j=0}^{k} r_{j,k,n} \alpha^{-2k+2j-1} u^{-n-2k-1} e^{-\alpha^2 u^2/2} \\
&\quad = (-1)^K \sum_{k=0}^{K-1} (-1)^k \left(\sum_{j=0}^{k} r_{j,k,n} \alpha^j I_{2k-j,n+j} + \sum_{j=0}^{k+1} r_{j,k+1,n} \alpha^j I_{2k-j+2,n+j} \right) \\
&\quad = (-1)^K \sum_{k=0}^{K-1} \left((-1)^k \sum_{j=0}^{k} r_{j,k,n} \alpha^j I_{2k-j,n+j} - (-1)^{k+1} \sum_{j=0}^{k+1} r_{j,k+1,n} \alpha^j I_{2k-j+2,n+j} \right) \\
&\quad = (-1)^K \left(r_{0,0,n} I_{0,n} - (-1)^K \sum_{j=0}^{K} r_{j,K,n} \alpha^j I_{2K-j,n+j} \right) \\
&\quad = (-1)^K I_{0,n} - \sum_{j=0}^{K} r_{j,K,n} \alpha^j I_{2K-j,n+j} \\
&\quad < (-1)^K I_{0,n}.
\end{aligned}
$$

Lemma 3.9. *If l and m are integers with $0 \le l \le m$, then*

$$\sum_{n=0}^{m-l} \frac{(m+l-n)!(m-l+n)!}{(m-l-n)!(2n)!!} = (2l-1)!!(2m)!!.$$

Proof. For each $l \in \mathbb{N}_0$, let P_l be the statement that the lemma is true for all $m \ge l$. We shall prove P_l by induction on l.

To establish P_0, we need to prove that

$$\sum_{n=0}^{m} \frac{(m+n)!}{(2n)!!} = (2m)!!$$

for all $m \ge 0$. If $m = 0$, then both sides are 1. Suppose that equality holds for m. Then

$$\begin{aligned}\sum_{n=0}^{m+1}\frac{(m+n+1)!}{(2n)!!} &= \sum_{n=0}^{m+1}\frac{(m+n)!}{(2n)!!}(m+n+1)\\ &= (m+1)\sum_{n=0}^{m+1}\frac{(m+n)!}{(2n)!!}+\frac{1}{2}\sum_{n=1}^{m+1}\frac{(m+n)!}{(2n-2)!!}\\ &= (m+1)\left((2m)!!+\frac{(2m+1)!}{(2m+2)!!}\right)+\frac{1}{2}\sum_{n=0}^{m}\frac{(m+n+1)!}{(2n)!!}\\ &= \frac{(2m+2)!!}{2}+\frac{1}{2}\sum_{n=0}^{m+1}\frac{(m+n+1)!}{(2n)!!},\end{aligned}$$

from which it follows that

$$\sum_{n=0}^{m+1}\frac{(m+n+1)!}{(2n)!!}=(2m+2)!!.$$

Therefore equality holds for $m+1$ as well. Hence P_0 has been verified.

Now suppose that P_l is true. Let $m \geq l+1$. Since

$$\begin{aligned}&\frac{(m-l+n+1)!}{(m-l-n+1)!}-\frac{(m-l+n-1)!}{(m-l-n-1)!}\\ &\quad=\frac{(m-l+n-1)!}{(m-l-n+1)!}\big((m-l+n)(m-l+n+1)-(m-l-n)(m-l-n+1)\big)\\ &\quad=\frac{(m-l+n-1)!}{(m-l-n+1)!}\cdot 2n(2m-2l+1)\end{aligned}$$

for $0 \leq n \leq m-l-1$, we have

$$\begin{aligned}&\sum_{n=0}^{m-l-1}\frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n-1)!(2n)!!}\\ &\quad=\sum_{n=0}^{m-l-1}\frac{(m+l-n+1)!(m-l+n+1)!}{(m-l-n+1)!(2n)!!}\\ &\qquad-\sum_{n=0}^{m-l-1}\frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n+1)!(2n)!!}2n(2m-2l+1).\end{aligned}$$

The inductive hypothesis shows that

$$\sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n+1)!}{(m-l-n+1)!(2n)!!}$$
$$= \sum_{n=0}^{m-l+1} \frac{(m+l-n+1)!(m-l+n+1)!}{(m-l-n+1)!(2n)!!} - \frac{(2l+1)!(2m-2l+1)!}{1!(2m-2l)!!} - \frac{(2l)!(2m-2l+2)!}{0!(2m-2l+2)!!}$$
$$= (2l-1)!!(2m+2)!! - \frac{(2l+1)!(2m-2l+1)!}{(2m-2l)!!} - \frac{(2l)!(2m-2l+1)!}{(2m-2l)!!}$$

and that

$$\sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n+1)!(2n)!!} 2n(2m-2l+1)$$
$$= (2m-2l+1) \sum_{n=1}^{m-l-1} \frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n+1)!(2n-2)!!}$$
$$= (2m-2l+1) \sum_{n=0}^{m-l-2} \frac{(m+l-n)!(m-l+n)!}{(m-l-n)!(2n)!!}$$
$$= (2m-2l+1) \left((2l-1)!!(2m)!! - \frac{(2l+1)!(2m-2l-1)!}{1!(2m-2l-2)!!} - \frac{(2l)!(2m-2l)!}{0!(2m-2l)!!} \right)$$
$$= (2l-1)!!(2m+2)!! - (2l+1)!!(2m)!! - \frac{(2l+1)!(2m-2l+1)!}{(2m-2l)!!} - \frac{(2l)!(2m-2l+1)!}{(2m-2l)!!}.$$

Therefore we have

$$\sum_{n=0}^{m-l-1} \frac{(m+l-n+1)!(m-l+n-1)!}{(m-l-n-1)!(2n)!!} = (2l+1)!!(2m)!!,$$

as required.

Proposition 3.10. *For every $N \in \mathbb{N}$, we have*

$$A = \frac{1}{2\pi} \sqrt{\frac{(1+\rho)^3}{1-\rho}}\, e^{-s^2/2} e^{-\frac{1-\rho}{2(1+\rho)} s^2} \left(\sum_{n=0}^{N-1} a_n s^{-2n-2} + O(s^{-2N-2}) \right).$$

Proof. Lemmas 3.5 and 3.8 show that

$$
\begin{aligned}
(-1)^K A &> (-1)^K \frac{\beta}{\pi} e^{-s^2/2} \sum_{n=0}^{K-1} b_n \int_{\alpha s/\beta}^{\infty} (\alpha w + \beta s)^{-2n-1} e^{-w^2/2}\, dw \\
&> (-1)^K \frac{\beta}{\pi} e^{-s^2/2} e^{-\alpha^2 s^2/2\beta^2} \\
&\quad \times \sum_{\substack{0\le n\le K-1\\ 0\le j\le k\le K-1}} (-1)^{n+k}(2n-1)!!\, r_{j,k,2n+1} \alpha^{-2k+2j-1} \beta^{2n+2k+2} s^{-2n-2k-2}
\end{aligned}
$$

for every $K \in \mathbb{N}$.

Now let $N \in \mathbb{N}$. If $K \ge N$, then

$$
\begin{aligned}
&\sum_{\substack{0\le n\le K-1\\ 0\le j\le k\le K-1}} (-1)^{n+k}(2n-1)!!\, r_{j,k,2n+1} \alpha^{-2k+2j-1} \beta^{2n+2k+2} s^{-2n-2k-2} \\
&= \sum_{m=0}^{N-1} (-1)^m \beta^{2m+2} s^{-2m-2} \sum_{\substack{n\ge 0,\ 0\le j\le k\\ n+k=m}} (2n-1)!!\, r_{j,k,2n+1} \alpha^{-2k+2j-1} + O(s^{-2N-2}) \\
&= \sum_{m=0}^{N-1} (-1)^m \beta^{2m+2} s^{-2m-2} \sum_{l=0}^{m} \alpha^{-2l-1} \sum_{n=0}^{m-l} (2n-1)!!\, r_{m-l-n,m-n,2n+1} + O(s^{-2N-2}) \\
&= \sum_{m=0}^{N-1} (-1)^m \beta^{2m+2} s^{-2m-2} \sum_{l=0}^{m} \frac{1}{(2l)!!} \alpha^{-2l-1} \sum_{n=0}^{m-l} \frac{(m+l-n)!(m-l+n)!}{(m-l-n)!(2n)!!} + O(s^{-2N-2}) \\
&= \sum_{m=0}^{N-1} (-1)^m (2m)!! \left(\sum_{l=0}^{m} \frac{(2l-1)!!}{(2l)!!} \alpha^{-2l-1} \right) \beta^{2m+2} s^{-2m-2} + O(s^{-2N-2})
\end{aligned}
$$

by Lemma 3.9, and so

$$\begin{aligned}(-1)^K A &> (-1)^K \frac{\beta}{\pi} e^{-s^2/2} e^{-\alpha^2 s^2/2\beta^2} \\ &\quad \times \left(\sum_{m=0}^{N-1} (-1)^m (2m)!! \left(\sum_{l=0}^{m} \frac{(2l-1)!!}{(2l)!!} \alpha^{-2l-1} \right) \beta^{2m+2} s^{-2m-2} + O(s^{-2N-2}) \right) \\ &= (-1)^K \frac{1}{2\pi} \sqrt{\frac{(1+\rho)^3}{1-\rho}}\, e^{-s^2/2} e^{-\frac{1-\rho}{2(1+\rho)} s^2} \\ &\quad \times \left(\sum_{m=0}^{N-1} (-1)^m \frac{(2m)!!}{2^m} (1+\rho)^m \left(\sum_{l=0}^{m} \frac{(2l-1)!!}{(2l)!!/2^l} (1-\rho)^{-l} \right) s^{-2m-2} + O(s^{-2N-2}) \right) \\ &= (-1)^K \frac{1}{2\pi} \sqrt{\frac{(1+\rho)^3}{1-\rho}}\, e^{-s^2/2} e^{-\frac{1-\rho}{2(1+\rho)} s^2} \left(\sum_{m=0}^{N-1} a_m s^{-2m-2} + O(s^{-2N-2}) \right).\end{aligned}$$

By taking an odd K and an even K, we may obtain the proposition.

3.3. Proof of the main theorem

Proof (of Theorem 2.3). By Propositions 3.2 and 3.10, we have

$$\begin{aligned}\lambda(t) = \frac{A}{B} &= \frac{\frac{1}{2\pi}\sqrt{\frac{(1+\rho)^3}{1-\rho}}\, e^{-s^2/2} e^{-\frac{1-\rho}{2(1+\rho)}s^2} \left(\sum_{n=0}^{N-1} a_n s^{-2n-2} + O(s^{-2N-2}) \right)}{\frac{1}{\sqrt{2\pi}} e^{-s^2/2} \left(\sum_{n=0}^{N-1} b_n s^{-2n-1} + O(s^{-2N-1}) \right)} \\ &= \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, s^{-1} e^{-\frac{1-\rho}{2(1+\rho)}s^2} \frac{\sum_{n=0}^{N-1} a_n s^{-2n} + O(s^{-2N})}{\sum_{n=0}^{N-1} b_n s^{-2n} + O(s^{-2N})} \\ &= \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, s^{-1} e^{-\frac{1-\rho}{2(1+\rho)}s^2} \left(\sum_{n=0}^{N-1} c_n s^{-2n} + O(s^{-2N}) \right) \\ &= \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, e^{-\frac{1-\rho}{2(1+\rho)}s^2} \left(\sum_{n=0}^{N-1} c_n s^{-2n-1} + O(s^{-2N-1}) \right).\end{aligned}$$

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