CAP representations of inner forms of Sp(4) with respect to Klingen parabolic subgroup

Yasuda, Takanori
Faculty of Mathematics, Kyushu University

出版情報：MI Preprint Series. 2010-36, 2010-12-08. 九州大学大学院数理学研究院
バージョン：

MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

CAP representations of inner forms of $Sp(4)$ with respect to Klingen parabolic subgroup

Takanori Yasuda

MI 2010-36

(Received December 8, 2010)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

CAP representations of inner forms of $Sp(4)$ with respect to Klingen parabolic subgroup

Takanori Yasuda

Abstract

The unitary group of the hyperbolic hermitian space of dimension two over a quaternion division algebra over a number field is a non-quasisplit inner form of $Sp(4)$, and does not have a parabolic subgroup corresponding to the Klingen parabolic subgroup. However, it has CAP representations with respect to the Klingen parabolic subgroup. We construct them by using the theta lifting from the unitary groups of one-dimensional (-1)-hermitian spaces and estimate their multiplicities in the discrete spectrum. In many cases, their multiplicities become bigger than 1.

1 Introduction

The purposes of this paper are, for an inner form G of $Sp(4)$ defined over a number field k,

1. construction of a certain class of non-tempered automorphic representations and evaluation (of lower bounds) of their multiplicities in the discrete spectrum, and

2. the description of the multiplicities expected from the evaluation using the formulation of the Arthur's multiplicity conjecture.

According to the Arthur's conjecture [Art89], for any irreducible non-tempered automorphic representation π, there exists an A-parameter

$$\psi : \mathcal{L}_k \times SL(2, \mathbf{C}) \to {}^L G = SO(5, \mathbf{C}),$$

where $\mathcal{L}_k$ is the hypothetical Langlands group of k, satisfying $\psi|_{SL(2,\mathbf{C})} \neq \mathbf{1}$ such that π is expressed as an element of its global A-packet. When π is isomorphic to the restricted tensor product $\bigotimes_v \pi_v$ of local irreducible representations π_v, each π_v is also expressed as an element of the local A-packet of the local A-parameter ψ_v

obtained by ψ. Since $Sp(4)$ and G share the set of A-parameters there should exist the global and local A-packets for $Sp(4)$ associated to ψ too. At this time, at almost all place of k, the local A-packets of $Sp(4)$ and G coincide.

For a general reductive group, an irreducible representation appearing in the residual spectrum is a typical example of non-tempered automorphic representation. The residual spectrum is the subspace of the space of the L^2-automorphic forms generated by the non-cuspidal irreducible automorphic representations appearing in the discrete spectrum. The irreducible decomposition of the residual spectrum for $Sp(4)$ is completely determined [Kim95]. In the case where k is totally real, part of the irreducible representations appearing in the result of [Kim95] can be rewritten by representations given by theta lifting [KN94]. For comparison with the case of G we quote the latter here.

Theorem 1.1 ([KN94]). *Let k be totally real. An irreducible representation appearing in the residual spectrum for $Sp(4)$ is one of the following irreducible representations.*

(1) The trivial representation $\mathbf{1}_{Sp(4)}$ of $Sp(4, \mathbf{A})$. Here $\mathbf{A} = \mathbf{A}_k$ denotes the adele ring of k.

(2) the theta lift $R(V)$ from the trivial representation of the orthogonal group $O(V, \mathbf{A})$ of a 2-dimensional non-hyperbolic quadratic space V over k.

(3) The unique irreducible quotient of $\mathrm{Ind}_{P_S(\mathbf{A})}^{Sp(4,\mathbf{A})}(\sigma|\det|_{\mathbf{A}}^{1/2})$ for an irreducible self-dual cuspidal representation σ of $GL(2, \mathbf{A})$ whose standard L-functions $L(s, \pi)$ do not vanish at $s = 1/2$.

(4) The unique irreducible quotient of $\mathrm{Ind}_{P_K(\mathbf{A})}^{Sp(2,\mathbf{A})}(\omega_{k'/k}|\cdot|_{\mathbf{A}} \otimes \pi)$ for a quadratic character $\omega_{k'/k}$, which is associated to a non-trivial quadratic extension k' of k, and an irreducible unitary representation π of $SL(2, \mathbf{A})$ such that

$$\sigma \subset \pi(\Omega)|_{SL(2,\mathbf{A})}$$

for some character Ω of $\mathbf{A}_{k'}^\times/k^\times$ not isomorphic to its conjugate. Here $\pi(\Omega)$ is the automorphic representation of $GL(2, \mathbf{A})$ given in [LL79, Prop. 6.5] and [GJ78, p.491].

Here P_S and P_K are the Siegel and Klingen parabolic subgroups of $Sp(4)$, respectively. The respective Levi factors M_S and M_K satisfy that $M_S \simeq GL(2)$ and $M_K \simeq \mathbb{G}_m \times SL(2)$. Each irreducible representation in the above appears with multiplicity one in the discrete spectrum.

In this paper, we treat only the case where G is expressed by the unitary group of the 2-dimensional hyperbolic hermitian space over a quaternion division algebra D over k. In this case, as for the irreducible decomposition of the residual spectrum the following is obtained.

Theorem 1.2 ([Yas07]). *Let k be totally real. An irreducible representation appearing in the residual spectrum for G is one of the following irreducible representations.*

(1) The trivial representation $\mathbf{1}_G$ of $G(\mathbf{A})$.

(2) The theta lift $R(V)$ from the trivial representation of the group $U(V)(\mathbf{A})$ of adele points of the unitary group of a (-1)-hermitian (right) D-space V of dimension one.

(3) The unique irreducible quotient of $\mathrm{Ind}_{P'_S(\mathbf{A})}^{G(\mathbf{A})}(\sigma|\nu_D|_{\mathbf{A}}^{1/2})$ for an irreducible self-dual cuspidal representation σ of $D^\times(\mathbf{A})$ whose standard L-functions $L(s,\pi)$ do not vanish at $s = 1/2$.

Here P'_S is the Siegel parabolic subgroup of G, where its Levi factor M'_S is isomorphic to $D^\times$, and ν_D the reduced norm of D. In the case of (1) and (3), the multiplicity of each representation is one. In the case of (2), the multiplicity of each representation is $2^{\sharp S_D - 2}$ where S_D is the set of places of k at which D is ramified.

When comparing the irreducible decomposition of the residual spectra of $Sp(4)$ and G, it is noticed that similar forms of representations appear. However there are no representations for G corresponding to those in Theorem 1.1 (4) for $Sp(4)$. This is because there is no proper parabolic subgroup of G over k containing the correspondence of the Klingen parabolic subgroup of $Sp(4)$ by an inner twist. Nevertheless, the representations in Theorem 1.1 (4) must be expressed as elements of the A-packets of some A-parameters, and there should exist also the A-packets for G of these A-parameters. Therefore, if they exist, they consist of cuspidal representations of $G(\mathbf{A})$, which are expressed in the forms defined as follows.

Definition 1.3. *Let $\pi \simeq \bigotimes_v \pi_v$ be an irreducible cuspidal representation of $G(\mathbf{A})$. We say that π is a CAP representation with respect to P_K if there exist an irreducible cuspidal representation $\sigma \simeq \bigotimes_v \sigma_v$ of $SL(2,\mathbf{A})$ and a character $\omega = \prod_v \omega_v$ of $\mathbf{A}^\times/k^\times$ such that for almost all v, π_v is isomorphic to a composition factor of $\mathrm{Ind}_{P_K(k_v)}^{Sp(2,k_v)}(\omega_v|\cdot|_v \otimes \sigma_v)$.*

Remark that because G is isomorphic to $Sp(4)$ over k_v for almost all v, P_K is regarded as a subgroup over k_v of G for such v, and the above definition makes sense.

We make use of theta lifting to construct such CAP representations. The theta correspondences from $O(2)$ to $Sp(4)$ become automorphic representations appearing in Theorem 1.1 (4). Therefore, the target representations are constructed by the theta lifting from the unitary group of a skew-hermitian space over D of dimension one to G, which is an inner form version of the theta lifting from $O(2)$ to $Sp(4)$. In fact, writing $U_0(V)$ for the connected component of the unit of the unitary group

$U(V)$ of a skew-hermitian space V over D of dimension one, a non-zero irreducible automorphic representation $\theta(V,\chi,S)$ of G can be defined for an irreducible automorphic representation (character) of $U_0(V)$ and a set S of places of k which consists of finite elements and satisfies a certain condition (Theorem 6.1). As special cases, the representations in Theorem 1.2 (2) (the case that $\chi = \mathbf{1}, S = \emptyset$) and the inner form version of cuspidal representations constructed in [HPS79] (the case that $\chi = \mathbf{1}, S \neq \emptyset$) can be dealt with. As for the multiplicity $m(\theta(V,\chi,S))$ of $\theta(V,\chi,S)$ in the discrete spectrum, we have the following estimate.

Theorem 1.4.

$$m(\theta(V,\chi,S)) \geq \begin{cases} 2^{\sharp(S_\chi \cap S_D)-1} & S_D \not\subset S_\chi,\ \ S_D \cap S_\chi \neq \emptyset, \\ 2^{\sharp S_D - 2} & S_D \subset S_\chi, \\ 1 & S_D \cap S_\chi = \emptyset, \end{cases}$$

where $S_\chi = \{v \mid \chi_v^2 = \mathbf{1}\}$.

This estimate is obtained by the failure of the Hasse's principle for skew-hermitian spaces over D. In view of the result of the multiplicities of representations in Theorem 1.2 (2), it is expected that the above inequality sign is exchanged for the equal one. If the equality is satisfied then the value in the right hand side should be described by the formulation in the Arthur's multiplicity conjecture. Therefore, under this assumption the author inspected how local A-packets, S-groups and pairings between them, etc. should be described in the formulation. As a consequence, it turns out that they can be described as they satisfy some necessary conditions (§ 7).

2 Inner form G of $Sp(4)$

Let k be a number field, and $\mathbf{A} = \mathbf{A}_k$ its adele ring. We write $\mathbf{A}_\infty, \mathbf{A}_f$ for the infinite and finite components of $\mathbf{A}$, respectively. $|\ |_\mathbf{A}$ denotes the idele norm of $\mathbf{A}^\times$. For any place v of k, we write k_v for the completion of k at v and $|\ |_v$ for the v-adic norm. If v is non-archimedean, $\mathcal{O}_v$ denotes the maximal compact subring of k_v.

k-group $Sp(2)$ will be realized by

$$Sp(2) = \left\{ g \in GL(4) \;\middle|\; g \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} {}^t g = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \right\}.$$

Fix a minimal k-parabolic subgroup P_0 of $Sp(4)$ by

$$P_0 = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix} \in Sp(4) \right\},$$

and a Levi factor M_0 of P_0 consisting of diagonal matrices. The unipotent radical of P_0 is denoted by U_0. We define the Siegel parabolic subgroup $P_S = M_S U_S$ and the Klingen parabolic subgroup $P_K = M_K U_K$ of $Sp(4)$ as

$$M_S = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \,\middle|\, a \in GL(2) \right\},$$

$$U_S = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathrm{Sym}(2) \right\},$$

and

$$M_K = \left\{ \begin{pmatrix} t & & & \\ & a & & b \\ & & t^{-1} & \\ & c & & d \end{pmatrix} \,\middle|\, t \in \mathbb{G}_m, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2) \right\},$$

$$U_K = \left\{ \begin{pmatrix} 1 & a & b & c \\ & 1 & c & \\ & & 1 & \\ & & -a & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{G}_a \right\}.$$

Let R be a quaternion algebra over a local or global field F. We write ν_R, τ_R and $*$ for the reduced norm, the reduced trace and the main involution of R, respectively. We write $R_- = \{x \in R \mid \tau_R(x) = 0\}$. When $F = k$, we write S_R for the set of places v of k at which R is ramified. This set has finite and even elements. $\mathbb{M}(n, A)$ denotes the algebra of all $n \times n$-matrices over a ring A. For $a = (a_{i,j}) \in \mathbb{M}(n, R)$, write ${}^\tau a = ({}^* a_{j,i})$. Let D be a quaternion division algebra over k. On $W = D^{\oplus 2}$ viewed as a left D-module, a hyperbolic $(D, *)$-hermitian form h_W is defined by

$$h_W((x_1, x_2), (y_1, y_2)) = (x_1, x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} {}^* y_1 \\ {}^* y_2 \end{pmatrix} \quad (\forall x_1, x_2, y_1, y_2 \in D). \tag{2.1}$$

Its unitary group $G = G(W, h_W)$ is a k-group which associates

$$G(A) = \left\{ g \in GL(2, R \otimes_k A) \,\middle|\, g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^\tau g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

to each (abelian) k-algebra A.

There is a $p, q \in k^\times$ such that $D \simeq (p, q)_k$ [Sch85, p.75]. Then G and $Sp(4)$ are isomorphic over the quadratic extension $K = k(\sqrt{p})$ of k and moreover G is an inner form of $Sp(4)$. Define a k-parabolic subgroup $P_S = M_S U_S$ of G as

$$M_S = \left\{ m(x) := \begin{pmatrix} x & 0 \\ 0 & {}^* x^{-1} \end{pmatrix} \,\middle|\, x \in D^\times \right\},$$

$$U_S = \left\{ u(y) := \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \,\middle|\, y \in D_- \right\}.$$

Via an inner twist this parabolic subgroup coincides with the Siegel parabolic subgroup of $Sp(4)$. Therefore we use the same notation P_S as the case of $Sp(4)$. The character $M_S \ni m(x) \mapsto \nu_D(x) \in \mathbb{G}_m$ is again denoted by ν_D. P_S is a minimal and maximal proper parabolic subgroup of G defined over k.

The k-split component of the center of M_S is

$$A_S = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| \; t \in \mathbf{G}_m \right\}.$$

Let W_G denote the Weyl group of G, which consists of two elements. A representative of the non-trivial element of W_G is chosen by

$$w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G(k).$$

If $v \notin S_D$, $G(k_v)$ can be identified with $Sp(4, k_v)$. For every place v of k, we fix a maximal compact subgroup $\mathbf{K}_v$ of $G(k_v)$ and a Haar measure dg_v such that

$$\mathbf{K}_v = \begin{cases} Sp(4, \mathcal{O}_v) & \text{for a non-archimedean } v \notin S_D, \\ Sp(4, \mathbf{R}) \cap O(4) \simeq U(2) & \text{for a real } v \notin S_D, \\ Sp(1,1) \cap Sp(2) \simeq Sp(1) \times Sp(1) & \text{for a real } v \in S_D, \\ Sp(4, \mathbf{C}) \cap U(4) \simeq Sp(2) & \text{for a complex } v \notin S_D, \end{cases}$$

and the volume of $\mathbf{K}_v$ with respect to dg_v is equal to 1 for almost all non-archimedean v. We define a maximal compact subgroup $\mathbf{K}$ of $G(\mathbf{A})$ by $\prod_v \mathbf{K}_v$ and a Haar measure dg of $G(\mathbf{A})$ by $\prod_v dg_v$.

3 Construction of automorphic forms of G

3.1 Automorphic forms of the unitary group of a (-1)-hermitian space

Fix a non-zero $\eta_0 \in D_-$. Define a non-degenerate (-1)-hermitian right space (V, h_V) over D as $V = D$ and

$$h_V(x_1, x_2) = {}^*x_1 \cdot \eta_0 \cdot x_2 \quad (x_1, x_2 \in D).$$

$G(V)$ and $G_0(V)$ denote the unitary group and the special unitary group of V defined over k, respectively. When emphasizing V as a space of k-valued points, it will be written by V_k. Also V_v stands for the completion of V at a place v and $V_\mathbf{A} = V \otimes_k \mathbf{A}$. In particular, we will often use the notation $G(V_k), G(V_v), G(V_\mathbf{A})$ instead of $G(V)(k), G(V)(k_v), G(V)(\mathbf{A})$, respectively. Similar notation is also used for $G_0(V)$. It is known that $G(V_k) = G_0(V_k)$ and $G(V_v) = G_0(V_v)$ for any $v \in S_D$. Set $k' = k(\eta)$, which is a quadratic extension of k, and define a quadratic space (T, b_T) over k as $T = k'$ and $b_T = N_{k'/k}$ where $N_{k'/k}$ is the norm of k'/k. We may make the following identification:

$$G(V_v) = \begin{cases} O(T_v) & v \notin S_D, \\ SO(T_v) & v \in S_D, \end{cases}$$

$$G_0(V_v) = SO(T_v),$$

$$G(V_k) = G_0(V_k) = SO(T_k).$$

Here T_v is the completion of T at v and $SO(T_k) = SO(T)(k)$. T_v is isotropic if and only if $\eta^2(\in k^\times)$ is a quadratic residue in k_v, and in this case identify $O(T_v) = (1,1;k_v)$. A maximal compact subgroup $\mathbf{L}_v$ of $G(V_v)$ is defined as

$$\mathbf{L}_v = \begin{cases} O(T_v) & v \notin S_D \text{ and } T_v \text{ is anisotropic,} \\ O(1,1;\mathcal{O}_v) & v \notin S_D \text{ and } T_v \text{ is isotropic,} \\ SO(T_v) & v \in S_D, \end{cases}$$

Remark that if $v \in S_D$ then T_v is always anisotropic. A measure dh_v on $G(V_v)$ is chosen by the Haar measure such that the volume of $\mathbf{L}_v$ is 1, and a measure dh on $G(V_\mathbf{A})$ is defined by the product of dh_v.

Fix a $\gamma_0 \in O(T_k)\backslash SO(T_k)$, which always holds $\gamma_0^2 = 1$. Let $\chi = \prod_v \chi_v$ be an unitary character of $G_0(V_k)\backslash G_0(V_\mathbf{A})$. Consider the induced representation of $G(V_v)$ from χ_v. In the case of $v \in S_D$, χ_v is also recognized as a representation of $G(V_v)$ because $G(V_v) = G_0(V_v)$. In the case of $v \notin S_D$, if $\chi_v^2 \neq \mathbf{1}$ then $\mathrm{Ind}_{G_0(V_v)}^{G(V_v)}\chi_v$ is irreducible, which is denoted by $\tilde{\chi}_v^+$, and if $\chi_v^2 = \mathbf{1}$ then

$$\mathrm{Ind}_{G_0(V_v)}^{G(V_v)}\chi_v \simeq \tilde{\chi}_v^+ \oplus \tilde{\chi}_v^-. \tag{3.1}$$

Here $\tilde{\chi}_v^+, \tilde{\chi}_v^-$ are characters of $G(V_v)$ characterized by $\tilde{\chi}_v^\pm(\gamma_0) = \pm 1$. Write S_χ for the set of places v of k satisfying $\chi_v^2 = \mathbf{1}$. If χ_v is unramified, $f_{0,v} \in \tilde{\chi}_+$ is defined by

$$\phi_{0,v}(h) = \begin{cases} 1 & h \in \mathbf{L}_v \\ 0 & h \notin \mathbf{L}_v \end{cases}$$

Let S be a subset of $S_\chi \cap S_D^c$ with finite number of elements. An irreducible representation $\sigma(V,\chi,S)$ of $G(V_\mathbf{A})$ is defined by the restricted tensor product,

$$(\bigotimes_{v\in S} \tilde{\chi}_v^-) \otimes (\bigotimes_{v\notin S} \tilde{\chi}_v^+)$$

with respect to $\phi_{0,v}$. In particular, if $\chi^2 = 1$ then $\sigma(V,\chi,S)$ becomes a character of $G(V_\mathbf{A})$. The v-component of $\sigma(V,\chi,S)$ for a place v is denoted by $\sigma_v(V,\chi,S)$. Since $G(V_k) = G_0(V_k)$, we can define an injective intertwining operator characterized by

$$\sigma(V,\chi,S) \ni \bigotimes_v \phi_v \mapsto \prod_v \phi_v \in \mathcal{A}(G(V_k)\backslash G(V_\mathbf{A})).$$

Here $\mathcal{A}(G(V_k)\backslash G(V_\mathbf{A}))$ is the space of automorphic forms on $G(V_k)\backslash G(V_\mathbf{A})$ ([MW95] §I.2.17). $\sigma(V,\chi,S)$ is identified with the image of this intertwining operator. Conversely, any irreducible $G(V_\mathbf{A})$-subspace of $\mathcal{A}(G(V_k)\backslash G(V_\mathbf{A}))$ is expressed by the form $\sigma(V,\chi,S)$ for some χ and S because the restriction of an automorphic form on $G(V_\mathbf{A})$ to $G_0(V_\mathbf{A})$ is also automorphic form.

3.2 Theta correspondence

Let ψ be a non-trivial character of $\mathbf{A}/k$, and for any place v of k, the v-component of ψ is denoted by ψ_v. For a vector space X over k_v, $\mathcal{S}(X)$ denotes the space of the Schwartz-Bruhat functions on X. The Weil representation ω_{ψ_v,V_v} of $G(V_v)\times G(k_v)$ with respect to ψ_v is defined on $\mathcal{S}(V_v)$ as in [Yas07]. In particular, for $v \notin S_D$,

ω_{ψ_v,V_v} is identified with the Weil representation $\omega_{\psi_v,T_{V_v}}$ of $O(T_{V_v})\times Sp(4,k_v)$ defined on $\mathcal{S}(T_{V_v}^2)$ where T_{V_v} is the 2-dimensional quadratic space with the quadratic form given by the symmetric matrix

$$\begin{pmatrix}\gamma & -\alpha\\ -\alpha & -\beta\end{pmatrix}\quad \text{when}\quad \eta_0=\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix}\in \mathbf{M}(2,k_v)_-.$$

The explicit formula of ω_{ψ_v,V_v} is described as follows: For $f\in\mathcal{S}(V_v),\ x\in V_v$,

(1) $\omega_{\psi_v,V_v}(h,1)f(x)=f(h^{-1}x),\quad (h\in G(V_v))$,

(2) $\omega_{\psi_v,V_v}(1,m(a))f(x)=\zeta_{V_v}(\nu(a))|\nu_{D_v}(a)|_v f(xa),\quad (a\in D^\times(k_v))$,

(3) $\omega_{\psi_v,V_v}(1,u(b))f(x)=\psi_v(\frac{1}{4}\tau_{D_v}(b\,h_V(x,x)))f(x),\quad (b\in D_-(k_v))$,

(4) $\omega_{\psi_v,V_v}(1,w_0)f(x)=(-1,-\det V_v)_v\kappa(D_v)\,\mathcal{F}_{V_v}f(-x)$.

Here $(\cdot,\cdot)_v$ is the Hilbert symbol at v, ζ_{V_v} the quadratic character $(-\det V_v,\,\cdot\,)_v$ of $k_v^\times$, $\kappa(D_v)=-1$ if $v\in S_D$ and $\kappa(D_v)=1$ otherwise, and

$$\mathcal{F}_{V_v}f(x)=\int_{V_v} f(y)\psi(\frac{1}{2}\tau_{D_v}\circ\mathrm{tr}(y,x)_{V_v})d_{V_v}y$$

where $d_{V_v}y$ is the self-dual measure with respect to the bilinear form

$$V_v\times V_v\ni(x,y)\mapsto\psi(\frac{1}{2}\tau_{D_v}\circ\mathrm{tr}(y,x)_{V_v}).$$

For a non-archimedean $v\notin S_D$, $f_{0,v}\in\mathcal{S}(V_v)$ denotes the characteristic function of $\mathbb{M}(2,\mathcal{O}_v)\subset\mathbb{M}(2,k_v)=V_v$. $\mathcal{S}(V_\mathbf{A})$ is defined by the tensor product of $\mathcal{S}(V_v)$ with respect to $f_{0,v}$, which is identified with a space of functions on $V_\mathbf{A}$. The global Weil representation $\omega_{\psi,V}$ of $G(V_\mathbf{A})\times G(\mathbf{A})$ with respecto to ψ is defined on $\mathcal{S}(V_\mathbf{A})$ by the restricted tensor product of the local Weil representations. For $f\in\mathcal{S}(V_\mathbf{A})$, set

$$\theta(f;h,g)=\sum_{\xi\in V_k}\omega_{\psi,V}(h,g)f(\xi)\quad(h\in G(V_\mathbf{A}),g\in G(\mathbf{A})),$$

which converges absolutely and becomes $G(k)$-invariant. Since $G(V_k)\backslash G(V_\mathbf{A})$ is compact, for $\phi\in\sigma(V,\chi,S)$ and $f\in\mathcal{S}(V_\mathbf{A})$, the integral

$$\theta(f,\phi)(g)=\int_{G(V_k)\backslash G(V_\mathbf{A})}\theta(f;h,g)\overline{\phi(h)}dh$$

is defined and converges. Then $\theta(f,\phi)$ becomes an automorphic form on $G(\mathbf{A})$. We denote by $\Theta(V,\chi,S)$ the space generated by $\theta(f,\phi)$ for all $\phi\in\sigma(V,\chi,S)$ and $f\in\mathcal{S}(V_\mathbf{A})$.

Lemma 3.1. *If $(\chi,S)\neq(\mathbf{1},\emptyset)$ then $\theta(f,\phi)$ is cuspidal.*

Proof. The constant term $\theta(f,\phi)_{P_S}$ of $\theta(f,\phi)$ along to P_S is calculated as follows.

$$\begin{aligned}
\theta(f,\phi)_{P_S}(g) &= \int_{U_S(k)\backslash U_S(\mathbf{A})} \theta(f,\phi)(ug)du \\
&= \int_{U_S(k)\backslash U_S(\mathbf{A})} \int_{G(V_k)\backslash G(V_\mathbf{A})} \sum_{\xi\in V_k} \omega_{\psi,V}(h,ug)f(\xi)\overline{\phi(h)}dhdu \\
&= \int_{G(V_k)\backslash G(V_\mathbf{A})} \sum_{\xi\in V_k} \int_{U_S(k)\backslash U_S(\mathbf{A})} \omega_{\psi,V}(h,ug)f(\xi)\overline{\phi(h)}dudh \\
&= \int_{G(V_k)\backslash G(V_\mathbf{A})} \sum_{\xi\in V_k} \int_{U_S(k)\backslash U_S(\mathbf{A})} \psi(\frac{1}{4}\tau_D(bh_V(\xi,\xi)))\omega_{\psi,V}(h,g)f(\xi)\overline{\phi(h)}dbdh \\
&= \int_{G(V_k)\backslash G(V_\mathbf{A})} \omega_{\psi,V}(h,g)f(0)\overline{\phi(h)}dh \\
&= \omega_{\psi,V}(1,g)f(0)\int_{G(V_k)\backslash G(V_\mathbf{A})} \overline{\phi(h)}dh.
\end{aligned}$$

Since $(\chi,S)\neq(\mathbf{1},\emptyset)$, ϕ is orthogonal to a constant function. Therefore the last term is zero. □

For $\eta\in D_-$, the Fourier coefficient $\mathcal{F}_\eta f$ of an automorphic form f on $G(\mathbf{A})$ is defined by

$$\mathcal{F}_\eta f(g) = \int_{U_S(k)\backslash U_S(\mathbf{A})} f(ug)\psi_\eta(u)du \quad (g\in G(\mathbf{A})),$$
$$\text{where}\quad \psi_\eta(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}) = \psi(-\frac{1}{4}\tau_D(b\eta)) \quad (b\in D_-(\mathbf{A})).$$

Lemma 3.2. *Let $\eta\in D_-\backslash\{0\}$.*

$$\mathcal{F}_\eta\theta(f,\phi)(g) = \begin{cases} \mathcal{F}_{\eta_0}\theta(f,\phi)(m(\alpha)g) & \eta = {}^*\alpha\eta_0\alpha \text{ for some } \alpha\in D^\times, \\ 0 & \text{otherwise.}\end{cases}$$

Proof.

$$\begin{aligned}
\mathcal{F}_\eta\theta(f,\phi)(g) &= \int_{U_S(k)\backslash U_S(\mathbf{A})} \theta(ug)\psi_\eta(u)du \\
&= \int_{U_S(k)\backslash U_S(\mathbf{A})} \int_{G(V_k)\backslash G(V_\mathbf{A})} \sum_{\xi\in V_k} \omega_{\psi,V}(h,ug)f(\xi)\overline{\phi(h)}\psi_\eta(u)dhdu \\
&= \int_{G(V_k)\backslash G(V_\mathbf{A})} \int_{D_-(k)\backslash D_-(\mathbf{A})} \sum_{\xi\in V_k} \omega_{\psi,V}(h,g)f(\xi)\overline{\phi(h)}\psi(\frac{1}{4}\tau_D(b(h_V(\xi,\xi)-\eta)))dbdh \\
&= \int_{G(V_k)\backslash G(V_\mathbf{A})} \sum_{h_V(\xi,\xi)=\eta} \omega_{\psi,V}(h,g)f(\xi)\overline{\phi(h)}dh \qquad (3.2)
\end{aligned}$$

If η is not expressed by the form ${}^*\alpha\eta_0\alpha$ for some $\alpha \in D^\times$, (3.2) = 0. Assume $\eta = {}^*\alpha\eta_0\alpha$ for a $\alpha \in D^\times$.

$$\begin{aligned}(3.2) &= \int_{G(V_k)\backslash G(V_{\mathbf{A}})} \sum_{h'\in G(V_k)} \omega_{\psi,V}(h,g) f(h'^{-1}\alpha)\overline{\phi(h)}dh \\ &= \int_{G(V_k)\backslash G(V_{\mathbf{A}})} \sum_{h'\in G(V_k)} \omega_{\psi,V}(h'h,g) f(\alpha)\overline{\phi(h'h)}dh \\ &= \int_{G(V_{\mathbf{A}})} \omega_{\psi,V}(h,g) f(\alpha)\overline{\phi(h)}dh \\ &= \int_{G(V_{\mathbf{A}})} \omega_{\psi,V}(h,m(\alpha)g) f(1_D)\overline{\phi(h)}dh \qquad (3.3)\end{aligned}$$

In particular, if $\alpha = 1$,

$$\mathcal{F}_{\eta_0}\theta(f,\phi)(g) = \int_{G(V_{\mathbf{A}})} \omega_{\psi,V}(h,g) f(1_D)\overline{\phi(h)}dh.$$

Therefore

$$(3.3) = \mathcal{F}_{\eta_0}\theta(f,\phi)(m(\alpha)g) \qquad (3.4)$$

□

$\mathcal{U}(V,\chi,S)$ denotes the space generated by functions $\mathcal{F}_{\eta_0}f$ on $G(\mathbf{A})$ for all $f \in \Theta(V,\chi,S)$. The right regular action r of $G(\mathbf{A})$ defines a representation of $G(\mathbf{A})$ on $\mathcal{U}(V,\chi,S)$, and

$$\mathcal{F}_{\eta_0} : \Theta(V,\chi,S) \ni f \mapsto \mathcal{F}_{\eta_0}f \in \mathcal{U}(V,\chi,S)$$

becomes a surjective intertwining operator. From Lemma 3.1, Lemma 3.2 and the Fourier inversion formula, $\mathcal{F}_{\eta_0}$ is also injective. Therefore we have the following.

Proposition 3.3. *If $(\chi,S) \neq (\mathbf{1},\emptyset)$ then $\Theta(V,\chi,S)$ is isomorphic to $\mathcal{U}(V,\chi,S)$ as a representation of $G(\mathbf{A})$.*

Let v be a place and $\sigma_v = \tilde{\chi}_v^\epsilon$ for some χ_v and $\epsilon = \pm 1$. $\mathcal{U}(V_v,\sigma_v)$ is defined by the space generated by functions

$$\lambda_v(f_v,\phi_v) : G(k_v) \ni g_v \mapsto \int_{G(V_v)} \omega_{\psi_v,V_v}(h_v,g_v) f_v(1_D)\overline{\phi_v(h_v)}dh_v \qquad (3.5)$$

for all $f_v \in \mathcal{S}(V_v)$ and all $\phi_v \in \tilde{\chi}_v^\epsilon$. Similarly for the global case, a representation of $G(k_v)$ on $\mathcal{U}(V_v,\tilde{\chi}^\epsilon)$ is defined by the right regular action r.

Lemma 3.4. *(1) $\mathcal{U}(V_v,\sigma_v) \neq 0$ for all v.*

(2) For almost all v and $\sigma_v = \tilde{\mathbf{1}}^+ = \mathbf{1}$, $\lambda_v(f_{0,v},\phi_{0,v})$ is $\mathbf{K}_v$-invariant and $\lambda_v(f_{0,v},\phi_{0,v})|_{\mathbf{K}_v} \equiv 1$.

Proof. (1) It suffices to show that there are $f'_v \in \mathcal{S}(V_v)$ and $\phi'_v \in \sigma_v$ such that $\lambda_v(f'_v, \phi'_v)(1) \neq 0$. For a non-zero ϕ'_v there is a relatively compact open subset $\mathcal{O}$ of V_v such that

$$\int_{G(V_v)} ch_{\mathcal{O}}(h_v^{-1})\overline{\phi'_v(h_v)}dh_v \neq 0$$

where $ch_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$ defined on V_v. If v is non-archimedean we can take $ch_{\mathcal{O}}$ as f'_v. If v is archimedean then $f_{\mathcal{O}}$ is approximated by elements in $\mathcal{S}(V_v)$, so that we can choose an apropriate f'_v. (2) For any non-archimedean $v \notin S_D$ such that ψ_v is unramified and $v \nmid 2$, $f_{0,v}$ is $\mathbf{K}_v$-invariant, and so is $\lambda_v(f_{0,v}, \phi_{0,v})$. Since

$$\begin{aligned}
\lambda_v(f_{0,v}, \phi_{0,v})(1) &= \int_{G(V_v)} \omega_{\psi_v, V_v}(h_v, g_v) f_{0,v}(1_D)\overline{\phi_{0,v}(h_v)}dh_v \\
&= \int_{G(V_v)} \omega_{\psi_v, V_v}(1,1) f_{0,v}(h_v^{-1})\phi_{0,v}(h_v)dh_v \\
&= \int_{\mathbf{L}_v} 1 dh_v \\
&= 1,
\end{aligned}$$

and $\lambda_v(f_{0,v}, \phi_{0,v})$ is $\mathbf{K}_v$-invariant, $\lambda_v(f_{0,v}, \phi_{0,v})|_{\mathbf{K}_v} \equiv 1$. □

From the above lemma and (3.3), for $f(x) = \prod_v f_v(x_v) \in \mathcal{S}(\mathcal{V}_{\mathbf{A}})$ and $\phi(h) = \prod_v \phi_v(h_v) \in \sigma(V, \chi, S)$, we have a decomposition,

$$\mathcal{F}_{\eta_0}\theta(f, \phi)(g) = \prod_v \lambda_v(f_v, \phi_v)(g_v) \quad (g = (g_v) \in G(\mathbf{A})). \tag{3.6}$$

Therefore the next proposition follows.

Proposition 3.5. *$\Theta(V, \chi, S) \neq 0$ and $\Theta(V, \chi, S)$ is isomorphic to the restricted tensor product $\bigotimes_v \mathcal{U}(V_v, \sigma_v(V, \chi, S))$ with respect to $\lambda_v(f_{0,v}, \phi_{0,v})$.*

4 Review of Shalika-Tanaka lifting

Here we review the results of [ST69]. Let (T, q_T) be a non-degenerate and non-hyperbolic quadratic space over k of dimension 2 and $\chi = \prod_v \chi_v$ a character of $SO(T_{\mathbf{A}})$ invariant on $SO(T_k)$. For a place v, the Weil representaion $\omega^0_{\psi_v, T_v}$ of $O(T_v) \times SL(2, k_v)$ on $\mathcal{S}(T_v)$ is defined usually. For $u \in T_v$ such that $q_{T_v}(u) \neq 0$, $f \in \mathcal{S}(T_v)$, put

$$(P_{\chi_v} f)(u) = \int_{SO(T_v)} f(h^{-1}u)\overline{\chi_v(h)}dh.$$

This integral converges absolutely and defines a continuous function on the open subset of T_v which consists of $u \in T_v$ satisfying $q_{T_v}(u, u) \neq 0$. $S(\chi_v, T_v)$ denotes the image of $\mathcal{S}(T_v)$ by P_{χ_v}, which is regarded as a representation of $SL(2, k_v)$ by the transfer r of the action of the Weil representation. Fix $\gamma_0 \in O(T_k)\backslash SO(T_k)$ and write $\mathcal{S}^{\pm}(T_v)$ for the space consisting of $f \in \mathcal{S}(T_v)$ such that $f(\gamma_0 u) = \pm f(u)$. Clearly, $\mathcal{S}(T_v) = \mathcal{S}^{+}(T_v) \oplus \mathcal{S}^{-}(T_v)$. Elements $\phi_v^{\pm}$ of $\mathrm{Ind}_{SO(T_v)}^{O(T_v)}\chi_v$ are defined by

- $\phi_v^{\pm}(h) = \chi_v(h) \quad (h \in SO(V_v))$,
- $\phi_v^{\pm}(\gamma_0) = \pm 1$.

Then $\mathrm{Ind}_{SO(T_v)}^{O(T_v)}\chi_v$ is spaned by $\phi_v^{\pm}$. For any $f_+ \in \mathcal{S}^{+}(T_v),\ f_- \in \mathcal{S}^{-}(T_v)$,

$$\int_{O(T_v)} f_+(h^{-1}u)\overline{\phi_v^-(h)}dh = \int_{O(T_v)} f_-(h^{-1}u)\overline{\phi_v^+(h)}dh = 0. \tag{4.1}$$

And for $f = f_+ + f_- \in \mathcal{S}(T_v)$,

$$\begin{aligned}(P_{\chi_v}f)(u) &= \int_{SO(T_v)} f(h^{-1}u)\overline{\chi_v(h)}dh \\ &= \frac{1}{2}\int_{O(T_v)} f_+(h^{-1}u)\overline{\phi_v^+(h)}dh + \frac{1}{2}\int_{O(T_v)} f_-(h^{-1}u)\overline{\phi_v^-(h)}dh. \end{aligned} \tag{4.2}$$

Irreducible representations $\tilde{\chi}_v^{\pm}$ which appear as subrepresentations of $\mathrm{Ind}_{SO(T_v)}^{O(T_v)}\chi_v$ are defined as in §3.1. Fix a $u_0 \in T_v$ such that $q_{T_v}(u_0) \neq 0$. We denote by $\Theta_v^0(T_v, \tilde{\chi}_v^{\epsilon})$ ($\epsilon = \pm 1$) the space of functions

$$SL(2,k_v) \ni g \mapsto \int_{O(T_v)} \omega_{\psi_v,T_v}(h,g)f(u_0)\overline{\phi(h)}dh$$

for all $f \in \mathcal{S}(T_v)$ and all $\phi \in \tilde{\chi}_v^{\epsilon}$, which is regarded as a representation of $SL(2,k_v)$ by the right regular action. From (4.1) and (4.2), a surjective intertwining operator,

$$\Psi : S(\chi_v, T_v) \to \Theta_v^0(T_v, \tilde{\chi}_v^{\epsilon})$$

is defined as follows. If $\mathrm{Ind}_{SO(T_v)}^{O(T_v)}\chi_v$ is irreducible then $S(\chi_v, T_v) = \Theta_v^0(T_v, \tilde{\chi}_v^{+})$ and Ψ is taken by the identity map. If $\mathrm{Ind}_{SO(T_v)}^{O(T_v)}\chi_v$ is reducible then

$$\Psi(\tilde{f}) = \Big(SL(2,k_v) \ni g \mapsto r(g)\tilde{f}(u_0) + \epsilon \cdot r(g)\tilde{f}(\gamma_0 u_0)\Big) \quad (\text{for } \tilde{f} \in S(\chi_v, T_v)).$$

From the representation theory of $SL(2,k_v)$, the results of [ST69] with respect to $S(\chi_v, T_v)$, [MVW87] Chap.3 IV Th.4 and [Pau05] Th.15, the following is obtained.

Theorem 4.1. *(1)* $\Theta_v^0(T_v, \tilde{\mathbf{1}}_v^-(= \det)) = 0$.

(2) $\Theta_v^0(T_v, \tilde{\chi}_v^{\epsilon})$ *is non-zero and irreducible for* $\tilde{\chi}_v^{\epsilon} \neq \tilde{\mathbf{1}}_v^-$.

(3) If T_v *is hyperbolic then*

$$\Theta_v^0(T_v, \tilde{\chi}_v^{\epsilon}) \simeq \begin{cases} \mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\chi_v & \chi_v = \mathbf{1} \text{ or } \chi_v^2 \neq \mathbf{1}, \\ \text{an irreducible subrepresentation of } \mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\chi_v & \chi_v \neq \mathbf{1} \text{ and } \chi_v^2 = \mathbf{1}. \end{cases}$$

Here $B^{SL(2)}$ is the Borel subgroup of $SL(2)$ consisting of upper-triangular matrices and χ_v is also regarded as a character of the subgroup of $SL(2)$ consisting of diagonal matices in the natural way.

(4) *If T_v is non-hyperbolic and $\tilde{\chi}_v^\epsilon = \mathbf{1}$ then $\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$ is isomorphic to an irreducible subrepresentation of $\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)} \zeta_{T_v}$ where ζ_{T_v} is the quadratic character $(-\det T_v, \cdot)_v$ of $k_v^\times$. Suppose T_v is non-hyperbolic and $\chi_v \neq \mathbf{1}$. If v is non-archimedean, $\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$ is supercuspidal. If v is real, $\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$ is isomorphic to the discrete series representation $\delta(\epsilon\eta n)$ with the Harich-Chandra parameter $\epsilon\eta n$ where $\psi_v = \exp(\lambda\,\cdot), \lambda = \epsilon\sqrt{-1}|\lambda| \in \sqrt{-1}\mathbf{R},\ \chi_v = \exp(2\pi n\sqrt{-1}\,\cdot),\ n \in \mathbf{Z}$ and $\eta = 1$ if T_v is positive definite, -1 otherwise.*

For a subset $S \subset S_\chi$ with finite cardinality, an irreducible representation $\sigma(T, \chi, S)$ of $O(T_{\mathbf{A}})$ is defined as in § 3.1. Its component at v is denoted by $\tilde{\chi}_v^{\epsilon_v}$ for any v. An element of $\sigma(T, \chi, S)$ is identified with a function on $SO(T_k)\backslash O(T_{\mathbf{A}})$ by the correspondence,

$$\sigma(T, \chi, S) \ni \bigotimes_v \phi_v \longleftrightarrow \prod_v \phi_v.$$

Note that for $\phi \in \sigma(T, \chi, S)$, $\phi(h) + \phi(\gamma_0 h)$ becomes an automrphic form on $O(T_{\mathbf{A}})$. For $\phi(\neq 0) \in \sigma(T, \chi, S)$, we define

$$(P_{\chi,\phi} f)(u) = \int_{O(T_{\mathbf{A}})} f(h^{-1}u)\overline{\phi(h)}dh \quad (f \in \mathcal{S}(T_{\mathbf{A}}),\ u \in T_{\mathbf{A}}).$$

The image $S(T, \chi, S)$ of $\mathcal{S}(T_{\mathbf{A}})$ by $P_{\chi,\phi}$ is determined independently of choice of ϕ. It is an irreducible representation of $SL(2, \mathbf{A})$ by the right regular action r, which is isomorphic to the restricted tensor product

$$\bigotimes_v \Theta_v^0(T, \tilde{\chi}_v^{\epsilon_v}).$$

For $\tilde{f} \in S(T, \chi, S)$, define a function $I_\chi \tilde{f}$ on $SL(2, \mathbf{A})$ by

$$(I_\chi \tilde{f})(g) = \sum_{\xi \in SO(T_k)\backslash(T_k \backslash \{0\})} (r(g)\tilde{f})(\xi) \quad (g \in SL(2, \mathbf{A})).$$

The Weil representation $(\omega_{\psi,T}^0, \mathcal{S}(T_{\mathbf{A}}))$ of $O(T_{\mathbf{A}}) \times SL(2, \mathbf{A})$ is defined by the restricted tensor product of $(\omega_{\psi_v,T_v}^0, \mathcal{S}(T_v))$. For $f \in \mathcal{S}(T_{\mathbf{A}})$, set

$$\theta^0(f; h, g) = \sum_{\xi \in T_k \backslash \{0\}} \omega_{\psi,T}^0(h, g) f(\xi) \quad (h \in O(T_{\mathbf{A}}), g \in SL(2, \mathbf{A})),$$

$$\theta^0(f, \phi)(g) = \int_{O(T_k)\backslash O(T_{\mathbf{A}})} \theta^0(f; h, g)(\overline{\phi(h) + \phi(\gamma_0 h)})dh \quad (\phi \in \sigma(T, \chi, S)).$$

Then $\theta^0(f, \phi)$ becomes a cuspidal automorphic form on $SL(2, \mathbf{A})$. We denote by $\Theta^0(T, \chi, S)$ the space generated by $\theta^0(f, \phi)$ for all $\phi \in \sigma(T, \chi, S)$ and $f \in \mathcal{S}(T_{\mathbf{A}})$, which is a cuspidal representation of $SL(2, \mathbf{A})$.

Theorem 4.2 ([ST69]). *(1) The follwing diagram of intertwining operators is commutative.*

$$\begin{array}{ccc} \mathcal{S}(T_{\mathbf{A}}) & \xrightarrow{\theta^0(\cdot,\phi)} & \\ \downarrow{\scriptstyle P_{\chi,\phi}} & \searrow & \\ S(T,\chi,S) & \xrightarrow[I_\chi]{} & \Theta^0(T,\chi,S) \end{array}$$

In particular, if $\Theta^0(T,\chi,S)$ is non-zero then $\Theta^0(T,\chi,S)$ is isomorphic to $S(T,\chi,S) \simeq \bigotimes_v \Theta_v^0(T,\tilde{\chi}_v^{\epsilon_v})$ as a representation of $SL(2,\mathbf{A})$.

(2) For a non-hyperbolic quadratic space T of dimension 2 and a character χ of $SO(T_k)\backslash SO(T_{\mathbf{A}})$, there is a subset S of S_χ with finite cardinality such that $\Theta^0(T,\chi,S)$ is non-zero.

5 Local theory

5.1 Non-archimedean case

Let v be a non-archimedean place. For an irreducible admissible representation σ_v of $G(V_v)$, put

$$\mathcal{N}_{\sigma_v} = \bigcap_f \mathrm{Ker} f,$$

where f runs over $\mathrm{Hom}_{G(V_v)}(\omega_{\psi_v,V_v},\sigma_v)$. Then there is an unique admissible representation $\Omega(V_v,\sigma_v)$ of $G(k_v)$ such that

$$\mathcal{S}(V_v)/\mathcal{N}_{\sigma_v} \simeq \sigma_v \otimes \Omega(V_v,\sigma_v)$$

as a representation of $G(V_v)\times G(k_v)$ ([MVW87] Chap.2 III 5, Chap.3 IV 4). Let $\sigma_v = \tilde{\chi}_v^\epsilon$ for a character χ_v of $SO(k_v)$ and $\epsilon = \pm 1$. $\mathrm{Ind}_{SO(V_v)}^{O(V_v)}\chi_v$ is unitarizable by an inner product

$$\langle \phi,\phi'\rangle = \int_{G_0(V_v)\backslash G(V_v)} \phi(h)\overline{\phi'(h)}dh \quad (\phi,\phi' \in \mathrm{Ind}_{SO(V_v)}^{O(V_v)}\chi_v).$$

If $\phi_1,\dots,\phi_l$ form an orthonormal basis of $\tilde{\chi}_v^\epsilon$ with respect to this inner product,

$$\mathcal{S}(V_v) \ni f_v \mapsto \sum_{i=1}^{l} \phi_i \otimes \lambda_v(f_v,\phi_i) \in \tilde{\chi}_v^\epsilon \otimes \mathcal{U}(V_v,\tilde{\chi}_v^\epsilon)$$

becomes a surjective intertwining operator. This implies that $\mathcal{U}(V_v,\tilde{\chi}_v^\epsilon)$ is a quotient representation of $\Omega(V_v,\tilde{\chi}_v^\epsilon)$.

Proposition 5.1. *If V_v is anisotropic and either $v \notin S_D$ and $\sigma_v = \tilde{\mathbf{1}}^- = \det$ or $v \in S_D$ and $\sigma_v \neq \mathbf{1}$ then $\mathcal{U}(V_v,\sigma_v)$ is irreducible and supercuspidal. $\mathcal{U}(V_v,\sigma_v)$ will be also denoted by $\theta(V_v,\sigma_v)$.*

Proof. Any σ_v is supercuspidal and the Howe correspondent $\Omega(V_v, \sigma_v)$ of σ_v to $G(k_v)$ becomes the first occurrence in the Witt tower over W_v. This follows clearly in the case of $v \in S_D$ and from Theorem 4.1 (1) in the case of $v \notin S_D$. From [MVW87] Chap.3 IV Th.4 1), $\Omega(V_v, \sigma_v)$ is irreducible and supercuspidal. Since $\mathcal{U}(V_v, \sigma_v)$ is non-zero and a quotient of $\Omega(V_v, \sigma_v)$, $\mathcal{U}(V_v, \sigma_v)$ is also irreducible and supercuspidal. □

Let $v \notin S_D$ and $T_v = T_{V_v}$. From the definition of the Weil representation, there are ξ_1, ξ_2 in T_v linearly independent such that $\mathcal{U}(V_v, \sigma_v)$ is identified with the space $\mathcal{U}(T_v, \sigma_v)$ generated by functions

$$\lambda_v(f_v, \phi_v) : Sp(4, k_v) \ni g_v \mapsto \int_{O(T_v)} \omega_{\psi_v, T_v}(h_v, g_v) f_v(\xi_1, \xi_2) \overline{\phi_v(h_v)} dh_v$$

for all $f_v \in \mathcal{S}(T_v^2)$ and all $\phi_v \in \tilde{\chi}_v^\epsilon$. We may assume that both $q_{T_v}(\xi_1)$ and $q_{T_v}(\xi_2)$ are non-zero by acting of an element of $M_S(k_v)$. We denote by $\mathcal{U}'(T_v, \sigma_v)$ the space generated by functions

$$\lambda'_v(f_v, \phi_v) : Sp(4, k_v) \ni g_v \mapsto \int_{O(T_v)} \omega_{\psi_v, T_v}(h_v, g_v) f_v(0, \xi_2) \overline{\phi_v(h_v)} dh_v$$

for all $f_v \in \mathcal{S}(T_v^2)$ and all $\phi_v \in \tilde{\chi}_v^\epsilon$. It is easily checked that $\mathcal{U}'(T_v, \sigma_v)$ is a subrepresentation of $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(\zeta_{T_v} | \cdot |_v^{-1} \otimes \Theta_v^0(T_v, \sigma_v))$ and if $\Theta_v^0(T_v, \sigma_v)$ is non-zero then so is $\mathcal{U}'(T_v, \sigma_v)$. From the explicit formula of the Weil representation, if $\lambda_v(f_v, \phi_v)$ is zero then so is $\lambda'_v(f_v, \phi_v)$. This induces a surjective intertwining operator $\Psi_v ; \mathcal{U}(T_v, \sigma_v) \to \mathcal{U}'(T_v, \sigma_v)$ characterized by

$$\Psi_v(\lambda_v(f_v, \phi_v)) = \lambda'_v(f_v, \phi_v).$$

For a smooth representation π of $G(k_v)$ (or $G(V_v) \times G(k_v)$) and $P = P_0, P_S, P_K$, π_P denotes the normalized Jacquet module with respect to P. If $v \notin S_D$ and T_v is anisotropic we obtain directly

$$(\omega_{\psi_v, T_v})_{P_K} \simeq \zeta_{T_v} | \cdot |_v^{-1} \otimes \omega_{\psi_v, T_v}^0$$

as a $O(T_v) \times M_K(k_v)$-module where $\zeta_{T_v} |\cdot|_v^{-1}$ is a representation of the first component of $k_v^\times \times SL(2, k_v) \simeq M_K(k_v)$. From this isomorphism, we have

$$\mathcal{U}(T_v, \sigma_v)_{P_K} \simeq \zeta_{T_v} | \cdot |_v^{-1} \otimes \Theta^0(T_v, \sigma_v). \tag{5.1}$$

Proposition 5.2. *Suppose that T_v is anisotropic.*

(1) If $v \notin S_D$ and $\sigma_v \neq \det$ then $\mathcal{U}(T_v, \sigma_v)$ is isomorphic to the unique irreducible quotient $J_K(\zeta_{V_v} | \cdot | \otimes \Theta^0(T_v, \sigma_v))$ of $\mathrm{Ind}_{P_K(k_v)}^{G(k_v)}(\zeta_{V_v} | \cdot |_v \otimes \Theta^0(T_v, \sigma_v))$.

(2) If $v \in S_D$ and $\sigma_v = \mathbf{1}$ then $\mathcal{U}(T_v, \sigma_v)$ is isomorphic to the unique irreducible quotient $J_S((\zeta_{V_v} | \cdot |_v^{1/2}) \circ \nu_{D_v})$ of $\mathrm{Ind}_{P_S(k_v)}^{G(k_v)}((\zeta_{V_v} | \cdot |_v^{1/2}) \circ \nu_{D_v})$.

In both cases, $\mathcal{U}(T_v, \sigma_v)$ is denoted by $\theta(V_v, \sigma_v)$.

Proof. Assume $\sigma_v = \mathbf{1}$. Let $R(V_v)$ denote the image of the map

$$\mathcal{S}(V_v) \ni f \mapsto (G(k_v) \ni g \mapsto \omega_{\psi_v,V_v} f(0)) \in \mathrm{Ind}_{P_S(k_v)}^{G(k_v)}((\zeta_{V_v}|\cdot|_v^{-1/2}) \circ \nu_{D_v}).$$

Then the $G(V_v)$-coinvariant space $\mathcal{S}(V_v)_{G(V_v)}$ of $\mathcal{S}(V_v)$ is isomorphic to $R(V_v)$ as a representation of $G(k_v)$ ([MVW87] Chap.3 IV Th.7). It is known that $R(V_v)$ is irreducible for all v ([KRS92] Prop.1.1, [Yas07] Prop.4.5). Since $\mathcal{U}(V_v, \sigma_v)$ is non-zero and a quotient of $\mathcal{S}(V_v)_{G(V_v)}$, it is isomorphic to $R(V_v)$. If $v \in S_D$, $\mathcal{U}(V_v, \sigma_v)$ is isomorphic to $J_S((\zeta_{V_v}|\cdot|_v^{1/2}) \circ \nu_{D_v})$ by [Yas07] Prop.4.5. If $v \notin S_D$ then $R(V_v)$ is isomorphic to an irreducible subrepresentation of $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(\zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v))$ because $\Theta^0(T_v, \sigma_v)$ is non-zero. As for Jacquet module, from (5.1) and [BZ77] § 2.12, we have

$$\begin{aligned} R(V_v)_{P_K} &= \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v), \\ (\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)} \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v))_{P_K} &= \zeta_{T_v} \otimes \mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)} \zeta_{T_v}|\cdot|_v \\ &\quad + \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v) + \zeta_{T_v}|\cdot|_v \otimes \Theta^0(T_v, \sigma_v) \end{aligned}$$

in the Grothendieck group. From [ST93] Prop.5.4, $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)} \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v)$ has three irreducible constituents. More precisely, all the constituents consist of

$$J_K(\zeta_{T_v}|\cdot|_v \otimes \Theta^0(T_v, \sigma_v)), \quad J_S(|\cdot|_v^{1/2} \zeta_{T_v} St_{GL(2)}), \quad \delta(T_v, \sigma_v).$$

Here,

- $J_S(|\cdot|_v^{1/2} \zeta_{T_v} St_{GL(2)})$ is the unique irreducible quotient of $\mathrm{Ind}_{P_S(k_v)}^{Sp(4,k_v)}(|\cdot|_v^{1/2} \zeta_{T_v} St_{GL(2)})$ where $St_{GL(2)}$ is the Steinberg representation of $GL(2, k_v)$ and $J_S(|\cdot|_v^{1/2} \zeta_{T_v} St_{GL(2)})_{P_K} = \zeta_{T_v} \otimes \mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)} \zeta_{T_v}|\cdot|_v$,
- $\delta(T_v, \sigma_v)$ is an irreducible tempered representation with Jacquet module $\zeta_{T_v}|\cdot|_v \otimes \Theta^0(T_v, \sigma_v)$,
- $J_K(\zeta_{T_v}|\cdot|_v \otimes \Theta^0(T_v, \sigma_v))_{P_K} \simeq \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v)$.

Therefore, $R(V_v)$ is isomorphic to $J_K(\zeta_{T_v}|\cdot|_v \otimes \Theta^0(T_v, \sigma_v))$. Next assume $v \notin S_D$, $\sigma_v \neq \mathbf{1}, \det$. Since σ_v is supercuspidal, $\Omega(V_v, \sigma_v)$ is irreducible by [MVW87] Chap.3 IV Th.4 1) and so is $\mathcal{U}(T_v, \sigma_v)$. Since $\Theta^0(T_v, \sigma_v)$ is supercuspidal, as for Jacquet module we have

$$(\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)} \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v))_{P_K} = \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v) + \zeta_{T_v}|\cdot|_v \otimes \Theta^0(T_v, \sigma_v)$$

in the Grothendieck group. Since $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)} \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v)$ has length at most 2, from (5.1), $\Omega(V_v, \sigma_v)$ is isomorphic to the unique irreducible subrepresentation of $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)} \zeta_{T_v}|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v)$. □

If $v \notin S_D$ and T_v is isotropic, there is a smooth subrepresentation $\mathcal{F}$ of the

Jacquet module $(\omega_{\psi_v,T_v})_{P_K}$ as a representation of $O(1,1;k_v)\times M_K(k_v)$ such that

$$(\omega_{\psi_v,T_v})_{P_K}/\mathcal{F}\simeq|\cdot|_v^{-1}\otimes\omega^0_{\psi_v,T_v},$$
$$\mathcal{F}\simeq(\mathrm{Ind}^{O(1,1;k_v)\times\mathbb{G}_m(k_v)}_{SO(1,1;k_v)\times\mathbb{G}_m(k_v)}\tau_v)\otimes\mathbf{1}_{SL(2)}$$

where $|\cdot|_v^{-1}$ is a representation of the first component of $\mathbb{G}_m(k_v)\times SL(2,k_v)\simeq M_K(k_v)$ and τ_v is the representation of $SO(1,1;k_v)\times\mathbb{G}_m(k_v)\simeq k_v^\times\times k_v^\times$ defined on $\mathcal{S}(GL(1,k_v))$ by

$$\tau_v(c,a)\phi(y)=\phi(c^{-1}ya)\quad(c,a,y\in k_v^\times,\phi\in\mathcal{S}(GL(1,k_v)))$$

(See [MVW87] Chap.3 IV or [Kud86] Th.2.8). Note that

$$\mathrm{Ind}^{O(1,1;k_v)\times\mathbb{G}_m(k_v)}_{SO(1,1;k_v)\times\mathbb{G}_m(k_v)}\tau_v\simeq\mathcal{S}(O(1,1;k_v))$$

where a representation on $\mathcal{S}(O(1,1;k_v))$ is given by the left and right regular action of $O(1,1;k_v)\times\mathbb{G}_m(k_v)$ (or $O(1,1;k_v)\times O(1,1;k_v)$).

Lemma 5.3.

$$\begin{aligned}\mathcal{U}(T_v,\sigma_v)_{P_K}=&\sigma_v|_{\mathbb{G}_m(k_v)}\otimes\mathbf{1}_{SL(2)}+|\cdot|_v^{-1}\otimes\Theta^0(T_v,\sigma_v)\\&+|\cdot|_v^{-1}\otimes(\textit{a supercuspidal representation of }SL(2,k_v)),\end{aligned}$$

in the Grothendieck group.

Remark 5.4. *The supercuspidal representation appearing in the lemma may be zero.*

Proof. For a smooth representation ρ of $O(1,1;k_v)\times M_K(k_v)$, $\mathcal{N}(\rho,\sigma_v)$ denotes the intersection of the kernels of f which belong to $\mathrm{Hom}_{O(1,1;k_v)}(\rho,\sigma_v)$. Then $\rho_{\sigma_v}:=\rho/\mathcal{N}(\rho,\sigma_v)$ becomes the maximal σ_v-isotypic quotient of ρ, which is isomorphic to $\sigma_v\otimes\pi(\tau,\sigma_v)$ where $\pi(\tau,\sigma_v)$ is a smooth representation of $M_K(k_v)$ ([MVW87] Chap.2 III). In general, for an exact sequence of smooth representations of $O(1,1;k_v)\times M_K(k_v)$,

$$0\to\rho_1\to\rho_2\to\rho_3\to 0,$$

a following exact sequence is obtained.

$$\rho_{1,\sigma_v}\to\rho_{2,\sigma_v}\to\rho_{3,\sigma_v}\to 0.$$

This implies that there is an exact sequence,

$$\pi(\rho_1,\sigma_v)\xrightarrow{\iota}\pi(\rho_2,\sigma_v)\to\pi(\rho_3,\sigma_v)\to 0.$$

Now let us set $\rho_1=\mathcal{F}$, $\rho_2=(\omega_{\psi_v,T_v})_{P_K}$, $\rho_3=(\omega_{\psi_v,T_v})_{P_K}/\mathcal{F}$. For $\phi_v(\neq 0)\in\sigma_v$, a surjective map

$$\omega_{\psi_v,T_v}\ni f_v\mapsto\lambda'_v(f_v,\phi_v)\in\mathcal{U}(T_v,\sigma_v)$$

can be reduced to

$$\overline{\lambda}_{\phi_v}:(\omega_{\psi_v,T_v})_{P_K}\to\mathcal{U}(T_v,\sigma_v)_{P_K}$$

by definition. For $\xi \in T_v$ such that $q_{T_v}(\xi) \neq 0$, put

$$\lambda^1_{\phi_v}(f_v)(a) = \int_{O(1,1;k_v)} f_v(ha)\overline{\phi_v(h)}dh \quad (f_v \in \mathcal{S}(O(1,1;k_v)), a \in k_v^\times),$$

$$\lambda^3_{\phi_v}(f'_v)(g) = \int_{O(1,1;k_v)} \omega^0_{\psi_v,T_v}(h,g)f'_v(\xi)\overline{\phi_v(h)}dh \quad (f'_v \in \mathcal{S}(T_v), g \in SL(2,k_v)),$$

where these integral converge absolutely. $\mathcal{U}^1_K(T_v,\sigma_v)$ (resp. $\mathcal{U}^3_K(T_v,\sigma_v)$) denotes the space generated by functions $\lambda^1_{\phi_v}(f_v)$ for all $f_v \in \mathcal{S}(O(1,1;k_v))$ (resp. $\lambda^3_{\phi_v}(f'_v)$ for all $f'_v \in \mathcal{S}(T_v)$). ρ_1 and ρ_3 provide $\mathcal{U}^1_K(T_v,\sigma_v)$ and $\mathcal{U}^3_K(T_v,\sigma_v)$ with structure of representations of $M_K(k_v)$. Then $\mathcal{U}^3_K(T_v,\sigma_v)$, which is isomorphic to $|\cdot|_v^{-1} \otimes \Theta^0(T_v,\sigma_v)$, is a quotient of $\mathcal{U}(T_v,\sigma_v)_{P_K}$ and the following diagram is commutative;

$$\begin{array}{ccccc} 0 & \longrightarrow & \rho_1 & \longrightarrow & \rho_2 \\ & & \downarrow{\scriptstyle \lambda^1_{\phi_v}} & & \downarrow{\scriptstyle \bar{\lambda}_{\phi_v}} \\ 0 & \longrightarrow & \mathcal{U}^1_K(T_v,\sigma_v) & \longrightarrow & \mathcal{U}(T_v,\sigma_v)_{P_K}. \end{array} \tag{5.2}$$

Similarly to the observation before Proposition 5.1, there are surjective homomorphisms $\pi(\rho_i,\sigma_v) \to \mathcal{U}^i_K(T_v,\sigma_v)$ $(i = 1,3)$ and $\pi(\rho_2,\sigma_v) \to \mathcal{U}(T_v,\sigma_v)_{P_K}$. From (5.2), we have the following commutative diagram;

$$\begin{array}{ccccc} & & \pi(\rho_1,\sigma_v) & \longrightarrow & \pi(\rho_2,\sigma_v) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{U}^1_K(T_v,\sigma_v) & \longrightarrow & \mathcal{U}(T_v,\sigma_v)_{P_K}. \end{array}$$

Since

$$\dim_{\mathbf{C}} \mathrm{Hom}_{O(1,1;k_v)\times O(1,1;k_v)}(\tau_v, \sigma_v^\vee \otimes \pi_v) = \begin{cases} 1 & \pi_v \simeq \sigma_v, \\ 0 & \text{otherwise}, \end{cases}$$

and $\lambda^1_{\phi_v}$ is non-zero, one concludes that ι is injective and

$$\iota(\pi(\rho_1,\sigma_v)) \simeq \pi(\rho_1,\sigma_v) \simeq \mathcal{U}^1_K(T_v,\sigma_v) \simeq \sigma_v|_{\mathbb{G}_m(k_v)} \otimes \mathbf{1}_{SL(2)}.$$

Since $(\omega^0_{\psi_v,T_v})_{B^{SL(2)}} = \mathbf{1}\otimes\mathbf{1} + \tau_v$ (See [MVW87] Chap.3 IV or [Kud86] Th.2.8) ,

$$\pi((\omega^0_{\psi_v,T_v})_{B^{SL(2)}},\sigma_v) = \Theta^0(T_v,\sigma_v)_{B^{SL(2)}} = \begin{cases} \mathbf{1}+\mathbf{1} & \sigma_v = \mathbf{1}, \\ \sigma_v|_{\mathbb{G}_m(k_v)} & \sigma_v \neq \mathbf{1}, \end{cases}$$

in the Grothendieck group. Since $\Theta^0(T_v,\sigma_v)$ is a quotient of $\pi(\omega^0_{\psi_v,T_v},\sigma_v)$,

$$\pi(\omega^0_{\psi_v,T_v},\sigma_v) = \Theta^0(T_v,\sigma_v) \oplus (\text{a supercuspidal representation})$$

by (an analogy for $SL(2)$ of) [BZ76] Th.4.17. Consequently,

$$\begin{aligned}\pi((\omega_{\psi_v,T_v})_{P_K},\sigma_v) =& \sigma_v|_{\mathbb{G}_m(k_v)} \otimes \mathbf{1}_{SL(2)} + |\cdot|_v^{-1} \otimes \Theta^0(T_v,\sigma_v) \\ &+ |\cdot|_v^{-1} \otimes (\text{a supercuspidal representation of } SL(2,k_v))\end{aligned}$$

in the Grothendieck group. Since $\mathcal{U}(T_v,\sigma_v)_{P_K}$ is a quotient of $\pi((\omega_{\psi_v,T_v})_{P_K},\sigma_v)$ and $\mathcal{U}^1_K(T_v,\sigma_v), \mathcal{U}^3_K(T_v,\sigma_v)$ are subquotients of $\mathcal{U}(T_v,\sigma_v)_{P_K}$, the proposition is obtained. □

For a quasi-character μ_v of $k_v^\times$ and a smooth representation τ_v of $SL(2,k_v)$,

$$(\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(\mu_v\otimes\tau_v))_{P_K}=\mu_v\otimes\tau_v+\mu_v^\vee\otimes\tau_v+\tau_{v\,B^{SL(2)}}\otimes(\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\mu_v)$$

in the Grothendieck group ([BZ77] § 2.12).

Proposition 5.5. *Let $v\notin S_D$ and T_v is isotropic. Then $\mathcal{U}(T_v,\sigma_v)$ has an irreducible quotient isomorphic to*

$$\theta(T_v,\sigma_v):=\begin{cases} J_K(|\cdot|_v\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\chi_v) & \sigma_v=\mathrm{Ind}_{SO(1,1;k_v)}^{O(1,1;k_v)}\chi_v,\chi_v^2\neq\mathbf{1},\\ J_K(|\cdot|_v\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\mathbf{1}) & \sigma_v=\mathbf{1},\\ J_K(|\cdot|_v\otimes\Theta^0(T_v,\sigma_v)) & \sigma_v=\tilde{\chi}_v^\epsilon,\ \chi_v^2=\mathbf{1},\ \chi_v\neq\mathbf{1},\ \epsilon=\pm1,\\ J_S(|\cdot|_v^{1/2}St_{GL(2)}) & \sigma_v=\det,\end{cases}$$

where $St_{GL(2)}$ is the Steinberg representation of $GL(2,k_v)$ and $J_K(\tau)$ (resp. $J_S(\tau)$) denotes the Langlands quotient of $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}\tau$ (resp. $\mathrm{Ind}_{P_S(k_v)}^{G(k_v)}\tau$).

Proof. By Lemma 5.3 and [BZ76] Th.4.17, $\mathcal{U}(V_v,\sigma_v)_{P_K}$ has a quotient equal to

$$\begin{cases} |\cdot|_v^{-1}\otimes\Theta^0(V_v,\sigma_v)+\chi_v\otimes\mathbf{1}_{SL(2)}+\chi_v^{-1}\otimes\mathbf{1}_{SL(2)} & (\chi_v^2\neq\mathbf{1}),\\ |\cdot|_v^{-1}\otimes\Theta^0(V_v,\sigma_v)+\chi_v\otimes\mathbf{1}_{SL(2)} & (\chi_v^2=\mathbf{1})\end{cases}$$

in the Grothendieck group. If $\chi_v^2\neq\mathbf{1}$, $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}\chi_v\otimes\mathbf{1}_{SL(2)}$ is irreducible from [ST93] Th.5.4 (ii), and $J_K(|\cdot|_v\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\chi_v)$ is isomorphic to $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}\chi_v\otimes\mathbf{1}_{SL(2)}$. Therefore, in the Grothendieck group

$$\begin{aligned} J_K(|\cdot|_v\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\chi_v)_{P_K} &= (\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}\chi_v\otimes\mathbf{1}_{SL(2)})_{P_K}\\ &= |\cdot|_v^{-1}\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\chi_v+\chi_v\otimes\mathbf{1}_{SL(2)}+\chi_v^{-1}\otimes\mathbf{1}_{SL(2)} \quad (\chi_v^2\neq\mathbf{1}).\end{aligned}$$

If $\chi_v^2=\mathbf{1},\chi_v\neq\mathbf{1}$, in the Grothendieck group

$$\begin{aligned} \mathrm{Ind}_{P_0(k_v)}^{Sp(4,k_v)}(|\cdot|_v\otimes\chi_v) &= \mathrm{Ind}_{P_0(k_v)}^{Sp(4,k_v)}(|\cdot|_v\otimes\delta(\chi_v,+))+\mathrm{Ind}_{P_0(k_v)}^{Sp(4,k_v)}(|\cdot|_v\otimes\delta(\chi_v,-))\\ &= \mathrm{Ind}_{P_0(k_v)}^{Sp(4,k_v)}(\chi_v\otimes\mathbf{1}_{SL(2)})+\mathrm{Ind}_{P_0(k_v)}^{Sp(4,k_v)}(\chi_v\otimes St_{SL(2)})\end{aligned}$$

where $\delta(\chi_v,\pm)$ is irreducible representations of $SL(2,k_v)$ such that

$$\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\chi_v\simeq\delta(\chi_v,+)\oplus\delta(\chi_v,-)$$

and $St_{SL(2)}$ is the Steinberg representation of $SL(2,k_v)$. The length of $\mathrm{Ind}_{P_0(k_v)}^{Sp(4,k_v)}(|\cdot|_v\otimes\chi_v)$ is 4 from [ST93] Th.5.4. By comparison of Jacquet modules, we see that the semisimplification of $J_K(|\cdot|_v\otimes\delta(\chi_v,\pm))_{P_K}$ coinsides with the common composition factor of $(\mathrm{Ind}_{P_0(k_v)}^{Sp(4,k_v)}|\cdot|_v\otimes\delta(\chi_v,\pm))_{P_K}$ and $(\mathrm{Ind}_{P_0(k_v)}^{Sp(4,k_v)}\chi_v\otimes\mathbf{1}_{SL(2)})_{P_K}$. Therefore,

$$J_K(|\cdot|_v\otimes\delta(\chi_v,\pm))_{P_K}=|\cdot|_v^{-1}\otimes\delta(\chi_v,\pm)+\chi_v\otimes\mathbf{1}_{SL(2)}\ (\chi_v^2=\mathbf{1},\chi_v\neq\mathbf{1})$$

in the Grothendieck group. If $\sigma_v\neq\det$, a non-zero quotient of $\mathcal{U}(T_v,\sigma_v)$ is isomorphic to a subrepresentation of $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(|\cdot|_v^{-1}\otimes\Theta^0(T_v,\sigma_v))$ because $\Theta^0(T_v,\sigma_v)\neq0$. If $\chi_v\neq\mathbf{1}$, from the Jacquet module of $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(|\cdot|_v^{-1}\otimes\Theta^0(T_v,\sigma_v))$ and the description of the above Jacquet modules , we have that $J_K(|\cdot|_v\otimes\Theta^0(T_v,\sigma_v))$ is a quotient of $\mathcal{U}(T_v,\sigma_v)$. Assume $\sigma_v=\mathbf{1}$. $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(\mathbf{1}\otimes\mathbf{1}_{SL(2)})$ is a subrepresentation of $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(|\cdot|_v^{-1}\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\mathbf{1})$. From the comparison of Jacquet modules, there is a non-zero inetertwining operator from $\mathcal{U}(T_v,\sigma_v)$ to $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(\mathbf{1}\otimes\mathbf{1}_{SL(2)})$. From [ST93] Lem.3.8 and [Tad94] Lem.6.2,

$$\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(\mathbf{1}\otimes\mathbf{1}_{SL(2)})\simeq J_K(|\cdot|_v\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\mathbf{1})\oplus J_S(|\cdot|_v^{1/2}St_{GL(2)}). \tag{5.3}$$

Since the Jacquet module of the image of $\mathcal{U}(T_v,\sigma_v)$ does not contain $J_S(|\cdot|_v^{1/2}St_{GL(2)})_{P_K}$, $J_K(|\cdot|_v\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\mathbf{1})$ is a quotient of $\mathcal{U}(T_v,\sigma_v)$. Finally, consider $\sigma_v=\det$. Since $\mathcal{U}(T_v,\sigma_v)_{P_K}$ has a quotient isomorphic to $\mathbf{1}\otimes\mathbf{1}_{SL(2)}$, there is a non-zero intertwining operator from $\mathcal{U}(T_v,\sigma_v)$ to $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(\mathbf{1}\otimes\mathbf{1}_{SL(2)})$ by the Frobenius reciprocity. It is seen that $J_K(|\cdot|_v\otimes\mathrm{Ind}_{B^{SL(2)}(k_v)}^{SL(2,k_v)}\mathbf{1})$ is not an irreducible constituent of the image of $\mathcal{U}(T_v,\sigma_v)$ from the Jacquet modules. Therefore, by (5.3) one concludes that a quotient of $\mathcal{U}(T_v,\sigma_v)$ is isomorphic to $J_S(|\cdot|_v^{1/2}St_{GL(2)})$. □

5.2 Archimedean case

Let v be an archimedean place. $\mathfrak{g}(V_v)$ and $\mathfrak{g}_v$ denote the complexifications of Lie algebras of $G(V_v)$ and $G(k_v)$. Let $S(V_v)\subset\mathcal{S}(V_v)$ be the space of Schwartz functions which correspond to polynomials in the Fock model of ω_{ψ_v,V_v}. For an irreducible admissible $(\mathfrak{g}(V_v),\mathbf{L}_v)$-module σ_v, let $\mathcal{N}_{\sigma_v}$ be the intersection of all subspaces $\mathcal{N}\subset S(V_v)$ such that $S(V_v)/\mathcal{N}\simeq\sigma_v$. Then there exists an admissible $(\mathfrak{g}_v,\mathbf{K}_v)$-module $\rho(\sigma_v)$ such that

$$S(V_v)/\mathcal{N}_{\sigma_v}\simeq\sigma_v\otimes\rho(\sigma_v)$$

as a $(\mathfrak{g}(V_v)\oplus\mathfrak{g}_v,\mathbf{L}_v\times\mathbf{K}_v)$-module. Furthermore, if ρ_v is non-zero, $\rho(\sigma_v)$ has a unique irreducible $(\mathfrak{g}_v,\mathbf{K}_v)$-quotient $\theta(V_v,\sigma_v)$ ([How89] Th.2.1). For $\sigma_v=\tilde{\chi}_v^\epsilon$ for a character χ_v and $\epsilon=\pm1$, $U(V_v,\sigma_v)$ is defined by the space generated by $\lambda_v(f_v,\phi_v)$ for all $f_v\in S(V_v)$ and all $\phi_v\in\tilde{\chi}_v^\epsilon$. Similarly to the non-arhimedean case, $U(V_v,\sigma_v)$ is a quotient of $\rho(\sigma_v)$. Therefore $\theta(V_v,\sigma_v)$ is a quotient of $U(V_v,\sigma_v)$.

Proposition 5.6. *If $v\notin S_D$ and $\sigma_v\neq\det$ then $\theta(V_v,\sigma_v)$ is isomorphic to the unique irreducible quotient $J_K(|\cdot|_v\otimes\Theta^0(T_v,\sigma_v))$ of $\mathrm{Ind}_{P_K(k_v)}^{Sp(4,k_v)}(\zeta_{T_v}|\cdot|_v\otimes\Theta^0(T_v,\sigma_v))$ where $T_v=T_{V_v}$.*

This proposition is obtained by [Pau05] Th.38 if v is real and [AB95] Prop.21, Th.2.8 if v is complex.

Proposition 5.7 ([LZ97] Cor.3.2 and [AB95] Prop.2.1)**.** *If $v\notin S_D$ is real, T_{V_v} is isotypic and $\sigma_v=\det$ then $\theta(V_v,\sigma_v)$ is isomorphic to an irreducible consituent of $\mathrm{Ind}_{P_S(\mathbf{R})}^{Sp(4,\mathbf{R})}(\mathrm{sgn}|\cdot|^{1/2})\circ\det$ where sgn is the sign character of $\mathbf{R}^\times$. If v is complex and $\sigma_v=\det$ then $\theta(V_v,\sigma_v)$ is isomorphic to the unique irreducible constituent of $\mathrm{Ind}_{P_0(\mathbf{C})}^{Sp(4,\mathbf{C})}(|\cdot|_{\mathbf{C}}^{1/2}\otimes|\cdot|_{\mathbf{C}}^{-1/2})$ containing the lowest $\mathbf{K}_v$-type of the induction.*

When $v \notin S_D$ and v is real we choose a Cartan subgroup T of $G(k_v) = Sp(4, \mathbf{R})$ with Lie algebra $\mathfrak{t}_{v,0}$ and complexification $\mathfrak{t}_v$ as follows:

$$\mathfrak{t}_{v,0} = \mathfrak{t}_v \cap \mathbb{M}(4, \mathbf{R}) \subset \mathfrak{sp}(4, \mathbf{R}),$$

$$\mathfrak{t}_v = \left\{ t(a_1, a_2) = \left(\begin{array}{cc|cc} & & a_1 & \\ & & & a_2 \\ \hline -a_1 & & & \\ & -a_2 & & \end{array} \right) \middle| \, a_1, a_2 \in \mathbf{C} \right\}.$$

When writing $e_i : \mathfrak{t}_v \ni t(a_1, a_2) \mapsto a_i \in \mathbf{C}$ $(i = 1, 2)$, the roots of $\mathfrak{t}_v$ in $\mathfrak{g}_v = \mathfrak{sp}(4, \mathbf{C})$ are

$$\Delta(\mathfrak{g}_v, \mathfrak{t}_v) = \{\pm e_1 \pm e_2, \pm 2e_1, \pm 2e_2\}.$$

Ψ_+ (resp. Ψ_-) $\subset \Delta(\mathfrak{g}_v, \mathfrak{t}_v)$ denotes the set of positive roots defined by simple roots $e_1 - e_2, 2e_2$ (resp. $e_1 - e_2, -2e_1$). If $\lambda \in \sqrt{-1}\mathfrak{t}_{v,0}^*$ is dominant with respect to $\Psi_\pm$, the limit of discrete series defined by λ and $\Psi_\pm$ is denoted by $\delta(\lambda, \Psi_\pm)$. If $v \in S_D$, $G(\mathbf{R})$ is realized by

$$Sp(1,1) = \left\{ g \in Sp(4, \mathbf{C}) \,|\, g I_{1,1} {}^t\overline{g} = I_{1,1} \right\}, \quad I_{1,1} = \mathrm{diag}(1, -1, 1, -1).$$

We choose a Cartan subgroup T of $G(k_v)$ with Lie algebra $\mathfrak{t}_{v,0}$ and complexification $\mathfrak{t}_v$ as follows:

$$\mathfrak{t}_{v,0} = \mathfrak{t}_v \cap \sqrt{-1}\mathbb{M}(4, \mathbf{R}) \subset \mathfrak{g}_v,$$

$$\mathfrak{t}_v = \left\{ t(a_1, a_2) = \left(\begin{array}{cc|cc} a_1 & & & \\ & a_2 & & \\ \hline & & -a_1 & \\ & & & -a_2 \end{array} \right) \middle| \, a_1, a_2 \in \mathbf{C} \right\}.$$

the roots of $\mathfrak{t}_v$ in $\mathfrak{g}_v$ are

$$\Delta(\mathfrak{g}_v, \mathfrak{t}_v) = \{\pm e_1 \pm e_2, \pm 2e_1, \pm 2e_2\}$$

where $e_i : \mathfrak{t}_v \ni t(a_1, a_2) \mapsto a_i \in \mathbf{C}$ $(i = 1, 2)$. Ψ_+ (resp. Ψ_-) $\subset \Delta(\mathfrak{g}_v, \mathfrak{t}_v)$ denotes the set of positive roots defined by simple roots $e_1 - e_2, 2e_2$ (resp. $-e_1 + e_2, 2e_1$). If $\lambda \in \sqrt{-1}\mathfrak{t}_0^*$ is dominant with respect to $\Psi_\pm$, the limit of discrete series defined by λ and $\Psi_\pm$ is denoted by $\delta(\lambda, \Psi_\pm)$. If $v \in S_D$, $G(V_v)$ is isomorphic to $O^*(2) \simeq \mathbf{C}^1$, so that a character of $G(V_v)$ is identified with that of $\mathbf{C}^1$.

Proposition 5.8. *Let* $\psi_v = \exp(\lambda \cdot\,)$ *for* $\lambda = \epsilon\sqrt{-1}|\lambda| \in \sqrt{-1}\mathbf{R}$ $\epsilon = \pm 1$.

(1) If $v \notin S_D$, v *is real and* $T_v = T_{V_v}$ *is anisotropic then*

$$\theta(V_v, \det) \simeq \begin{cases} \delta((1,0), \Psi_+) & \epsilon\eta > 0, \\ \delta((0,-1), \Psi_-) & \epsilon\eta < 0. \end{cases}$$

where $\eta = 1$ *if* T_v *is positive definite,* -1 *otherwise.*

(2) Assume $v \in S_D$ *and* v *is real. If* $\sigma_v = \mathbf{1}$ *then* $\theta(V_v, \sigma_v) \simeq \mathrm{Ind}_{P_S(\mathbf{R})}^{G(\mathbf{R})}(|\nu_{D_v}|_{\mathbf{R}}^{1/2})$. *If*

$\sigma_v = \exp(2\pi\sqrt{-1}n\,\cdot\,)$ $(n \in \mathbf{Z}\backslash\{0\})$ *then*

$$\theta(V_v,\sigma_v) \simeq \begin{cases} \delta((|n|,1),\Psi_+) & \epsilon n > 0, \\ \delta((1,|n|),\Psi_-) & \epsilon n < 0. \end{cases}$$

In particular, if $|n|>1$, $\theta(V_v,\sigma_v)$ *is in the discrete series.*

To prove this proposition we recall the Fock model. For $n \in \mathbf{N}$, we define the Fock space

$$\mathcal{F} = \left\{ F : \mathbf{C}^n \to \mathbf{C},\ \text{entire} \;\middle|\; \|F\|_c^2 := \int_{\mathbf{C}^n} |F(z)|^2 e^{-\frac{1}{2}|z|^2} d_c z < +\infty \right\}$$

where $d_c z$ is the self-dual measure with respect to $\mathbf{C}^n \times \mathbf{C}^n \ni (z_1,z_2) \mapsto e^{\sqrt{-1}\mathrm{Re}(z_1\cdot\overline{z_2})} \in \mathbf{C}^1$. Then $\mathcal{F}$ is a Hilbert space by the inner product given by $\|\cdot\|_c$. For $f \in L^2(\mathbf{R}^n)$, put

$$\mathcal{B}f(z) = \int_{\mathbf{R}^n} f(x)\,B(x,z)\,d_\psi x \qquad (z \in \mathbf{C}^n),$$

$$B(x,z) = 2^{\frac{n}{4}} \cdot \exp\left(-\frac{1}{2}|\lambda|x^2 + |\lambda|^{\frac{1}{2}}x\cdot z - \frac{1}{4}z^2\right) \qquad (x \in \mathbf{R}^n,\ z \in \mathbf{C}^n)$$

where $d_\psi x$ is the self-dual measure with respect to $X \times X \ni (x,y) \mapsto \psi(x\cdot {}^t y)$. The L^2-norm $\|\cdot\|_{L^2}$ on $L^2(\mathbf{R}^n)$ is defined by $d_\psi x$. Then $\mathcal{B}$ becomes an isomorphism from $(L(\mathbf{R}^n), \|\cdot\|_{L^2})$ to $(\mathcal{F}, \|\cdot\|_c)$ ([Fol89] Chap.1 § 6). The Weil representation ω_{ψ_v} of the metaplectic cover of $Sp(2n,\mathbf{R})$ is defined on $L(\mathbf{R}^n)$ as a unitary representation. By the transfer by $\mathcal{B}$, the Weil representation can be also realized on $\mathcal{F}$, which is denoted by $\hat{\omega}_{\psi_v}$. The Harish-Chandra module of $\hat{\omega}_{\psi_v}$ is $\mathbf{C}[z_1,\ldots,z_n] \subset \mathcal{F}$ and the explicit formula of $\hat{\omega}_{\psi_v}$ is described as a $(\mathfrak{sp}(2n,\mathbf{C}) \oplus \mathbf{C}, U(n) \times \mathbf{C}^1)$-module as follows:

(1) $\hat{\omega}_{\psi_v}((0,t)) = t \cdot \mathrm{Id}_{\mathbf{C}[z_1,\ldots,z_n]} \qquad (t \in \mathbf{C})$,

(2) $\hat{\omega}_{\psi_v}(\frac{1}{2}\left(\begin{array}{c|c} E_{i,j}-E_{j,i} & -\epsilon\sqrt{-1}(E_{i,j}+E_{j,i}) \\ \hline \epsilon\sqrt{-1}(E_{i,j}+E_{j,i}) & E_{i,j}-E_{j,i} \end{array}\right)) = z_i\dfrac{\partial}{\partial z_j} + \dfrac{1}{2}\ \delta_{i,j} \quad (1 \le i,j \le n)$,

(3) $\hat{\omega}_{\psi_v}(\frac{1}{2}\left(\begin{array}{c|c} -\epsilon\sqrt{-1}(E_{i,j}+E_{j,i}) & E_{i,j}+E_{j,i} \\ \hline E_{i,j}+E_{j,i} & \epsilon\sqrt{-1}(E_{i,j}+E_{j,i}) \end{array}\right)) = \epsilon\sqrt{-1}\ z_i z_j \qquad (1 \le i,j \le n)$,

(4) $\hat{\omega}_{\psi_v}(\frac{1}{2}\left(\begin{array}{c|c} \epsilon\sqrt{-1}(E_{i,j}+E_{j,i}) & E_{i,j}+E_{j,i} \\ \hline E_{i,j}+E_{j,i} & -\epsilon\sqrt{-1}(E_{i,j}+E_{j,i}) \end{array}\right)) = \epsilon\sqrt{-1}\dfrac{\partial^2}{\partial z_i \partial z_j} \quad (1 \le i,j \le n)$,

(5) $\hat{\omega}_{\psi_v}(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \eta) f(z) = \eta f(z(a+\epsilon\sqrt{-1}b)) \quad ((\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \eta) \in U(n) \times \mathbf{C}^1,\ f \in \mathbf{C}[z_1,\ldots,z_n])$.

Here $E_{i,j} = (\delta_{i,k}\delta_{j,l})_{k,l}$ and (2), (3), (4) are the actions with respect to a basis to

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| A \in \mathrm{Alt}(n, \mathbf{C}), B \in \mathrm{Sym}(n, \mathbf{C}) \right\},$$

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} -\epsilon\sqrt{-1}B & B \\ B & \epsilon\sqrt{-1}B \end{pmatrix} \middle| B \in \mathrm{Sym}(n, \mathbf{C}) \right\},$$

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} \epsilon\sqrt{-1}B & B \\ B & -\epsilon\sqrt{-1}B \end{pmatrix} \middle| B \in \mathrm{Sym}(n, \mathbf{C}) \right\},$$

where $\mathfrak{sp}(2n, \mathbf{C}) = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ and $\mathfrak{k}$ is the complexification of the Lie algebra of a maximal compact subgroup of $Sp(2n, \mathbf{R})$.

In our case, $n = 8$ and the transfer $\hat{\omega}_{\psi_v,V_v}$ on $\mathbf{C}[z_1, \ldots, z_4]$ of ω_{ψ_v,V_v} as a $(\mathfrak{g}(V_v) \oplus \mathfrak{g}_v, \mathbf{L}_v \times \mathbf{K}_v)$-module coincides with the composition of $\hat{\omega}_{\psi_v}$ and the splitting,

- $d\iota : \mathfrak{g}(V_v) \oplus \mathfrak{g}_v \to \mathfrak{sp}(8, \mathbf{R})$,
- $\mathbf{L}_v \times \mathbf{K}_v \ni (l, k) \mapsto (\iota(l, k), 1) \in U(4) \times \mathbf{C}^1$,

where $\iota : G(V_v) \times G(k_v) \to Sp(8, \mathbf{R})$ is the natural homomorphism, $d\iota$ its differential, and $U(4)$ is identified with the maximal compact subgroup of $Sp(8, \mathbf{R})$.

Let us prove Proposition 5.8. If $v \in S_D$ and $\sigma_v = \mathbf{1}$, the statement has already been shown by [Yas07] Prop.4.7. In the other case, we will prove only the case of $v \in S_D$. For the case of $v \notin S_D$ is similarly proven to the case of $v \in S_D$ and $\sigma_v = \exp(\pm 2\pi\sqrt{-1}\,\cdot)$. (And it is proven for $\epsilon > 0, \eta > 0$ in [Ada04] Th.4.1.) Since for

$$Y_t = \begin{pmatrix} & -t \\ t & \end{pmatrix} \in \mathrm{Lie}\, O^*(2) \ (t \in \mathbf{R}),$$

$$\hat{\omega}_{\psi_v,V_v}(Y_t) = \sqrt{-1}\epsilon\eta t(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4}),$$

we have an isotypic decomposition of $\mathbf{C}[z_1, \ldots, z_4]$ as $\mathfrak{g}(V_v)$-modules,

$$\mathbf{C}[z_1, \ldots, z_4] = \oplus_{k \in \mathbf{Z}} \mathcal{P}_k,$$

$$\mathcal{P}_k = \bigoplus_{k_1 - k_2 = \epsilon\eta k} \mathbf{C}[z_1, z_2]^{(k_1)} \otimes \mathbf{C}[z_3, z_4]^{(k_2)},$$

where $\mathcal{P}_k$ is the isotypic space of a character $\exp(2\pi\sqrt{-1}k\,\cdot)$ of $G(V_v) \simeq \mathbf{C}^1$ and $\mathbf{C}[z_i, z_j]^{(l)}$ denotes the set of homogenuous polynomials of degree l. The actions of a basis of $\mathfrak{g}_v$ is described as follows:

(1) $$\hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}) = z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2},$$

(2) $$\hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}) = z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4},$$

$$(3)\quad \hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} & & 2 & \\ & & & 0 \\ 0 & & & \\ & 0 & & \end{pmatrix}) = 2\epsilon\eta\sqrt{-1}z_1\frac{\partial}{\partial z_2},$$

$$(4)\quad \hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} & & 0 & \\ & & & 2 \\ 0 & & & \\ & 0 & & \end{pmatrix}) = 2\epsilon\eta\sqrt{-1}z_3\frac{\partial}{\partial z_4},$$

$$(5)\quad \hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} & & 0 & \\ & & & 0 \\ 2 & & & \\ & 0 & & \end{pmatrix}) = -2\epsilon\eta\sqrt{-1}z_2\frac{\partial}{\partial z_1},$$

$$(6)\quad \hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} & & 0 & \\ & & & 0 \\ 0 & & & \\ & 2 & & \end{pmatrix}) = -2\epsilon\eta\sqrt{-1}z_4\frac{\partial}{\partial z_3},$$

$$(7)\quad \hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & -1 & 0 \end{pmatrix}) = -\frac{1}{2}(z_1z_4 - 4\frac{\partial^2}{\partial z_2 \partial z_3}),$$

$$(8)\quad \hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 0 & 0 \end{pmatrix}) = -\frac{1}{2}(z_2z_3 - 4\frac{\partial^2}{\partial z_1 \partial z_4}),$$

$$(9)\quad \hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ 0 & 0 & & \\ 0 & 0 & & \end{pmatrix}) = \frac{1}{2}\epsilon\eta\sqrt{-1}(z_3z_4 + 4\frac{\partial^2}{\partial z_1 \partial z_2}),$$

$$(10)\quad \hat{\omega}_{\psi_v,V_v}(\begin{pmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 1 & & \\ 1 & 0 & & \end{pmatrix}) = \frac{1}{2}\epsilon\eta\sqrt{-1}(z_1z_2 + 4\frac{\partial^2}{\partial z_3 \partial z_4}).$$

Using this description one can demonstrate the following:

- Every $\mathcal{P}_k$ is irreducible as a $\mathfrak{g}_v$-module.
- The set of $\mathbf{K}_v$-types in $\mathcal{P}_k$ is
$$\begin{cases} \{(\epsilon\eta k + l, l) \mid l \in \mathbf{Z}_{\geq 0}\} & \epsilon\eta k > 0, \\ \{(l, -\epsilon\eta k + l) \mid l \in \mathbf{Z}_{\geq 0}\} & \epsilon\eta k < 0. \end{cases}$$
The multiplicity of each $\mathbf{K}_v$-type is one.
- The infinitesimal character of $\mathcal{P}_k$ is associated to
$$\begin{cases} (\epsilon\eta k, 1) \in \mathfrak{t}^* & \epsilon\eta k > 0, \\ (1, -\epsilon\eta k) \in \mathfrak{t}^* & \epsilon\eta k < 0. \end{cases}$$
by the Harish-Chandra isomorphism.

In particular, if $|k| > 1$, $\mathcal{P}_k$ is the discrete series representation with the Harish-Chandra parameter $(\epsilon\eta k, 1)$ if $\epsilon\eta k > 0$ and $(1, -\epsilon\eta k)$ otherwise. If $\epsilon\eta k = \pm 1$, $\delta((1,1), \Omega_\pm)$ and $\mathcal{P}_k$ are irreducible highest weight modules with the same highest weight. Therefore, they are isomorphic.

6 Main theorem

Let V, χ, S be as in § 3.

Theorem 6.1. *There is an irreducible $G(\mathbf{A})$-subspace $\Theta^1(V,\chi,S)$ of $\Theta(V,\chi,S)$ such that*

$$\Theta^1(V,\chi,S) \simeq \bigotimes_v \theta(V_v, \sigma_v(V,\chi,S)).$$

Proof. From § 5, $\Theta(V,\sigma,S)$ has a quotient isomorphic to

$$\bigotimes_v \theta(V_v, \sigma_v(V,\chi,S)).$$

Since $\Theta(V,\sigma,S)$ is completely reducible, it must contain a subspace isomorphic to this representation. □

Theorem 6.2. *$\Theta^1(V,\chi,S)$ is a CAP representation with respect to P_K. More concretely, there is a set S' of places of k with finite cadinarity such that*

1. *$\Theta^0(T,\chi,S')$ is non-zero and irreducible cuspidal representation of $SL(2,\mathbf{A})$,*
2. *For almost all v, $\theta(V_v, \sigma_v(V,\chi,S))$ is isomorphic to the unique irreducible quotient of $\mathrm{Ind}_{P_K(k_v)}^{G(k_v)} \zeta_{V_v} |\cdot|_v \otimes \Theta_v^0(T_v, \sigma_v(V,\chi,S'))$.*

where T is the quadratic space on $k(\eta)$ defined by the norm form.

Proof. We can choose S' satisfying the first condition by Theorem 4.2. For almost all $v \notin S_D$, T_v is isometric to T_{V_v} because both the quadratic spaces have the same determinant and the Hasse invariant 1. From this and results of § 5, the second condition follows. □

We write $m(\Theta^1(V,\chi,S))$ for the multiplicity of $\Theta^1(V,\chi,S)$ in the discrete spectrum of $L^2(G(k)\backslash G(\mathbf{A}))$.

Theorem 6.3.

$$m(\Theta^1(V,\chi,S)) \geq \begin{cases} 2^{\sharp(S_\chi \cap S_D)-1} & S_D \not\subset S_\chi,\ S_D \cap S_\chi \neq \emptyset \\ 2^{\sharp S_D - 2} & S_D \subset S_\chi, \\ 1 & S_D \cap S_\chi = \emptyset. \end{cases}$$

To show this theorem, we make use of the failure of the Hasse's principle for (-1)-hermitian spaces over a quaternion division algebra. For $\eta \in D_- \backslash \{0\}$ let $k_{D,\eta}^\times = \{c \in k^\times \mid (\eta^2, c)_v = 1 \text{ for all } v \notin S_D\}$. A group homomorphism λ is defined by

$$k_{D,\eta}^\times \ni c \mapsto \{(\eta^2, c)_v\}_{v \in S_D} \in \{\pm 1\}^{\sharp S_D}.$$

Let $\{\pm 1\}$ be regarded as the subgroup of $\{\pm 1\}^{\sharp S_D}$ via the diagonal embedding. Note that the number of elements of $k_{D,\eta}^\times / \lambda^{-1}(\{\pm 1\})$ is $2^{\sharp S_D - 2}$.

Proposition 6.4. *[Sch85, Theorem 10.4.6, Remark 10.4.6] Let $\langle\eta\rangle$ be a (-1)-hermitian right D-space of dimension 1 defined by $\eta \in D_-\backslash\{0\}$. Then for any $c \in k_{D,\eta}^\times$, $\langle c\eta\rangle$ is locally isometric to $\langle\eta\rangle$. For any $a \in \lambda^{-1}(\{\pm 1\})$, $\langle a\eta\rangle$ is globally isometric to $\langle\eta\rangle$. $\{\langle a\eta\rangle \mid a \in k_{D,\eta}^\times/\lambda^{-1}(\{\pm 1\})\}$ is the set of classes locally isometric to $\langle\eta\rangle$, so that this set contains $2^{\sharp S_D - 2}$ elements.*

In the case of $v \in S_D$, the Weil representation ω_{ψ_v,V_v} of $G(V_v) \times G(k_v)$ is determined by not the isometry class of V_v but the choice of a basis of V_v. In fact, if x_1 and x_2 are elements in V_v not equal to each other, x_1 and x_2 define different Weil representations and the Howe correspondents of an irreducible representation of $G(V_v)$ with respect to these Weil representations do not need to be isomorphic. Here if $\eta_v = h_{V_v}(x,x)$ for $x \in V_v$, we write ω_{ψ_v,η_v} for the Weil representation defined by v instead of ω_{ψ_v,V_v}. Also write $\lambda_v(\eta_v, f_v, \phi_v)$ instead of $\lambda_v(f_v,\phi_v)$ of (3.5), and $\theta(\eta_v,\chi_v)$ instead of $\theta(\langle\eta_v\rangle,\chi_v)$.

Lemma 6.5. *Let $v \in S_D$ and $V_v \simeq \langle\eta_v\rangle$. For $c \in k_v^\times$, $f_v \in \mathcal{S}(V_v)$ and a unitary character χ_v of $G(V_v)$, there exists $f'_v \in \mathcal{S}(V_v)$ such that*

$$\lambda_v(\eta_v, f_v, \chi_v) = \lambda_v(c\eta_v, f'_v, \chi_v^c)$$

where

$$\chi_v^c = \begin{cases} \chi_v & (\eta_v^2, c)_v = 1, \\ \overline{\chi_v} & (\eta_v^2, c)_v = -1. \end{cases}$$

In particular, $\theta(\eta_v,\chi_v) \simeq \theta(c\eta_v, \chi_v^c)$.

Proof. If $(c,\eta_v^2) = 1$, there is a $\xi \in k(\eta) \subset D$ such that ${}^*\xi\,\xi = c$. If $(c,\eta_v^2) = -1$, there is a $\xi \in D_-$ such that $\xi^2 = c$ and $\xi\eta = -\eta\xi$. In both cases, $(\xi,\xi)_{V_v} = c\eta_v$, so that ξ defines the Weil representation $\omega_{\psi_v,c\eta_v}$ of $G(V_v)\times G(k_v)$. It is easily checked that $\omega_{\psi_v,c\eta_v}$ is realized on $\mathcal{S}(V_v)$ by

$$\omega_{\psi_v,c\eta_v}(h,g) = \omega_{\psi_v,\eta_v}(\xi^{-1}h\xi, g) \quad (h \in G(V_v), g \in G(k_v)).$$

Therefore for $g \in G(k_v)$,

$$\begin{aligned} \lambda(\eta_v, f_v, \chi_v)(g) &= \int_{G(V_v)} \omega_{\psi_v,\eta_v}(h,g) f(1_{D_v}) \chi_v(h)\,dh \\ &= \int_{G(V_v)} \omega_{\psi_v,\eta_v}(\xi^{-1}h'\xi, g) f_v(1_{D_v}) \chi_v(\xi^{-1}h'\xi)\,dh' \\ &= \int_{G(V_v)} \omega_{\psi_v,a\eta_v}(h', g) f_v(1_{D_v}) \chi_v^c(h')\,dh' \\ &= \lambda(a\eta_v, f_v, \chi_v^c)(g). \end{aligned}$$

□

Proof of Theorem 6.3.
If $v \in S_D$ and $\chi_v^2 = \mathbf{1}$, $\theta(\eta_0,\chi_v) \simeq \theta(c\eta_0,\chi_v)$ for any $c \in k^\times$ from Lemma 6.5. Therefore, if $a \in k_{D,\eta_0}^\times$ holds $(a,\eta_0^2)_v = 1$ for all $v \in S_D\backslash S_\chi$, $\Theta^1(V,\chi,S) \simeq \Theta^1(aV,\chi,S)$.

From Lemma 3.2, if $V_1, \dots, V_n$ are (-1)-hermitian spaces over D of dimension 1 not isometric to each other, in the space of automorphic space of $G(\mathbf{A})$

$$\Theta^1(V_1,\chi,S) + \cdots + \Theta^1(V_n,\chi,S) = \Theta^1(V_1,\chi,S) \oplus \cdots \oplus \Theta^1(V_n,\chi,S).$$

The image into $k_{D,\eta}^\times/\lambda^{-1}(\{\pm 1\})$ of the set of $a \in k_{D,\eta_0}^\times$ holding $(a,\eta_0^2)_v = 1$ for all $v \in S_D \backslash S_\chi$ is isomorphic to

$$\begin{cases} \{a \in k_{D,\eta}^\times/\ker\lambda \,|\, (a,\eta^2)_v = 1 \text{ for } v \in S_D\backslash S_\chi\} & S_D \not\subset S_\chi, \\ k_{D,\eta}^\times/\lambda^{-1}(\{\pm 1\}) & S_D \subset S_\chi. \end{cases}$$

The cardinality of this set is equal to the number in the right hand side of inequality in the theorem.

7 Multiplicity conjecture

7.1 Arthur's conjecture

In this section we attempt to explain expected value of the multiplicity appearing in Theorem 6.3 in view of the Arthur's multiplicity conjecture. Therefore, the argument in this section supposes some conjectures.

Let F be a local field of characteristic 0 and $\Gamma = \mathrm{Gal}(\overline{F}/F)$. The quasisplit inner form of G is $G^* = Sp(4)$. We have the following bijection [PR91].

$$\begin{array}{ccc} \{\text{inner forms of } G^*\}/\sim & \approx & H^1(F, {G^*}_{\mathrm{ad}}) \\ \cup & & \cup \\ G' & \longleftrightarrow & u_{G'} : \Gamma \ni \gamma \mapsto \eta_{G'}^{-1} \circ {}^\gamma\eta_{G'} \end{array}$$

Here $\sim$ means isomorphic equivalence and $\eta_{G'} : G^*(\overline{F}) \to G'(\overline{F})$ is an inner twist. In addition, if F is non-archimedean then from [Kot84] Prop.6.4

$$\begin{array}{ccc} H^1(F, {G^*}_{\mathrm{ad}}) & \approx & \pi_0(Z(\widehat{G^*}_{\mathrm{sc}})^\Gamma)^D \\ \cup & & \cup \\ u_{G'} & \longleftrightarrow & \hat{\zeta}_{G'}. \end{array}$$

Here $\widehat{G^*}_{\mathrm{sc}}$ is the simply connected cover of $\widehat{G^*} = \widehat{G} = SO(5,\mathbf{C})$ so that $\widehat{G^*}_{\mathrm{sc}} = Sp(4,\mathbf{C})$ and $(\ )^D$ means Pontrjagin dual. Write $j_{\mathrm{sc}} : \widehat{G^*}_{\mathrm{sc}} \to \widehat{G^*}$ for the covering map. The local Langlands group $\mathcal{L}_F$ is defined by

$$\mathcal{L}_F = \begin{cases} W_F \times SU(2,\mathbf{R}) & F : \text{non-archimedean}, \\ W_F & F : \text{archimedean}, \end{cases}$$

where W_F is the Weil group of F. By a (local) A-parameter is meant a continuous homomorphism $\phi : \mathcal{L}_F \times SL(2,\mathbf{C}) \to {}^LG = \widehat{G} \times W_F$ such that

(i) writing $p_F : \mathcal{L}_F \to W_F$ for the conjectural homomorphism and $p_2 : {}^LG \to W_F$ the projection to the second component, $p_2 \circ \phi = p_F$,

(ii) its restriction to $\mathcal{L}_F$ is a Langlands parameter with bounded image [Bor79], and

(iii) its restriction to $SL(2,\mathbf{C})$ is analytic.

We write C_ψ for the centralizer of the image of ψ in $\widehat{G}$. For a local A-parameter ψ and an inner form G' of G^* suppose the existence of local A-packet $\Pi_\psi^{G'}$ [Art89], which becomes a finite set of irreducible admissible representations of $G'(F)$. For a global or local A-parameter ψ, S_ψ denotes $j_{\mathrm{sc}}^{-1}(C_\psi)$. $\mathcal{S}_\psi$ is defined by $\pi_0(S_\psi)=S_\psi/S_\psi^0$. For an inner form G' of G^* the following condition is called the relevance condition for (G',ψ):

$$\operatorname{Ker}\hat{\zeta}_{G'}\supset Z(\widehat{G^*}_{\mathrm{sc}})^\Gamma\cap S_\psi^0.$$

Since

$$Z_\psi^\Gamma:=\operatorname{Im}(Z(\widehat{G^*}_{\mathrm{sc}})^\Gamma\to\mathcal{S}_\psi)\simeq Z(\widehat{G^*}_{\mathrm{sc}})^\Gamma/(Z(\widehat{G^*}_{\mathrm{sc}})^\Gamma\cap S_\psi^0),$$

if (G',ψ) satisfies the relevance condition then $\hat{\zeta}_{G'}$ can be regarded as a character of Z_ψ^Γ.

Conjecture 7.1 ([Art06] § 3)**.** *Let F be non-archimedean. For a local A-parameter $\psi:\mathcal{L}_F\times SL(2,\mathbf{C})\to {}^LG^*$ there exists a pairing*

$$\langle\ ,\ \rangle_F:\mathcal{S}_\psi\times\Big(\coprod_{G'\in H^1(F,G^*{}_{\mathrm{ad}})}\Pi_\psi^{G'}\Big)\to\mathbf{C}$$

which satisfies the following condition:

For any inner form G' of G^*, if (G',ψ) does not satisfiy relevance condition, $\Pi_\psi^{G'}=\emptyset$ and otherwise there exists

$$\begin{array}{cccc}\rho: & \Pi_\psi^{G'} & \longrightarrow & \Pi(\mathcal{S}_\psi,\hat{\zeta}_{G'})=\{\textit{irred. repre.}\,\sigma\textit{ of }\mathcal{S}_\psi\mid \sigma|_{Z_\psi^\Gamma}=\hat{\zeta}_{G'}\}/\!\sim\\ & \Psi & & \Psi\\ & \pi & \longmapsto & \rho_\pi\end{array}$$

such that $\langle s,\pi\rangle_F=\operatorname{Tr}\rho_\pi(s)$ for all $s\in\mathcal{S}_\psi$.

If F is non-archimedean then the set of inner forms of G^* consists of $G_F^s=Sp(4)$ and non-split group G_F^{ns}. If F is real it consists of G_F^s, $G_F^{ns}=Sp(1,1)$ and the compact group $Sp(4)$, and if F is complex it consists of only G_F^s. In any case put $\Pi_\psi^{\mathrm{s}}=\Pi_\psi^{G_F^s}$, $\Pi_\psi^{\mathrm{ns}}=\Pi_\psi^{G_F^{ns}}$ ($\Pi_\psi^{\mathrm{ns}}=\emptyset$ if F is complex). Since we do not treat the case that G coincides with the compact $Sp(4)$ at a real place, we will not consider the A-packet in the case.

Next consider the global case. Assume existence of the hypothetical Langlands group $\mathcal{L}_k$ of k. A global A-parameter is defined similarly to the local one. Two A-parameters are equivalent if they are $\widehat{G}$-conjugate. An A-parameter ψ is said to be elliptic if the centralizer C_ψ of the image of ψ into $\widehat{G}$ is contained in the center $Z(\widehat{G})$ of $\widehat{G}$. The set of equivalence classes of elliptic A-parameters is denoted by $\Psi_0(G)$. For an elliptic A-parameter ψ, the associated local A-parameter ψ_v is given for any place v by the hypothetical homomorphism $\mathcal{L}_{k_v} \to \mathcal{L}_k$. $\mathcal{S}_\psi$ is defined similarly to the local one. Then a homomorphism $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$ is given. Assume that the pairing $\langle\, ,\, \rangle_v : \mathcal{S}_{\psi_v} \times (\Pi^{\mathrm{s}}_{\psi_v} \sqcup \Pi^{\mathrm{ns}}_{\psi_v}) \to \mathbf{C}$ satisfying Conjecture 7.1 is given for any v . Then the global pairing $\langle\, ,\, \rangle = \prod_v \langle\, ,\, \rangle_v : \mathcal{S}_\psi \times \Pi^G_\psi \to \mathbf{C}$ is defined if the product exist. Let $\epsilon_\psi : \mathcal{S}_\psi \to \{\pm 1\}$ be the character defined in [Art89]. Π^G_ψ is defined by the set of $\pi = \otimes \pi_v$ such that $\pi_v \in \Pi^{G(k_v)}_{\psi_v}$ and π_v is unramified for almost all v. For $\pi \in \Pi^G_\psi$ set

$$m_\psi(\pi) = \frac{1}{|\mathcal{S}_\psi|} \sum_{s \in \mathcal{S}_\psi} \epsilon_\psi(s) \langle s, \pi \rangle.$$

The Arthur's multiplicity conjecture is described as follows.

Conjecture 7.2 ([Art89] Conj.8.1)**.** *The multiplicity of π in the discrete spectrum of $L^2(G(k)\backslash G(\mathbf{A}))$ is equal to $\sum_{\psi \in \Psi_0(G)} m_\psi(\pi)$.*

7.2 Multiplicity of Klingen CAP representation

From the Adams' conjecture ([Ada89] § 4, I), it is expected that $\Theta^1(V, \chi, S)$ belongs to the global A-packet associated to the A-parameter $\psi = \psi_{k',\chi} : W_k \times SL(2, \mathbf{C}) \to SO(5, \mathbf{C}) \times W_k$ defined by

$$\psi_{k',\chi} = \left(\mathrm{Ind}^{W_k}_{W_{k'}} \mu(\chi) \otimes \mathbf{1}_{SL(2,\mathbf{C})} \right) \oplus \left(\omega_{k'/k} \otimes \mathrm{Sym}^2 \right) \times \mathrm{Id}_{W_k}.$$

Here, k' is the quadratic extension $k(\eta)$ of k with discriminant δ, $W_k, W_{k'}$ the Weil groups of k, k', $\omega_{k'/k}$ the quadratic character associated to the quadratic extension k'/k, Sym^2 the second symmetric power of the standard representation of $SL(2, \mathbf{C})$ and $\mu(\chi) : W_{k'} \to \mathbf{C}^\times$ the image of the element of $H^1(W_k, \widehat{U_{k'/k}(1)})$ associated to χ by the Langlands' class field theory via the restriction map $H^1(W_k, \widehat{U_{k'/k}(1)}) \to H^1(W_{k'}, \widehat{U_{k'/k}(1)})$.

We need description of local and global A-packets, S-groups and pairings to describe the conjectural multiplicity. To obtain local and global A-packets, we set an assumption (c.f. [Ada89] § 4.5 Conj.B).

Assumption Let F be a local field of a form k_v for some v, D_F the quaternion algebra $D(F)$ and K a quadratic algebra of F with descriminant δ_K. For a unitary character χ_K of the group of norm 1 of $K^\times$, set an A-parameter ψ_{K,χ_K} for $G(F)$ by

$$\psi_{K,\chi_K} = \left(\mathrm{Ind}^{W_F}_{W_K} \mu(\chi_K) \otimes \mathbf{1}_{SL(2,\mathbf{C})} \right) \oplus \left(\omega_{K/F} \otimes \mathrm{Sym}^2 \right) \times \mathrm{Id}_{W_F}$$

where $\mu(\chi_K)$ is defined as before. Then the A-packet $\Pi^{G(F)}_{\psi_{K,\chi_K}}$ associated to ψ_{K,χ_K} coincides with

$$\{\theta(V_F, \sigma) \,|\, V_F, \sigma\}$$

where V_F runs over non-degenerate (-1)-hermitian right spaces over D_F with dimension 1 and determinant $-\delta_K$, and σ is in the L-packet $\Pi_{\chi_K}^{V_F}$ associated to a L-parameter $\phi_{\chi_K}^{V_F}$ defined by the composition of the embedding ${}^LG_0(V_F) \to {}^LG(V_F)$ and the L-parameter $\mathcal{L}_F \to {}^LG_0(V_F)$ associated to χ_K.

By the equivalent relation of L-parameters for $G(V_v)$ ([Ada89] § 3.4), we obtain

$$\Pi_{\chi_v}^{V_v} = \begin{cases} \{\text{irreducible constituents of } \mathrm{Ind}_{G_0(V_v)}^{G(V_v)} \chi_v\} & v \notin S_D, \\ \{\chi_v, \chi_v^{-1}\} & v \in S_D. \end{cases}$$

From our assumption, $\Pi_{\psi_v}^{G(k_v)}$ is described as follows:
(1) The case of $v \notin S_D$

$$\Pi_{\psi_v}^{\mathrm{s}} = \begin{cases} \{\theta(V_v^{\pm}, \tilde{\chi}_v^{+})\} & \chi_v^2 \neq 1 \text{ and } \delta \notin (k_v^{\times})^2, \\ \{\theta(\mathbb{H}_v, \tilde{\chi}_v^{+})\} & \chi_v^2 \neq 1 \text{ and } \delta \in (k_v^{\times})^2, \\ \{\theta(V_v^{\pm}, \tilde{\chi}_v^{\pm})\} & \chi_v^2 = 1 \text{ and } \delta \notin (k_v^{\times})^2, \\ \{\theta(\mathbb{H}_v, \tilde{\chi}_v^{\pm})\} & \chi_v^2 = 1 \text{ and } \delta \in (k_v^{\times})^2. \end{cases}$$

Here $V_v^{\pm}$ is the two-dimensional quadratic space over k_v with determinant $-\delta$ and Hasse invariant ± 1, $\mathbb{H}_v$ is the 2-dimensional hyperbolic space over k_v. Note that $\theta(V_v, \sigma_v)$ appearing in the above packets is of the form of an irreducible quotient of $\mathrm{Ind}_{P_K(k_v)}^{Sp(2,k_v)}(\omega_{k_v'/k_v}|\cdot|_v \otimes \tau_v)$ for some irreducible representation τ_v of $SL(2,k_v)$ and the character ω_{k_v'/k_v} associated to the extension k_v'/k_v except for $\sigma_v = \det$.
(2) The case of $v \in S_D$

$$\Pi_{\psi_v}^{\mathrm{ns}} = \begin{cases} \{\theta(V_v, \chi_v), \theta(V_v, \chi_v^{-1})\} & \chi_v^2 \neq 1, \\ \{\theta(V_v, \chi_v)\} & \chi_v^2 = 1, \end{cases}$$

where V_v is the 1-dimensional (-1)-hermitian space over D_v with determinant $-\delta$. Note that elements of $\Pi_{\psi_v}^{\mathrm{ns}}$ are supercuspidal except for $\chi_v = 1$.

Next describe global and local S-groups for ψ.

$$\mathcal{S}_{\psi} \simeq \begin{cases} 2\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \chi^2 \neq 1, \\ D_4 & \chi^2 = 1, \end{cases}$$

where D_4 is the dihedral group with 8 elements. If k_v' is a quadratic extension of k_v then

$$\mathcal{S}_{\psi_v} \simeq \begin{cases} 2\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \chi_v^2 \neq 1, \\ D_4 & \chi_v^2 = 1, \end{cases}$$

and if $k_v' \simeq k_v \oplus k_v$ then

$$\mathcal{S}_{\psi_v} \simeq \begin{cases} \{1\} \times \{1\} & \chi_v^2 \neq 1, \\ \mathbf{Z}/2\mathbf{Z} \times \{1\} & \chi_v^2 = 1. \end{cases}$$

The homomorphism $\mathcal{S}_{\psi} \to \mathcal{S}_{\psi_v}$ is determined by the above description of S-groups and the following diagram:

$$\begin{array}{ccccc} D_4 = \mathbf{Z}/4\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z} & \stackrel{\kappa}{\to} & \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \to & \mathbf{Z}/2\mathbf{Z} \times \{1\} \\ \cup & & \cup & & \cup \\ 2\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \to & \{1\} \times \mathbf{Z}/2\mathbf{Z} & \to & \{1\} \times \{1\} \end{array}$$

Finally, define a pairing $\langle\ ,\ \rangle_v$ as follows. If $v \in S_D$ and $\chi_v^2 = 1$,

$$\langle s, \theta(V_v, \chi_v)\rangle_v = \begin{cases} \pm 2 & s = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

if $v \in S_D$ and $\chi_v^2 \neq 1$,

$$\langle \cdot\, , \theta(V_v, \chi_v^\epsilon)\rangle_v = \mathrm{sgn}^{(\epsilon-1)/2} \otimes \mathbf{1},$$

and otherwise

$$\langle \cdot\, , \theta(V_v^\eta, \tilde{\chi}_v^\epsilon)\rangle_v = \mathrm{sgn}^{(\epsilon-1)/2} \otimes \mathrm{sgn}^{(\eta-1)/2},$$

where $-1 = (2,0) \in \mathbf{Z}/4\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$, we regard $\mathbb{H}_v = V_v^+$ and if $\mathcal{S}_{\psi_v} \simeq D_4$ it is reduced to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ via κ. Remark that our definition of local pairings satisfies Conjecture 7.1.

We calculate $\epsilon_\psi = \mathbf{1}$ by definition. Therefore, for an irreducible automorphic representation $\pi = \Theta^1(V.\chi, S) \in \Pi_\psi^G$ for some V, S, the Arthur's conjectural multiplicity is described by

$$m_\psi(\pi) = \begin{cases} 2^{\sharp(S_\chi \cap S_D)-1} & \chi^2 \neq \mathbf{1},\ S_D \cap S_\chi \neq \emptyset, \\ 2^{\sharp S_D - 2} & \chi^2 = \mathbf{1},\ S_D \cap S_\chi \neq \emptyset \\ 1 & S_D \cap S_\chi = \emptyset. \end{cases}$$

References

[AB95] Jeffrey Adams and Dan Barbasch. Reductive dual pair correspondence for complex groups. *J. Funct. Anal.*, 132(1):1–42, 1995.

[Ada89] J. Adams. *L*-functoriality for dual pairs. *Astérisque*, (171-172):85–129, 1989. Orbites unipotentes et représentations, II.

[Ada04] Jeffrey Adams. Theta-10. In *Contributions to automorphic forms, geometry, and number theory*, pages 39–56. Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[Art89] James Arthur. Unipotent automorphic representations: conjectures. *Astérisque*, (171-172):13–71, 1989. Orbites unipotentes et représentations, II.

[Art06] James Arthur. A note on *L*-packets. *Pure Appl. Math. Q.*, 2(1, part 1):199–217, 2006.

[Bor79] A. Borel. Automorphic L-functions. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979.

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ. Representations of the group $GL(n, F)$, where F is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[Fol89] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[GJ78] Stephen Gelbart and Hervé Jacquet. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.

[How89] Roger Howe. Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2(3):535–552, 1989.

[HPS79] R. Howe and I. I. Piatetski-Shapiro. A counterexample to the "generalized Ramanujan conjecture" for (quasi-) split groups. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 315–322. Amer. Math. Soc., Providence, R.I., 1979.

[Kim95] Henry H. Kim. The residual spectrum of Sp_4. *Compositio Math.*, 99(2):129–151, 1995.

[KN94] Takuya Kon-No. The residual spectrum of Sp(2). *Proc. Japan Acad. Ser. A Math. Sci.*, 70(6):204–207, 1994.

[Kot84] Robert E. Kottwitz. Stable trace formula: cuspidal tempered terms. *Duke Math. J.*, 51(3):611–650, 1984.

[KRS92] Stephen S. Kudla, Stephen Rallis, and David Soudry. On the degree 5 L-function for Sp(2). *Invent. Math.*, 107(3):483–541, 1992.

[Kud86] Stephen S. Kudla. On the local theta-correspondence. *Invent. Math.*, 83(2):229–255, 1986.

[LL79] J.-P. Labesse and R. P. Langlands. L-indistinguishability for SL(2). *Canad. J. Math.*, 31(4):726–785, 1979.

[LZ97] Soo Teck Lee and Chen-Bo Zhu. Degenerate principal series and local theta correspondence. II. *Israel J. Math.*, 100:29–59, 1997.

[MVW87] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger. *Correspondances de Howe sur un corps p-adique*, volume 1291 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[MW95] C. Mœglin and J.-L. Waldspurger. *Spectral decomposition and Eisenstein series*, volume 113 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1995. Une paraphrase de l'Écriture [A paraphrase of Scripture].

[Pau05] Annegret Paul. On the Howe correspondence for symplectic-orthogonal dual pairs. *J. Funct. Anal.*, 228(2):270–310, 2005.

[PR91] V. P. Platonov and A. S. Rapinchuk. *Algebraicheskie gruppy i teoriya chisel*. "Nauka", Moscow, 1991. With an English summary.

[Sch85] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[ST69] J. A. Shalika and S. Tanaka. On an explicit construction of a certain class of automorphic forms. *Amer. J. Math.*, 91:1049–1076, 1969.

[ST93] Paul J. Sally, Jr. and Marko Tadić. Induced representations and classifications for $\mathrm{GSp}(2,F)$ and $\mathrm{Sp}(2,F)$. *Mém. Soc. Math. France (N.S.)*, (52):75–133, 1993.

[Tad94] Marko Tadić. Representations of p-adic symplectic groups. *Compositio Math.*, 90(2):123–181, 1994.

[Yas07] Takanori Yasuda. The residual spectrum of inner forms of Sp(2). *Pacific J. Math.*, 232(2):471–490, 2007.

List of MI Preprint Series, Kyushu University

The Global COE Program
Math-for-Industry Education & Research Hub

MI

MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata

MI2008-2 Eiji ONODERA
The intial value problem for a third-order dispersive flow into compact almost Hermitian manifolds

MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space

MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme

MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in nfinite extensions over a p-adic field

MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields

MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited

MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition

MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization

MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12 Tetsu MASUDA
Hypergeometric τ-functions of the q-Painlevé system of type $E_8^{(1)}$

MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications

MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force

MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application

MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions

MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map

MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2-projection

MI2009-26 Manabu YOSHIDA
Ramification of local fields and Fontaine's property (Pm)

MI2009-27 Yu KAWAKAMI
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space

MI2009-28 Masahisa TABATA
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme

MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance

MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis

MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
Hecke's zeros and higher depth determinants

MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type

MI2009-33 Chikashi ARITA
Queueing process with excluded-volume effect

MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA
Projective reduction of the discrete Painlevé system of type$(A_2 + A_1)^{(1)}$

MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI
Finite element computation for scattering problems of micro-hologram using DtN map

MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes

MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model

MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes

MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations

MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models

MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons

MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine

MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization

MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation

MI2010-8 Takahiro ITO
Abstract collision systems on groups

MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments

MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models

MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems

MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow

MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates

MI2010-14 Yoshiyasu OZEKI
Non-existence of certain Galois representations with a uniform tame inertia weight

MI2010-15 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency

MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO
The value distribution of the Gauss map of improper affine spheres

MI2010-17 Kazunori YASUTAKE
On the classification of rank 2 almost Fano bundles on projective space

MI2010-18 Toshimitsu TAKAESU
Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field

MI2010-19 Reiichiro KAWAI & Hiroki MASUDA
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling

MI2010-20 Yasuhide FUKUMOTO, Makoto HIROTA & Youichi MIE
Lagrangian approach to weakly nonlinear stability of an elliptical flow

MI2010-32 Nobutaka NAKAZONO
Hypergeometric τ functions of the q-Painlevé systems of type $(A_2 + A_1)^{(1)}$

MI2010-33 Yoshiyuki KAGEI
Global existence of solutions to the compressible Navier-Stokes equation around parallel flows

MI2010-34 Nobushige KUROKAWA, Masato WAKAYAMA & Yoshinori YAMASAKI
Milnor-Selberg zeta functions and zeta regularizations

MI2010-35 Kissani PERERA & Yoshihiro MIZOGUCHI
Laplacian energy of directed graphs and minimizing maximum outdegree algorithms

MI2010-36 Takanori Yasuda
CAP representations of inner forms of $Sp(4)$ with respect to Klingen parabolic subgroup