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<https://hdl.handle.net/2324/18725>

出版情報 : MI Preprint Series. 2010-36, 2010-12-08. 九州大学大学院数理学研究院
バージョン :
権利関係 :

MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

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MI 2010-36

(Received December 8, 2010)

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Abstract

The unitary group of the hyperbolic hermitian space of dimension two over a quaternion division algebra over a number field is a non-quasisplit inner form of $Sp(4)$, and does not have a parabolic subgroup corresponding to the Klingen parabolic subgroup. However, it has CAP representations with respect to the Klingen parabolic subgroup. We construct them by using the theta lifting from the unitary groups of one-dimensional (-1)-hermitian spaces and estimate their multiplicities in the discrete spectrum. In many cases, their multiplicities become bigger than 1.

1 Introduction

The purposes of this paper are, for an inner form G of $Sp(4)$ defined over a number field k ,

1. construction of a certain class of non-tempered automorphic representations and evaluation (of lower bounds) of their multiplicities in the discrete spectrum, and
2. the description of the multiplicities expected from the evaluation using the formulation of the Arthur's multiplicity conjecture.

According to the Arthur's conjecture [Art89], for any irreducible non-tempered automorphic representation π , there exists an A -parameter

$$\psi : \mathcal{L}_k \times SL(2, \mathbf{C}) \rightarrow {}^L G = SO(5, \mathbf{C}),$$

where \mathcal{L}_k is the hypothetical Langlands group of k , satisfying $\psi|_{SL(2, \mathbf{C})} \neq \mathbf{1}$ such that π is expressed as an element of its global A -packet. When π is isomorphic to the restricted tensor product $\bigotimes_v \pi_v$ of local irreducible representations π_v , each π_v is also expressed as an element of the local A -packet of the local A -parameter ψ_v

obtained by ψ . Since $Sp(4)$ and G share the set of A -parameters there should exist the global and local A -packets for $Sp(4)$ associated to ψ too. At this time, at almost all place of k , the local A -packets of $Sp(4)$ and G coincide.

For a general reductive group, an irreducible representation appearing in the residual spectrum is a typical example of non-tempered automorphic representation. The residual spectrum is the subspace of the space of the L^2 -automorphic forms generated by the non-cuspidal irreducible automorphic representations appearing in the discrete spectrum. The irreducible decomposition of the residual spectrum for $Sp(4)$ is completely determined [Kim95]. In the case where k is totally real, part of the irreducible representations appearing in the result of [Kim95] can be rewritten by representations given by theta lifting [KN94]. For comparison with the case of G we quote the latter here.

Theorem 1.1 ([KN94]). *Let k be totally real. An irreducible representation appearing in the residual spectrum for $Sp(4)$ is one of the following irreducible representations.*

- (1) *The trivial representation $\mathbf{1}_{Sp(4)}$ of $Sp(4, \mathbf{A})$. Here $\mathbf{A} = \mathbf{A}_k$ denotes the adèle ring of k .*
- (2) *the theta lift $R(V)$ from the trivial representation of the orthogonal group $O(V, \mathbf{A})$ of a 2-dimensional non-hyperbolic quadratic space V over k .*
- (3) *The unique irreducible quotient of $\text{Ind}_{P_S(\mathbf{A})}^{Sp(4, \mathbf{A})}(\sigma | \det |_{\mathbf{A}}^{1/2})$ for an irreducible self-dual cuspidal representation σ of $GL(2, \mathbf{A})$ whose standard L -functions $L(s, \pi)$ do not vanish at $s = 1/2$.*
- (4) *The unique irreducible quotient of $\text{Ind}_{P_K(\mathbf{A})}^{Sp(2, \mathbf{A})}(\omega_{k'/k} | \cdot |_{\mathbf{A}} \otimes \pi)$ for a quadratic character $\omega_{k'/k}$, which is associated to a non-trivial quadratic extension k' of k , and an irreducible unitary representation π of $SL(2, \mathbf{A})$ such that*

$$\sigma \subset \pi(\Omega)|_{SL(2, \mathbf{A})}$$

for some character Ω of $\mathbf{A}_{k'}^\times/k^\times$ not isomorphic to its conjugate. Here $\pi(\Omega)$ is the automorphic representation of $GL(2, \mathbf{A})$ given in [LL79, Prop. 6.5] and [GJ78, p.491].

Here P_S and P_K are the Siegel and Klingen parabolic subgroups of $Sp(4)$, respectively. The respective Levi factors M_S and M_K satisfy that $M_S \simeq GL(2)$ and $M_K \simeq \mathbb{G}_m \times SL(2)$. Each irreducible representation in the above appears with multiplicity one in the discrete spectrum.

In this paper, we treat only the case where G is expressed by the unitary group of the 2-dimensional hyperbolic hermitian space over a quaternion division algebra D over k . In this case, as for the irreducible decomposition of the residual spectrum the following is obtained.

Theorem 1.2 ([Yas07]). *Let k be totally real. An irreducible representation appearing in the residual spectrum for G is one of the following irreducible representations.*

- (1) *The trivial representation $\mathbf{1}_G$ of $G(\mathbf{A})$.*
- (2) *The theta lift $R(V)$ from the trivial representation of the group $U(V)(\mathbf{A})$ of adèle points of the unitary group of a (-1) -hermitian (right) D -space V of dimension one.*
- (3) *The unique irreducible quotient of $\text{Ind}_{P'_S(\mathbf{A})}^{G(\mathbf{A})}(\sigma|\nu_D|_{\mathbf{A}}^{1/2})$ for an irreducible self-dual cuspidal representation σ of $D^\times(\mathbf{A})$ whose standard L -functions $L(s, \pi)$ do not vanish at $s = 1/2$.*

Here P'_S is the Siegel parabolic subgroup of G , where its Levi factor M'_S is isomorphic to D^\times , and ν_D the reduced norm of D . In the case of (1) and (3), the multiplicity of each representation is one. In the case of (2), the multiplicity of each representation is $2^{\#S_D-2}$ where S_D is the set of places of k at which D is ramified.

When comparing the irreducible decomposition of the residual spectra of $Sp(4)$ and G , it is noticed that similar forms of representations appear. However there are no representations for G corresponding to those in Theorem 1.1 (4) for $Sp(4)$. This is because there is no proper parabolic subgroup of G over k containing the correspondence of the Klingen parabolic subgroup of $Sp(4)$ by an inner twist. Nevertheless, the representations in Theorem 1.1 (4) must be expressed as elements of the A -packets of some A -parameters, and there should exist also the A -packets for G of these A -parameters. Therefore, if they exist, they consist of cuspidal representations of $G(\mathbf{A})$, which are expressed in the forms defined as follows.

Definition 1.3. *Let $\pi \simeq \bigotimes_v \pi_v$ be an irreducible cuspidal representation of $G(\mathbf{A})$. We say that π is a CAP representation with respect to P_K if there exist an irreducible cuspidal representation $\sigma \simeq \bigotimes_v \sigma_v$ of $SL(2, \mathbf{A})$ and a character $\omega = \prod_v \omega_v$ of $\mathbf{A}^\times/k^\times$ such that for almost all v , π_v is isomorphic to a composition factor of $\text{Ind}_{P_K(k_v)}^{Sp(2, k_v)}(\omega_v | \cdot |_v \otimes \sigma_v)$.*

Remark that because G is isomorphic to $Sp(4)$ over k_v for almost all v , P_K is regarded as a subgroup over k_v of G for such v , and the above definition makes sense.

We make use of theta lifting to construct such CAP representations. The theta correspondences from $O(2)$ to $Sp(4)$ become automorphic representations appearing in Theorem 1.1 (4). Therefore, the target representations are constructed by the theta lifting from the unitary group of a skew-hermitian space over D of dimension one to G , which is an inner form version of the theta lifting from $O(2)$ to $Sp(4)$. In fact, writing $U_0(V)$ for the connected component of the unit of the unitary group

$U(V)$ of a skew-hermitian space V over D of dimension one, a non-zero irreducible automorphic representation $\theta(V, \chi, S)$ of G can be defined for an irreducible automorphic representation (character) of $U_0(V)$ and a set S of places of k which consists of finite elements and satisfies a certain condition (Theorem 6.1). As special cases, the representations in Theorem 1.2 (2) (the case that $\chi = \mathbf{1}, S = \emptyset$) and the inner form version of cuspidal representations constructed in [HPS79] (the case that $\chi = \mathbf{1}, S \neq \emptyset$) can be dealt with. As for the multiplicity $m(\theta(V, \chi, S))$ of $\theta(V, \chi, S)$ in the discrete spectrum, we have the following estimate.

Theorem 1.4.

$$m(\theta(V, \chi, S)) \geq \begin{cases} 2^{\#(S_\chi \cap S_D) - 1} & S_D \not\subset S_\chi, S_D \cap S_\chi \neq \emptyset, \\ 2^{\#S_D - 2} & S_D \subset S_\chi, \\ 1 & S_D \cap S_\chi = \emptyset, \end{cases}$$

where $S_\chi = \{v \mid \chi_v^2 = \mathbf{1}\}$.

This estimate is obtained by the failure of the Hasse's principle for skew-hermitian spaces over D . In view of the result of the multiplicities of representations in Theorem 1.2 (2), it is expected that the above inequality sign is exchanged for the equal one. If the equality is satisfied then the value in the right hand side should be described by the formulation in the Arthur's multiplicity conjecture. Therefore, under this assumption the author inspected how local A -packets, S -groups and pairings between them, etc. should be described in the formulation. As a consequence, it turns out that they can be described as they satisfy some necessary conditions (§ 7).

2 Inner form G of $Sp(4)$

Let k be a number field, and $\mathbf{A} = \mathbf{A}_k$ its adèle ring. We write $\mathbf{A}_\infty, \mathbf{A}_f$ for the infinite and finite components of \mathbf{A} , respectively. $|\cdot|_{\mathbf{A}}$ denotes the idele norm of \mathbf{A}^\times . For any place v of k , we write k_v for the completion of k at v and $|\cdot|_v$ for the v -adic norm. If v is non-archimedean, \mathcal{O}_v denotes the maximal compact subring of k_v .

k -group $Sp(2)$ will be realized by

$$Sp(2) = \left\{ g \in GL(4) \mid g \begin{pmatrix} & & & 1_2 \\ & & & \\ & & & \\ -1_2 & & & \end{pmatrix} {}^t g = \begin{pmatrix} & & & 1_2 \\ & & & \\ & & & \\ -1_2 & & & \end{pmatrix} \right\}.$$

Fix a minimal k -parabolic subgroup P_0 of $Sp(4)$ by

$$P_0 = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * & * \end{pmatrix} \in Sp(4) \right\},$$

and a Levi factor M_0 of P_0 consisting of diagonal matrices. The unipotent radical of P_0 is denoted by U_0 . We define the Siegel parabolic subgroup $P_S = M_S U_S$ and

the Klingen parabolic subgroup $P_K = M_K U_K$ of $Sp(4)$ as

$$M_S = \left\{ \begin{pmatrix} a & 0 \\ 0 & t a^{-1} \end{pmatrix} \middle| a \in GL(2) \right\},$$

$$U_S = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \text{Sym}(2) \right\},$$

and

$$M_K = \left\{ \begin{pmatrix} t & & & \\ & a & b & \\ & & t^{-1} & \\ & c & & d \end{pmatrix} \middle| t \in \mathbb{G}_m, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2) \right\},$$

$$U_K = \left\{ \begin{pmatrix} 1 & a & b & c \\ & 1 & c & \\ & & 1 & \\ & & & -a & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{G}_a \right\}.$$

Let R be a quaternion algebra over a local or global field F . We write ν_R , τ_R and $*$ for the reduced norm, the reduced trace and the main involution of R , respectively. We write $R_- = \{x \in R \mid \tau_R(x) = 0\}$. When $F = k$, we write S_R for the set of places v of k at which R is ramified. This set has finite and even elements. $\mathbb{M}(n, A)$ denotes the algebra of all $n \times n$ -matrices over a ring A . For $a = (a_{i,j}) \in \mathbb{M}(n, R)$, write $\tau a = (*a_{j,i})$. Let D be a quaternion division algebra over k . On $W = D^{\oplus 2}$ viewed as a left D -module, a hyperbolic $(D, *)$ -hermitian form h_W is defined by

$$h_W((x_1, x_2), (y_1, y_2)) = (x_1, x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} *y_1 \\ *y_2 \end{pmatrix} \quad (\forall x_1, x_2, y_1, y_2 \in D). \quad (2.1)$$

Its unitary group $G = G(W, h_W)$ is a k -group which associates

$$G(A) = \left\{ g \in GL(2, R \otimes_k A) \middle| g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tau g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

to each (abelian) k -algebra A .

There is a $p, q \in k^\times$ such that $D \simeq (p, q)_k$ [Sch85, p.75]. Then G and $Sp(4)$ are isomorphic over the quadratic extension $K = k(\sqrt{p})$ of k and moreover G is an inner form of $Sp(4)$. Define a k -parabolic subgroup $P_S = M_S U_S$ of G as

$$M_S = \left\{ m(x) := \begin{pmatrix} x & 0 \\ 0 & *x^{-1} \end{pmatrix} \middle| x \in D^\times \right\},$$

$$U_S = \left\{ u(y) := \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \middle| y \in D_- \right\}.$$

Via an inner twist this parabolic subgroup coincides with the Siegel parabolic subgroup of $Sp(4)$. Therefore we use the same notation P_S as the case of $Sp(4)$. The character $M_S \ni m(x) \mapsto \nu_D(x) \in \mathbb{G}_m$ is again denoted by ν_D . P_S is a minimal and maximal proper parabolic subgroup of G defined over k .

The k -split component of the center of M_S is

$$A_S = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbf{G}_m \right\}.$$

Let W_G denote the Weyl group of G , which consists of two elements. A representative of the non-trivial element of W_G is chosen by

$$w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G(k).$$

If $v \notin S_D$, $G(k_v)$ can be identified with $Sp(4, k_v)$. For every place v of k , we fix a maximal compact subgroup \mathbf{K}_v of $G(k_v)$ and a Haar measure dg_v such that

$$\mathbf{K}_v = \begin{cases} Sp(4, \mathcal{O}_v) & \text{for a non-archimedean } v \notin S_D, \\ Sp(4, \mathbf{R}) \cap O(4) \simeq U(2) & \text{for a real } v \notin S_D, \\ Sp(1, 1) \cap Sp(2) \simeq Sp(1) \times Sp(1) & \text{for a real } v \in S_D, \\ Sp(4, \mathbf{C}) \cap U(4) \simeq Sp(2) & \text{for a complex } v \notin S_D, \end{cases}$$

and the volume of \mathbf{K}_v with respect to dg_v is equal to 1 for almost all non-archimedean v . We define a maximal compact subgroup \mathbf{K} of $G(\mathbf{A})$ by $\prod_v \mathbf{K}_v$ and a Haar measure dg of $G(\mathbf{A})$ by $\prod_v dg_v$.

3 Construction of automorphic forms of G

3.1 Automorphic forms of the unitary group of a (-1)-hermitian space

Fix a non-zero $\eta_0 \in D_-$. Define a non-degenerate (-1)-hermitian right space (V, h_V) over D as $V = D$ and

$$h_V(x_1, x_2) = {}^*x_1 \cdot \eta_0 \cdot x_2 \quad (x_1, x_2 \in D).$$

$G(V)$ and $G_0(V)$ denote the unitary group and the special unitary group of V defined over k , respectively. When emphasizing V as a space of k -valued points, it will be written by V_k . Also V_v stands for the completion of V at a place v and $V_{\mathbf{A}} = V \otimes_k \mathbf{A}$. In particular, we will often use the notation $G(V_k), G(V_v), G(V_{\mathbf{A}})$ instead of $G(V)(k), G(V)(k_v), G(V)(\mathbf{A})$, respectively. Similar notation is also used for $G_0(V)$. It is known that $G(V_k) = G_0(V_k)$ and $G(V_v) = G_0(V_v)$ for any $v \in S_D$. Set $k' = k(\eta)$, which is a quadratic extension of k , and define a quadratic space (T, b_T) over k as $T = k'$ and $b_T = N_{k'/k}$ where $N_{k'/k}$ is the norm of k'/k . We may make the following identification:

$$\begin{aligned} G(V_v) &= \begin{cases} O(T_v) & v \notin S_D, \\ SO(T_v) & v \in S_D, \end{cases} \\ G_0(V_v) &= SO(T_v), \\ G(V_k) &= G_0(V_k) = SO(T_k). \end{aligned}$$

Here T_v is the completion of T at v and $SO(T_k) = SO(T)(k)$. T_v is isotropic if and only if $\eta^2(\in k^\times)$ is a quadratic residue in k_v , and in this case identify $O(T_v) = (1, 1; k_v)$. A maximal compact subgroup \mathbf{L}_v of $G(V_v)$ is defined as

$$\mathbf{L}_v = \begin{cases} O(T_v) & v \notin S_D \text{ and } T_v \text{ is anisotropic,} \\ O(1, 1; \mathcal{O}_v) & v \notin S_D \text{ and } T_v \text{ is isotropic,} \\ SO(T_v) & v \in S_D, \end{cases}$$

Remark that if $v \in S_D$ then T_v is always anisotropic. A measure dh_v on $G(V_v)$ is chosen by the Haar measure such that the volume of \mathbf{L}_v is 1, and a measure dh on $G(V_{\mathbf{A}})$ is defined by the product of dh_v .

Fix a $\gamma_0 \in O(T_k) \backslash SO(T_k)$, which always holds $\gamma_0^2 = 1$. Let $\chi = \prod_v \chi_v$ be an unitary character of $G_0(V_k) \backslash G_0(V_{\mathbf{A}})$. Consider the induced representation of $G(V_v)$ from χ_v . In the case of $v \in S_D$, χ_v is also recognized as a representation of $G(V_v)$ because $G(V_v) = G_0(V_v)$. In the case of $v \notin S_D$, if $\chi_v^2 \neq \mathbf{1}$ then $\text{Ind}_{G_0(V_v)}^{G(V_v)} \chi_v$ is irreducible, which is denoted by $\tilde{\chi}_v^+$, and if $\chi_v^2 = \mathbf{1}$ then

$$\text{Ind}_{G_0(V_v)}^{G(V_v)} \chi_v \simeq \tilde{\chi}_v^+ \oplus \tilde{\chi}_v^-. \quad (3.1)$$

Here $\tilde{\chi}_v^+, \tilde{\chi}_v^-$ are characters of $G(V_v)$ characterized by $\tilde{\chi}_v^\pm(\gamma_0) = \pm 1$. Write S_χ for the set of places v of k satisfying $\chi_v^2 = \mathbf{1}$. If χ_v is unramified, $f_{0,v} \in \tilde{\chi}_+$ is defined by

$$\phi_{0,v}(h) = \begin{cases} 1 & h \in \mathbf{L}_v \\ 0 & h \notin \mathbf{L}_v \end{cases}$$

Let S be a subset of $S_\chi \cap S_D^c$ with finite number of elements. An irreducible representation $\sigma(V, \chi, S)$ of $G(V_{\mathbf{A}})$ is defined by the restricted tensor product,

$$\left(\bigotimes_{v \in S} \tilde{\chi}_v^- \right) \otimes \left(\bigotimes_{v \notin S} \tilde{\chi}_v^+ \right)$$

with respect to $\phi_{0,v}$. In particular, if $\chi^2 = 1$ then $\sigma(V, \chi, S)$ becomes a character of $G(V_{\mathbf{A}})$. The v -component of $\sigma(V, \chi, S)$ for a place v is denoted by $\sigma_v(V, \chi, S)$. Since $G(V_k) = G_0(V_k)$, we can define an injective intertwining operator characterized by

$$\sigma(V, \chi, S) \ni \bigotimes_v \phi_v \mapsto \prod_v \phi_v \in \mathcal{A}(G(V_k) \backslash G(V_{\mathbf{A}})).$$

Here $\mathcal{A}(G(V_k) \backslash G(V_{\mathbf{A}}))$ is the space of automorphic forms on $G(V_k) \backslash G(V_{\mathbf{A}})$ ([MW95] §I.2.17). $\sigma(V, \chi, S)$ is identified with the image of this intertwining operator. Conversely, any irreducible $G(V_{\mathbf{A}})$ -subspace of $\mathcal{A}(G(V_k) \backslash G(V_{\mathbf{A}}))$ is expressed by the form $\sigma(V, \chi, S)$ for some χ and S because the restriction of an automorphic form on $G(V_{\mathbf{A}})$ to $G_0(V_{\mathbf{A}})$ is also automorphic form.

3.2 Theta correspondence

Let ψ be a non-trivial character of \mathbf{A}/k , and for any place v of k , the v -component of ψ is denoted by ψ_v . For a vector space X over k_v , $\mathcal{S}(X)$ denotes the space of the Schwartz-Bruhat functions on X . The Weil representation ω_{ψ_v, V_v} of $G(V_v) \times G(k_v)$ with respect to ψ_v is defined on $\mathcal{S}(V_v)$ as in [Yas07]. In particular, for $v \notin S_D$,

ω_{ψ_v, V_v} is identified with the Weil representation $\omega_{\psi_v, T_{V_v}}$ of $O(T_{V_v}) \times Sp(4, k_v)$ defined on $\mathcal{S}(T_{V_v}^2)$ where T_{V_v} is the 2-dimensional quadratic space with the quadratic form given by the symmetric matrix

$$\begin{pmatrix} \gamma & -\alpha \\ -\alpha & -\beta \end{pmatrix} \quad \text{when } \eta_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathbf{M}(2, k_v)_-.$$

The explicit formula of ω_{ψ_v, V_v} is described as follows: For $f \in \mathcal{S}(V_v)$, $x \in V_v$,

- (1) $\omega_{\psi_v, V_v}(h, 1)f(x) = f(h^{-1}x)$, $(h \in G(V_v))$,
- (2) $\omega_{\psi_v, V_v}(1, m(a))f(x) = \zeta_{V_v}(\nu(a))|\nu_{D_v}(a)|_v f(xa)$, $(a \in D^\times(k_v))$,
- (3) $\omega_{\psi_v, V_v}(1, u(b))f(x) = \psi_v(\frac{1}{4}\tau_{D_v}(b h_V(x, x)))f(x)$, $(b \in D_-(k_v))$,
- (4) $\omega_{\psi_v, V_v}(1, w_0)f(x) = (-1, -\det V_v)_v \kappa(D_v) \mathcal{F}_{V_v} f(-x)$.

Here $(\cdot, \cdot)_v$ is the Hilbert symbol at v , ζ_{V_v} the quadratic character $(-\det V_v, \cdot)_v$ of k_v^\times , $\kappa(D_v) = -1$ if $v \in S_D$ and $\kappa(D_v) = 1$ otherwise, and

$$\mathcal{F}_{V_v} f(x) = \int_{V_v} f(y) \psi(\frac{1}{2}\tau_{D_v} \circ \text{tr}(y, x)_{V_v}) d_{V_v} y$$

where $d_{V_v} y$ is the self-dual measure with respect to the bilinear form

$$V_v \times V_v \ni (x, y) \mapsto \psi(\frac{1}{2}\tau_{D_v} \circ \text{tr}(y, x)_{V_v}).$$

For a non-archimedean $v \notin S_D$, $f_{0,v} \in \mathcal{S}(V_v)$ denotes the characteristic function of $\mathbf{M}(2, \mathcal{O}_v) \subset \mathbf{M}(2, k_v) = V_v$. $\mathcal{S}(V_{\mathbf{A}})$ is defined by the tensor product of $\mathcal{S}(V_v)$ with respect to $f_{0,v}$, which is identified with a space of functions on $V_{\mathbf{A}}$. The global Weil representation $\omega_{\psi, V}$ of $G(V_{\mathbf{A}}) \times G(\mathbf{A})$ with respect to ψ is defined on $\mathcal{S}(V_{\mathbf{A}})$ by the restricted tensor product of the local Weil representations. For $f \in \mathcal{S}(V_{\mathbf{A}})$, set

$$\theta(f; h, g) = \sum_{\xi \in V_k} \omega_{\psi, V}(h, g) f(\xi) \quad (h \in G(V_{\mathbf{A}}), g \in G(\mathbf{A})),$$

which converges absolutely and becomes $G(k)$ -invariant. Since $G(V_k) \backslash G(V_{\mathbf{A}})$ is compact, for $\phi \in \sigma(V, \chi, S)$ and $f \in \mathcal{S}(V_{\mathbf{A}})$, the integral

$$\theta(f, \phi)(g) = \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \theta(f; h, g) \overline{\phi(h)} dh$$

is defined and converges. Then $\theta(f, \phi)$ becomes an automorphic form on $G(\mathbf{A})$. We denote by $\Theta(V, \chi, S)$ the space generated by $\theta(f, \phi)$ for all $\phi \in \sigma(V, \chi, S)$ and $f \in \mathcal{S}(V_{\mathbf{A}})$.

Lemma 3.1. *If $(\chi, S) \neq (\mathbf{1}, \emptyset)$ then $\theta(f, \phi)$ is cuspidal.*

Proof. The constant term $\theta(f, \phi)_{P_S}$ of $\theta(f, \phi)$ along to P_S is calculated as follows.

$$\begin{aligned}
\theta(f, \phi)_{P_S}(g) &= \int_{U_S(k) \backslash U_S(\mathbf{A})} \theta(f, \phi)(ug) du \\
&= \int_{U_S(k) \backslash U_S(\mathbf{A})} \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \sum_{\xi \in V_k} \omega_{\psi, V}(h, ug) f(\xi) \overline{\phi(h)} dh du \\
&= \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \sum_{\xi \in V_k} \int_{U_S(k) \backslash U_S(\mathbf{A})} \omega_{\psi, V}(h, ug) f(\xi) \overline{\phi(h)} dudh \\
&= \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \sum_{\xi \in V_k} \int_{U_S(k) \backslash U_S(\mathbf{A})} \psi\left(\frac{1}{4} \tau_D(bh_V(\xi, \xi))\right) \omega_{\psi, V}(h, g) f(\xi) \overline{\phi(h)} db dh \\
&= \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \omega_{\psi, V}(h, g) f(0) \overline{\phi(h)} dh \\
&= \omega_{\psi, V}(1, g) f(0) \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \overline{\phi(h)} dh.
\end{aligned}$$

Since $(\chi, S) \neq (\mathbf{1}, \emptyset)$, ϕ is orthogonal to a constant function. Therefore the last term is zero. \square

For $\eta \in D_-$, the Fourier coefficient $\mathcal{F}_\eta f$ of an automorphic form f on $G(\mathbf{A})$ is defined by

$$\begin{aligned}
\mathcal{F}_\eta f(g) &= \int_{U_S(k) \backslash U_S(\mathbf{A})} f(ug) \psi_\eta(u) du \quad (g \in G(\mathbf{A})), \\
\text{where } \psi_\eta\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) &= \psi\left(-\frac{1}{4} \tau_D(b\eta)\right) \quad (b \in D_-(\mathbf{A})).
\end{aligned}$$

Lemma 3.2. *Let $\eta \in D_- \setminus \{0\}$.*

$$\mathcal{F}_\eta \theta(f, \phi)(g) = \begin{cases} \mathcal{F}_{\eta_0} \theta(f, \phi)(m(\alpha)g) & \eta = {}^* \alpha \eta_0 \alpha \text{ for some } \alpha \in D^\times, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned}
\mathcal{F}_\eta \theta(f, \phi)(g) &= \int_{U_S(k) \backslash U_S(\mathbf{A})} \theta(ug) \psi_\eta(u) du \\
&= \int_{U_S(k) \backslash U_S(\mathbf{A})} \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \sum_{\xi \in V_k} \omega_{\psi, V}(h, ug) f(\xi) \overline{\phi(h)} \psi_\eta(u) dh du \\
&= \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \int_{D_-(k) \backslash D_-(\mathbf{A})} \sum_{\xi \in V_k} \omega_{\psi, V}(h, g) f(\xi) \overline{\phi(h)} \psi\left(\frac{1}{4} \tau_D(b(h_V(\xi, \xi) - \eta))\right) db dh \\
&= \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \sum_{h_V(\xi, \xi) = \eta} \omega_{\psi, V}(h, g) f(\xi) \overline{\phi(h)} dh \tag{3.2}
\end{aligned}$$

If η is not expressed by the form ${}^*\alpha\eta_0\alpha$ for some $\alpha \in D^\times$, (3.2) = 0. Assume $\eta = {}^*\alpha\eta_0\alpha$ for a $\alpha \in D^\times$.

$$\begin{aligned}
(3.2) &= \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \sum_{h' \in G(V_k)} \omega_{\psi, V}(h, g) f(h'^{-1}\alpha) \overline{\phi(h)} dh \\
&= \int_{G(V_k) \backslash G(V_{\mathbf{A}})} \sum_{h' \in G(V_k)} \omega_{\psi, V}(h'h, g) f(\alpha) \overline{\phi(h'h)} dh \\
&= \int_{G(V_{\mathbf{A}})} \omega_{\psi, V}(h, g) f(\alpha) \overline{\phi(h)} dh \\
&= \int_{G(V_{\mathbf{A}})} \omega_{\psi, V}(h, m(\alpha)g) f(1_D) \overline{\phi(h)} dh \tag{3.3}
\end{aligned}$$

In particular, if $\alpha = 1$,

$$\mathcal{F}_{\eta_0}\theta(f, \phi)(g) = \int_{G(V_{\mathbf{A}})} \omega_{\psi, V}(h, g) f(1_D) \overline{\phi(h)} dh.$$

Therefore

$$(3.3) = \mathcal{F}_{\eta_0}\theta(f, \phi)(m(\alpha)g) \tag{3.4}$$

□

$\mathcal{U}(V, \chi, S)$ denotes the space generated by functions $\mathcal{F}_{\eta_0}f$ on $G(\mathbf{A})$ for all $f \in \Theta(V, \chi, S)$. The right regular action r of $G(\mathbf{A})$ defines a representation of $G(\mathbf{A})$ on $\mathcal{U}(V, \chi, S)$, and

$$\mathcal{F}_{\eta_0} : \Theta(V, \chi, S) \ni f \mapsto \mathcal{F}_{\eta_0}f \in \mathcal{U}(V, \chi, S)$$

becomes a surjective intertwining operator. From Lemma 3.1, Lemma 3.2 and the Fourier inversion formula, \mathcal{F}_{η_0} is also injective. Therefore we have the following.

Proposition 3.3. *If $(\chi, S) \neq (\mathbf{1}, \emptyset)$ then $\Theta(V, \chi, S)$ is isomorphic to $\mathcal{U}(V, \chi, S)$ as a representation of $G(\mathbf{A})$.*

Let v be a place and $\sigma_v = \tilde{\chi}_v^\epsilon$ for some χ_v and $\epsilon = \pm 1$. $\mathcal{U}(V_v, \sigma_v)$ is defined by the space generated by functions

$$\lambda_v(f_v, \phi_v) : G(k_v) \ni g_v \mapsto \int_{G(V_v)} \omega_{\psi_v, V_v}(h_v, g_v) f_v(1_D) \overline{\phi_v(h_v)} dh_v \tag{3.5}$$

for all $f_v \in \mathcal{S}(V_v)$ and all $\phi_v \in \tilde{\chi}_v^\epsilon$. Similarly for the global case, a representation of $G(k_v)$ on $\mathcal{U}(V_v, \tilde{\chi}^\epsilon)$ is defined by the right regular action r .

Lemma 3.4. (1) $\mathcal{U}(V_v, \sigma_v) \neq 0$ for all v .

(2) For almost all v and $\sigma_v = \tilde{\mathbf{1}}^+ = \mathbf{1}$, $\lambda_v(f_{0,v}, \phi_{0,v})$ is \mathbf{K}_v -invariant and $\lambda_v(f_{0,v}, \phi_{0,v})|_{\mathbf{K}_v} \equiv 1$.

Proof. (1) It suffices to show that there are $f'_v \in \mathcal{S}(V_v)$ and $\phi'_v \in \sigma_v$ such that $\lambda_v(f'_v, \phi'_v)(1) \neq 0$. For a non-zero ϕ'_v there is a relatively compact open subset \mathcal{O} of V_v such that

$$\int_{G(V_v)} ch_{\mathcal{O}}(h_v^{-1}) \overline{\phi'_v(h_v)} dh_v \neq 0$$

where $ch_{\mathcal{O}}$ is the characteristic function of \mathcal{O} defined on V_v . If v is non-archimedean we can take $ch_{\mathcal{O}}$ as f'_v . If v is archimedean then $f_{\mathcal{O}}$ is approximated by elements in $\mathcal{S}(V_v)$, so that we can choose an appropriate f'_v . (2) For any non-archimedean $v \notin S_D$ such that ψ_v is unramified and $v \nmid 2$, $f_{0,v}$ is \mathbf{K}_v -invariant, and so is $\lambda_v(f_{0,v}, \phi_{0,v})$. Since

$$\begin{aligned} \lambda_v(f_{0,v}, \phi_{0,v})(1) &= \int_{G(V_v)} \omega_{\psi_v, V_v}(h_v, g_v) f_{0,v}(1_D) \overline{\phi_{0,v}(h_v)} dh_v \\ &= \int_{G(V_v)} \omega_{\psi_v, V_v}(1, 1) f_{0,v}(h_v^{-1}) \phi_{0,v}(h_v) dh_v \\ &= \int_{\mathbf{L}_v} 1 dh_v \\ &= 1, \end{aligned}$$

and $\lambda_v(f_{0,v}, \phi_{0,v})$ is \mathbf{K}_v -invariant, $\lambda_v(f_{0,v}, \phi_{0,v})|_{\mathbf{K}_v} \equiv 1$. \square

From the above lemma and (3.3), for $f(x) = \prod_v f_v(x_v) \in \mathcal{S}(\mathcal{V}_{\mathbf{A}})$ and $\phi(h) = \prod_v \phi_v(h_v) \in \sigma(V, \chi, S)$, we have a decomposition,

$$\mathcal{F}_{\eta_0} \theta(f, \phi)(g) = \prod_v \lambda_v(f_v, \phi_v)(g_v) \quad (g = (g_v) \in G(\mathbf{A})). \quad (3.6)$$

Therefore the next proposition follows.

Proposition 3.5. $\Theta(V, \chi, S) \neq 0$ and $\Theta(V, \chi, S)$ is isomorphic to the restricted tensor product $\bigotimes_v \mathcal{U}(V_v, \sigma_v(V, \chi, S))$ with respect to $\lambda_v(f_{0,v}, \phi_{0,v})$.

4 Review of Shalika-Tanaka lifting

Here we review the results of [ST69]. Let (T, q_T) be a non-degenerate and non-hyperbolic quadratic space over k of dimension 2 and $\chi = \prod_v \chi_v$ a character of $SO(T_{\mathbf{A}})$ invariant on $SO(T_k)$. For a place v , the Weil representation ω_{ψ_v, T_v}^0 of $O(T_v) \times SL(2, k_v)$ on $\mathcal{S}(T_v)$ is defined usually. For $u \in T_v$ such that $q_{T_v}(u) \neq 0$, $f \in \mathcal{S}(T_v)$, put

$$(P_{\chi_v} f)(u) = \int_{SO(T_v)} f(h^{-1}u) \overline{\chi_v(h)} dh.$$

This integral converges absolutely and defines a continuous function on the open subset of T_v which consists of $u \in T_v$ satisfying $q_{T_v}(u, u) \neq 0$. $S(\chi_v, T_v)$ denotes the image of $\mathcal{S}(T_v)$ by P_{χ_v} , which is regarded as a representation of $SL(2, k_v)$ by

the transfer r of the action of the Weil representation. Fix $\gamma_0 \in O(T_k) \backslash SO(T_k)$ and write $\mathcal{S}^\pm(T_v)$ for the space consisting of $f \in \mathcal{S}(T_v)$ such that $f(\gamma_0 u) = \pm f(u)$. Clearly, $\mathcal{S}(T_v) = \mathcal{S}^+(T_v) \oplus \mathcal{S}^-(T_v)$. Elements ϕ_v^\pm of $\text{Ind}_{SO(T_v)}^{O(T_v)} \chi_v$ are defined by

$$\begin{aligned} \cdot \quad & \phi_v^\pm(h) = \chi_v(h) \quad (h \in SO(V_v)), \\ \cdot \quad & \phi_v^\pm(\gamma_0) = \pm 1. \end{aligned}$$

Then $\text{Ind}_{SO(T_v)}^{O(T_v)} \chi_v$ is spanned by ϕ_v^\pm . For any $f_+ \in \mathcal{S}^+(T_v)$, $f_- \in \mathcal{S}^-(T_v)$,

$$\int_{O(T_v)} f_+(h^{-1}u) \overline{\phi_v^-(h)} dh = \int_{O(T_v)} f_-(h^{-1}u) \overline{\phi_v^+(h)} dh = 0. \quad (4.1)$$

And for $f = f_+ + f_- \in \mathcal{S}(T_v)$,

$$\begin{aligned} (P_{\chi_v} f)(u) &= \int_{SO(T_v)} f(h^{-1}u) \overline{\chi_v(h)} dh \\ &= \frac{1}{2} \int_{O(T_v)} f_+(h^{-1}u) \overline{\phi_v^+(h)} dh + \frac{1}{2} \int_{O(T_v)} f_-(h^{-1}u) \overline{\phi_v^-(h)} dh. \end{aligned} \quad (4.2)$$

Irreducible representations $\tilde{\chi}_v^\pm$ which appear as subrepresentations of $\text{Ind}_{SO(T_v)}^{O(T_v)} \chi_v$ are defined as in §3.1. Fix a $u_0 \in T_v$ such that $q_{T_v}(u_0) \neq 0$. We denote by $\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$ ($\epsilon = \pm 1$) the space of functions

$$SL(2, k_v) \ni g \mapsto \int_{O(T_v)} \omega_{\psi_v, T_v}(h, g) f(u_0) \overline{\phi(h)} dh$$

for all $f \in \mathcal{S}(T_v)$ and all $\phi \in \tilde{\chi}_v^\epsilon$, which is regarded as a representation of $SL(2, k_v)$ by the right regular action. From (4.1) and (4.2), a surjective intertwining operator,

$$\Psi : S(\chi_v, T_v) \rightarrow \Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$$

is defined as follows. If $\text{Ind}_{SO(T_v)}^{O(T_v)} \chi_v$ is irreducible then $S(\chi_v, T_v) = \Theta_v^0(T_v, \tilde{\chi}_v^+)$ and Ψ is taken by the identity map. If $\text{Ind}_{SO(T_v)}^{O(T_v)} \chi_v$ is reducible then

$$\Psi(\tilde{f}) = \left(SL(2, k_v) \ni g \mapsto r(g) \tilde{f}(u_0) + \epsilon \cdot r(g) \tilde{f}(\gamma_0 u_0) \right) \quad (\text{for } \tilde{f} \in S(\chi_v, T_v)).$$

From the representation theory of $SL(2, k_v)$, the results of [ST69] with respect to $S(\chi_v, T_v)$, [MVW87] Chap.3 IV Th.4 and [Pau05] Th.15, the following is obtained.

Theorem 4.1. (1) $\Theta_v^0(T_v, \tilde{\mathbf{1}}_v^- (= \det)) = 0$.

(2) $\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$ is non-zero and irreducible for $\tilde{\chi}_v^\epsilon \neq \tilde{\mathbf{1}}_v^-$.

(3) If T_v is hyperbolic then

$$\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon) \simeq \begin{cases} \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \chi_v & \chi_v = \mathbf{1} \text{ or } \chi_v^2 \neq \mathbf{1}, \\ \text{an irreducible subrepresentation} & \\ \text{of } \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \chi_v & \chi_v \neq \mathbf{1} \text{ and } \chi_v^2 = \mathbf{1}. \end{cases}$$

Here $B^{SL(2)}$ is the Borel subgroup of $SL(2)$ consisting of upper-triangular matrices and χ_v is also regarded as a character of the subgroup of $SL(2)$ consisting of diagonal matrices in the natural way.

- (4) If T_v is non-hyperbolic and $\tilde{\chi}_v^\epsilon = \mathbf{1}$ then $\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$ is isomorphic to an irreducible subrepresentation of $\text{Ind}_{B^{SL(2)}(k_v)}^{SL(2, k_v)} \zeta_{T_v}$ where ζ_{T_v} is the quadratic character $(-\det T_v, \cdot)_v$ of k_v^\times . Suppose T_v is non-hyperbolic and $\chi_v \neq \mathbf{1}$. If v is non-archimedean, $\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$ is supercuspidal. If v is real, $\Theta_v^0(T_v, \tilde{\chi}_v^\epsilon)$ is isomorphic to the discrete series representation $\delta(\epsilon\eta n)$ with the Harich-Chandra parameter $\epsilon\eta n$ where $\psi_v = \exp(\lambda \cdot)$, $\lambda = \epsilon\sqrt{-1}|\lambda| \in \sqrt{-1}\mathbf{R}$, $\chi_v = \exp(2\pi n\sqrt{-1} \cdot)$, $n \in \mathbf{Z}$ and $\eta = 1$ if T_v is positive definite, -1 otherwise.

For a subset $S \subset S_\chi$ with finite cardinality, an irreducible representation $\sigma(T, \chi, S)$ of $O(T_{\mathbf{A}})$ is defined as in §3.1. Its component at v is denoted by $\tilde{\chi}_v^{\epsilon_v}$ for any v . An element of $\sigma(T, \chi, S)$ is identified with a function on $SO(T_k) \backslash O(T_{\mathbf{A}})$ by the correspondence,

$$\sigma(T, \chi, S) \ni \bigotimes_v \phi_v \longleftrightarrow \prod_v \phi_v.$$

Note that for $\phi \in \sigma(T, \chi, S)$, $\phi(h) + \phi(\gamma_0 h)$ becomes an automorphic form on $O(T_{\mathbf{A}})$. For $\phi(\neq 0) \in \sigma(T, \chi, S)$, we define

$$(P_{\chi, \phi} f)(u) = \int_{O(T_{\mathbf{A}})} f(h^{-1}u) \overline{\phi(h)} dh \quad (f \in \mathcal{S}(T_{\mathbf{A}}), u \in T_{\mathbf{A}}).$$

The image $S(T, \chi, S)$ of $\mathcal{S}(T_{\mathbf{A}})$ by $P_{\chi, \phi}$ is determined independently of choice of ϕ . It is an irreducible representation of $SL(2, \mathbf{A})$ by the right regular action r , which is isomorphic to the restricted tensor product

$$\bigotimes_v \Theta_v^0(T, \tilde{\chi}_v^{\epsilon_v}).$$

For $\tilde{f} \in S(T, \chi, S)$, define a function $I_\chi \tilde{f}$ on $SL(2, \mathbf{A})$ by

$$(I_\chi \tilde{f})(g) = \sum_{\xi \in SO(T_k) \backslash (T_k \setminus \{0\})} (r(g)\tilde{f})(\xi) \quad (g \in SL(2, \mathbf{A})).$$

The Weil representation $(\omega_{\psi, T}^0, \mathcal{S}(T_{\mathbf{A}}))$ of $O(T_{\mathbf{A}}) \times SL(2, \mathbf{A})$ is defined by the restricted tensor product of $(\omega_{\psi_v, T_v}^0, \mathcal{S}(T_v))$. For $f \in \mathcal{S}(T_{\mathbf{A}})$, set

$$\theta^0(f; h, g) = \sum_{\xi \in T_k \setminus \{0\}} \omega_{\psi, T}^0(h, g) f(\xi) \quad (h \in O(T_{\mathbf{A}}), g \in SL(2, \mathbf{A})),$$

$$\theta^0(f, \phi)(g) = \int_{O(T_k) \backslash O(T_{\mathbf{A}})} \theta^0(f; h, g) \overline{(\phi(h) + \phi(\gamma_0 h))} dh \quad (\phi \in \sigma(T, \chi, S)).$$

Then $\theta^0(f, \phi)$ becomes a cuspidal automorphic form on $SL(2, \mathbf{A})$. We denote by $\Theta^0(T, \chi, S)$ the space generated by $\theta^0(f, \phi)$ for all $\phi \in \sigma(T, \chi, S)$ and $f \in \mathcal{S}(T_{\mathbf{A}})$, which is a cuspidal representation of $SL(2, \mathbf{A})$.

Theorem 4.2 ([ST69]). (1) The following diagram of intertwining operators is commutative.

$$\begin{array}{ccc} \mathcal{S}(T_{\mathbf{A}}) & & \\ P_{\chi, \phi} \downarrow & \searrow^{\theta^0(\cdot, \phi)} & \\ S(T, \chi, S) & \xrightarrow{I_{\chi}} & \Theta^0(T, \chi, S) \end{array}$$

In particular, if $\Theta^0(T, \chi, S)$ is non-zero then $\Theta^0(T, \chi, S)$ is isomorphic to $S(T, \chi, S) \simeq \bigotimes_v \Theta_v^0(T, \tilde{\chi}_v^{\epsilon_v})$ as a representation of $SL(2, \mathbf{A})$.

(2) For a non-hyperbolic quadratic space T of dimension 2 and a character χ of $SO(T_k) \backslash SO(T_{\mathbf{A}})$, there is a subset S of S_{χ} with finite cardinality such that $\Theta^0(T, \chi, S)$ is non-zero.

5 Local theory

5.1 Non-archimedean case

Let v be a non-archimedean place. For an irreducible admissible representation σ_v of $G(V_v)$, put

$$\mathcal{N}_{\sigma_v} = \bigcap_f \text{Ker } f,$$

where f runs over $\text{Hom}_{G(V_v)}(\omega_{\psi_v, V_v}, \sigma_v)$. Then there is a unique admissible representation $\Omega(V_v, \sigma_v)$ of $G(k_v)$ such that

$$\mathcal{S}(V_v) / \mathcal{N}_{\sigma_v} \simeq \sigma_v \otimes \Omega(V_v, \sigma_v)$$

as a representation of $G(V_v) \times G(k_v)$ ([MVW87] Chap.2 III 5, Chap.3 IV 4). Let $\sigma_v = \tilde{\chi}_v^{\epsilon}$ for a character χ_v of $SO(k_v)$ and $\epsilon = \pm 1$. $\text{Ind}_{SO(V_v)}^{O(V_v)} \chi_v$ is unitarizable by an inner product

$$\langle \phi, \phi' \rangle = \int_{G_0(V_v) \backslash G(V_v)} \phi(h) \overline{\phi'(h)} dh \quad (\phi, \phi' \in \text{Ind}_{SO(V_v)}^{O(V_v)} \chi_v).$$

If ϕ_1, \dots, ϕ_l form an orthonormal basis of $\tilde{\chi}_v^{\epsilon}$ with respect to this inner product,

$$\mathcal{S}(V_v) \ni f_v \mapsto \sum_{i=1}^l \phi_i \otimes \lambda_v(f_v, \phi_i) \in \tilde{\chi}_v^{\epsilon} \otimes \mathcal{U}(V_v, \tilde{\chi}_v^{\epsilon})$$

becomes a surjective intertwining operator. This implies that $\mathcal{U}(V_v, \tilde{\chi}_v^{\epsilon})$ is a quotient representation of $\Omega(V_v, \tilde{\chi}_v^{\epsilon})$.

Proposition 5.1. *If V_v is anisotropic and either $v \notin S_D$ and $\sigma_v = \tilde{\mathbf{1}}^- = \det$ or $v \in S_D$ and $\sigma_v \neq \mathbf{1}$ then $\mathcal{U}(V_v, \sigma_v)$ is irreducible and supercuspidal. $\mathcal{U}(V_v, \sigma_v)$ will be also denoted by $\theta(V_v, \sigma_v)$.*

Proof. Any σ_v is supercuspidal and the Howe correspondent $\Omega(V_v, \sigma_v)$ of σ_v to $G(k_v)$ becomes the first occurrence in the Witt tower over W_v . This follows clearly in the case of $v \in S_D$ and from Theorem 4.1 (1) in the case of $v \notin S_D$. From [MVW87] Chap.3 IV Th.4 1), $\Omega(V_v, \sigma_v)$ is irreducible and supercuspidal. Since $\mathcal{U}(V_v, \sigma_v)$ is non-zero and a quotient of $\Omega(V_v, \sigma_v)$, $\mathcal{U}(V_v, \sigma_v)$ is also irreducible and supercuspidal. \square

Let $v \notin S_D$ and $T_v = T_{V_v}$. From the definition of the Weil representation, there are ξ_1, ξ_2 in T_v linearly independent such that $\mathcal{U}(V_v, \sigma_v)$ is identified with the space $\mathcal{U}(T_v, \sigma_v)$ generated by functions

$$\lambda_v(f_v, \phi_v) : Sp(4, k_v) \ni g_v \mapsto \int_{O(T_v)} \omega_{\psi_v, T_v}(h_v, g_v) f_v(\xi_1, \xi_2) \overline{\phi_v(h_v)} dh_v$$

for all $f_v \in \mathcal{S}(T_v^2)$ and all $\phi_v \in \tilde{\chi}_v^\epsilon$. We may assume that both $q_{T_v}(\xi_1)$ and $q_{T_v}(\xi_2)$ are non-zero by acting of an element of $M_S(k_v)$. We denote by $\mathcal{U}'(T_v, \sigma_v)$ the space generated by functions

$$\lambda'_v(f_v, \phi_v) : Sp(4, k_v) \ni g_v \mapsto \int_{O(T_v)} \omega_{\psi_v, T_v}(h_v, g_v) f_v(0, \xi_2) \overline{\phi_v(h_v)} dh_v$$

for all $f_v \in \mathcal{S}(T_v^2)$ and all $\phi_v \in \tilde{\chi}_v^\epsilon$. It is easily checked that $\mathcal{U}'(T_v, \sigma_v)$ is a subrepresentation of $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(\zeta_{T_v} | \cdot |^{-1} \otimes \Theta_v^0(T_v, \sigma_v))$ and if $\Theta_v^0(T_v, \sigma_v)$ is non-zero then so is $\mathcal{U}'(T_v, \sigma_v)$. From the explicit formula of the Weil representation, if $\lambda_v(f_v, \phi_v)$ is zero then so is $\lambda'_v(f_v, \phi_v)$. This induces a surjective intertwining operator $\Psi_v; \mathcal{U}(T_v, \sigma_v) \rightarrow \mathcal{U}'(T_v, \sigma_v)$ characterized by

$$\Psi_v(\lambda_v(f_v, \phi_v)) = \lambda'_v(f_v, \phi_v).$$

For a smooth representation π of $G(k_v)$ (or $G(V_v) \times G(k_v)$) and $P = P_0, P_S, P_K$, π_P denotes the normalized Jacquet module with respect to P . If $v \notin S_D$ and T_v is anisotropic we obtain directly

$$(\omega_{\psi_v, T_v})_{P_K} \simeq \zeta_{T_v} | \cdot |^{-1} \otimes \omega_{\psi_v, T_v}^0$$

as a $O(T_v) \times M_K(k_v)$ -module where $\zeta_{T_v} | \cdot |^{-1}$ is a representation of the first component of $k_v^\times \times SL(2, k_v) \simeq M_K(k_v)$. From this isomorphism, we have

$$\mathcal{U}(T_v, \sigma_v)_{P_K} \simeq \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v). \quad (5.1)$$

Proposition 5.2. *Suppose that T_v is anisotropic.*

- (1) *If $v \notin S_D$ and $\sigma_v \neq \det$ then $\mathcal{U}(T_v, \sigma_v)$ is isomorphic to the unique irreducible quotient $J_K(\zeta_{V_v} | \cdot | \otimes \Theta^0(T_v, \sigma_v))$ of $\text{Ind}_{P_K(k_v)}^{G(k_v)}(\zeta_{V_v} | \cdot | \otimes \Theta^0(T_v, \sigma_v))$.*
- (2) *If $v \in S_D$ and $\sigma_v = \mathbf{1}$ then $\mathcal{U}(T_v, \sigma_v)$ is isomorphic to the unique irreducible quotient $J_S((\zeta_{V_v} | \cdot |^{1/2}) \circ \nu_{D_v})$ of $\text{Ind}_{P_S(k_v)}^{G(k_v)}((\zeta_{V_v} | \cdot |^{1/2}) \circ \nu_{D_v})$.*

In both cases, $\mathcal{U}(T_v, \sigma_v)$ is denoted by $\theta(V_v, \sigma_v)$.

Proof. Assume $\sigma_v = \mathbf{1}$. Let $R(V_v)$ denote the image of the map

$$\mathcal{S}(V_v) \ni f \mapsto (G(k_v) \ni g \mapsto \omega_{\psi_v, V_v} f(0)) \in \text{Ind}_{P_S(k_v)}^{G(k_v)} ((\zeta_{V_v} | \cdot |^{-1/2}) \circ \nu_{D_v}).$$

Then the $G(V_v)$ -coinvariant space $\mathcal{S}(V_v)_{G(V_v)}$ of $\mathcal{S}(V_v)$ is isomorphic to $R(V_v)$ as a representation of $G(k_v)$ ([MVW87] Chap.3 IV Th.7). It is known that $R(V_v)$ is irreducible for all v ([KRS92] Prop.1.1, [Yas07] Prop.4.5). Since $\mathcal{U}(V_v, \sigma_v)$ is non-zero and a quotient of $\mathcal{S}(V_v)_{G(V_v)}$, it is isomorphic to $R(V_v)$. If $v \in S_D$, $\mathcal{U}(V_v, \sigma_v)$ is isomorphic to $J_S((\zeta_{V_v} | \cdot |^{-1/2}) \circ \nu_{D_v})$ by [Yas07] Prop.4.5. If $v \notin S_D$ then $R(V_v)$ is isomorphic to an irreducible subrepresentation of $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} (\zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v))$ because $\Theta^0(T_v, \sigma_v)$ is non-zero. As for Jacquet module, from (5.1) and [BZ77] § 2.12, we have

$$\begin{aligned} R(V_v)_{P_K} &= \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v), \\ (\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v))_{P_K} &= \zeta_{T_v} \otimes \text{Ind}_{B^{SL(2)}(k_v)}^{SL(2, k_v)} \zeta_{T_v} | \cdot |_v \\ &\quad + \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v) + \zeta_{T_v} | \cdot |_v \otimes \Theta^0(T_v, \sigma_v) \end{aligned}$$

in the Grothendieck group. From [ST93] Prop.5.4, $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v)$ has three irreducible constituents. More precisely, all the constituents consist of

$$J_K(\zeta_{T_v} | \cdot |_v \otimes \Theta^0(T_v, \sigma_v)), \quad J_S(| \cdot |_v^{1/2} \zeta_{T_v} St_{GL(2)}), \quad \delta(T_v, \sigma_v).$$

Here,

- $J_S(| \cdot |_v^{1/2} \zeta_{T_v} St_{GL(2)})$ is the unique irreducible quotient of $\text{Ind}_{P_S(k_v)}^{Sp(4, k_v)} (| \cdot |_v^{1/2} \zeta_{T_v} St_{GL(2)})$ where $St_{GL(2)}$ is the Steinberg representation of $GL(2, k_v)$ and $J_S(| \cdot |_v^{1/2} \zeta_{T_v} St_{GL(2)})_{P_K} = \zeta_{T_v} \otimes \text{Ind}_{B^{SL(2)}(k_v)}^{SL(2, k_v)} \zeta_{T_v} | \cdot |_v$,
- $\delta(T_v, \sigma_v)$ is an irreducible tempered representation with Jacquet module $\zeta_{T_v} | \cdot |_v \otimes \Theta^0(T_v, \sigma_v)$,
- $J_K(\zeta_{T_v} | \cdot |_v \otimes \Theta^0(T_v, \sigma_v))_{P_K} \simeq \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v)$.

Therefore, $R(V_v)$ is isomorphic to $J_K(\zeta_{T_v} | \cdot |_v \otimes \Theta^0(T_v, \sigma_v))$. Next assume $v \notin S_D$, $\sigma_v \neq \mathbf{1}$, \det . Since σ_v is supercuspidal, $\Omega(V_v, \sigma_v)$ is irreducible by [MVW87] Chap.3 IV Th.41) and so is $\mathcal{U}(T_v, \sigma_v)$. Since $\Theta^0(T_v, \sigma_v)$ is supercuspidal, as for Jacquet module we have

$$(\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v))_{P_K} = \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v) + \zeta_{T_v} | \cdot |_v \otimes \Theta^0(T_v, \sigma_v)$$

in the Grothendieck group. Since $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v)$ has length at most 2, from (5.1), $\Omega(V_v, \sigma_v)$ is isomorphic to the unique irreducible subrepresentation of $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \zeta_{T_v} | \cdot |^{-1} \otimes \Theta^0(T_v, \sigma_v)$. \square

If $v \notin S_D$ and T_v is isotropic, there is a smooth subrepresentation \mathcal{F} of the

Jacquet module $(\omega_{\psi_v, T_v})_{P_K}$ as a representation of $O(1, 1; k_v) \times M_K(k_v)$ such that

$$\begin{aligned} (\omega_{\psi_v, T_v})_{P_K} / \mathcal{F} &\simeq |\cdot|_v^{-1} \otimes \omega_{\psi_v, T_v}^0, \\ \mathcal{F} &\simeq (\text{Ind}_{SO(1, 1; k_v) \times \mathbb{G}_m(k_v)}^{O(1, 1; k_v) \times \mathbb{G}_m(k_v)} \tau_v) \otimes \mathbf{1}_{SL(2)} \end{aligned}$$

where $|\cdot|_v^{-1}$ is a representation of the first component of $\mathbb{G}_m(k_v) \times SL(2, k_v) \simeq M_K(k_v)$ and τ_v is the representation of $SO(1, 1; k_v) \times \mathbb{G}_m(k_v) \simeq k_v^\times \times k_v^\times$ defined on $\mathcal{S}(GL(1, k_v))$ by

$$\tau_v(c, a)\phi(y) = \phi(c^{-1}ya) \quad (c, a, y \in k_v^\times, \phi \in \mathcal{S}(GL(1, k_v)))$$

(See [MVW87] Chap.3 IV or [Kud86] Th.2.8). Note that

$$\text{Ind}_{SO(1, 1; k_v) \times \mathbb{G}_m(k_v)}^{O(1, 1; k_v) \times \mathbb{G}_m(k_v)} \tau_v \simeq \mathcal{S}(O(1, 1; k_v))$$

where a representation on $\mathcal{S}(O(1, 1; k_v))$ is given by the left and right regular action of $O(1, 1; k_v) \times \mathbb{G}_m(k_v)$ (or $O(1, 1; k_v) \times O(1, 1; k_v)$).

Lemma 5.3.

$$\begin{aligned} \mathcal{U}(T_v, \sigma_v)_{P_K} &= \sigma_v|_{\mathbb{G}_m(k_v)} \otimes \mathbf{1}_{SL(2)} + |\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v) \\ &\quad + |\cdot|_v^{-1} \otimes (a \text{ supercuspidal representation of } SL(2, k_v)), \end{aligned}$$

in the Grothendieck group.

Remark 5.4. *The supercuspidal representation appearing in the lemma may be zero.*

Proof. For a smooth representation ρ of $O(1, 1; k_v) \times M_K(k_v)$, $\mathcal{N}(\rho, \sigma_v)$ denotes the intersection of the kernels of f which belong to $\text{Hom}_{O(1, 1; k_v)}(\rho, \sigma_v)$. Then $\rho_{\sigma_v} := \rho / \mathcal{N}(\rho, \sigma_v)$ becomes the maximal σ_v -isotypic quotient of ρ , which is isomorphic to $\sigma_v \otimes \pi(\tau, \sigma_v)$ where $\pi(\tau, \sigma_v)$ is a smooth representation of $M_K(k_v)$ ([MVW87] Chap.2 III). In general, for an exact sequence of smooth representations of $O(1, 1; k_v) \times M_K(k_v)$,

$$0 \rightarrow \rho_1 \rightarrow \rho_2 \rightarrow \rho_3 \rightarrow 0,$$

a following exact sequence is obtained.

$$\rho_{1, \sigma_v} \rightarrow \rho_{2, \sigma_v} \rightarrow \rho_{3, \sigma_v} \rightarrow 0.$$

This implies that there is an exact sequence,

$$\pi(\rho_1, \sigma_v) \xrightarrow{L} \pi(\rho_2, \sigma_v) \rightarrow \pi(\rho_3, \sigma_v) \rightarrow 0.$$

Now let us set $\rho_1 = \mathcal{F}$, $\rho_2 = (\omega_{\psi_v, T_v})_{P_K}$, $\rho_3 = (\omega_{\psi_v, T_v})_{P_K} / \mathcal{F}$. For $\phi_v (\neq 0) \in \sigma_v$, a surjective map

$$\omega_{\psi_v, T_v} \ni f_v \mapsto \lambda'_v(f_v, \phi_v) \in \mathcal{U}(T_v, \sigma_v)$$

can be reduced to

$$\bar{\lambda}_{\phi_v} : (\omega_{\psi_v, T_v})_{P_K} \rightarrow \mathcal{U}(T_v, \sigma_v)_{P_K}$$

by definition. For $\xi \in T_v$ such that $q_{T_v}(\xi) \neq 0$, put

$$\begin{aligned}\lambda_{\phi_v}^1(f_v)(a) &= \int_{O(1,1;k_v)} f_v(ha) \overline{\phi_v(h)} dh \quad (f_v \in \mathcal{S}(O(1,1;k_v)), a \in k_v^\times), \\ \lambda_{\phi_v}^3(f'_v)(g) &= \int_{O(1,1;k_v)} \omega_{\psi_v, T_v}^0(h, g) f'_v(\xi) \overline{\phi_v(h)} dh \quad (f'_v \in \mathcal{S}(T_v), g \in SL(2, k_v)),\end{aligned}$$

where these integrals converge absolutely. $\mathcal{U}_K^1(T_v, \sigma_v)$ (resp. $\mathcal{U}_K^3(T_v, \sigma_v)$) denotes the space generated by functions $\lambda_{\phi_v}^1(f_v)$ for all $f_v \in \mathcal{S}(O(1,1;k_v))$ (resp. $\lambda_{\phi_v}^3(f'_v)$ for all $f'_v \in \mathcal{S}(T_v)$). ρ_1 and ρ_3 provide $\mathcal{U}_K^1(T_v, \sigma_v)$ and $\mathcal{U}_K^3(T_v, \sigma_v)$ with structure of representations of $M_K(k_v)$. Then $\mathcal{U}_K^3(T_v, \sigma_v)$, which is isomorphic to $|\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v)$, is a quotient of $\mathcal{U}(T_v, \sigma_v)_{P_K}$ and the following diagram is commutative;

$$\begin{array}{ccccc} 0 & \longrightarrow & \rho_1 & \longrightarrow & \rho_2 & (5.2) \\ & & \downarrow \lambda_{\phi_v}^1 & & \downarrow \bar{\lambda}_{\phi_v} & \\ 0 & \longrightarrow & \mathcal{U}_K^1(T_v, \sigma_v) & \longrightarrow & \mathcal{U}(T_v, \sigma_v)_{P_K} & \end{array}$$

Similarly to the observation before Proposition 5.1, there are surjective homomorphisms $\pi(\rho_i, \sigma_v) \rightarrow \mathcal{U}_K^i(T_v, \sigma_v)$ ($i = 1, 3$) and $\pi(\rho_2, \sigma_v) \rightarrow \mathcal{U}(T_v, \sigma_v)_{P_K}$. From (5.2), we have the following commutative diagram;

$$\begin{array}{ccc} \pi(\rho_1, \sigma_v) & \longrightarrow & \pi(\rho_2, \sigma_v) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{U}_K^1(T_v, \sigma_v) \longrightarrow \mathcal{U}(T_v, \sigma_v)_{P_K}. \end{array}$$

Since

$$\dim_{\mathbf{C}} \text{Hom}_{O(1,1;k_v) \times O(1,1;k_v)}(\tau_v, \sigma_v^\vee \otimes \pi_v) = \begin{cases} 1 & \pi_v \simeq \sigma_v, \\ 0 & \text{otherwise,} \end{cases}$$

and $\lambda_{\phi_v}^1$ is non-zero, one concludes that ι is injective and

$$\iota(\pi(\rho_1, \sigma_v)) \simeq \pi(\rho_1, \sigma_v) \simeq \mathcal{U}_K^1(T_v, \sigma_v) \simeq \sigma_v|_{\mathbb{G}_m(k_v)} \otimes \mathbf{1}_{SL(2)}.$$

Since $(\omega_{\psi_v, T_v}^0)_{BSL(2)} = \mathbf{1} \otimes \mathbf{1} + \tau_v$ (See [MVW87] Chap.3 IV or [Kud86] Th.2.8) ,

$$\pi((\omega_{\psi_v, T_v}^0)_{BSL(2)}, \sigma_v) = \Theta^0(T_v, \sigma_v)_{BSL(2)} = \begin{cases} \mathbf{1} + \mathbf{1} & \sigma_v = \mathbf{1}, \\ \sigma_v|_{\mathbb{G}_m(k_v)} & \sigma_v \neq \mathbf{1}, \end{cases}$$

in the Grothendieck group. Since $\Theta^0(T_v, \sigma_v)$ is a quotient of $\pi(\omega_{\psi_v, T_v}^0, \sigma_v)$,

$$\pi(\omega_{\psi_v, T_v}^0, \sigma_v) = \Theta^0(T_v, \sigma_v) \oplus (\text{a supercuspidal representation})$$

by (an analogy for $SL(2)$ of) [BZ76] Th.4.17. Consequently,

$$\begin{aligned}\pi((\omega_{\psi_v, T_v})_{P_K}, \sigma_v) &= \sigma_v|_{\mathbb{G}_m(k_v)} \otimes \mathbf{1}_{SL(2)} + |\cdot|_v^{-1} \otimes \Theta^0(T_v, \sigma_v) \\ &\quad + |\cdot|_v^{-1} \otimes (\text{a supercuspidal representation of } SL(2, k_v))\end{aligned}$$

in the Grothendieck group. Since $\mathcal{U}(T_v, \sigma_v)_{P_K}$ is a quotient of $\pi((\omega_{\psi_v, T_v})_{P_K}, \sigma_v)$ and $\mathcal{U}_K^1(T_v, \sigma_v), \mathcal{U}_K^3(T_v, \sigma_v)$ are subquotients of $\mathcal{U}(T_v, \sigma_v)_{P_K}$, the proposition is obtained. \square

For a quasi-character μ_v of k_v^\times and a smooth representation τ_v of $SL(2, k_v)$,

$$(\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(\mu_v \otimes \tau_v))_{P_K} = \mu_v \otimes \tau_v + \mu_v^\vee \otimes \tau_v + \tau_v_{BSL(2)} \otimes (\text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \mu_v)$$

in the Grothendieck group ([BZ77] § 2.12).

Proposition 5.5. *Let $v \notin S_D$ and T_v is isotropic. Then $\mathcal{U}(T_v, \sigma_v)$ has an irreducible quotient isomorphic to*

$$\theta(T_v, \sigma_v) := \begin{cases} J_K(| \cdot |_v \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \chi_v) & \sigma_v = \text{Ind}_{SO(1,1; k_v)}^{O(1,1; k_v)} \chi_v, \chi_v^2 \neq \mathbf{1}, \\ J_K(| \cdot |_v \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \mathbf{1}) & \sigma_v = \mathbf{1}, \\ J_K(| \cdot |_v \otimes \Theta^0(T_v, \sigma_v)) & \sigma_v = \tilde{\chi}_v^\epsilon, \chi_v^2 = \mathbf{1}, \chi_v \neq \mathbf{1}, \epsilon = \pm 1, \\ J_S(| \cdot |_v^{1/2} St_{GL(2)}) & \sigma_v = \det, \end{cases}$$

where $St_{GL(2)}$ is the Steinberg representation of $GL(2, k_v)$ and $J_K(\tau)$ (resp. $J_S(\tau)$) denotes the Langlands quotient of $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \tau$ (resp. $\text{Ind}_{P_S(k_v)}^{G(k_v)} \tau$).

Proof. By Lemma 5.3 and [BZ76] Th.4.17, $\mathcal{U}(V_v, \sigma_v)_{P_K}$ has a quotient equal to

$$\begin{cases} | \cdot |_v^{-1} \otimes \Theta^0(V_v, \sigma_v) + \chi_v \otimes \mathbf{1}_{SL(2)} + \chi_v^{-1} \otimes \mathbf{1}_{SL(2)} & (\chi_v^2 \neq \mathbf{1}), \\ | \cdot |_v^{-1} \otimes \Theta^0(V_v, \sigma_v) + \chi_v \otimes \mathbf{1}_{SL(2)} & (\chi_v^2 = \mathbf{1}) \end{cases}$$

in the Grothendieck group. If $\chi_v^2 \neq \mathbf{1}$, $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \chi_v \otimes \mathbf{1}_{SL(2)}$ is irreducible from [ST93] Th.5.4 (ii), and $J_K(| \cdot |_v \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \chi_v)$ is isomorphic to $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \chi_v \otimes \mathbf{1}_{SL(2)}$. Therefore, in the Grothendieck group

$$\begin{aligned} J_K(| \cdot |_v \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \chi_v)_{P_K} &= (\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)} \chi_v \otimes \mathbf{1}_{SL(2)})_{P_K} \\ &= | \cdot |_v^{-1} \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \chi_v + \chi_v \otimes \mathbf{1}_{SL(2)} + \chi_v^{-1} \otimes \mathbf{1}_{SL(2)} \quad (\chi_v^2 \neq \mathbf{1}). \end{aligned}$$

If $\chi_v^2 = \mathbf{1}, \chi_v \neq \mathbf{1}$, in the Grothendieck group

$$\begin{aligned} \text{Ind}_{P_0(k_v)}^{Sp(4, k_v)}(| \cdot |_v \otimes \chi_v) &= \text{Ind}_{P_0(k_v)}^{Sp(4, k_v)}(| \cdot |_v \otimes \delta(\chi_v, +)) + \text{Ind}_{P_0(k_v)}^{Sp(4, k_v)}(| \cdot |_v \otimes \delta(\chi_v, -)) \\ &= \text{Ind}_{P_0(k_v)}^{Sp(4, k_v)}(\chi_v \otimes \mathbf{1}_{SL(2)}) + \text{Ind}_{P_0(k_v)}^{Sp(4, k_v)}(\chi_v \otimes St_{SL(2)}) \end{aligned}$$

where $\delta(\chi_v, \pm)$ is irreducible representations of $SL(2, k_v)$ such that

$$\text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \chi_v \simeq \delta(\chi_v, +) \oplus \delta(\chi_v, -)$$

and $St_{SL(2)}$ is the Steinberg representation of $SL(2, k_v)$. The length of $\text{Ind}_{P_0(k_v)}^{Sp(4, k_v)}(| \cdot |_v \otimes \chi_v)$ is 4 from [ST93] Th.5.4. By comparison of Jacquet modules, we see that the semisimplification of $J_K(| \cdot |_v \otimes \delta(\chi_v, \pm))_{P_K}$ coincides with the common composition factor of $(\text{Ind}_{P_0(k_v)}^{Sp(4, k_v)} | \cdot |_v \otimes \delta(\chi_v, \pm))_{P_K}$ and $(\text{Ind}_{P_0(k_v)}^{Sp(4, k_v)} \chi_v \otimes \mathbf{1}_{SL(2)})_{P_K}$. Therefore,

$$J_K(| \cdot |_v \otimes \delta(\chi_v, \pm))_{P_K} = | \cdot |_v^{-1} \otimes \delta(\chi_v, \pm) + \chi_v \otimes \mathbf{1}_{SL(2)} \quad (\chi_v^2 = \mathbf{1}, \chi_v \neq \mathbf{1})$$

in the Grothendieck group. If $\sigma_v \neq \det$, a non-zero quotient of $\mathcal{U}(T_v, \sigma_v)$ is isomorphic to a subrepresentation of $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(| \cdot |_v^{-1} \otimes \Theta^0(T_v, \sigma_v))$ because $\Theta^0(T_v, \sigma_v) \neq 0$. If $\chi_v \neq \mathbf{1}$, from the Jacquet module of $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(| \cdot |_v^{-1} \otimes \Theta^0(T_v, \sigma_v))$ and the description of the above Jacquet modules, we have that $J_K(| \cdot |_v \otimes \Theta^0(T_v, \sigma_v))$ is a

quotient of $\mathcal{U}(T_v, \sigma_v)$. Assume $\sigma_v = \mathbf{1}$. $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(\mathbf{1} \otimes \mathbf{1}_{SL(2)})$ is a subrepresentation of $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(|\cdot|_v^{-1} \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \mathbf{1})$. From the comparison of Jacquet modules, there is a non-zero intertwiner operator from $\mathcal{U}(T_v, \sigma_v)$ to $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(\mathbf{1} \otimes \mathbf{1}_{SL(2)})$. From [ST93] Lem.3.8 and [Tad94] Lem.6.2,

$$\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(\mathbf{1} \otimes \mathbf{1}_{SL(2)}) \simeq J_K(|\cdot|_v \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \mathbf{1}) \oplus J_S(|\cdot|_v^{1/2} St_{GL(2)}). \quad (5.3)$$

Since the Jacquet module of the image of $\mathcal{U}(T_v, \sigma_v)$ does not contain $J_S(|\cdot|_v^{1/2} St_{GL(2)})_{P_K}$, $J_K(|\cdot|_v \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \mathbf{1})$ is a quotient of $\mathcal{U}(T_v, \sigma_v)$. Finally, consider $\sigma_v = \det$. Since $\mathcal{U}(T_v, \sigma_v)_{P_K}$ has a quotient isomorphic to $\mathbf{1} \otimes \mathbf{1}_{SL(2)}$, there is a non-zero intertwiner operator from $\mathcal{U}(T_v, \sigma_v)$ to $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(\mathbf{1} \otimes \mathbf{1}_{SL(2)})$ by the Frobenius reciprocity. It is seen that $J_K(|\cdot|_v \otimes \text{Ind}_{BSL(2)(k_v)}^{SL(2, k_v)} \mathbf{1})$ is not an irreducible constituent of the image of $\mathcal{U}(T_v, \sigma_v)$ from the Jacquet modules. Therefore, by (5.3) one concludes that a quotient of $\mathcal{U}(T_v, \sigma_v)$ is isomorphic to $J_S(|\cdot|_v^{1/2} St_{GL(2)})$. \square

5.2 Archimedean case

Let v be an archimedean place. $\mathfrak{g}(V_v)$ and \mathfrak{g}_v denote the complexifications of Lie algebras of $G(V_v)$ and $G(k_v)$. Let $S(V_v) \subset \mathcal{S}(V_v)$ be the space of Schwartz functions which correspond to polynomials in the Fock model of ω_{ψ_v, V_v} . For an irreducible admissible $(\mathfrak{g}(V_v), \mathbf{L}_v)$ -module σ_v , let \mathcal{N}_{σ_v} be the intersection of all subspaces $\mathcal{N} \subset S(V_v)$ such that $S(V_v)/\mathcal{N} \simeq \sigma_v$. Then there exists an admissible $(\mathfrak{g}_v, \mathbf{K}_v)$ -module $\rho(\sigma_v)$ such that

$$S(V_v)/\mathcal{N}_{\sigma_v} \simeq \sigma_v \otimes \rho(\sigma_v)$$

as a $(\mathfrak{g}(V_v) \oplus \mathfrak{g}_v, \mathbf{L}_v \times \mathbf{K}_v)$ -module. Furthermore, if ρ_v is non-zero, $\rho(\sigma_v)$ has a unique irreducible $(\mathfrak{g}_v, \mathbf{K}_v)$ -quotient $\theta(V_v, \sigma_v)$ ([How89] Th.2.1). For $\sigma_v = \tilde{\chi}_v^\epsilon$ for a character χ_v and $\epsilon = \pm 1$, $U(V_v, \sigma_v)$ is defined by the space generated by $\lambda_v(f_v, \phi_v)$ for all $f_v \in S(V_v)$ and all $\phi_v \in \tilde{\chi}_v^\epsilon$. Similarly to the non-archimedean case, $U(V_v, \sigma_v)$ is a quotient of $\rho(\sigma_v)$. Therefore $\theta(V_v, \sigma_v)$ is a quotient of $U(V_v, \sigma_v)$.

Proposition 5.6. *If $v \notin S_D$ and $\sigma_v \neq \det$ then $\theta(V_v, \sigma_v)$ is isomorphic to the unique irreducible quotient $J_K(|\cdot|_v \otimes \Theta^0(T_v, \sigma_v))$ of $\text{Ind}_{P_K(k_v)}^{Sp(4, k_v)}(\zeta_{T_v} |\cdot|_v \otimes \Theta^0(T_v, \sigma_v))$ where $T_v = T_{V_v}$.*

This proposition is obtained by [Pau05] Th.38 if v is real and [AB95] Prop.21, Th.2.8 if v is complex.

Proposition 5.7 ([LZ97] Cor.3.2 and [AB95] Prop.2.1). *If $v \notin S_D$ is real, T_{V_v} is isotypic and $\sigma_v = \det$ then $\theta(V_v, \sigma_v)$ is isomorphic to an irreducible constituent of $\text{Ind}_{P_S(\mathbf{R})}^{Sp(4, \mathbf{R})}(\text{sgn} |\cdot|^{1/2}) \circ \det$ where sgn is the sign character of \mathbf{R}^\times . If v is complex and $\sigma_v = \det$ then $\theta(V_v, \sigma_v)$ is isomorphic to the unique irreducible constituent of $\text{Ind}_{P_0(\mathbf{C})}^{Sp(4, \mathbf{C})}(|\cdot|_{\mathbf{C}}^{1/2} \otimes |\cdot|_{\mathbf{C}}^{-1/2})$ containing the lowest \mathbf{K}_v -type of the induction.*

When $v \notin S_D$ and v is real we choose a Cartan subgroup T of $G(k_v) = Sp(4, \mathbf{R})$ with Lie algebra $\mathfrak{t}_{v,0}$ and complexification \mathfrak{t}_v as follows:

$$\mathfrak{t}_{v,0} = \mathfrak{t}_v \cap \mathbb{M}(4, \mathbf{R}) \subset \mathfrak{sp}(4, \mathbf{R}),$$

$$\mathfrak{t}_v = \left\{ t(a_1, a_2) = \left(\begin{array}{c|c} & a_1 \\ \hline -a_1 & \\ \hline & a_2 \\ & \hline & -a_2 \end{array} \right) \middle| a_1, a_2 \in \mathbf{C} \right\}.$$

When writing $e_i : \mathfrak{t}_v \ni t(a_1, a_2) \mapsto a_i \in \mathbf{C}$ ($i = 1, 2$), the roots of \mathfrak{t}_v in $\mathfrak{g}_v = \mathfrak{sp}(4, \mathbf{C})$ are

$$\Delta(\mathfrak{g}_v, \mathfrak{t}_v) = \{\pm e_1 \pm e_2, \pm 2e_1, \pm 2e_2\}.$$

Ψ_+ (resp. Ψ_-) $\subset \Delta(\mathfrak{g}_v, \mathfrak{t}_v)$ denotes the set of positive roots defined by simple roots $e_1 - e_2, 2e_2$ (resp. $e_1 - e_2, -2e_1$). If $\lambda \in \sqrt{-1}\mathfrak{t}_{v,0}^*$ is dominant with respect to Ψ_{\pm} , the limit of discrete series defined by λ and Ψ_{\pm} is denoted by $\delta(\lambda, \Psi_{\pm})$. If $v \in S_D$, $G(\mathbf{R})$ is realized by

$$Sp(1, 1) = \{g \in Sp(4, \mathbf{C}) \mid gI_{1,1} \bar{g} = I_{1,1}\}, \quad I_{1,1} = \text{diag}(1, -1, 1, -1).$$

We choose a Cartan subgroup T of $G(k_v)$ with Lie algebra $\mathfrak{t}_{v,0}$ and complexification \mathfrak{t}_v as follows:

$$\mathfrak{t}_{v,0} = \mathfrak{t}_v \cap \sqrt{-1}\mathbb{M}(4, \mathbf{R}) \subset \mathfrak{g}_v,$$

$$\mathfrak{t}_v = \left\{ t(a_1, a_2) = \left(\begin{array}{c|c} a_1 & \\ \hline & a_2 \\ \hline & -a_1 \\ & \hline & -a_2 \end{array} \right) \middle| a_1, a_2 \in \mathbf{C} \right\}.$$

the roots of \mathfrak{t}_v in \mathfrak{g}_v are

$$\Delta(\mathfrak{g}_v, \mathfrak{t}_v) = \{\pm e_1 \pm e_2, \pm 2e_1, \pm 2e_2\}$$

where $e_i : \mathfrak{t}_v \ni t(a_1, a_2) \mapsto a_i \in \mathbf{C}$ ($i = 1, 2$). Ψ_+ (resp. Ψ_-) $\subset \Delta(\mathfrak{g}_v, \mathfrak{t}_v)$ denotes the set of positive roots defined by simple roots $e_1 - e_2, 2e_2$ (resp. $-e_1 + e_2, 2e_1$). If $\lambda \in \sqrt{-1}\mathfrak{t}_0^*$ is dominant with respect to Ψ_{\pm} , the limit of discrete series defined by λ and Ψ_{\pm} is denoted by $\delta(\lambda, \Psi_{\pm})$. If $v \in S_D$, $G(V_v)$ is isomorphic to $O^*(2) \simeq \mathbf{C}^1$, so that a character of $G(V_v)$ is identified with that of \mathbf{C}^1 .

Proposition 5.8. *Let $\psi_v = \exp(\lambda \cdot)$ for $\lambda = \epsilon \sqrt{-1}|\lambda| \in \sqrt{-1}\mathbf{R}$ $\epsilon = \pm 1$.*

(1) *If $v \notin S_D$, v is real and $T_v = T_{V_v}$ is anisotropic then*

$$\theta(V_v, \det) \simeq \begin{cases} \delta((1, 0), \Psi_+) & \epsilon\eta > 0, \\ \delta((0, -1), \Psi_-) & \epsilon\eta < 0. \end{cases}$$

where $\eta = 1$ if T_v is positive definite, -1 otherwise.

(2) *Assume $v \in S_D$ and v is real. If $\sigma_v = \mathbf{1}$ then $\theta(V_v, \sigma_v) \simeq \text{Ind}_{P_S(\mathbf{R})}^{G(\mathbf{R})} (|\nu_{D_v}|_{\mathbf{R}}^{1/2})$. If*

$$\sigma_v = \exp(2\pi\sqrt{-1}n \cdot) \quad (n \in \mathbf{Z} \setminus \{0\}) \text{ then}$$

$$\theta(V_v, \sigma_v) \simeq \begin{cases} \delta(|n|, 1), \Psi_+ & \epsilon n > 0, \\ \delta(1, |n|), \Psi_- & \epsilon n < 0. \end{cases}$$

In particular, if $|n| > 1$, $\theta(V_v, \sigma_v)$ is in the discrete series.

To prove this proposition we recall the Fock model. For $n \in \mathbf{N}$, we define the Fock space

$$\mathcal{F} = \left\{ F : \mathbf{C}^n \rightarrow \mathbf{C}, \text{ entire} \left| \|F\|_c^2 := \int_{\mathbf{C}^n} |F(z)|^2 e^{-\frac{1}{2}|z|^2} d_c z < +\infty \right. \right\}$$

where $d_c z$ is the self-dual measure with respect to $\mathbf{C}^n \times \mathbf{C}^n \ni (z_1, z_2) \mapsto e^{\sqrt{-1}\operatorname{Re}(z_1 \cdot \bar{z}_2)} \in \mathbf{C}^1$. Then \mathcal{F} is a Hilbert space by the inner product given by $\|\cdot\|_c$. For $f \in L^2(\mathbf{R}^n)$, put

$$\mathcal{B}f(z) = \int_{\mathbf{R}^n} f(x) B(x, z) d_\psi x \quad (z \in \mathbf{C}^n),$$

$$B(x, z) = 2^{\frac{n}{4}} \cdot \exp\left(-\frac{1}{2}|\lambda|x^2 + |\lambda|^{\frac{1}{2}}x \cdot z - \frac{1}{4}z^2\right) \quad (x \in \mathbf{R}^n, z \in \mathbf{C}^n)$$

where $d_\psi x$ is the self-dual measure with respect to $X \times X \ni (x, y) \mapsto \psi(x \cdot y)$. The L^2 -norm $\|\cdot\|_{L^2}$ on $L^2(\mathbf{R}^n)$ is defined by $d_\psi x$. Then \mathcal{B} becomes an isomorphism from $(L(\mathbf{R}^n), \|\cdot\|_{L^2})$ to $(\mathcal{F}, \|\cdot\|_c)$ ([Fol89] Chap.1 § 6). The Weil representation ω_{ψ_v} of the metaplectic cover of $Sp(2n, \mathbf{R})$ is defined on $L(\mathbf{R}^n)$ as a unitary representation. By the transfer by \mathcal{B} , the Weil representation can be also realized on \mathcal{F} , which is denoted by $\hat{\omega}_{\psi_v}$. The Harish-Chandra module of $\hat{\omega}_{\psi_v}$ is $\mathbf{C}[z_1, \dots, z_n] \subset \mathcal{F}$ and the explicit formula of $\hat{\omega}_{\psi_v}$ is described as a $(\mathfrak{sp}(2n, \mathbf{C}) \oplus \mathbf{C}, U(n) \times \mathbf{C}^1)$ -module as follows:

- (1) $\hat{\omega}_{\psi_v}((0, t)) = t \cdot \operatorname{Id}_{\mathbf{C}[z_1, \dots, z_n]} \quad (t \in \mathbf{C}),$
- (2) $\hat{\omega}_{\psi_v}\left(\frac{1}{2} \left(\frac{E_{i,j} - E_{j,i}}{\epsilon\sqrt{-1}(E_{i,j} + E_{j,i})} \middle| \frac{-\epsilon\sqrt{-1}(E_{i,j} + E_{j,i})}{E_{i,j} - E_{j,i}} \right)\right) = z_i \frac{\partial}{\partial z_j} + \frac{1}{2} \delta_{i,j} \quad (1 \leq i, j \leq n),$
- (3) $\hat{\omega}_{\psi_v}\left(\frac{1}{2} \left(\frac{-\epsilon\sqrt{-1}(E_{i,j} + E_{j,i})}{E_{i,j} + E_{j,i}} \middle| \frac{E_{i,j} + E_{j,i}}{\epsilon\sqrt{-1}(E_{i,j} + E_{j,i})} \right)\right) = \epsilon\sqrt{-1} z_i z_j \quad (1 \leq i, j \leq n),$
- (4) $\hat{\omega}_{\psi_v}\left(\frac{1}{2} \left(\frac{\epsilon\sqrt{-1}(E_{i,j} + E_{j,i})}{E_{i,j} + E_{j,i}} \middle| \frac{E_{i,j} + E_{j,i}}{-\epsilon\sqrt{-1}(E_{i,j} + E_{j,i})} \right)\right) = \epsilon\sqrt{-1} \frac{\partial^2}{\partial z_i \partial z_j} \quad (1 \leq i, j \leq n),$
- (5) $\hat{\omega}_{\psi_v}\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \eta\right) f(z) = \eta f(z(a + \epsilon\sqrt{-1}b)) \quad \left(\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \eta\right) \in U(n) \times \mathbf{C}^1, f \in \mathbf{C}[z_1, \dots, z_n]\right).$

Here $E_{i,j} = (\delta_{i,k}\delta_{j,l})_{k,l}$ and (2), (3), (4) are the actions with respect to a basis to

$$\begin{aligned}\mathfrak{k} &= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| A \in \text{Alt}(n, \mathbf{C}), B \in \text{Sym}(n, \mathbf{C}) \right\}, \\ \mathfrak{p}^+ &= \left\{ \begin{pmatrix} -\epsilon\sqrt{-1}B & B \\ B & \epsilon\sqrt{-1}B \end{pmatrix} \middle| B \in \text{Sym}(n, \mathbf{C}) \right\}, \\ \mathfrak{p}^- &= \left\{ \begin{pmatrix} \epsilon\sqrt{-1}B & B \\ B & -\epsilon\sqrt{-1}B \end{pmatrix} \middle| B \in \text{Sym}(n, \mathbf{C}) \right\},\end{aligned}$$

where $\mathfrak{sp}(2n, \mathbf{C}) = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ and \mathfrak{k} is the complexification of the Lie algebra of a maximal compact subgroup of $Sp(2n, \mathbf{R})$.

In our case, $n = 8$ and the transfer $\hat{\omega}_{\psi_v, V_v}$ on $\mathbf{C}[z_1, \dots, z_4]$ of ω_{ψ_v, V_v} as a $(\mathfrak{g}(V_v) \oplus \mathfrak{g}_v, \mathbf{L}_v \times \mathbf{K}_v)$ -module coincides with the composition of $\hat{\omega}_{\psi_v}$ and the splitting,

$$\begin{aligned}\cdot d\iota &: \mathfrak{g}(V_v) \oplus \mathfrak{g}_v \rightarrow \mathfrak{sp}(8, \mathbf{R}), \\ \cdot \mathbf{L}_v \times \mathbf{K}_v &\ni (l, k) \mapsto (\iota(l, k), 1) \in U(4) \times \mathbf{C}^1,\end{aligned}$$

where $\iota : G(V_v) \times G(k_v) \rightarrow Sp(8, \mathbf{R})$ is the natural homomorphism, $d\iota$ its differential, and $U(4)$ is identified with the maximal compact subgroup of $Sp(8, \mathbf{R})$.

Let us prove Proposition 5.8. If $v \in S_D$ and $\sigma_v = \mathbf{1}$, the statement has already been shown by [Yas07] Prop.4.7. In the other case, we will prove only the case of $v \in S_D$. For the case of $v \notin S_D$ is similarly proven to the case of $v \in S_D$ and $\sigma_v = \exp(\pm 2\pi\sqrt{-1}\cdot)$. (And it is proven for $\epsilon > 0, \eta > 0$ in [Ada04] Th.4.1.) Since for

$$Y_t = \begin{pmatrix} & -t \\ t & \end{pmatrix} \in \text{Lie } O^*(2) \quad (t \in \mathbf{R}),$$

$$\hat{\omega}_{\psi_v, V_v}(Y_t) = \sqrt{-1}\epsilon\eta t \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - z_4 \frac{\partial}{\partial z_4} \right),$$

we have an isotypic decomposition of $\mathbf{C}[z_1, \dots, z_4]$ as $\mathfrak{g}(V_v)$ -modules,

$$\begin{aligned}\mathbf{C}[z_1, \dots, z_4] &= \bigoplus_{k \in \mathbf{Z}} \mathcal{P}_k, \\ \mathcal{P}_k &= \bigoplus_{k_1 - k_2 = \epsilon\eta k} \mathbf{C}[z_1, z_2]^{(k_1)} \otimes \mathbf{C}[z_3, z_4]^{(k_2)},\end{aligned}$$

where \mathcal{P}_k is the isotypic space of a character $\exp(2\pi\sqrt{-1}k\cdot)$ of $G(V_v) \simeq \mathbf{C}^1$ and $\mathbf{C}[z_i, z_j]^{(l)}$ denotes the set of homogenous polynomials of degree l . The actions of a basis of \mathfrak{g}_v is described as follows:

$$\begin{aligned}(1) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} \right) &= z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}, \\ (2) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} \right) &= z_3 \frac{\partial}{\partial z_3} - z_4 \frac{\partial}{\partial z_4},\end{aligned}$$

$$\begin{aligned}
(3) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} & 2 & \\ 0 & & 0 \end{pmatrix} \right) &= 2\epsilon\eta\sqrt{-1}z_1 \frac{\partial}{\partial z_2}, \\
(4) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} & 0 & \\ 0 & & 2 \end{pmatrix} \right) &= 2\epsilon\eta\sqrt{-1}z_3 \frac{\partial}{\partial z_4}, \\
(5) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} & 0 & \\ 2 & & 0 \end{pmatrix} \right) &= -2\epsilon\eta\sqrt{-1}z_2 \frac{\partial}{\partial z_1}, \\
(6) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} & 0 & \\ 0 & & 2 \end{pmatrix} \right) &= -2\epsilon\eta\sqrt{-1}z_4 \frac{\partial}{\partial z_3}, \\
(7) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & 0 & 0 \\ -1 & & 0 \end{pmatrix} \right) &= -\frac{1}{2}(z_1 z_4 - 4 \frac{\partial^2}{\partial z_2 \partial z_3}), \\
(8) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ 0 & 0 & -1 \\ 0 & & 0 \end{pmatrix} \right) &= -\frac{1}{2}(z_2 z_3 - 4 \frac{\partial^2}{\partial z_1 \partial z_4}), \\
(9) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} & 0 & 1 & \\ 0 & 0 & 1 & 0 \end{pmatrix} \right) &= \frac{1}{2}\epsilon\eta\sqrt{-1}(z_3 z_4 + 4 \frac{\partial^2}{\partial z_1 \partial z_2}), \\
(10) \quad \hat{\omega}_{\psi_v, V_v} \left(\begin{pmatrix} & 0 & 0 & \\ 0 & 1 & 0 & 0 \end{pmatrix} \right) &= \frac{1}{2}\epsilon\eta\sqrt{-1}(z_1 z_2 + 4 \frac{\partial^2}{\partial z_3 \partial z_4}).
\end{aligned}$$

Using this description one can demonstrate the following:

- Every \mathcal{P}_k is irreducible as a \mathfrak{g}_v -module.
- The set of \mathbf{K}_v -types in \mathcal{P}_k is
$$\begin{cases} \{(\epsilon\eta k + l, l) \mid l \in \mathbf{Z}_{\geq 0}\} & \epsilon\eta k > 0, \\ \{(l, -\epsilon\eta k + l) \mid l \in \mathbf{Z}_{\geq 0}\} & \epsilon\eta k < 0. \end{cases}$$

The multiplicity of each \mathbf{K}_v -type is one.

- The infinitesimal character of \mathcal{P}_k is associated to
$$\begin{cases} (\epsilon\eta k, 1) \in \mathfrak{t}^* & \epsilon\eta k > 0, \\ (1, -\epsilon\eta k) \in \mathfrak{t}^* & \epsilon\eta k < 0. \end{cases}$$

by the Harish-Chandra isomorphism.

In particular, if $|k| > 1$, \mathcal{P}_k is the discrete series representation with the Harish-Chandra parameter $(\epsilon\eta k, 1)$ if $\epsilon\eta k > 0$ and $(1, -\epsilon\eta k)$ otherwise. If $\epsilon\eta k = \pm 1$, $\delta((1, 1), \Omega_{\pm})$ and \mathcal{P}_k are irreducible highest weight modules with the same highest weight. Therefore, they are isomorphic.

6 Main theorem

Let V, χ, S be as in § 3.

Theorem 6.1. *There is an irreducible $G(\mathbf{A})$ -subspace $\Theta^1(V, \chi, S)$ of $\Theta(V, \chi, S)$ such that*

$$\Theta^1(V, \chi, S) \simeq \bigotimes_v \theta(V_v, \sigma_v(V, \chi, S)).$$

Proof. From § 5, $\Theta(V, \sigma, S)$ has a quotient isomorphic to

$$\bigotimes_v \theta(V_v, \sigma_v(V, \chi, S)).$$

Since $\Theta(V, \sigma, S)$ is completely reducible, it must contain a subspace isomorphic to this representation. \square

Theorem 6.2. *$\Theta^1(V, \chi, S)$ is a CAP representation with respect to P_K . More concretely, there is a set S' of places of k with finite cardinality such that*

1. $\Theta^0(T, \chi, S')$ is non-zero and irreducible cuspidal representation of $SL(2, \mathbf{A})$,
2. For almost all v , $\theta(V_v, \sigma_v(V, \chi, S))$ is isomorphic to the unique irreducible quotient of $\text{Ind}_{P_K(k_v)}^{G(k_v)} \zeta_{V_v} | \cdot |_v \otimes \Theta_v^0(T_v, \sigma_v(V, \chi, S'))$.

where T is the quadratic space on $k(\eta)$ defined by the norm form.

Proof. We can choose S' satisfying the first condition by Theorem 4.2. For almost all $v \notin S_D$, T_v is isometric to T_{V_v} because both the quadratic spaces have the same determinant and the Hasse invariant 1. From this and results of § 5, the second condition follows. \square

We write $m(\Theta^1(V, \chi, S))$ for the multiplicity of $\Theta^1(V, \chi, S)$ in the discrete spectrum of $L^2(G(k) \backslash G(\mathbf{A}))$.

Theorem 6.3.

$$m(\Theta^1(V, \chi, S)) \geq \begin{cases} 2^{\#(S_\chi \cap S_D) - 1} & S_D \not\subset S_\chi, S_D \cap S_\chi \neq \emptyset \\ 2^{\#S_D - 2} & S_D \subset S_\chi, \\ 1 & S_D \cap S_\chi = \emptyset. \end{cases}$$

To show this theorem, we make use of the failure of the Hasse's principle for (-1)-hermitian spaces over a quaternion division algebra. For $\eta \in D_- \setminus \{0\}$ let $k_{D, \eta}^\times = \{c \in k^\times \mid (\eta^2, c)_v = 1 \text{ for all } v \notin S_D\}$. A group homomorphism λ is defined by

$$k_{D, \eta}^\times \ni c \mapsto \{(\eta^2, c)_v\}_{v \in S_D} \in \{\pm 1\}^{\#S_D}.$$

Let $\{\pm 1\}$ be regarded as the subgroup of $\{\pm 1\}^{\#S_D}$ via the diagonal embedding. Note that the number of elements of $k_{D, \eta}^\times / \lambda^{-1}(\{\pm 1\})$ is $2^{\#S_D - 2}$.

Proposition 6.4. [Sch85, Theorem 10.4.6, Remark 10.4.6] Let $\langle \eta \rangle$ be a (-1)-hermitian right D -space of dimension 1 defined by $\eta \in D_- \setminus \{0\}$. Then for any $c \in k_{D,\eta}^\times$, $\langle c\eta \rangle$ is locally isometric to $\langle \eta \rangle$. For any $a \in \lambda^{-1}(\{\pm 1\})$, $\langle a\eta \rangle$ is globally isometric to $\langle \eta \rangle$. $\{\langle a\eta \rangle \mid a \in k_{D,\eta}^\times / \lambda^{-1}(\{\pm 1\})\}$ is the set of classes locally isometric to $\langle \eta \rangle$, so that this set contains $2^{\#S_D-2}$ elements.

In the case of $v \in S_D$, the Weil representation ω_{ψ_v, V_v} of $G(V_v) \times G(k_v)$ is determined by not the isometry class of V_v but the choice of a basis of V_v . In fact, if x_1 and x_2 are elements in V_v not equal to each other, x_1 and x_2 define different Weil representations and the Howe correspondents of an irreducible representation of $G(V_v)$ with respect to these Weil representations do not need to be isomorphic. Here if $\eta_v = h_{V_v}(x, x)$ for $x \in V_v$, we write ω_{ψ_v, η_v} for the Weil representation defined by v instead of ω_{ψ_v, V_v} . Also write $\lambda_v(\eta_v, f_v, \phi_v)$ instead of $\lambda_v(f_v, \phi_v)$ of (3.5), and $\theta(\eta_v, \chi_v)$ instead of $\theta(\langle \eta_v \rangle, \chi_v)$.

Lemma 6.5. Let $v \in S_D$ and $V_v \simeq \langle \eta_v \rangle$. For $c \in k_v^\times$, $f_v \in \mathcal{S}(V_v)$ and a unitary character χ_v of $G(V_v)$, there exists $f'_v \in \mathcal{S}(V_v)$ such that

$$\lambda_v(\eta_v, f_v, \chi_v) = \lambda_v(c\eta_v, f'_v, \chi_v^c)$$

where

$$\chi_v^c = \begin{cases} \chi_v & (\eta_v^2, c)_v = 1, \\ \overline{\chi_v} & (\eta_v^2, c)_v = -1. \end{cases}$$

In particular, $\theta(\eta_v, \chi_v) \simeq \theta(c\eta_v, \chi_v^c)$.

Proof. If $(c, \eta_v^2) = 1$, there is a $\xi \in k(\eta) \subset D$ such that $*\xi\xi = c$. If $(c, \eta_v^2) = -1$, there is a $\xi \in D_-$ such that $\xi^2 = c$ and $\xi\eta = -\eta\xi$. In both cases, $(\xi, \xi)_{V_v} = c\eta_v$, so that ξ defines the Weil representation $\omega_{\psi_v, c\eta_v}$ of $G(V_v) \times G(k_v)$. It is easily checked that $\omega_{\psi_v, c\eta_v}$ is realized on $\mathcal{S}(V_v)$ by

$$\omega_{\psi_v, c\eta_v}(h, g) = \omega_{\psi_v, \eta_v}(\xi^{-1}h\xi, g) \quad (h \in G(V_v), g \in G(k_v)).$$

Therefore for $g \in G(k_v)$,

$$\begin{aligned} \lambda(\eta_v, f_v, \chi_v)(g) &= \int_{G(V_v)} \omega_{\psi_v, \eta_v}(h, g) f(1_{D_v}) \chi_v(h) dh \\ &= \int_{G(V_v)} \omega_{\psi_v, \eta_v}(\xi^{-1}h'\xi, g) f_v(1_{D_v}) \chi_v(\xi^{-1}h'\xi) dh' \\ &= \int_{G(V_v)} \omega_{\psi_v, a\eta_v}(h', g) f_v(1_{D_v}) \chi_v^c(h') dh' \\ &= \lambda(a\eta_v, f_v, \chi_v^c)(g). \end{aligned}$$

□

Proof of Theorem 6.3.

If $v \in S_D$ and $\chi_v^2 = \mathbf{1}$, $\theta(\eta_0, \chi_v) \simeq \theta(c\eta_0, \chi_v)$ for any $c \in k^\times$ from Lemma 6.5. Therefore, if $a \in k_{D,\eta_0}^\times$ holds $(a, \eta_0^2)_v = 1$ for all $v \in S_D \setminus S_\chi$, $\Theta^1(V, \chi, S) \simeq \Theta^1(aV, \chi, S)$.

From Lemma 3.2, if V_1, \dots, V_n are (-1)-hermitian spaces over D of dimension 1 not isometric to each other, in the space of automorphic space of $G(\mathbf{A})$

$$\Theta^1(V_1, \chi, S) + \dots + \Theta^1(V_n, \chi, S) = \Theta^1(V_1, \chi, S) \oplus \dots \oplus \Theta^1(V_n, \chi, S).$$

The image into $k_{D,\eta}^\times/\lambda^{-1}(\{\pm 1\})$ of the set of $a \in k_{D,\eta_0}^\times$ holding $(a, \eta_0^2)_v = 1$ for all $v \in S_D \setminus S_\chi$ is isomorphic to

$$\begin{cases} \{a \in k_{D,\eta}^\times/\ker \lambda \mid (a, \eta^2)_v = 1 \text{ for } v \in S_D \setminus S_\chi\} & S_D \not\subset S_\chi, \\ k_{D,\eta}^\times/\lambda^{-1}(\{\pm 1\}) & S_D \subset S_\chi. \end{cases}$$

The cardinality of this set is equal to the number in the right hand side of inequality in the theorem.

7 Multiplicity conjecture

7.1 Arthur's conjecture

In this section we attempt to explain expected value of the multiplicity appearing in Theorem 6.3 in view of the Arthur's multiplicity conjecture. Therefore, the argument in this section supposes some conjectures.

Let F be a local field of characteristic 0 and $\Gamma = \text{Gal}(\overline{F}/F)$. The quasisplit inner form of G is $G^* = Sp(4)$. We have the following bijection [PR91].

$$\begin{array}{ccc} \{\text{inner forms of } G^*\}/\sim & \approx & H^1(F, G^*_{\text{ad}}) \\ \Psi & & \Psi \\ G' & \longleftrightarrow & u_{G'} : \Gamma \ni \gamma \mapsto \eta_{G'}^{-1} \circ \gamma \eta_{G'} \end{array}$$

Here \sim means isomorphic equivalence and $\eta_{G'} : G^*(\overline{F}) \rightarrow G'(\overline{F})$ is an inner twist. In addition, if F is non-archimedean then from [Kot84] Prop.6.4

$$\begin{array}{ccc} H^1(F, G^*_{\text{ad}}) & \approx & \pi_0(Z(\widehat{G^*_{\text{sc}}})^\Gamma)^D \\ \Psi & & \Psi \\ u_{G'} & \longleftrightarrow & \widehat{\zeta}_{G'}. \end{array}$$

Here $\widehat{G^*_{\text{sc}}}$ is the simply connected cover of $\widehat{G^*} = \widehat{G} = SO(5, \mathbf{C})$ so that $\widehat{G^*_{\text{sc}}} = Sp(4, \mathbf{C})$ and $(\)^D$ means Pontrjagin dual. Write $j_{\text{sc}} : \widehat{G^*_{\text{sc}}} \rightarrow \widehat{G^*}$ for the covering map. The local Langlands group \mathcal{L}_F is defined by

$$\mathcal{L}_F = \begin{cases} W_F \times SU(2, \mathbf{R}) & F : \text{non-archimedean,} \\ W_F & F : \text{archimedean,} \end{cases}$$

where W_F is the Weil group of F . By a (local) A-parameter is meant a continuous homomorphism $\phi : \mathcal{L}_F \times SL(2, \mathbf{C}) \rightarrow {}^L G = \widehat{G} \times W_F$ such that

- (i) writing $p_F : \mathcal{L}_F \rightarrow W_F$ for the conjectural homomorphism and $p_2 : {}^L G \rightarrow W_F$ the projection to the second component, $p_2 \circ \phi = p_F$,
- (ii) its restriction to \mathcal{L}_F is a Langlands parameter with bounded image [Bor79],

and

(iii) its restriction to $SL(2, \mathbf{C})$ is analytic.

We write C_ψ for the centralizer of the image of ψ in \widehat{G} . For a local A -parameter ψ and an inner form G' of G^* suppose the existence of local A -packet $\Pi_\psi^{G'}$ [Art89], which becomes a finite set of irreducible admissible representations of $G'(F)$. For a global or local A -parameter ψ , \mathcal{S}_ψ denotes $j_{\text{sc}}^{-1}(C_\psi)$. \mathcal{S}_ψ is defined by $\pi_0(\mathcal{S}_\psi) = \mathcal{S}_\psi / \mathcal{S}_\psi^0$. For an inner form G' of G^* the following condition is called the relevance condition for (G', ψ) :

$$\text{Ker } \hat{\zeta}_{G'} \supset Z(\widehat{G}_{\text{sc}}^*)^\Gamma \cap \mathcal{S}_\psi^0.$$

Since

$$Z_\psi^\Gamma := \text{Im}(Z(\widehat{G}_{\text{sc}}^*)^\Gamma \rightarrow \mathcal{S}_\psi) \simeq Z(\widehat{G}_{\text{sc}}^*)^\Gamma / (Z(\widehat{G}_{\text{sc}}^*)^\Gamma \cap \mathcal{S}_\psi^0),$$

if (G', ψ) satisfies the relevance condition then $\hat{\zeta}_{G'}$ can be regarded as a character of Z_ψ^Γ .

Conjecture 7.1 ([Art06] §3). *Let F be non-archimedean. For a local A -parameter $\psi : \mathcal{L}_F \times SL(2, \mathbf{C}) \rightarrow {}^L G^*$ there exists a pairing*

$$\langle \cdot, \cdot \rangle_F : \mathcal{S}_\psi \times \left(\prod_{G' \in H^1(F, G_{\text{ad}}^*)} \Pi_\psi^{G'} \right) \rightarrow \mathbf{C}$$

which satisfies the following condition:

For any inner form G' of G^* , if (G', ψ) does not satisfy relevance condition, $\Pi_\psi^{G'} = \emptyset$ and otherwise there exists

$$\begin{array}{ccc} \rho : \Pi_\psi^{G'} & \rightarrow & \Pi(\mathcal{S}_\psi, \hat{\zeta}_{G'}) = \{ \text{irred. repre. } \sigma \text{ of } \mathcal{S}_\psi \mid \sigma|_{Z_\psi^\Gamma} = \hat{\zeta}_{G'} \} / \sim \\ \cup & & \cup \\ \pi & \mapsto & \rho_\pi \end{array}$$

such that $\langle s, \pi \rangle_F = \text{Tr } \rho_\pi(s)$ for all $s \in \mathcal{S}_\psi$.

If F is non-archimedean then the set of inner forms of G^* consists of $G_F^s = Sp(4)$ and non-split group G_F^{ns} . If F is real it consists of G_F^s , $G_F^{ns} = Sp(1, 1)$ and the compact group $Sp(4)$, and if F is complex it consists of only G_F^s . In any case put $\Pi_\psi^s = \Pi_\psi^{G_F^s}$, $\Pi_\psi^{ns} = \Pi_\psi^{G_F^{ns}}$ ($\Pi_\psi^{ns} = \emptyset$ if F is complex). Since we do not treat the case that G coincides with the compact $Sp(4)$ at a real place, we will not consider the A -packet in the case.

Next consider the global case. Assume existence of the hypothetical Langlands group \mathcal{L}_k of k . A global A -parameter is defined similarly to the local one. Two A -parameters are equivalent if they are \widehat{G} -conjugate. An A -parameter ψ is said to be elliptic if the centralizer C_ψ of the image of ψ into \widehat{G} is contained in the center $Z(\widehat{G})$ of \widehat{G} . The set of equivalence classes of elliptic A -parameters is denoted by $\Psi_0(G)$. For an elliptic A -parameter ψ , the associated local A -parameter ψ_v is given for any

place v by the hypothetical homomorphism $\mathcal{L}_{k_v} \rightarrow \mathcal{L}_k$. \mathcal{S}_ψ is defined similarly to the local one. Then a homomorphism $\mathcal{S}_\psi \rightarrow \mathcal{S}_{\psi_v}$ is given. Assume that the pairing $\langle \cdot, \cdot \rangle_v : \mathcal{S}_{\psi_v} \times (\Pi_{\psi_v}^s \sqcup \Pi_{\psi_v}^{\text{ns}}) \rightarrow \mathbf{C}$ satisfying Conjecture 7.1 is given for any v . Then the global pairing $\langle \cdot, \cdot \rangle = \prod_v \langle \cdot, \cdot \rangle_v : \mathcal{S}_\psi \times \Pi_\psi^G \rightarrow \mathbf{C}$ is defined if the product exist. Let $\epsilon_\psi : \mathcal{S}_\psi \rightarrow \{\pm 1\}$ be the character defined in [Art89]. Π_ψ^G is defined by the set of $\pi = \otimes \pi_v$ such that $\pi_v \in \Pi_{\psi_v}^{G(k_v)}$ and π_v is unramified for almost all v . For $\pi \in \Pi_\psi^G$ set

$$m_\psi(\pi) = \frac{1}{|\mathcal{S}_\psi|} \sum_{s \in \mathcal{S}_\psi} \epsilon_\psi(s) \langle s, \pi \rangle.$$

The Arthur's multiplicity conjecture is described as follows.

Conjecture 7.2 ([Art89] Conj.8.1). *The multiplicity of π in the discrete spectrum of $L^2(G(k) \backslash G(\mathbf{A}))$ is equal to $\sum_{\psi \in \Psi_0(G)} m_\psi(\pi)$.*

7.2 Multiplicity of Klingen CAP representation

From the Adams' conjecture ([Ada89] § 4, I), it is expected that $\Theta^1(V, \chi, S)$ belongs to the global A -packet associated to the A -parameter $\psi = \psi_{k', \chi} : W_k \times SL(2, \mathbf{C}) \rightarrow SO(5, \mathbf{C}) \times W_k$ defined by

$$\psi_{k', \chi} = \left(\text{Ind}_{W_{k'}}^{W_k} \mu(\chi) \otimes \mathbf{1}_{SL(2, \mathbf{C})} \right) \oplus (\omega_{k'/k} \otimes \text{Sym}^2) \times \text{Id}_{W_k}.$$

Here, k' is the quadratic extension $k(\eta)$ of k with discriminant δ , $W_k, W_{k'}$ the Weil groups of k, k' , $\omega_{k'/k}$ the quadratic character associated to the quadratic extension k'/k , Sym^2 the second symmetric power of the standard representation of $SL(2, \mathbf{C})$ and $\mu(\chi) : W_{k'} \rightarrow \mathbf{C}^\times$ the image of the element of $H^1(W_k, \widehat{U_{k'/k}(1)})$ associated to χ by the Langlands' class field theory via the restriction map $H^1(W_k, \widehat{U_{k'/k}(1)}) \rightarrow H^1(W_{k'}, \widehat{U_{k'/k}(1)})$.

We need description of local and global A -packets, S -groups and pairings to describe the conjectural multiplicity. To obtain local and global A -packets, we set an assumption (c.f. [Ada89] § 4.5 Conj.B).

Assumption Let F be a local field of a form k_v for some v , D_F the quaternion algebra $D(F)$ and K a quadratic algebra of F with discriminant δ_K . For a unitary character χ_K of the group of norm 1 of K^\times , set an A -parameter ψ_{K, χ_K} for $G(F)$ by

$$\psi_{K, \chi_K} = \left(\text{Ind}_{W_K}^{W_F} \mu(\chi_K) \otimes \mathbf{1}_{SL(2, \mathbf{C})} \right) \oplus (\omega_{K/F} \otimes \text{Sym}^2) \times \text{Id}_{W_F}$$

where $\mu(\chi_K)$ is defined as before. Then the A -packet $\Pi_{\psi_{K, \chi_K}}^{G(F)}$ associated to ψ_{K, χ_K} coincides with

$$\{\theta(V_F, \sigma) \mid V_F, \sigma\}$$

where V_F runs over non-degenerate (-1)-hermitian right spaces over D_F with dimension 1 and determinant $-\delta_K$, and σ is in the L -packet $\Pi_{\chi_K}^{V_F}$ associated to a L -parameter $\phi_{\chi_K}^{V_F}$ defined by the composition of the embedding ${}^L G_0(V_F) \rightarrow {}^L G(V_F)$ and the L -parameter $\mathcal{L}_F \rightarrow {}^L G_0(V_F)$ associated to χ_K .

By the equivalent relation of L -parameters for $G(V_v)$ ([Ada89] § 3.4), we obtain

$$\Pi_{\chi_v}^{V_v} = \begin{cases} \{\text{irreducible constituents of } \text{Ind}_{G_0(V_v)}^{G(V_v)} \chi_v\} & v \notin S_D, \\ \{\chi_v, \chi_v^{-1}\} & v \in S_D. \end{cases}$$

From our assumption, $\Pi_{\psi_v}^{G(k_v)}$ is described as follows:

(1) The case of $v \notin S_D$

$$\Pi_{\psi_v}^s = \begin{cases} \{\theta(V_v^\pm, \tilde{\chi}_v^+)\} & \chi_v^2 \neq 1 \text{ and } \delta \notin (k_v^\times)^2, \\ \{\theta(\mathbb{H}_v, \tilde{\chi}_v^+)\} & \chi_v^2 \neq 1 \text{ and } \delta \in (k_v^\times)^2, \\ \{\theta(V_v^\pm, \tilde{\chi}_v^\pm)\} & \chi_v^2 = 1 \text{ and } \delta \notin (k_v^\times)^2, \\ \{\theta(\mathbb{H}_v, \tilde{\chi}_v^\pm)\} & \chi_v^2 = 1 \text{ and } \delta \in (k_v^\times)^2. \end{cases}$$

Here V_v^\pm is the two-dimensional quadratic space over k_v with determinant $-\delta$ and Hasse invariant ± 1 , \mathbb{H}_v is the 2-dimensional hyperbolic space over k_v . Note that $\theta(V_v, \sigma_v)$ appearing in the above packets is of the form of an irreducible quotient of $\text{Ind}_{P_K(k_v)}^{Sp(2, k_v)}(\omega_{k'_v/k_v} | \cdot |_v \otimes \tau_v)$ for some irreducible representation τ_v of $SL(2, k_v)$ and the character $\omega_{k'_v/k_v}$ associated to the extension k'_v/k_v except for $\sigma_v = \det$.

(2) The case of $v \in S_D$

$$\Pi_{\psi_v}^{\text{ns}} = \begin{cases} \{\theta(V_v, \chi_v), \theta(V_v, \chi_v^{-1})\} & \chi_v^2 \neq 1, \\ \{\theta(V_v, \chi_v)\} & \chi_v^2 = 1, \end{cases}$$

where V_v is the 1-dimensional (-1)-hermitian space over D_v with determinant $-\delta$. Note that elements of $\Pi_{\psi_v}^{\text{ns}}$ are supercuspidal except for $\chi_v = 1$.

Next describe global and local S -groups for ψ .

$$\mathcal{S}_\psi \simeq \begin{cases} 2\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \chi^2 \neq 1, \\ D_4 & \chi^2 = 1, \end{cases}$$

where D_4 is the dihedral group with 8 elements. If k'_v is a quadratic extension of k_v then

$$\mathcal{S}_{\psi_v} \simeq \begin{cases} 2\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \chi_v^2 \neq 1, \\ D_4 & \chi_v^2 = 1, \end{cases}$$

and if $k'_v \simeq k_v \oplus k_v$ then

$$\mathcal{S}_{\psi_v} \simeq \begin{cases} \{1\} \times \{1\} & \chi_v^2 \neq 1, \\ \mathbf{Z}/2\mathbf{Z} \times \{1\} & \chi_v^2 = 1. \end{cases}$$

The homomorphism $\mathcal{S}_\psi \rightarrow \mathcal{S}_{\psi_v}$ is determined by the above description of S -groups

and the following diagram:

$$\begin{array}{ccccc} D_4 = \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \xrightarrow{\kappa} & \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \rightarrow & \mathbf{Z}/2\mathbf{Z} \times \{1\} \\ \cup & & \cup & & \cup \\ 2\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \rightarrow & \{1\} \times \mathbf{Z}/2\mathbf{Z} & \rightarrow & \{1\} \times \{1\} \end{array}$$

Finally, define a pairing $\langle \cdot, \cdot \rangle_v$ as follows. If $v \in S_D$ and $\chi_v^2 = 1$,

$$\langle s, \theta(V_v, \chi_v) \rangle_v = \begin{cases} \pm 2 & s = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

if $v \in S_D$ and $\chi_v^2 \neq 1$,

$$\langle \cdot, \theta(V_v, \chi_v^\epsilon) \rangle_v = \text{sgn}^{(\epsilon-1)/2} \otimes \mathbf{1},$$

and otherwise

$$\langle \cdot, \theta(V_v^\eta, \tilde{\chi}_v^\epsilon) \rangle_v = \text{sgn}^{(\epsilon-1)/2} \otimes \text{sgn}^{(\eta-1)/2},$$

where $-1 = (2, 0) \in \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, we regard $\mathbb{H}_v = V_v^+$ and if $\mathcal{S}_{\psi_v} \simeq D_4$ it is reduced to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ via κ . Remark that our definition of local pairings satisfies Conjecture 7.1.

We calculate $\epsilon_\psi = \mathbf{1}$ by definition. Therefore, for an irreducible automorphic representation $\pi = \Theta^1(V, \chi, S) \in \Pi_\psi^G$ for some V, S , the Arthur's conjectural multiplicity is described by

$$m_\psi(\pi) = \begin{cases} 2^{\#(S_\chi \cap S_D)-1} & \chi^2 \neq \mathbf{1}, S_D \cap S_\chi \neq \emptyset, \\ 2^{\#S_D-2} & \chi^2 = \mathbf{1}, S_D \cap S_\chi \neq \emptyset \\ 1 & S_D \cap S_\chi = \emptyset. \end{cases}$$

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