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https://doi.org/10.15017/1866255

出版情報：九州大学, 2017, 博士（数理学）, 課程博士
バージョン：
権利関係：

Noncommutative Bass–Serre trees and their applications

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Abstract

Any amalgamated free product of discrete groups acts on its associated Bass–Serre tree. In this paper, we consider an analogue of the Bass–Serre trees for reduced amalgamated free products of C*-algebras.

In the first part of the paper, we introduce a unital C*-algebra $\Delta\mathbf{T}(A, E)$ for a given reduced amalgamated free product $(A, E) = (A_1, E_1) \star_D (A_2, E_2)$, which generalizes the crossed product of the Bowditch compactification of the Bass–Serre tree by an amalgamated free product group. We then show that our C*-algebra is isomorphic to an explicit Cuntz–Pimsner algebra and has a universal property. This result allows us to show a "boundary amenability" result for $\Delta\mathbf{T}(A, E)$.

In the second part, we prove that any reduced amalgamated free product of separable C*-algebras is KK-equivalent to the corresponding full amalgamated free product via the canonical surjection. Our proof is based on Julg and Valette's geometric argument for groups acting on trees. We also give a new proof of Fima and Germain's six-term exact sequences of KK-groups using the Pimsner algebra structure of $\Delta\mathbf{T}(A, E)$.

Contents

1 Introduction

The amalgamated free product is an operation to produce a new group $\Gamma_1 *_\Lambda \Gamma_2$ from two groups Γ_1 and Γ_2 with a common subgroup Λ. This is a fundamental construction in not only combinatorial group theory, but also geometric group theory. In fact, amalgamated free product groups admit canonical actions on their associated Bass–Serre trees and this type of actions is one of two fundamental examples of group actions on trees in the Bass–Serre theory [46] (the other arises from HNN extensions). In the study of those groups, actions on trees are powerful tools and have been applied extensively.

In C*-algebra theory, there are two notions of amalgamated free products; the full and the reduced amalgamated free products [1] [53]. These are analogues of the full and the reduced group C*-algebra constructions, and have been seriously studied so far (see e.g. [13][15][14][44][50]). However, geometric aspects of these constructions like the group case have never been studied seriously so far. The purpose of the present paper is to investigate a C*-analogue of Bass–Serre trees and apply it to the study of amalgamated free product C*-algebras themselves.

For a given amalgamated free product group $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$, the action of Γ on the associated Bass–Serre tree $\mathbf{T} = (\mathbf{V}, \mathbf{E})$ induces two unitary representations on $\ell^2(\mathbf{V})$ and $\ell^2(\mathbf{E})$. Our key observation is that one can construct two C*-correspondences which are natural counterparts of these unitary representations for general reduced amalgamated free products (Remark 3.1.2). This is based on an idea in the previous paper [26], where we developed a representation theory of C*-algebras by C*-correspondences. Based on this observation, we will investigate C*-analogues of actions on compactifications of trees, and Julg–Valette's work for K-theory of groups acting on trees.

For any group Γ acting on a tree $\mathbf{T}$ (more generally, a uniformly fine hyperbolic graph), Bowditch [4] introduced the compactification $\Delta\mathbf{T}$ and the induced action $\Gamma \curvearrowright \Delta\mathbf{T}$. The Γ-space $\Delta\mathbf{T}$ can be viewed as an analogue of the Gromov boundary for hyperbolic groups and captures information about the original action on $\mathbf{T}$. In fact, $\Delta\mathbf{T}$ (or its suitable quotient) is a Γ-boundary in the sense of Furstenberg [21] and Ozawa proved that the action $\Gamma \curvearrowright \Delta\mathbf{T}$ is amenable if and only if all the stabilizer subgroups of $\Gamma \curvearrowright \mathbf{T}$ are amenable [39].

Motived by these facts, in §3 we introduce a unital C*-algebra $\Delta\mathbf{T}(A, E)$ for a given reduced amalgamated free product $(A, E) = (A_1, E_1) \star_D (A_2, E_2)$. The C*-algebra $\Delta\mathbf{T}(A, E)$ includes A as a unital C*-subalgebra and generalizes crossed products in the following sense: when (A, E) comes from the reduced group C*-algebra $\mathrm{C}^*_{\mathrm{red}}(\Gamma)$ of $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ and $\mathbf{T}$ is the associated Bass–Serre tree, one has

$$(\mathrm{C}^*_{\mathrm{red}}(\Gamma) \subset \Delta\mathbf{T}(\mathrm{C}^*_{\mathrm{red}}(\Gamma), E)) \cong (\mathrm{C}^*_{\mathrm{red}}(\Gamma) \subset C(\Delta\mathbf{T}) \rtimes_{\mathrm{red}} \Gamma).$$

Our main result in §3 is a structural theorem (Theorem 3.3.5) that $\Delta\mathbf{T}(A, E)$ is isomorphic to both an explicit Cuntz–Pimsner algebra and the universal C*-algebra generated by a unital copy of the *algebraic* amalgamated free product of A_1 and A_2 over D and projections e_1 and e_2 such that $e_1 + e_2 = 1$ and

$$e_k a e_k = E_k(a) e_k \quad \text{for} \quad a \in A_k,\ k = 1, 2.$$

The isomorphism with a Cuntz–Pimsner algebra is inspired by [51][37] and the universality can be viewed as an analogue of isomorphisms between full and reduced crossed products for amenable actions. As a consequence, we show that $\Delta\mathbf{T}(A, E)$ has one of the following properties nuclearity/exactness/completely bounded approximation property (CBAP)/weak expectation property (WEP)/ local lifting property (LLP) if and only if both A_1 and A_2 have the same property (Corollary 3.4.4). This covers Ozawa's result mentioned above in the case of amalgamated free product groups acting on Bass–Serre trees. As applications, we give simple and conceptual proofs of Dykema's result [14] for the stability of exactness and Dykema–Blanchard's result [3] for embeddability of reduced amalagamated free products and generalizes Ozawa's result [38] about the stability of nuclearity to CBAP, WEP and LLP. Also, we prove that local embeddability into the full group C*-algebra of the free group $\mathbb{F}_\infty$ studied by Junge and Pisier [30] is stable under the reduced amalgamated free product.

We next turn to the KK-theory of amalgamated free products. The study of K-theory of amalgamated free product groups dates back to Cuntz's paper [9] in the early 80s. In [9][10] Cuntz suggested the following strategy of computing the K-theory of the reduced C*-algebra $\mathrm{C}^*_{\mathrm{red}}(\Gamma)$ of a given discrete group Γ:

(1) proving that the canonical surjection $\lambda : \mathrm{C}^*(\Gamma) \to \mathrm{C}^*_{\mathrm{red}}(\Gamma)$ gives a KK-equivalence, and

(2) computing the K-theory of $\mathrm{C}^*(\Gamma)$.

In fact, usual computations of K-groups are consequences of suitable exact sequences, and universal objects are easier to handle than reduced ones in K-theory. By the strategy, Cuntz indeed gave an elegent proof of Pimsner–Voiculescu's result of the K-theory of $\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_n)$ ([42]). Then Julg and Valette [29] achieved part (1) of the strategy when Γ acts on a tree with amenable stabilizers. In the direction of groups acting on trees, Pimsner [40] obtained an optimal result.

It is natural to try to apply the strategy to amalgamated free products of C*-algebras. In [22][23] Germain obtained striking results which solve both parts of the strategy for plain free products of nuclear C*-algebras. Following Germain's idea in [22][24] we proved in [26] (also see [25]) the KK-equivalence between full and reduced amalgamated free products under the assumption of "strong relative nuclearity", which can be applied to amalgamated free products of nuclear C*-algebras over finite dimensional subalgebras. However, this is still unsatisfactory, because there are inclusions of nuclear C*-algebras which are not strongly relative nuclear.

In §§4.1 we consider part (1) of the strategy. We follow Julg–Valette's idea [29] unlike the previous works [22][24][26] based on the "vertex" and the "edge" C*-correspondences mentioned above. Translating the geometric construction of Fredholm modules due to Julg–Valette (and its quantum group analogue due to Vergnioux [52]) into a C*-algebraic language, we prove the optimal KK-equivalence result that for any reduced amalgamated free product $(A, E) = \bigstar_D(A_k, E_k)$ of countable family of separable C*-algebras, the canonical surjection from the full amalgamated free product $\bigstar_D A_k$ onto A always gives a KK-equivalence. We note that in [19] Fima and Germain also reached independently the same KK-equivalence result only in the case of two free components. However, in their paper, they also established exact sequences of KK-groups (i.e., part (2) of the strategy) under the very weak assumption of presence of conditional expectations.

In §§4.2 we will give a new, simpler proof of Fima and Germain's exact sequences based on the C*-algebra $\Delta\mathbf{T}(A, E)$ and K-theory of Pimsner algebras. In the course of our proof, we first show that the embedding $A \hookrightarrow \Delta\mathbf{T}(A, E)$ is right invertible in KK-theory by using the analogue of Julg–Valette construction. Then, the desired sequences will follow from the six-term exact sequences of KK-groups ([11]) induced from the Toeplitz extension of the Cuntz–Pimsner algebra $\Delta\mathbf{T}(A, E)$. As a by-product of our approach, we show that $\Delta\mathbf{T}(A, E)$ is KK-equivalent to $A \oplus D$. In particular, this implies that the KK-class of $\Delta\mathbf{T}(A, E)$ is independent of the choice of conditional expectations.

This paper basically follows the author's two papers [27] and [28], but some new results are added and some proofs are improved. One of our new results is Theorem 3.4.3. Also, we present a simplified proof of the KK-equivalence result (Theorem 4.1.1), which heavily relies on the universal property of $\Delta\mathbf{T}(A, E)$.

The paper is organized as follows. In §2 we fix notation and terminologies and collect necessary facts on Hilbert C*-modules, amalgamated free products of C*-algebras, Pimsner algebras, and KK-theory. §3 is devoted to the compactifications of Bass–Serre trees. In §§3.1 we first investigate Bass–Serre trees and their compactifications in the group case. Before the construction and investigation of the C*-algebra $\Delta\mathbf{T}(A, E)$, we prepare a general theory of extensions associated with conditional expectations in §§3.2. We then prove the structural theorem in §§3.3 and its consequences in §§3.4. In §4 we study KK-theory, and prove the KK-equivalence result in §§4.1 and give an alternative proof of Fima and Germain's result in §§4.2.

Acknowledgment

The author wishes to express his deep gratitude to his supervisor, Yoshimichi Ueda, for his constant encouragement and variable comments. He wishes to appreciate Yuhei Suzuki and Narutaka Ozawa for stimulating conversations and helpful comments. Also, he wishes to thank Gilles Pisier and Texas A&M University, where a part of this research was done, for the invitation and hospitality. This work was supported by the Research Fellow of the Japan Society for the Promotion of Science.

2 Preliminaries

2.1 Notations

For any Hilbert space $\mathcal{H}$, we denote by $\mathbb{B}(\mathcal{H})$ and $\mathbb{K}(\mathcal{H})$ the set of bounded linear operators and compact operators on $\mathcal{H}$, respectively. For vector spaces X and Y over $\mathbb{C}$, we denote by $X \odot Y$ the algebraic tensor product over $\mathbb{C}$. When X and Y are C*-algebras, $X \otimes Y$ denotes the minimal tensor product. When X and Y are Hilbert spaces, $X \otimes Y$ is the tensor product Hilbert space. For any subset S of a normed space X, we denote by $\overline{\operatorname{span}}\, S$ the closed linear span of S.

2.2 Hilbert C*-modules

We refer the reader to Lance's book [35] for Hilbert C*-module theory.

Definition 2.2.1. Let A be a C*-algebra. An *inner product* A-module is a linear space X with a right A-action which is compatible with scalar multiplication, i.e., $\lambda(\xi a) = (\lambda\xi)a = \xi(\lambda a)$ for $\lambda \in \mathbb{C}, \xi \in X, a \in A$ and an A-valued inner product $\langle \cdot, \cdot \rangle : X \times X \to A$ satisfying the following conditions:

(1) $\langle \xi, \lambda\eta + \mu\zeta \rangle = \lambda\langle \xi, \eta \rangle + \mu\langle \xi, \zeta \rangle$ for $\xi, \eta, \zeta \in X$ and $\lambda, \mu \in \mathbb{C}$,

(2) $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$ for $\xi, \eta \in X$ and $a \in A$,

(3) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for $\xi, \eta \in X$,

(4) $\langle \xi, \xi \rangle \geq 0$ for $\xi \in X$,

(5) $\xi = 0$ if and only if $\langle \xi, \xi \rangle = 0$ for $\xi \in X$.

When X is complete with respect to the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$, we call X a *Hilbert A-module* or *Hilbert* C*-*module over A*. We say that X is *full* if $\overline{\operatorname{span}}\{\langle \xi, \eta \rangle \mid \xi, \eta \in X\} = A$ and *countably generated* if there exists a countable subset $\{\xi_n\}_{n=1}^{\infty} \subset X$ such that $\overline{\operatorname{span}}\{\xi_n a \mid a \in A, n \geq 1\} = X$.

Let X and Y be Hilbert A-modules. A linear map $x : X \to Y$ is said to be *adjointable* if there exists a linear map $x^* : Y \to X$ satisfying $\langle \eta, x\xi \rangle = \langle x^*\eta, \xi \rangle$ for all $\xi \in X, \eta \in Y$. We denote by $\mathbb{L}(X, Y)$ the set of adjointable linear maps from X into Y and set $\mathbb{L}(X) := \mathbb{L}(X, X)$. Any adjointable linear map is automatically bounded and right A-linear, and $\mathbb{L}(X)$ equipped with the operator norm and the involution $x \mapsto x^*$ forms a unital C*-algebra.

For given vectors $\xi, \eta \in X$ we define the operator $\theta_{\xi,\eta} \in \mathbb{L}(X)$ by $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle$. We denote by $\mathbb{K}(X)$ the C*-subalgebra of $\mathbb{L}(X)$ generated by $\{\theta_{\xi,\eta} \mid \xi, \eta \in X\}$ and call operators in $\mathbb{K}(X)$ compact operators. It is known that

$\mathbb{K}(X) = \overline{\mathrm{span}}\{\theta_{\xi,\eta} \mid \xi, \eta \in X\}$ is a closed two-sided ideal of $\mathbb{L}(X)$. The *strict topology* on $\mathbb{L}(X)$ is the topology given by the family of semi-norms $\varphi_y(x) = \|xy\|$ for $y \in \mathbb{K}(X)$.

Definition 2.2.2. Let A and B be C*-algebras. An A-B C*-*correspondence* is a pair (X, ϕ_X) consisting of a Hilbert B-module X and a $*$-homomorphism $\phi_X : A \to \mathbb{L}(X)$, called the left action. A-A C*-correspondences are also called C*-*correspondences over* A. An A-B C*-correspondence (X, ϕ_X) is said to be *unital* if A is unital and ϕ_X is a unital map, *countably generated* if X is countably generated as a Hilbert B-module, and *injective* if ϕ_X is injective.

Every C*-algebra A forms a Hilbert A-module with the inner product $\langle a, b\rangle = a^*b$. It is not hard to see that $A \cong \mathbb{K}(A)$. Let $L_A : A \to \mathbb{K}(A)$ be the canonical $*$-homomorphism given by the left multiplication. The pair (A, L_A) is called the *identity* C*-*correspondence over* A.

Another example of C*-correspondences is the GNS-representation associated with a conditional expectation. Let $D \subset A$ be a unital inclusion of C*-algebras with conditional expectation $E\colon A \to D$. We denote by $L^2(A, E)$ the Hilbert D-module given by separation and completion of A with respect to the D-valued inner product $\langle x, y\rangle = E(x^*y)$ for $x, y \in A$, and by $\phi_E : A \to \mathbb{L}(L^2(A, E))$ the $*$-homomorphism induced from the left multiplication. The conditional expectation E is said to be *nondegenerate* if ϕ_E is faithful. Let ξ_E denote the vector in $L^2(A, E)$ corresponding to 1_A, and we call the triplet $(L^2(A, E), \phi_E, \xi_E)$ the *GNS-representation* associated with the conditional expectation E. The projection $e_D = \theta_{\xi_E,\xi_E}$ is called the *Jones projection*.

We will use the internal and the external tensor products of Hilbert C*-modules. Let X and Y be Hilbert C*-modules over A and B, respectively, and $\varphi : A \to \mathbb{L}(Y)$ be a $*$-homomorphism. Then we can construct the Hilbert B-module $X \otimes_\varphi Y$ by separation and completion of $X \odot Y$ with respect to the B-valued semi-inner product $\langle \xi \otimes \eta, \xi' \otimes \eta'\rangle := \langle \eta, \varphi(\langle \xi, \xi'\rangle)\eta'\rangle$ for $\xi, \xi' \in X$ and $\eta, \eta' \in Y$. There are two $*$-homomorphisms:

$$\mathbb{L}(X) \to \mathbb{L}(X \otimes_\varphi Y); \quad x \mapsto x \otimes 1_Y$$
$$\varphi(A)' \cap \mathbb{L}(Y) \to \mathbb{L}(X \otimes_\varphi Y); \quad y \mapsto 1_X \otimes y$$

satisfying that $(x \otimes 1_Y)(\xi \otimes \eta) = (x\xi) \otimes \eta$ and $(1_X \otimes y)(\xi \otimes \eta) = \xi \otimes (y\eta)$, for $\xi \in X$ and $\eta \in Y$. Since these $*$-homomorphisms have mutually commuting ranges, we will write $x \otimes y := (x \otimes 1_Y)(1_X \otimes y) = (1_X \otimes y)(x \otimes 1_Y)$. We call the module $X \otimes_\varphi Y$ the *interior tensor product* of X and (Y, φ). When no confusion may arise, we may also write $X \otimes_A Y = X \otimes_\varphi Y$. Further assume that $Y = B$ and $\varphi : A \to B$ is surjective. In this case, $X \otimes_\varphi B$ is called the *pushout* of X by φ and denoted by X_φ. We also write $x_\varphi := x \otimes 1_B$ for $x \in \mathbb{L}(X)$.

Next, assume that X and Y be Hilbert C*-modules over C and D, respectively. Then, the *external tensor product* of X and Y is the Hilbert $C \otimes D$-module $X \otimes Y$, which is the completion of $X \odot Y$ with respect to the inner product $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \otimes \langle \eta_1, \eta_2 \rangle \in C \otimes D$ for $\xi_1, \xi_2 \in X$ and $\eta_1, \eta_2 \in Y$. When (X, ϕ_X) and (Y, ϕ_Y) are A-C and B-D C*-correspondences, respectively, it is known that there exists a natural $*$-homomorphism $\phi_X \otimes \phi_Y \colon A \otimes B \to \mathbb{L}(X \otimes Y)$.

Let $A_i, i \in \mathcal{I}$ be a family of C*-algebras and X_i be a Hilbert A_i-module for $i \in \mathcal{I}$. Consider the direct product $A = \prod_{i \in \mathcal{I}} A_i$ and define $\boxplus_{i \in I} X_i$ by $\bigoplus_{i \in \mathcal{I}} X_i \otimes_{A_i} A$.

The next technical lemma will be used later.

Lemma 2.2.3. *Let A and B be* C*-*algebras, X be Hilbert A-module and (Y, ϕ_Y) be an injective A-B* C*-*correspondence. For any $x \in \mathbb{L}(X)$, if $x \otimes 1 \in \mathbb{L}(X \otimes_A Y)$ is compact, then so is x.*

Proof. Take an approximate unit $(e_i)_i$ of $\mathbb{K}(X)$. Since $\|e_i \xi - \xi\| \to 0$ holds for $\xi \in X$, if $x \otimes 1$ is compact, then $e_i x \otimes 1$ converges to $x \otimes 1$ in norm. Since ϕ_Y is injective, this implies that $x = \lim_i e_i x \in \mathbb{K}(X)$. □

2.3 Amalgamated free products

Let $\{D \subset A_k\}_{k \in \mathcal{I}}$ be a family of unital inclusions of C*-algebras.

Definition 2.3.1 ([1])**.** The *full amalgamated free product* of $A_k, k \in \mathcal{I}$ over D is the universal C*-algebra $\bigstar_D A_k$ generated by the images of injective $*$-homomorphisms $f_k \colon A_k \to \bigstar_D A_k$ such that $f_k = f_l$ on D for all $k, l \in \mathcal{I}$. We may omit f_k and assume that $A_k \subset \bigstar_D A_k$.

Now further assume that we have a conditional expectation $E_k \colon A_k \to D$ for $k \in \mathcal{I}$. For $n \geq 1$ we set $\mathcal{I}_n = \{\iota \colon \{1, \cdots, n\} \to \mathcal{I} \mid \iota(k) \neq \iota(k+1) \text{ for } k = 1, \cdots, n-1\}$. Also, we set $A_k^\circ = \ker E_k$ and $a^\circ = a - E_k(a)$ for $a \in A_k$. We first assume that E_k is *nondegenerate* for every $k \in \mathcal{I}$.

Definition 2.3.2 ([53])**.** The *reduced amalgamated free product* of $(A_k, E_k), k \in \mathcal{I}$ over D, is a pair $(A, E) = \bigstar_D (A_k, E_k)$ such that A is a C*-algebra generated by the images of $*$-homomorphisms $j_k \colon A_k \to A$ with $j_k = j_l$ on D equipped with a nondegenerate conditional expectation $E \colon A \to j_k(A_k)$ satisfying the freeness condition:

$$E(j_{\iota(1)}(a_1) j_{\iota(2)}(a_2) \cdots j_{\iota(n)}(a_n)) = 0$$

for any $n \geq 1$, $\iota \in \mathcal{I}_n$ and $a_k \in A_{\iota(k)}^\circ$ for $k = 1, \cdots, n$.

See [53] for the construction. Note that the pair (A, E) satisfying the above property is unique up to isomorphism. We denote by (X, ϕ_X, ξ_0) and (X_k, ϕ_{X_k}, ξ_k)

the GNS-representations associated with E and E_k for $k \in \mathcal{I}$, respectively. Then X is identified with

$$\xi_0 D \oplus \bigoplus_{m\geq 1} \bigoplus_{\iota\in\mathcal{I}_m} X^\circ_{\iota(1)} \otimes_D \cdots \otimes_D X^\circ_{\iota(m)}.$$

The compression map by the projection onto $\xi_0 D \oplus X_k^\circ \cong X_k$ gives a UCP map $E_{A_k}\colon A \to \phi_{X_k}(A_k)$ such that $E_{A_k} \circ j_k = \phi_{X_k}$ on A_k for every $k \in \mathcal{I}$. Since E_k is nondegenerate, $j_k\colon A_k \to A$ is injective. Thus, omitting j_k and ϕ_{X_k} we may assume that $A_k \subset A$ and $E_{A_k}\colon A \to A_k$ is a conditional expectation. Since $E = E_k \circ E_{A_k}$ holds, E_{A_k} is also nondegenerate. The GNS-representation associated with E_{A_k} will be denoted by $(Y_k, \phi_{Y_k}, \eta_k)$.

When E_k is "degenerate" for some k, one can still construct the pair (A, E) satisfying the above property. However, the natural map $j_k\colon A_k \to A$ is not injective. In order to avoid this, we use the vertex reduced amalgamated free product introduced by Fima and Germain [19].

Definition 2.3.3. The *vertex reduced amalgamated free product* of $(A_k, E_k), k \in \mathcal{I}$ over D is a C*-algebra A generated by the images of injective $*$-homomorphisms $j_k\colon A_k \to A, k \in \mathcal{I}$ with $j_k = j_l$ on D for $k, l \in \mathcal{I}$, equipped with a family of conditional expectations $E_{A_k}\colon A \to j_k(A_k)$ such that

- $E_{A_k}(j_{\iota(1)}(a_1)j_{\iota(2)}(a_2)\cdots j_{\iota(n)}(a_n)) = 0$ for $n \geq 1$, $\iota \in \mathcal{I}_n$ and $a_l \in A^\circ_{\iota(l)}$ for $l = 1, \ldots, n$ with $\iota(n) \neq k$;
- the direct sum of all the GNS-representations $(Y_k, \phi_{Y_k}, \eta_k)$ associated with E_{A_k} for $k \in \mathcal{I}$ is faithful.

Since $j_k\colon A_k \to A$ is injective, we may assume that $A_k \subset A$ for $k \in \mathcal{I}$. Note that $E := E_k \circ E_{A_k}$ is a conditional expectation which is independent of the choice of $k \in \mathcal{I}$, but possibly degenerate. When all E_k's are nondegenerate, $E = E_k \circ E_{A_k}$ is nondegenerate. In this case, the vertex reduced amalgamated free product is identical to the original reduced amalgamated free product. Thus, *throughout this paper, we mean the reduced amalgamated free product by the vertex reduced amalgamated free product* and still denote it by $(A, E) = \bigstar_D(A_k, E_k)$.

For each $k \in \mathcal{I}$, we denote by $P_{(\ell,k)}$ and $P_{(r,k)}$ the projections onto the following submodules, respectively:

$$X(\ell, k) := \xi_0 D \oplus \bigoplus_{m\geq 1} \bigoplus_{\substack{\iota\in\mathcal{I}_m \\ \iota(1)\neq k}} X^\circ_{\iota(1)} \otimes_D \cdots \otimes_D X^\circ_{\iota(m)},$$

$$X(r, k) := \xi_0 D \oplus \bigoplus_{m\geq 1} \bigoplus_{\substack{\iota\in\mathcal{I}_m \\ \iota(m)\neq k}} X^\circ_{\iota(1)} \otimes_D \cdots \otimes_D X^\circ_{\iota(m)}.$$

Lemma 2.3.4 (cf. [52, Lemma 3.1]). *For each $k \in \mathcal{I}$, there exists a unitary $S_k \colon X(r,k) \otimes_D A_k \to Y_k$ such that $S_k x_1 \cdots x_n \xi_0 \otimes a = x_1 \cdots x_n \eta_k a$ for all $n \geq 1$ and any reduced word $x_1 \cdots x_n$ with $x_n \notin A_k$ and $a \in A_k$.*

Proof. Note that if S_k has closed range, then it must be surjective. Thus, it suffices to show that S_k is an isometry. We only have to verify that $E_{A_k}(x^*y) = E(x^*y)$ for all reduced words $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m$ with $n, m \geq 1$ and $x_n, y_m \notin A_k$. When $n = m = 1$, this is trivial. Assume that we have shown $E_{A_k}(x^*y) = E(x^*y)$ for $n, m = 1, \dots, N$. Take arbitrary reduced words $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m$ as above with $n, m \leq N+1$. Suppose that $n \geq 2$ and set $z = x_2 \cdots x_n$ and $w := y_2 \cdots y_n$. Then the induction hypothesis implies that $E_{A_k}(z^* E(x_1^* y_1) w) = E(z^* E(x_1^* y_1) w)$, and thus we have

$$\begin{aligned} E_{A_k}(x^*y) &= E_{A_k}(z^* E(x_1^* y_1) w) + E_{A_k}(y^*(x_1^* y_1 - E(x_1^* y_1)) z) \\ &= E(z^* E(x_1^* y_1) w) \\ &= E(z^* E(x_1^* y_1) w) + E(y^*(x_1^* y_1 - E(x_1^* y_1)) z) \\ &= E(x^*y). \end{aligned}$$

When $n = 1$, a similar argument shows that $E_{A_k}(x^*y) = E(x^*y)$. Hence, the assertion follows by induction. □

We use the following A-$\prod_{k \in \mathcal{I}} A_k$ and A-A C*-correspondences

$$(Y, \phi_Y) = \boxplus_{k \in \mathcal{I}} (Y_k, \phi_{Y_k}), \quad (Z, \phi_Z) = \bigoplus_{k \in \mathcal{I}} (Y_k \otimes_{A_k} A, \phi_{Y_k} \otimes 1), \tag{2.1}$$

and the $*$-homomorphism $\Phi_Z \colon \mathbb{L}(Z) \to \mathbb{L}(Y)$ induced from the natural $*$-homomorphisms $\mathbb{L}(Y_k) \to \mathbb{L}(Y_k \otimes_{A_k} A)$, $k \in \mathcal{I}$.

The next proposition is probably well-known, but we give its proof for the reader's convenience.

Proposition 2.3.5. *Let $(A, E) = \bigstar_D (A_k, E_k)$ be any reduced amalgamated free product and C be any unital* C*-*algebra. Then, the pair $(A \otimes C, E \otimes \mathrm{id}_C)$ is naturally identical to the reduced amalgamated free product $\bigstar_{D \otimes C} (A_k \otimes C, E_k \otimes \mathrm{id})$.*

Proof. Let $\phi_Y \colon A \to \mathbb{L}(Y)$ be as above. Since ϕ_Y is faithful, we have $A \otimes C \subset \mathbb{L}(Y) \otimes C \subset \prod_k \mathbb{L}(Y_k \otimes C)$. It is easy to check that the GNS representation associated with $E_{A_k} \otimes \mathrm{id} \colon A \otimes C \to A_k \otimes C$ is given by $(Y_k \otimes C, \phi_{Y_k} \otimes L_C, \eta_k \otimes 1)$. Thus, we only have to check the freeness condition in Definition 2.3.3 for $\{E_{A_k} \otimes \mathrm{id}\}_k$. This will immediately follow once we proved that $\ker(E_k \otimes \mathrm{id})$ is the norm closure of $(\ker E_k) \odot C$. Indeed, for any $x \in \ker(E_k \otimes \mathrm{id})$ and any $\varepsilon > 0$ there exists $y = \sum_{i=1}^n a_i \otimes c_i \in A_k \odot C$ such that $\|x - y\| < \varepsilon$. We may assume that $c_1, \ldots, c_n$ are linearly independent. Since $\|x-(y-(E_k\otimes \mathrm{id})(y))\| \leq \|x-y\|+\|(E_k\otimes \mathrm{id})(x-y)\| < 2\varepsilon$, we may assume that $(E_k\otimes \mathrm{id})(y)=0$. Then, we have $\sum_{i=1}^n E_k(a_i)\otimes c_i = 0$, implying $E_k(a_i)=0$ for $i=1,\cdots,n$. Thus, $y\in A_k^\circ \odot C$. Since ε is arbitrary, we are done. □

2.4 Pimsner algebras

We fix notations and terminologies on Pimsner algebras following Katsura's paper [31]. Let (X,ϕ_X) be a C*-correspondence over a C*-algebra A. Recall that a *representation* of X on a C*-algebra B is a pair (π,t) such that $\pi\colon A\to B$ is a $*$-homomorphism and $t\colon X\to B$ is a linear map satisfying $t(\xi)^*t(\eta)=\pi(\langle \xi,\eta\rangle)$ and $\pi(a)t(\xi)\pi(b)=t(\phi_X(a)\xi b)$ for $\xi,\eta\in X$ and $a,b\in A$. We denote by $\mathrm{C}^*(\pi,t)$ the C*-subalgebra of B generated by $\pi(A)$ and $t(X)$. Any representation (π,t) induces a $*$-homomorphism $\psi_t\colon \mathbb{K}(X)\to B$ such that $\psi_t(\theta_{\xi,\eta})=t(\xi)t(\eta)^*$. We define the ideal J_X of A by

$$\phi_X^{-1}(\mathbb{K}(X))\cap(\ker\phi_X)^\perp = \{a\in\phi_X^{-1}(\mathbb{K}(X)) \mid ax=0 \text{ for } x\in\ker\phi_X\}$$

and say that (π,t) is *covariant* if $\pi=\psi_t\circ\phi_X$ holds on J_X.

A (resp. covariant) representation (π,t) is said to be *universal* if for any (resp. covariant) representation (π',t') of X, there exists a $*$-homomorphism $\rho\colon \mathrm{C}^*(\pi,t)\to \mathrm{C}^*(\pi',t')$ such that $\rho\circ\pi=\pi'$ and $\rho\circ t=t'$. Note that if (π,t) and (π',t') are universal (covariant) representations, then $\mathrm{C}^*(\pi,t)\cong \mathrm{C}^*(\pi',t')$ canonically by universality.

A representation (π,t) is said to *admit a gauge action* if there exists a continuous action γ of $\mathbb{T}=\{z\in\mathbb{C}\mid |z|=1\}$ on $\mathrm{C}^*(\pi,t)$ such that $\gamma_z\circ\pi=\pi$ and $\gamma_z(t(\xi))=zt(\xi)$ for $z\in\mathbb{T}$ and $\xi\in X$. Note that any universal (covariant) representation admits a gauge-action by universality. We will use the next gauge-invariant uniqueness theorem.

Theorem 2.4.1 ([31, Theorem 6.2, Theorem 6.4])**.** *Let (π,t) be a representation of X. Then, (π,t) is universal if and only if (π,t) is injective and admits a gauge action, and $\pi(J_X)\cap\psi_t(\mathbb{K}(X))=\{0\}$. Further assume that (π,t) is covariant. Then (π,t) is universal if and only if π is injective and (π,t) admits a gauge action.*

We will use the following concrete universal representation, called the Fock representation. We set $(X^{\otimes 0},\phi_{X^{\otimes 0}})=(A,L_A)$ and $(X^{\otimes 1},\phi_{X^{\otimes 1}})=(X,\phi_X)$. For each $n\geq 2$, we define the C*-correspondence $(X^{\otimes n},\phi_{X^{\otimes n}})$ by

$$X^{\otimes n}=\overbrace{X\otimes_A X\otimes_A\cdots\otimes_A X}^{n},\quad \phi_{X^{\otimes n}}=\phi_X\otimes\overbrace{1_X\otimes\cdots\otimes 1_X}^{n-1}.$$

Then the full Fock space $\mathcal{F}(X) = \bigoplus_{n\geq 0} X^{\otimes n}$ over X together with $\varphi_\infty = \bigoplus_{n\geq 0} \phi_{X^{\otimes n}}$ is a C*-correspondence over A. For each ξ, we define the creation operator $\tau_\infty(\xi)$ on $\mathcal{F}(X)$ by $\tau_\infty(\xi)\eta = \xi \otimes \eta$ for $\eta \in X^{\otimes n}$ and $n \geq 0$. Then the pair $(\varphi_\infty, \tau_\infty)$ is a representation of X. The compression map by the projection onto $X^{\otimes 0}$ defines a conditional expectation $E_X \colon \mathrm{C}^*(\varphi_\infty, \tau_\infty) \to A$ which vanishes on $\overline{\mathrm{span}}\{\tau_\infty(\xi)\tau_\infty(\eta)^* \mid \xi, \eta \in X\} = \psi_{\tau_\infty}(\mathbb{K}(X))$. Also, the direct sum of the unitary representations $\mathbb{T} \ni z \mapsto z^n 1 \in \mathbb{L}(X^{\otimes n})$ implements a gauge action on $\mathrm{C}^*(\varphi_\infty, \tau_\infty)$. Thus, by Theorem 2.4.1 $(\varphi_\infty, \tau_\infty)$ is universal and we call $\mathcal{T}(X) := \mathrm{C}^*(\varphi_\infty, \tau_\infty)$ the *Toeplitz–Pimsner* algebra of X.

In order to construct a universal covariant representation we next consider the ideal of $\mathcal{T}(X)$ generated by $\{\varphi_\infty(x) - \psi_{\tau_\infty}(\phi_X(x)) \mid x \in J_X\}$, which is naturally isomorphic to $\mathbb{K}(\mathcal{F}(X)J_X)$. The quotient of $\mathcal{T}(X)$ by $\mathbb{K}(\mathcal{F}(X)J_X)$ is called the *Cuntz–Pimsner algebra* of X and denoted by $\mathcal{O}(X)$. Note that the representation of X on $\mathcal{O}(X)$ given by $(\varphi_\infty, \tau_\infty)$ and the quotient map is covariant and injective. Moreover, since $\mathbb{K}(\mathcal{F}(X)J_X)$ is invariant under the gauge action, this covariant representation is universal by Theorem 2.4.1. Note that the definition of $\mathcal{O}(X)$ is different from Pimsner's original one in [41] when ϕ_X is not injective.

2.5 KK-theory

Throughout this subsection, our C*-algebras are all assumed to be separable. We refer the reader to [2] for KK-theory.

Definition 2.5.1. For (trivially graded) C*-algebras A and B, a *Kasparov A-B bimodule* is a triplet (X, ϕ, F) such that X is a countably generated graded Hilbert B-module, $\phi : A \to \mathbb{L}(X)$ is a $*$-homomorphism of degree 0, and $F \in \mathbb{L}(X)$ is of degree 1 and satisfies the following condition:

- $[F, \phi(a)] \in \mathbb{K}(X)$ for $a \in A$,
- $(F - F^*)\phi(a) \in \mathbb{K}(X)$ for $a \in A$,
- $(1 - F^2)\phi(a) \in \mathbb{K}(X)$ for $a \in A$.

When $[F, \phi(a)] = (F - F^*)\phi(a) = (1 - F^2)\phi(a) = 0$ holds for every $a \in A$, we say that (X, ϕ, F) is *degenerate*. We denote by $\mathbb{E}(A, B)$ and $\mathbb{D}(A, B)$ the corrections of Kasparov A-B bimodules and degenerate ones, respectively.

We say that two Kasparov A-B bimodules (X, ϕ, F) and (Y, ψ, G) are *unitarily equivalent*, denoted by $(X, \phi, F) \cong (Y, \psi, G)$, if there exists a unitary $U \in \mathbb{L}(X, Y)$ of degree 0 such that $\psi = \mathrm{Ad}\, U \circ \phi$ and $G = UFU^*$.

For any Hilbert B-module X, we set $IX := C([0, 1]) \otimes X$. In particular, we set $IB = C([0, 1]) \otimes B$. For each $t \in [0, 1]$ we still denote by t the surjectiove $*$-homomorphism $IB \cong C([0, 1], B) \ni f \mapsto f(t) \in B$. Note that we have a natural isomorphism $IX \otimes_t B \cong X$ for every $t \in [0, 1]$.

Definition 2.5.2. Two Kasparov A-B bimodules (X_0, ϕ_0, F_0) and (X_1, ϕ_1, F_1) are said to be *homotopic* if there exists a Kasparov A-IB bimodule (Y, ψ, G) such that $(Y \otimes_t B, \psi \otimes 1_B, G \otimes 1_B) \cong (X_t, \phi_t, F_t)$ for $t = 0, 1$. The *KK-group* $KK(A, B)$ is the set of homotopy equivalence classes of Kasparov A-B bimodules.

The next technical lemma will be used later.

Lemma 2.5.3. *Let P, Q and R be separable* C*-*algebras and let $(X, \psi_i, F) \in \mathbb{E}(Q, R)$ be given for $i = 0, 1$. Suppose that there exist a surjective $*$-homomorphism $\pi : P \to Q$ and a family of Kasparov P-R bimodules (X, ϕ_t, F) for $t \in [0, 1]$ satisfying*

(i) the function $[0, 1] \ni t \mapsto \phi_t(a)$ is strictly continuous for each $a \in P$;

(ii) the functions sending t to $[F, \phi_t(a)], (F - F^*)\phi_t(a)$ and $(1 - F^2)\phi_t(a)$ are norm continuous for each $a \in P$;

(iii) ϕ_t factors through $\pi : P \to Q$ for every $t \in [0, 1]$;

(iv) $\phi_i = \psi_i \circ \pi$ holds for $i = 0, 1$.

Then, (X, ψ_0, F) and (X, ψ_1, F) are homotopic.

Proof. By assumption, there exists a $*$-homomorphism $\phi : P \to \mathbb{L}(IX)$ such that $(IX, \phi, F \otimes 1_{C[0,1]}) \in \mathbb{E}(P, IR)$ and (X, ϕ_t) is the push out of (IX, ϕ) by $t \colon C[0, 1] \to \mathbb{C}$ for $t \in [0, 1]$. Since one has $\|\phi(a)\| = \sup_{0 \le t \le 1} \|\phi_t(a)\| \le \|\pi(a)\|$ for $a \in P$, there exists $\psi : Q \to \mathbb{L}(IX)$ such that $\phi = \psi \circ \pi$. We then have $(IX, \psi, F \otimes 1_{C([0,1])}) \in \mathbb{E}(Q, IR)$ and the evaluations of this Kasparov bimodule at endpoints are exactly (X, ψ_i, F), $i = 0, 1$. □

The KK-group becomes an additive group in the following way: For $\alpha, \beta \in KK(A, B)$ implemented by $(X, \phi, F), (Y, \psi, G)$, respectively, $\alpha + \beta$ is the element implemented by $(X \oplus Y, \phi \oplus \psi, F \oplus G)$. All degenerate Kasparov bimodules are homotopic to the trivial bimodule $0 = (0, 0, 0)$ and define the zero element in $KK(A, B)$. Let X_0 and X_1 be the even and odd parts of X so that $X = X_0 \oplus X_1$ and let $-X$ be the graded Hilbert B-module with the even part X_1 and the odd part X_0. The inverse of α is implemented by $(-X, \operatorname{Ad} U \circ \phi, UFU^*)$, where $U : X \to -X$ is the natural unitary.

For any $*$-homomorphism $\phi : A \to B$, we have $(B \oplus 0, \phi \oplus 0, 0) \in \mathbb{E}(A, B)$ and still denote by ϕ the corresponding element in $KK(A, B)$.

For $\alpha \in KK(A, B)$ and $\gamma \in KK(B, C)$, the *Kasparov product* of α and γ is denoted by $\alpha \otimes_B \gamma$. When one of α and β comes from a $*$-homomorphism, the construction of the Kasparov product is very simple. Indeed, if γ comes from a $*$-homomorphism $\gamma : B \to C$ with $[\gamma(B)C] = C$ and α is implemented by (X, ϕ, F), then the Kasparov product$\alpha \otimes_B \gamma$ is implemented by $(X \otimes_\gamma C, \phi \otimes 1_C, F \otimes 1_C)$. Similarly, when α is a $*$-homomorphism from A into B and γ is implemented by (Y, ψ, G) with $[\psi(B)Y] = Y$, the Kasparov product $\alpha \otimes_B \gamma$ is implemented by $(Y, \psi \circ \alpha, G)$.

Definition 2.5.4. An element $\alpha \in KK(A, B)$ is said to be a KK-*equivalence* if there exists $\beta \in KK(B, A)$ such that $\mathrm{id}_A = \alpha \otimes_B \gamma$ and $\mathrm{id}_B = \beta \otimes_A \alpha$. In this case, A and B are said to be KK-*equivalent*.

Note that KK-equivalence between A and B implies $KK(A, C) \cong KK(B, C)$ and $KK(C, A) \cong KK(C, B)$ for any separable C*-algebra C.

Definition 2.5.5 ([10])**.** A countable discrete group Γ is said to be K-*amenable* if the canonical surjection from the full group C*-algebra $C^*(\Gamma)$ onto the reduced one $C^*_{\mathrm{red}}(\Gamma)$ gives a KK-equivalence.

All countable amenable groups are K-amenable, but there are many non-amenable, K-amenable groups. Indeed, by Pimsner's result [40] K-amenability is stable under the amalgamated free product. Motivated by Cuntz's K-amenability, in [48] Skandalis introduced the notion of K-nucleariry for C*-algebras. One of merits of being K-nuclear is that if A is K-nuclear, then the functor $KK(A, \cdot)$ is half-exact, that is, for any exact sequence of C*-algebras $0 \longrightarrow J \longrightarrow B \longrightarrow C \longrightarrow 0$, the induced sequence

$$KK(A, J) \longrightarrow KK(A, B) \longrightarrow KK(A, C)$$

is exact in the middle.

Theorem 2.5.6 ([48, Theoreme 1.5])**.** *Let A and B be separable* C*-*algebras and let $\pi : A \to \mathbb{B}(H)$ be a faithful and essential representation on a separable Hilbert space H. For a given A-B* C*-*correspondence (X, σ) with X countably generated, the following are equivalent:*

(i) For any unit vector $\xi \in X$ the CCP map $A \ni a \mapsto \langle \xi, \sigma(a)\xi \rangle \in B$ is nuclear.

(ii) For any $x \in \mathbb{K}(X)$ of norm 1, the CCP map $A \ni a \mapsto x^*\sigma(a)x \in \mathbb{K}(X)$ is nuclear.

(iii) There exists a sequence of isometries $V_n \in \mathbb{L}(X, H \otimes B)$ such that $\sigma(a) - V_n^*(\pi(a) \otimes 1_A)V_n \in \mathbb{K}(X)$ and $\lim_{n\to\infty} \|\sigma(a) - V_n^*(\pi(a) \otimes 1_A)V_n\| = 0$ for all $a \in A$.

When any of these three conditions holds, we say that (X, σ) is nuclear*.*

Definition 2.5.7. A separable C*-algebra A is said to be K-*nuclear* if id_A in $KK(A, A)$ is implemented by a Kasparov bimodule (X, ϕ, F) such that (X, ϕ) is nuclear.

3 Compactifications of Bass–Serre trees

In this section, we study the reduced crossed product of the compactification of the Bass–Serre tree associated with an amalgamated free product group, and its analogue for general reduced amalgamated free products.

3.1 Bass–Serre trees and compactifications

Let $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ be an amalgamated free product of discrete groups and put $\mathcal{I} = \{1, 2\}$. The *Bass–Serre tree* associated with Γ is the graph $\mathbf{T} = (\mathbf{V}, \mathbf{E})$, of which the vertex set is $\mathbf{V} = \Gamma/\Gamma_1 \sqcup \Gamma/\Gamma_2$ and the edge set is $\mathbf{E} = \Gamma/\Lambda$ such that the edge $g\Lambda$ relates $g\Gamma_1$ and $g\Gamma_2$ (see [46]). Note that Γ acts on $\mathbf{V}$ and $\mathbf{E}$ by left multiplication, which are compatible with the graph structure of $\mathbf{T}$. Notice that the unitary representation induced from $\Gamma \curvearrowright \mathbf{E}$ is nothing but the quasi-regular representation $\lambda_{\Gamma/\Lambda}$. Also, the unitary representation $(\ell^2(\mathbf{V}), \pi)$ induced from $\Gamma \curvearrowright \mathbf{V}$ is unitarily equivalent to $\lambda_{\Gamma/\Gamma_1} \oplus \lambda_{\Gamma/\Gamma_2}$.

We next consider the compactification $\Delta\mathbf{T}$ of $\mathbf{T}$ introduced by Bowditch [4] (see [6, §§5.2] for details). For any $x, y \in \mathbf{V}$ we denote the graph distance of x and y by $d(x, y)$. A sequence $(x(n))_{n=1}^\infty$ in $\mathbf{V}$ is called a *geodesic path* if there exists $N \in \mathbb{N} \cup \{\infty\}$ such that $d(x(n), x(m)) = |n - m|$ for any $n, m < N$ and $x(n) = x(N)$ for $n \geq N$. When N is finite (resp. infinite), we call $(x(n))_{n=1}^\infty$ a finite (resp. infinite) geodesic path. We denote by $\Delta\mathbf{T}$ the set of equivalence classes of geodesic paths. The set $\partial\mathbf{T}$ of all equivalence classes of infinite geodesic paths is called the *ideal boundary* of $\mathbf{T}$. Note that Γ acts on $\Delta\mathbf{T}$ by $g[(x(n))_n] = [(gx(n))_n]$.

Let Ω be the set of all finite or infinite geodesic path starting at $e\Gamma_1$. We can identify $\mathbf{V}$ with the finite geodesic paths in Ω in such a way that each vertex $x \in \mathbf{V}$ corresponds to the unique finite geodesic path from $e\Gamma_1$ to x. Since the canonical map from Ω onto $\Delta\mathbf{T}$ is bijective, we also identify Ω with $\Delta\mathbf{T}$ so that $\Delta\mathbf{T} = \mathbf{V} \sqcup \partial\mathbf{T}$. For each $x, y \in \Delta\mathbf{T}$ there exists a unique bi-infinite sequence $(z(n))_{n=-\infty}^\infty$ such that $(z(n))_{n=0}^\infty$ and $(z(-n))_{n=0}^\infty$ are geodesic paths representing x and y, respectively. We set $[x, y] := \{z(n) \mid n \in \mathbb{Z}\} \cup \{x, y\} \subset \Omega = \Delta\mathbf{T}$. For each $x \in \Delta\mathbf{T}$ and each finite subset F of V, we set

$$U(x, F) := \{x\} \cup \{y \in \Delta\mathbf{T} \mid [x, y] \cap F = \emptyset\}.$$

Then, $\{U(x, F) \mid F \subset \mathbf{V} \text{ finite}\}_{x \in \Delta\mathbf{T}}$ forms an open neighborhood system for a topology on $\Delta\mathbf{T}$. It is known that $\Delta\mathbf{T}$ equipped with this topology is compact and Hausdorff and the action $\Gamma \curvearrowright \Delta\mathbf{T}$ defined above is continuous. Let $\alpha \colon \Gamma \curvearrowright C(\Delta\mathbf{T})$ be the induced action given by $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in \Gamma, f \in C(\Delta\mathbf{T}), x \in \Delta\mathbf{T}$. Since $\mathbf{V}$ is dense in $\Delta\mathbf{T}$, we have natural inclusions $C(\Delta\mathbf{T}) \subset$

$\ell^\infty(\mathbf{V}) \subset \mathbb{B}(\ell^2(\mathbf{V}))$. Observe that α is implemented by the unitary representation $\pi\colon \Gamma \curvearrowright \ell^2(\mathbf{V})$. Thus, the reduced crossed product of $C(\Delta\mathbf{T})$ by α is given by

$$C(\Delta\mathbf{T}) \rtimes_{\mathrm{red}} \Gamma \cong \mathrm{C}^*\{C(\Delta\mathbf{T}) \otimes 1 \cup (\pi \otimes \lambda)(\Gamma)\} \subset \mathbb{B}(\ell^2(\mathbf{V}) \otimes \ell^2(\Gamma)). \tag{3.1}$$

We observe that for any $x \in \mathbf{V}$, the one point set $\{x\}$ is open in $\Delta\mathbf{T}$ if and only if x has finite degree. Also, it is easy to see that $\partial\mathbf{T}$ is closed if and only if $\mathbf{T}$ is locally finite, equivalently Λ is a finite index subgroup of Γ_k for $k = 1, 2$. This is the reason why we work on not the ideal boundary but the whole compactification.

Our next goal is to give a C*-algebraic description of the reduced crossed product $C(\Delta\mathbf{T}) \rtimes_{\mathrm{red}} \Gamma$. Our key machinery is the next elementary proposition. This is a C*-algebraic analogue of so-called Connes's viewpoint [12], asserting that bimodules over von Neumann algebras could play a role of unitary representations of groups (see also [26]).

Let Γ be a discrete group, $u\colon \Gamma \curvearrowright \mathcal{H}$ be either the left regular representation or the universal representation, and A be the corresponding group C*-algebra $\mathrm{C}^*(u(\Gamma))$. For a unitary representation $\pi\colon \Gamma \curvearrowright H_\pi$, consider the C*-correspondence (X_π, ϕ_π) over A defined by

$$X_\pi = H_\pi \otimes A, \quad \phi_\pi\colon \lambda(g) \mapsto \pi(g) \otimes L_A(u(g)).$$

Here the well-definedness of ϕ_π follows from Fell's absorption principle.

Proposition 3.1.1. *Let Γ and A be as above. The following hold true:*

(i) For the trivial representation $1_\Gamma\colon \Gamma \to \mathbb{C}$, one has $(X_{1_\Gamma}, \phi_{1_\Gamma}) = (A, L_A)$.

(ii) For the left regular representation $\lambda\colon \Gamma \curvearrowright \ell^2(\Gamma)$, one has $(X_\lambda, \phi_\lambda) \cong (\ell^2(\Gamma) \otimes A, \lambda \otimes 1)$.

(iii) Let $\Lambda \leq \Gamma$ be a subgroup and E be the canonical conditional expectation from A onto $D = \mathrm{C}^*(u(\Lambda))$. For the quasi-regular representation $\lambda_{\Gamma/\Lambda}\colon \Gamma \curvearrowright \ell^2(\Gamma/\Lambda)$, one has $(X_{\lambda_{\Gamma/\Lambda}}, \phi_{\lambda_{\Gamma/\Lambda}}) \cong (L^2(A, E) \otimes_D A, \phi_E \otimes 1)$.

Proof. Since (i) and (ii) are particular cases of (iii), we prove only (ii). Define an operator $U\colon \ell^2(\Gamma/\Lambda) \otimes A \to L^2(A, E) \otimes_D A$ by $U(\delta_{g\Lambda} \otimes a) = u(g)\xi_E \otimes u(g)^* a$ for $g \in \Gamma$ and $a \in A$. Then, U is well-defined and gives the desired unitary equivalence. Indeed, for any $g, h \in \Gamma$ and $a, b \in A$ we have

$$\begin{aligned}
\langle U(\delta_{g\Lambda} \otimes a), U(\delta_{h\Lambda} \otimes b)\rangle &= \langle u(g)\xi_E \otimes u(g)^* a, u(h)\xi_E \otimes u(h)^* b\rangle \\
&= \langle u(g)^* a, E(u(g^* h))u(h)^* b\rangle \\
&= \delta_{g\Lambda, h\Lambda} a^* b \\
&= \langle \delta_{g\Lambda} \otimes a, \delta_{h\Lambda} \otimes b\rangle
\end{aligned}$$

and also $U(\delta_{gh\Lambda} \otimes u(g)a) = u(g)\xi_E \otimes a = \phi_E(u(g))U(\delta_{h\Gamma} \otimes a)$. □

Remark 3.1.2. Let $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ be an amalgamated free product group. Denote by A, A_k $(k = 1, 2)$ and D be the reduced group C*-algebras of Γ, Γ_k $(k = 1, 2)$ and Λ, respectively, and by $E\colon A \to D$, $E_k\colon A_k \to D$ and $E_{A_k}\colon A \to A_k$ the canonical conditional expectations. Then, it follows that $(A, E) \cong (A_1, E_1) \star_D (A_2, E_2)$. The previous proposition implies that the C*-correspondences corresponding to the vertex and the edge sets are given by

$$\bigoplus_{k=1,2} (Y_k \otimes_{A_k} A, \phi_{Y_k} \otimes 1) \quad \text{and} \quad (X \otimes_D A, \phi_X \otimes 1), \tag{3.2}$$

respectively. Notice that these C*-correspondences can be defined for arbitrary reduced amalgamated free products. In fact, in §§ 4.1 we will see that they indeed play a role of Bass–Serre trees (see Remark 4.1.3). For related topics, we refer the reader to [18, 20] in which C*-algebraic analogues of graph of groups, called graph of C*-algebras are studied.

We set $\Omega_1 := U(e\Gamma_1, \{e\Gamma_2\}) = \{(x(n))_{n=1}^\infty \in \Omega \mid x(2) \neq e\Gamma_2\}$ and $\Omega_2 := U(e\Gamma_2, \{e\Gamma_1\}) = \{(x(n))_{n=1}^\infty \in \Omega \mid x(2) = e\Gamma_2\}$. Since $\Omega_1 \sqcup \Omega_2 = \Delta\mathbf{T}$, Ω_1 and Ω_2 are clopen subsets of $\Delta\mathbf{T}$.

Let $Y = Y_1 \boxplus Y_2$ and $\phi_Y\colon A \hookrightarrow \mathbb{L}(Y)$ be as in Eq. (2.1) and S_k be as in Lemma 2.3.4. Let $P_k \in \mathbb{L}(Y)$ be the projection onto the closed submodule generated by the set

$$\{a_1 a_2 \cdots a_n \eta_j \mid n \geq 1, a_1 a_2 \cdots a_n \text{ reduced word with } a_1 \in A_k^\circ, j = 1, 2\}$$

In other words, we set

$$P_k^\circ := \sum_{j=1,2} S_j (P_{(\ell,k)}^\perp P_{(r,j)} \otimes 1) S_j^*, \qquad P_k := e_{A_k} + P_k^\circ, \tag{3.3}$$

where $e_{A_k} = \theta_{\eta_k,\eta_k} \in \mathbb{K}(Y_k) \subset \mathbb{K}(Y)$ is the Jones projection of E_{A_k}.

Proposition 3.1.3. *There exists a $*$-isomorphism from $C(\Delta\mathbf{T}) \rtimes_{\rm red} \Gamma$ onto $\mathrm{C}^*(\phi_Y(A), P_1, P_2)$ sending $\chi_{\Omega_k} \otimes 1$ to P_k for $k = 1, 2$ and $\pi \otimes \lambda(g)$ to $\phi_Y(\lambda(g))$ for $g \in \Gamma$, respectively.*

Proof. Let $C(\Delta\mathbf{T}) \rtimes_{\rm red} \Gamma \subset \mathbb{B}(\ell^2(\mathbf{V}) \otimes \ell^2(\Gamma))$ be the faithful representation as in Eq. (3.1). Since the map $\mathbb{L}(\ell^2(\mathbf{V}) \otimes A) \to \mathbb{B}(\ell^2(\mathbf{V}) \otimes \ell^2(\Gamma))$ induced from $A \hookrightarrow \mathbb{B}(\ell^2(\Gamma))$ is injective, we may assume that $C(\Delta\mathbf{T}) \rtimes_{\rm red} \Gamma = \mathrm{C}^*(\{\pi(g) \otimes L_A \circ \lambda(g)\}_{g\in\Gamma} \cup C(\Delta\mathbf{T}) \otimes 1)$. We claim that $C(\Delta\mathbf{T}) \rtimes_{\rm red} \Gamma$ is generated by $\{\pi(g) \otimes L_A \circ \lambda(g)\}_{g\in\Gamma}$ and $\{\chi_{\Omega_k} \otimes 1\}_{k=1,2}$. For this, it suffices to show that the Γ-orbits of $\chi_{\Omega_k}, k = 1, 2$ generate $C(\Delta\mathbf{T})$. Take $g \in \Gamma \setminus \Lambda$ arbitrarily and let $g = g_1 \cdots g_n$ be in reduced form. If $g_n \notin \Gamma_k$, we have $\alpha_g(\chi_{\Omega_k}) = \chi_{g\Omega_k} = \chi_{\Omega(g\Gamma_k)}$, where

$$\Omega(x) := \{(y(n))_{n=1}^\infty \in \Omega \mid y(d(e\Gamma_1, x)) = x\} \quad \text{for} \quad x \in \mathbf{V}$$

under the identification $\Delta\mathbf{T} = \Omega$. Now let $x = (x(n))_{n=1}^{\infty}, y = (y(n))_{n=1}^{\infty} \in \Omega$ be distinct elements. Take the minimal $N \geq 1$ such that $x(n) \neq y(n)$ for $n \geq N$. Without loss of generality we may assume that $y(N) \notin \{x(n) \mid n \geq 1\}$. Then, we have $y \in \Omega(y(N))$ and $x \notin \Omega(y(N))$. Therefore, the claim follows from the Stone–Weiertstarss theorem.

Let $U_k \colon \ell^2(\Gamma/\Gamma_k) \otimes A \to Y_k \otimes_{A_k} A$ be the unitary given by Proposition 3.1.1 and set $U := U_1 \oplus U_2 \colon \ell^2(\mathbf{V}) \otimes A \to \bigoplus_{k=1,2} Y_k \otimes_{A_k} A$. Let $(Z, \phi_Z) := \bigoplus_{k=1,2}(Y_k \otimes_{A_k} A, \phi_{Y_k} \otimes 1)$ and $\Phi_Z \colon \mathbb{L}(Y) \to \mathbb{L}(Z)$ be as in Eq. (2.1) so that $\Phi_Z \circ \phi_Y = \phi_Z$. Then, Proposition 3.1.1 implies that $U(\pi \otimes \lambda(g))U^* = \phi_Z(\lambda(g))$ for $g \in \Gamma$. Also, it follows from the definitions of Ω_k and P_k that

$$\Phi_Z(P_k) = U(\chi_{\Omega_k} \otimes 1)U^* \quad \text{for } k = 1, 2.$$

By the first paragraph of the proof, we conclude that $\Phi_Z^{-1} \circ \operatorname{Ad} U$ gives the desired isomorphism. □

Remark 3.1.4. Here is another representation of $C(\Delta\mathbf{T}) \rtimes_{\mathrm{red}} \Gamma$ on $\ell^2(\Gamma) \oplus \ell^2(\Gamma)$. Let $P_\Lambda \in \mathbb{B}(\ell^2(\Gamma))$ be the projection onto $\ell^2(\Lambda)$ and q_k be the projection onto the closed span of the vectors δ_g such that g is a reduced word beginning with an element in Γ_k. Define $Q_1, Q_2 \in \mathbb{B}(\ell^2(\Gamma) \oplus \ell^2(\Gamma))$ by

$$Q_1 = (P_\Lambda + q_1) \oplus q_1, \quad Q_2 = q_2 \oplus (P_\Lambda + q_2).$$

Then, $C(\Delta\mathbf{T}) \rtimes_{\mathrm{red}} \Gamma$ is isomorphic to $\mathrm{C}^*(Q_1, Q_2, \lambda \oplus \lambda(\Gamma))$. To see this, we first observe that the unitary representation $\phi_{Y_k}(\lambda(\cdot)) \otimes 1 \colon \Gamma \curvearrowright Y_k \otimes_{A_k} \ell^2(\Gamma_k)$ is unitarily equivalent to $\lambda \colon \Gamma \curvearrowright \ell^2(\Gamma)$. Then, the composition

$$\begin{aligned} \mathbb{L}(Y) = \mathbb{L}(Y_1) \oplus \mathbb{L}(Y_2) &\to \mathbb{B}(Y_1 \otimes_{A_1} \ell^2(\Gamma_1)) \oplus \mathbb{B}(Y_1 \otimes_{A_1} \ell^2(\Gamma_1)) \\ &\cong \mathbb{B}(\ell^2(\Gamma)) \oplus \mathbb{B}(\ell^2(\Gamma)) \\ &\hookrightarrow \mathbb{B}(\ell^2(\Gamma) \oplus \ell^2(\Gamma)) \end{aligned}$$

sends $\phi_Y \circ \lambda(g)$ to $\lambda \oplus \lambda(g)$ and P_k to Q_k for $g \in \Gamma$ and $k = 1, 2$.

3.2 Extensions associated with conditional expectations

Let $D \subset A$ be a unital inclusion of C*-algebras with conditional expectation $E \colon A \to D$. we define the *split extension associated with* $(D \subset A, E)$ to be the universal unital C*-algebra $\langle\!\langle A, E \rangle\!\rangle$ generated by A and a projection e such that $eae = E(a)e$ for $a \in A$ and $1_A e = e 1_A = e$. Note that e commutes with D since $(1 - e)de = de - E(d)e = 0$ for $d \in D$.

Lemma 3.2.1. *If a unital $*$-representation $\rho \colon \langle\!\langle A, E \rangle\!\rangle \to \mathbb{B}(\mathcal{H})$ satisfies that the restrictions $\rho|_A$ and $\rho|_{De}$ are faithful and $\rho(A) \cap \overline{\operatorname{span}}\, \rho(AeA) = \{0\}$, then ρ is faithful.*

Proof. Since $A + \operatorname{span} AeA$ is norm dense in $\langle\langle A, E\rangle\rangle$, it suffices to show that $\|a + K\| \leq 3\|\rho(a + K)\|$ for all $a \in A$ and $K \in \operatorname{span} AeA$. By assumption, $\overline{\operatorname{span}}\, \rho(AeA)$ is a non-trivial closed ideal of $\rho(\langle\langle A, E\rangle\rangle)$ and the corresponding quotient is isomorphic to A. This implies that $\|a\| \leq \|\rho(a + K)\|$. Also, for any $\sum_{i=1}^{n} a_i e b_i^* \in \operatorname{span} AeA$ it follows from the proof of [6, Proposition 4.6.3] that

$$\|\sum_{i=1}^{n} \rho(a_i e b_i^*)\| = \|[\rho(E(a_i^* a_j)e)]_{i,j}^{1/2} [\rho(E(b_i^* b_j)e)]_{i,j}^{1/2}\|$$
$$= \|[(E(a_i^* a_j)e)]_{i,j}^{1/2} [(E(b_i^* b_j)e)]_{i,j}^{1/2}\| = \|\sum_{i=1}^{n} a_i e b_i^*\|.$$

Thus, we have $\|\rho(K)\| = \|K\|$. Therefore, we obtain $\|a + K\| \leq \|\rho(a)\| + \|\rho(K)\| \leq \|\rho(a + K)\| + \|\rho(a + K) - \rho(a)\| \leq 2\|\rho(a + K)\| + \|\rho(a)\| \leq 3\|\rho(a + K)\|$. □

Let (X, ϕ_X, ξ_0) be the GNS representation associated with E and $e_D = \theta_{\xi_0,\xi_0} \in \mathbb{K}(X)$ be the Jones projection. Since $e_D \phi_X(a) e_D = \phi_X(E(a)) e_D$ holds for $a \in A$, thanks to the above lemma, we may use the following identification

$$\langle\langle A, E\rangle\rangle = \mathrm{C}^*(\{0 \oplus e_D\} \cup \{a \oplus \phi_X(a) \mid a \in A\}) \subset A \oplus \mathbb{L}(X).$$

Note that the ideal $\overline{\operatorname{span}}\, AeA$ of $\langle\langle A, E\rangle\rangle$ is isomorphic to $\mathbb{K}(X)$ and the quotient by this ideal is isomorphic to A. Also we remark that $\langle\langle A, E\rangle\rangle$ is *not* isomorphic to $\mathrm{C}^*(\phi_X(A), e_D) \subset \mathbb{L}(X)$ in general. For example, if A is a crossed product of D by a finite group and E is the canonical one, then we have $\mathrm{C}^*(\phi_X(A), e_D) = \mathbb{K}(X) = \mathbb{L}(X)$.

Proposition 3.2.2. *The corner $e^\perp \langle\langle A, E\rangle\rangle e^\perp$ of $\langle\langle A, E\rangle\rangle$ is a semisplit extension of A by $\mathbb{K}(X^\circ)$ with UCP cross section $\Psi\colon a \mapsto e^\perp a e^\perp$. Moreover, the image of Ψ generates $e^\perp \langle\langle A, E\rangle\rangle e^\perp$.*

Proof. Via the above representation $\langle\langle A, E\rangle\rangle \subset A \oplus \mathbb{L}(X)$, the corner $e^\perp \langle\langle A, E\rangle\rangle e^\perp$ is faithfully represented in $A \oplus \mathbb{L}(X^\circ)$ and the ideal $e^\perp(\overline{\operatorname{span}}\, AeA)e^\perp$ is isomorphic to $0 \oplus \mathbb{K}(X^\circ)$. Let $\rho\colon e^\perp \langle\langle A, E\rangle\rangle e^\perp \to \langle\langle A, E\rangle\rangle / e^\perp(\overline{\operatorname{span}}\, AeA)e^\perp$ be the quotient map. We claim that $\rho \circ \Psi$ is a bijective $*$-homomorphism. The injectivity follows from the above representation. To see the multiplicativity, take $a, b \in A^\circ$ arbitrarily. Then, we have

$$\rho(\Psi(ab^*) - \Psi(a)\Psi(b)^*) = \rho(e^\perp ab^* e^\perp - e^\perp a e^\perp b^* e^\perp) = \rho(e^\perp a e b^* e^\perp) = 0.$$

The surjectivity follows from the fact that every element in $e^\perp \langle\langle A, E\rangle\rangle e^\perp$ is of the form $\Psi(a) + K$ for some $a \in A$ and $K \in \overline{\operatorname{span}}\, e^\perp AeAe^\perp$. Finally, the above computation show that $\mathrm{C}^*(\Psi(A))$ contains $\mathbb{K}(X^\circ)$, and thus equals $e^\perp \langle\langle A, E\rangle\rangle e^\perp$. □

By the proposition, we have the following commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{K}(X) & \longrightarrow & \langle\!\langle A, E\rangle\!\rangle & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathbb{K}(X^\circ) & \longrightarrow & e^\perp \langle\!\langle A, E\rangle\!\rangle e^\perp & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

such that the upper exact sequence is split and the lower one is semisplit with the UCP cross section Ψ. We call $(e^\perp \langle\!\langle A, E\rangle\!\rangle e^\perp, \Psi)$ the *semisplit extension associated with* $(D \subset A, E)$ and may assume that $e^\perp \langle\!\langle A, E\rangle\!\rangle e^\perp$ is a C*-subalgebra of $A \oplus \mathbb{L}(X^\circ)$.

Lemma 3.2.3. *The kernel of the left action* $\phi_{X^\circ} \colon e^\perp \langle\!\langle A, E\rangle\!\rangle e^\perp \to \mathbb{L}(X^\circ)$ *given by the projection* $A \oplus \mathbb{L}(X^\circ) \to \mathbb{L}(X^\circ)$ *is* $\{a \oplus 0 \mid \phi_X(a) \in \mathbb{K}(X)\} \cong \phi_X^{-1}(\mathbb{K}(X))$.

Proof. Note that every element x in $e^\perp \langle\!\langle A, E\rangle\!\rangle e^\perp$ is of the form $a \oplus (e_D^\perp \phi_X(a) e_D^\perp + K)$ for some $a \in A$ and $K \in \mathbb{K}(X^\circ)$. Then, x is in $\ker \phi_{X^\circ}$ if and only if $e_D^\perp \phi_X(a) e_D^\perp = -K$ if and only if $x = a \oplus 0$ and $a \in \phi_X^{-1}(\mathbb{K}(X))$. □

The following lemma easily follows from the definition of $\langle\!\langle A, E\rangle\!\rangle$:

Lemma 3.2.4. *There exists an isometric bijective linear map* $t^\circ \colon X^\circ \to e^\perp \langle\!\langle A, E\rangle\!\rangle e$ *such that*

- $t^\circ(a\xi_0) = ae = (1-e)ae$ *for* $a \in A^\circ$;
- $t^\circ(\xi)^* t^\circ(\eta) = \langle \xi, \eta\rangle e$ *for* $\xi, \eta \in X^\circ$;
- $t^\circ(\phi_{X^\circ}(b)\xi d) = bt^\circ(\xi)d$ *for* $b \in e^\perp \langle\!\langle A, E\rangle\!\rangle e^\perp, \xi \in X^\circ$ *and* $d \in D$.

This lemma says that the inclusion $\mathbb{K}(X) \subset \langle\!\langle A, E\rangle\!\rangle$ has the following matrix representation:

$$\begin{bmatrix} \mathbb{K}(X^\circ) & X^\circ \\ (X^\circ)^* & De \end{bmatrix} \quad \subset \quad \begin{bmatrix} e^\perp \langle\!\langle A, E\rangle\!\rangle e^\perp & X^\circ \\ (X^\circ)^* & De \end{bmatrix}.$$

Remark 3.2.5. Assume that $(D \subset A, E)$ comes from the reduced group C*-algebras of discrete groups $\Lambda \leq \Gamma$. Then, $[\Lambda : \Gamma] = \infty$ if and only if $\phi_X^{-1}(\mathbb{K}(X)) = \{0\}$. Indeed, if $[\Lambda : \Gamma] < \infty$, then one has $A = \mathbb{K}(X)$. Note that this the case when $\langle\!\langle A, E\rangle\!\rangle = A \oplus \mathbb{K}(X)$ and $e^\perp \langle\!\langle A, E\rangle\!\rangle e^\perp = A \oplus \mathbb{K}(X^\circ)$. Conversely, if $x \in \phi_X^{-1}(\mathbb{K}(X))$ is nonzero, then $x \otimes 1 \in \mathbb{K}(X \otimes_D A)$ is also nonzero. By the natural isomorphism $\mathbb{K}(X \otimes_D A) \cong c_0(\Gamma/\Lambda) \rtimes_{\mathrm{red}} \Gamma$, we have $(1 \otimes \mathrm{C}^*_{\mathrm{red}}(\Gamma)) \cap (c_0(\Gamma/\Lambda) \rtimes_{\mathrm{red}} \Gamma) \neq \{0\}$. This implies that $c_0(\Gamma/\Lambda)$ is unital, so we have $[\Gamma : \Lambda] < \infty$. Note that when $\phi_X(A) \cap \mathbb{K}(X) = \{0\}$, we have $\langle\!\langle A, E\rangle\!\rangle \cong \mathrm{C}^*(\phi_X(A), e_D) \cong (\mathbb{C}1 + c_0(\Gamma/\Lambda)) \rtimes_{\mathrm{red}} \Gamma$.

3.3 Construction and Cuntz–Pimsner algebras

Let $(A, E) = \bigstar_D(A_k, E_k)$ be the reduced amalgamated free product of $\{(D \subset A_k, E_k)\}_{k\in\mathcal{I}}$ and $(Y, \phi_Y) = \boxplus_{k\in\mathcal{I}}(Y_k, \phi_{Y_k})$ be as in Eq. (2.1). Also, let $S_k\colon X(r,k)\otimes_D A_k \to Y_k$ be as in Lemma 2.3.4. As in the group case, we define the projection $P_k \in \mathbb{L}(Y)$ by

$$P_k^\circ := \sum_{j\in\mathcal{I}} S_j(P_{(\ell,k)}^\perp P_{(r,k)} \otimes 1)S_j^*, \qquad P_k := e_{A_k} + P_k^\circ, \tag{3.4}$$

which is the projection onto the closed submodule

$$\overline{\mathrm{span}}\{a_1a_2\cdots a_n\eta_j \mid n \geq 1, a_1a_2\cdots a_n \text{ reduced word with } a_1 \in A_k^\circ, j \in \mathcal{I}\}.$$

Note that the projections $P_k, k \in \mathcal{I}$ are mutually orthogonal and satisfies that $\sum_{k\in\mathcal{I}} P_k = 1$.

Definition 3.3.1. For any reduced amalgamated free product $(A, E) = \bigstar_D(A_k, E_k)$ we define $\Delta\mathbf{T}(A, E)$ by the C*-algebra generated by $\phi_Y(A)$ and $\{P_k\}_{k\in\mathcal{I}}$ inside $\mathbb{L}(Y)$.

We may identify A with $\phi_Y(A)$ so that $A \subset \Delta\mathbf{T}(A, E)$.

Remark 3.3.2 (cf. Remark 3.1.4)**.** We will use the following representation of $\Delta\mathbf{T}(A, E)$. For each $k \in \mathcal{I}$, we consider the $\Delta\mathbf{T}(A, E)$-D C*-correspondence $X^{(k)} = Y_k \otimes_{A_k} X_k \cong X$ with the left action σ_k defined by the composition of the quotient map $\mathbb{L}(Y) \cong \prod_{i\in\mathcal{I}} \mathbb{L}(Y_i) \to \mathbb{L}(Y_k)$ and the map $\mathbb{L}(Y_k) \to \mathbb{L}(Y_k \otimes_{A_k} X_k)$ induced from the interior tensor product. Note that for each $k, j \in \mathcal{I}$ with $k \neq j$ one has

$$\sigma_k|_A = \phi_X, \quad \sigma_k(P_k) = e_D + P_{(\ell,k)}^\perp, \quad \sigma_j(P_k) = P_{(\ell,k)}^\perp, \tag{3.5}$$

and that $\bigoplus_k(X^{(k)}, \sigma_k)$ is faithful if E_k is nondegenerate for all $k \in \mathcal{I}$.

By Proposition 3.1.3, when (A, E) comes from the reduced group C*-algebra of $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$, we have $\Delta\mathbf{T}(A, E) \cong C(\Delta\mathbf{T}) \rtimes_{\mathrm{red}} \Gamma$. Thus, for general reduced amalgamated free products, one can view $\Delta\mathbf{T}(A, E)$ as an analogue of the "crossed product algebra of $C(\Delta\mathbf{T})$ by (A, E)", but we do not have any counterparts of the Cartan subalgebra $C(\Delta\mathbf{T})$ in general.

Proposition 3.3.3. *The following hold true:*

(i) The projections $P_k, k \in \mathcal{I}$ commute with D.

(ii) Any element $a \in A_k$ enjoys $P_k^\perp a P_k^\perp = E(a)P_k^\perp$ and $(a - E_k(a))P_k^\perp = P_k^\circ a P_k^\perp$.

(iii) *The compression* $\mathbb{L}(Y) \to \mathbb{L}(\eta_k A_k) \cong A_k$ *by* e_{A_k} *defines a conditional expectation from* $\Delta\mathbf{T}(A, E)$ *onto* A_k *extending* E_{A_k}.

(iv) *For any unital* C*-*algebra* C *and the reduced amalgamated free product* $(A \otimes C, E \otimes C) = \bigstar_{D\otimes C}(A_k \otimes C, E_k \otimes \mathrm{id})$ *(cf. 2.3.5), one has* $\Delta\mathbf{T}(A \otimes C, E \otimes \mathrm{id}) = \Delta\mathbf{T}(A, E) \otimes C$.

Proof. Since (i), (ii) and (iii) are obvious, we prove only (iv). Let $\phi_Y \colon A \to \mathbb{L}(Y)$ be as above and assume that $A \otimes C \subset \mathbb{L}(Y) \otimes C \subset \prod_k \mathbb{L}(Y_k \otimes C)$. Note that the left action of $A \otimes C$ on $Y_k \otimes C$ is the GNS representation associated with $E_{A_k} \otimes \mathrm{id}$ for $k \in \mathcal{I}$. Then, by definition, $P_k \otimes 1$ is nothing but the projection for $\bigstar_{D\otimes C}(A_k \otimes C, E_k \otimes \mathrm{id})$ given by Eq. (3.3). Thus, the assertion follows from the definition of $\Delta\mathbf{T}(A, E)$. □

We next prove that $\Delta\mathbf{T}(A, E)$ is identified with a Cuntz–Pimsner algebra. Let $\langle\!\langle A_k, E_k\rangle\!\rangle$ and (B_k, Ψ_k) be the split and semisplit extension associated with $(D \subset A_k, E_k)$ as in §§3.2. We consider the unital embedding $\Psi \colon D \to \prod_{k\in\mathcal{I}} B_k; d \mapsto (\Psi_k(d))_{k\in\mathcal{I}}$, and the C*-algebra $B := \bigoplus_{k\in\mathcal{I}} B_k + \Psi(D)$. We denote the support projection of B_k in B by 1_{B_k} and set $B_k^\perp = 1_{B_k}^\perp B$. Define the C*-correspondence $(\mathfrak{X}, \phi_{\mathfrak{X}})$ by

$$\bigoplus_{k\in\mathcal{I}} X_k^\circ \otimes_D B_k^\perp, \qquad \phi_{\mathfrak{X}} = \bigoplus_{k\in\mathcal{I}} \phi_{X_k^\circ} \otimes 1,$$

where the interior tensor product $X_k \otimes_D B_k^\perp$ is with respect to $D \ni d \mapsto \Psi(d)1_{B_k}^\perp \in B_k^\perp$ and $\phi_{X_k^\circ} \otimes 1$ is given by

$$B_k \hookrightarrow A_k \oplus \mathbb{L}(X_k^\circ) \to \mathbb{L}(X_k^\circ) \to \mathbb{L}(X_k^\circ \otimes_D B_k^\perp).$$

In the case when $\mathcal{I} = \{1, 2\}$, we have much simpler descriptions: $B = B_1 \oplus B_2$ and $\mathfrak{X} = (X_1^\circ \otimes_D B_2) \boxplus (X_2^\circ \otimes_D B_1)$.

We set $\xi_{k\bar{k}} := \xi_k \otimes 1_{B_k}^\perp \in \mathfrak{X}$. Recall that $\phi_{X_k}^{-1}(\mathbb{K}(X_k)) \oplus 0 \subset B_k$ (Lemma 3.2.3). We may use the identification $\mathbb{K}(X_k^\circ) \cong 0 \oplus \mathbb{K}(X_k^\circ) \subset B_k$.

Lemma 3.3.4. *The kernel of* $\phi_{\mathfrak{X}}$ *includes* $\bigoplus_{k\in\mathcal{I}} \phi_{X_k}^{-1}(\mathbb{K}(X_k)) \oplus 0$ *and we have* $J_{\mathfrak{X}} = \bigoplus_{k\in\mathcal{I}} \mathbb{K}(X_k^\circ)$.

Proof. The first assertion follows from Lemma 3.2.3. Take $x \in J_{\mathfrak{X}}$ arbitrarily. Then there exist $a_k \in A_k$ and $K_k \in \mathbb{K}(X_k^\circ)$ such that $x = (\Psi_k(a_k) + K_k)_{k\in\mathcal{I}}$. For each $k \in \mathcal{I}$, it follows from Lemma 2.2.3 that $\phi_{X_k}(a_k) \in \mathbb{K}(X_k)$, and so $a_k \in \phi_{X_k}^{-1}(\mathbb{K}(X_k))$. Thus, we obtain $(\Psi_k(a_k) + K_k)(a_k \oplus 0) = 0$, implying $a_k = 0$. Since $k \in \mathcal{I}$ is arbitrary, we have $x = (K_k)_k \in \bigoplus_k \mathbb{K}(X_k^\circ)$. The opposite inclusion follows from that $\phi_{\mathfrak{X}}(\theta_{a\xi_k, b\xi_k}) = \theta_{a\xi_{k\bar{k}}, b\xi_{k\bar{k}}}$ for $a, b \in A_k^\circ$ and $k \in \mathcal{I}$. □

Let $\{\delta_k\}_{k\in\mathcal{I}}$ be the canonical minimal projections in $c_0(\mathcal{I})$ and $c_0(\mathcal{I})^\sim := c_0(\mathcal{I}) + \mathbb{C}1$ be the unitization so that $c_0(\mathcal{I})^\sim = c_0(\mathcal{I})$ when $\mathcal{I}$ is a finite set. Let $\mathcal{C}$ be the universal C*-algebra generated by unital copies of *algebraic* (or full) amalgamated free product of A_i's over D and $c_0(\mathcal{I})^\sim$ such that $(1-\delta_k)a(1-\delta_k) = E_k(a)(1-\delta_k)$ for $k \in \mathcal{I}$ and $a \in A_k$. More precisely, consider the collection $\mathcal{F}$ of all cyclic representations $\pi\colon (\bigstar_D A_k) \star c_0(\mathcal{I})^\sim \to \mathbb{B}(\mathcal{H})$ satisfying

$$\pi((1-\delta_k)a(1-\delta_k)) = \pi(E_k(a)(1-\delta_k)) \quad \text{for} \quad k \in \mathcal{I}, a \in A_k.$$

Then, $\mathcal{C}$ is the image of $(\bigstar_D A_k) \star c_0(\mathcal{I})^\sim$ under the representation $\bigoplus_{(\pi,\mathcal{H})\in\mathcal{F}} \pi$.

Theorem 3.3.5. *Let $(\pi_{\mathfrak{X}}, t_{\mathfrak{X}})$ be a universal covariant representation of $\mathfrak{X}$ so that $\mathrm{C}^*(\pi_{\mathfrak{X}}, t_{\mathfrak{X}}) = \mathcal{O}(\mathfrak{X})$. Then, there exist bijective $*$-isomorphisms $\rho\colon \mathcal{O}(\mathfrak{X}) \to \mathcal{C}$ and $\rho'\colon \mathcal{C} \to \Delta\mathbf{T}(A,E)$ such that $\rho(\pi_{\mathfrak{X}}(\Psi_k(a))) = \delta_k a \delta_k$ and $\rho(t_{\mathfrak{X}}(b\xi_{k\bar{k}})) = b(1-\delta_k)$, and $\rho'(a) = a$ and $\rho'(\delta_k) = P_k$ for $a \in A_k, b \in A_k^\circ$ and $k \in \mathcal{I}$.*

Proof. By Proposition 3.3.3 there exists a surjective $*$-homomorphism $\rho'\colon \mathcal{C} \to \Delta\mathbf{T}(A,E)$ such that $\rho'(a) = a$ and $\rho'(\delta_k) = P_k$ for $k \in \mathcal{I}$ and $a \in A_k$.

We next construct a covariant representation (π, t) of $\mathfrak{X}$ on $\mathcal{C}$ such that $(\rho' \circ \pi, \rho' \circ t)$ is universal. For each $k \in \mathcal{I}$, by the universality of $\langle\!\langle A_k, E_k\rangle\!\rangle$, there exists a surjective $*$-homomorphism $\rho_k\colon \langle\!\langle A_k, E_k\rangle\!\rangle \to \mathcal{C}$ sending e_k to $1-\delta_k$ and being identical on A_k. We observe that $e_{A_k} A_k (1-P_k) A_k e_{A_k} = \{0\}$, implying that $A_k \cap \overline{\mathrm{span}}\, A_k(1-P_k)A_k = \{0\}$. Thus, Lemma 3.2.1 implies that $\rho' \circ \rho_k\colon \langle\!\langle A_k, E_k\rangle\!\rangle \to \Delta\mathbf{T}(A,E)$ is injective. Therefore, $\pi = \bigoplus_{k\in\mathcal{I}} \rho_k|_{B_k}\colon B \to \mathcal{C}$ and $\rho' \circ \pi\colon B \to \Delta\mathbf{T}(A,E)$ are unital injective $*$-homomorphisms.

Let $t_k^\circ\colon X_k^\circ \to e_k^\perp \langle\!\langle A_k, E_k\rangle\!\rangle e_k$ be as in Lemma 3.2.4 and define $t\colon \mathfrak{X} \to \mathcal{C}$ by $t(\xi \otimes x) = \rho_k(t_k^\circ(\xi))\pi(x)$ for $\xi \in X_k^\circ$ and $x \in B_k^\perp$. For any $\xi \in X_k^\circ, \eta \in X_j^\circ, x \in B_k^\perp, y \in B_j^\perp$, by Lemma 3.2.4 and the fact that $\rho_k(e_k) = 1-\delta_k = 1-\pi(1_{B_k})$ we have

$$\begin{aligned} t(\xi\otimes\pi(x))^* t(\eta\otimes\pi(y)) &= \pi(x^*)\rho_k(t_k^\circ(\xi))^*\rho_j(t_j^\circ(\eta))\pi(y) \\ &= \delta_{k,j}\pi(x^*)\rho_k(t_k^\circ(\xi)^* t_k^\circ(\eta))\pi(y) \\ &= \delta_{k,j}\pi(x^*)\rho_k(\langle\xi,\eta\rangle e_k)\pi(y) \\ &= \delta_{k,j}\pi(x^*\langle\xi,\eta\rangle 1_{B_k}^\perp y) \\ &= \pi(\langle \xi\otimes\pi(x), \eta\otimes\pi(y)\rangle). \end{aligned}$$

Also, for any $z \in B_k$ we have

$$\begin{aligned} t(\phi_{\mathfrak{X}}(z)\xi\otimes x) &= \rho_k(t_k^\circ(\phi_{X_k^\circ}(z\xi)))\pi(x) = \rho_k(z t_k^\circ(\xi))\pi(x) \\ &= \pi(z)\rho_k(t_k^\circ(\xi))\pi(x) = \pi(z)t(\xi\otimes x). \end{aligned}$$

Therefore, (π, t) is a representation of $\mathfrak{X}$. We claim that (π, t) is covariant. By Lemma 3.3.4 it is sufficient to show that $\pi(K) = \psi_t \circ \phi_{\mathfrak{X}}(K)$ for all $K \in$

$\bigoplus_{k\in\mathcal{I}}\mathbb{K}(X_k^\circ)$. We may assume that $K = \theta_{\xi,\eta}$ for some $\xi,\eta \in X_k^\circ$ and $k \in \mathcal{I}$. Then it follows from Lemma 3.2.4 that

$$\pi(\theta_{\xi,\eta}) = \rho_k(\theta_{\xi,\eta}) = \rho_k(t_k^\circ(\xi)t_k^\circ(\eta)^*) = t(\xi\otimes 1_{B_k^\perp})t(\eta\otimes 1_{B_k^\perp})^* = \psi_t(\phi_{\mathfrak{X}}(\theta_{\xi,\eta})).$$

To see the universality, it suffices to show that $(\rho'\circ\pi, \rho'\circ t)$ admits a gauge action thanks to Theorem 2.4.1. For each $n \in \mathcal{I}$, let Q_n be the projection in $\mathbb{L}(Y)$ onto the closed submodule generated by all vectors of the form $a_1\cdots a_n\eta_k$ for some $k \in \mathcal{I}$ and some reduced word $a_1\cdots a_n$ with $a_n \notin A_k$, and set $Q_0 = \sum_{k\in\mathcal{I}} e_{A_k}$. Letting $U_z := \bigoplus_{n\geq 0} z^n Q_n$ for $z \in \mathbb{C}$ with $|z| = 1$ we have $\operatorname{Ad} U_z(\rho'\circ\pi(x)) = \rho'\circ\pi(x)$ for $x \in B$ and $\operatorname{Ad} U_z(\rho'\circ t(\xi)) = z\rho'\circ t(\xi)$ for $\xi \in \mathfrak{X}$. Finally, the surjectivity of ρ' follows from the decomposition $a = P_k a P_k + P_k a^\circ P_k^\perp + P_k^\perp a^\circ P_k + E(a)P_k^\perp$ for $a \in A_k$. □

We next show that the Toeplitz extension

$$0 \longrightarrow \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}) \longrightarrow \mathcal{T}(\mathfrak{X}) \longrightarrow \mathcal{O}(\mathfrak{X}) \longrightarrow 0$$

is semisplit. Let (π, t) be as in the proof of Theorem 3.3.5. We may identify $\Delta\mathbf{T}(A,E)$ with $\mathcal{C}$ so that $\Delta\mathbf{T}(A,E) = \mathrm{C}^*(\pi,t)$. Let $\sigma_k\colon \Delta\mathbf{T}(A,E) \to \mathbb{L}(X^{(k)})$ be as in Eq. (3.5) and set $(X^{\mathcal{I}},\sigma) := \boxplus_{k\in\mathcal{I}}(X^{(k)},\sigma_k)$. We denote the GNS vector in $X^{(k)}$ by $\xi_0^{(k)}$. We fix a fixed-point free bijection τ on $\mathcal{I}$. To simplify the notation, we will write $\tau(k) = k+1$ for $k \in \mathcal{I}$. Let $Q \in \mathbb{L}(X^{\mathcal{I}})$ be the projection onto $\boxplus_{k\in\mathcal{I}} P_{(r,k)}^\perp X^{(k+1)}$. Note that $P_{(r,k)}^\perp X^{(k+1)} \cong X(r,k)\otimes_D X_k^\circ$ contains a copy of X_k°, denoted by $X_k^{\circ(k+1)}$. Since $QX^{\mathcal{I}}$ is invariant under $\sigma\circ\pi(B)$ and $\sigma\circ t(\mathfrak{X})$, the pair $(\pi',t') := (\sigma\circ\pi(\cdot)Q, \sigma\circ t(\cdot)Q)$ is a representation of $\mathfrak{X}$ on $\mathbb{L}(QX^{\mathcal{I}})$. Notice that Q does not commute with $\sigma\circ t(\mathfrak{X})$.

Proposition 3.3.6. *The representation $(\Pi,T) := (\pi\oplus\pi', t\oplus t')$ of $\mathfrak{X}$ is universal.*

Proof. Since (π,t) is injective, so is (Π,T). Consider the unitary representation $U'\colon \mathbb{T} \curvearrowright X^{\mathcal{I}}$ such that U'_z acts on the space of reduced words of length n by z^n. Then, $\{U'_z\}_z$ commute with Q and $\operatorname{Ad} U'_z(\cdot)Q$ defines a gauge action for (π',t'). Thus, we only have to check that $\Pi(J_{\mathfrak{X}}) \cap \psi_T(\mathbb{K}(\mathfrak{X})) = \{0\}$ by Theorem 2.4.1. Assume that $x \in \mathbb{K}(X_k^\circ)$ satisfies $\Pi(x) \in \psi_T(\mathbb{K}(\mathfrak{X}))$. Observe that $t'(\xi)t'(\eta)^* = \bigoplus_{j\in\mathcal{I}}\sigma_{j+1}(t(\xi))P_{(r,j)}^\perp\sigma_{j+1}(t(\eta))^*$ vanishes on $\boxplus_{j\in\mathcal{I}} X_j^{\circ(j+1)}$ for all $\xi,\eta \in \mathfrak{X}$, and hence so does $\pi'(x)$. On the other hand, the restriction of $\pi'(x)$ to $X_k^{\circ(k+1)}$ is unitarily equivalent to x itself on X_k°. Thus, x must be zero. □

Since (Π,T) is universal, there is a surjective $*$-homomorphism $p\colon \mathrm{C}^*(\Pi,T) \to \mathrm{C}^*(\pi,t)$ such that $p\circ\Pi = \pi$ and $p\circ T = t$. Note that the kernel of p is generated by $\{\Pi(\theta_{a\xi_k,b\xi_k}) - \psi_T(\phi_{\mathfrak{X}}(\theta_{a\xi_k,b\xi_k})) \mid k \in \mathcal{I}, a,b \in A_k^\circ\}$ by Lemma 3.3.4. We denote by θ the compression map $\mathbb{L}(X^{\mathcal{I}}) \to \mathbb{L}(QX^{\mathcal{I}})$ by Q.

Theorem 3.3.7. *The UCP map* $\Theta := \mathrm{id} \oplus (\theta \circ \sigma)\colon \Delta\mathbf{T}(A,E) \to \Delta\mathbf{T}(A,E) \oplus \mathbb{L}(QX^{\mathcal{I}})$ *maps into* $\mathrm{C}^*(\Pi, T)$ *and satisfies that* $p \circ \Theta = \mathrm{id}$.

Proof. Since DP_k sits in the multiplicative domain of Θ and $\Theta(c) = \Pi(\Psi_k(c)) + T(c^\circ \xi_{k\bar{k}}) + T(c^{*\circ}\xi_{k\bar{k}})^* + \Pi(\Psi(E(c))1_{B_k}^\perp) \in \mathrm{C}^*(\Pi, T)$ for $c \in A_k$ and $k \in \mathcal{I}$, it suffices to show that $\Theta(ab^*) - \Theta(a)\Theta(b^*) \in \ker p$ for $a, b \in A_k^\circ$ and $k \in \mathcal{I}$ and $\Theta(a_1 a_2 \cdots a_{n+1}) = \Theta(a_1 a_2 \cdots a_n)\Theta(a_{n+1})$ for all reduced words $a_1 \cdots a_n$ with $n \geq 1$. Indeed, it follows from the above decomposition of $\Theta(c)$ that

$$\begin{aligned}\Theta(ab^*) - \Theta(a)\Theta(b^*) &= \Pi(\Psi_k(ab^*)) - \Pi(\Psi_k(a)\Psi_k(b^*)) - T(a\xi_{k\bar{k}})T(b\xi_{k\bar{k}})^* \\ &= \Pi(\theta_{a\xi_k, b\xi_k}) - \psi_T(\phi_{\mathfrak{X}}(\theta_{a\xi_k, b\xi_k})),\end{aligned}$$

which belongs to $\ker p$. We next show the multiplicativity on reduced words. Take $k \in \mathcal{I}$ and a reduced word $a_1 \cdots a_n$ with $a_n \in A_k^\circ$ arbitrarily. Since $\pi(\Psi_k(A_k))$ and $t(\mathfrak{X})$ sit in the right multiplicative domain of Θ, we have

$$\begin{aligned}&\Theta(a_1 \cdots a_n) - \Theta(a_1 \cdots a_{n-1})\Theta(a_n) \\ &\quad = \Theta(a_1 \cdots a_{n-1} t(a_n^* \xi_{k\bar{k}})^*) - \Theta(a_1 \cdots a_{n-1}) T(a_n^* \xi_{k\bar{k}})^* \\ &\quad = 0 \oplus Q\sigma(a_1 \cdots a_{n-1})(1-Q)\sigma \circ t(a_n^* \xi_{k\bar{k}})^* Q.\end{aligned}$$

Note that $(1-Q)\sigma \circ t(a_n^*\xi_{k\bar{k}})^* Q$ is supported on $X_k^{(k+1)\circ}$. For any $x \in A_k^\circ$, since $\sigma = \phi_X$ on A, we have $Q\sigma(a_1 \cdots a_{n-1})(1-Q)\sigma \circ t(a_n^*\xi_{k\bar{k}})^* Q x \xi_0^{(k+1)} = Q a_1 \cdots a_{n-1} \xi_0^{(k+1)} E(a_n x) = 0$. □

Remark 3.3.8. The above proof shows that $\mathrm{C}^*(\Theta(A_k))$ is a semisplit extension of A_k by $\mathbb{K}(X_k^\circ)$. The restriction of $E_{\mathfrak{X}}(\cdot)1_{B_k}\colon \mathrm{C}^*(\Pi, T) \cong \mathcal{T}(\mathfrak{X}) \to B_k$ to $\mathrm{C}^*(\Theta(A_k))$ is a $*$-isomorphism onto B_k such that $E_{\mathfrak{X}}(\Theta(a))1_{B_k} = \Psi_k(a)$ for $a \in A_k$.

3.4 Consequences

We note that the definition of the universal C*-algebra $\mathcal{C}$ does not involve the reduced amalgamated free product. Thus, the isomorphism between $\mathcal{C}$ and $\Delta\mathbf{T}(A,E)$ implies the following:

Corollary 3.4.1. *Let* $(A, E) = \bigstar_D (A_k, E_k)$ *be any reduced amalgamated free product of* $\{(D \subset A_k, E_k)\}_{k \in \mathcal{I}}$ *and* $\bigstar_D A_k$ *be the corresponding full one. For any unital* C*-*algebra* $\mathcal{B}$ *and any unital* $*$-*homomorphism* $\phi\colon \bigstar_D A_k \to \mathcal{B}$, *the following hold true:*

(i) *Assume that* $\mathcal{I}$ *is infinite and there exists a family of mutually orthogonal projections* $\{p_k\}_{k \in \mathcal{I}}$ *in* $\mathcal{B}$ *such that* $(1-p_k)\phi(a)(1-p_k) = \phi(E_k(a))(1-p_k)$ *for* $a \in A_k$ *and* $k \in \mathcal{I}$, *then* ϕ *factors through the canonical surjection* $\bigstar_D A_k \to A$.

(ii) *Assume that $\mathcal{I}$ is finite and there exists a family of mutually orthogonal projections $\{p_k\}_{k\in\mathcal{I}}$ in $\mathcal{B}$ such that $\sum_{k\in\mathcal{I}} p_k = 1$ and $(1-p_k)\phi(a)(1-p_k) = \phi(E_k(a))(1-p_k)$ for $a \in A_k$ and $k \in \mathcal{I}$, then ϕ factors through the canonical surjection $\bigstar_D A_k \to A$.*

When $\mathcal{I} = \{1, 2\}$, we have $P_1 = 1 - P_2$. We next show that, in this case $\Delta\mathbf{T}(A, E)$ also has a full/reduced amalgamated free product structure. Define the conditional expectation $\mathcal{E}_k\colon \langle\langle A_k, E_k\rangle\rangle \to D(1-e_k) \oplus De_k$ by $\mathcal{E}_k(x) = E_k(\pi_k(a))(1-e_k) + e_k x e_k$, where π_k denotes the quotient map $\langle\langle A_k, E_k\rangle\rangle \to A_k$.

Corollary 3.4.2. *Let $(A, E) = (A_1, E_1) \star_D (A_2, E_2)$ be a reduced amalgamated free product and $\mathcal{E}_k\colon \langle\langle A_k, E_k\rangle\rangle \to D(1-e_k) \oplus De_k$ be as above. Then, $\Delta\mathbf{T}(A, E)$ is isomorphic to the reduced amalgamated free product of $(\langle\langle A_1, E_1\rangle\rangle, \mathcal{E}_1)$ and $(\langle\langle A_2, E_2\rangle\rangle, \mathcal{E}_2)$ over $D\oplus D$, where e_1 is identified with $1-e_2$. Moreover, the canonical surjection from the full amalgamated free product of $\langle\langle A_1, E_1\rangle\rangle \star_{D\oplus D} \langle\langle A_2, E_2\rangle\rangle$ onto $\Delta\mathbf{T}(A, E)$ is a $*$-isomorphism.*

Proof. By the previous corollary, $\langle\langle A_1, E_1\rangle\rangle \star_{D\oplus D} \langle\langle A_2, E_2\rangle\rangle$ is isomorphic to $\Delta\mathbf{T}(A, E)$ naturally. Thus it suffices to show that this isomorphism factors through the reduced amalgamated free product. We may identify $\langle\langle A_k, E_k\rangle\rangle$ with $\mathrm{C}^*(A_k, 1-P_k)$. Let $\sigma_k\colon \Delta\mathbf{T}(A, E) \to \mathbb{L}(X^{(k)})$ be as in Eq. (3.5) and set $\theta := \mathrm{id} \oplus \sigma_1 \oplus \sigma_2\colon \Delta\mathbf{T}(A, E) \to \mathbb{L}(Y \boxplus X^{(1)} \boxplus X^{(2)})$. Define the projection $f_1 \in \mathbb{L}(Y \boxplus X^{(1)} \boxplus X^{(2)})$ by $P_2 \oplus (e_D + P_{(\ell,2)}^\perp) \oplus P_{(\ell,2)}^\perp$ and put $f_2 = 1 - f_1$. We prove that $f_1\theta(x)f_1 = \theta(\mathcal{E}_1(x))f_1$ for $x \in \langle\langle A_1, E_1\rangle\rangle$. For any $x \in \langle\langle A_1, E_1\rangle\rangle$, it is clear that $P_2 x P_2 = \mathcal{E}_1(x)P_2$. By Eq. (3.5), we have

$$
\begin{aligned}
&(e_D + P_{(\ell,2)}^\perp)\sigma_1(x)(e_D + P_{(\ell,2)}^\perp)\\
&= e_D\sigma_1(x)e_D + e_D\sigma_1(x)P_{(\ell,2)}^\perp + P_{(\ell,2)}^\perp\sigma_1(x)e_D + P_{(\ell,2)}^\perp\sigma_1(x)P_{(\ell,2)}^\perp\\
&= e_D\sigma_1(P_1xP_1)e_D + e_D\sigma_1(P_1xP_2) + \sigma_1(P_2xP_1)e_D + \sigma_1(P_2xP_2)\\
&= \sigma_1(\mathcal{E}_1(x))e_D + \sigma_1(\mathcal{E}_1(x))P_{(\ell,2)}^\perp\\
&= \sigma_1(\mathcal{E}_1(x))(e_D + P_{(\ell,2)}^\perp).
\end{aligned}
$$

Here in the third equality we used that $e_D\sigma_1(P_1xP_2) = \sigma_1(P_2xP_1)e_D = 0$. This follows from the fact that $\sigma_1(P_1xP_2)$ is a norm limit of "creation operators" $P_{(\ell,1)}^\perp a_n P_{(\ell,2)}^\perp$ with $a_n \in A_1^\circ$. Similarly, one has $P_{(\ell,2)}^\perp\sigma_2(x)P_{(\ell,2)}^\perp = \sigma_2(\mathcal{E}_1(x))P_{(\ell,2)}^\perp$. Combing these we obtain that $f_1\theta(x)f_1 = \theta(\mathcal{E}_1(x))f_1$. By the same argument, we have $f_2\theta(y)f_2 = \theta(\mathcal{E}_2(y))f_2$ for $y \in \langle\langle A_2, E_2\rangle\rangle$. Thus, the previous corollary implies that θ factors through the reduced amalgamated free product. Since θ is injective, we are done. □

Our next goal is to characterize when $\Delta\mathbf{T}(A, E)$ is nuclear or exact. We say that a linear map from the algebraic tensor product of two C*-algebras into a

C*-algebra is *min-bounded* if it is bounded with respect to the minimal tensor norm.

Theorem 3.4.3. *Let $(A, E) = \bigstar_D(A_k, E_k)$ be a reduced amalgamated free product, P, Q be any unital* C*-*algebras, and $\varphi\colon \Delta\mathbf{T}(A, E) \to P$ be any unital $*$-homomorphism. If $(\varphi|_{A_k} \otimes \mathrm{id})\colon A_k \odot Q \to P \otimes_{\max} Q$ is min-bounded for each $k \in \mathcal{I}$, then $\varphi \otimes \mathrm{id}\colon \Delta\mathbf{T}(A, E) \odot Q \to P \otimes_{\max} Q$ is min-bounded.*

Proof. Thanks to Proposition 3.3.3 (iv) it suffices to construct a suitable $*$-homomorphism from $\Delta\mathbf{T}(A \otimes Q, E \otimes \mathrm{id})$ to $P \otimes_{\max} Q$. For each $k \in \mathcal{I}$, let $\psi_k\colon A_k \otimes Q \to P \otimes_{\max} Q$ be the bounded extension of $(\varphi|_{A_k}) \otimes \mathrm{id}$. Then, we have $(\varphi(1 - P_k) \otimes 1)\psi_k(x)(\varphi(1 - P_k) \otimes 1) = \psi_k(E_k \otimes \mathrm{id}(x))$ for $k \in \mathcal{I}$. Thus, thanks to Corollary 3.4.1 ψ_k's extend to a $*$-homomorphism from $\Delta\mathbf{T}(A \otimes Q, E \otimes \mathrm{id})$ to $P \otimes_{\max} Q$. ☐

Corollary 3.4.4. *Let $(\star)$ be one of the following properties: nuclearity, exactness, WEP, and LLP. If A_k has the property $(\star)$ for every $k \in \mathcal{I}$, then so does $\Delta\mathbf{T}(A, E)$.*

Proof. For the nuclearity, set $P = \Delta\mathbf{T}(A, E)$ and $\phi = \mathrm{id}$. If A_k is nuclear for every $k \in \mathcal{I}$, then the embedding $A_k \odot Q \to \Delta\mathbf{T}(A, E) \otimes_{\max} Q$ is min-bounded for any C*-algebra Q. The previous theorem then implies that $\Delta\mathbf{T}(A, E) \otimes Q = \Delta\mathbf{T}(A, E) \otimes_{\max} Q$. Since Q is arbitrary, $\Delta\mathbf{T}(A, E)$ is nuclear.

The assertion for the exactness can be shown in the same manner by using some faithful representation $\varphi\colon \Delta\mathbf{T}(A, E) \to P := \mathbb{B}(\mathcal{H})$. By Kirchberg's result [32] (see also [43]), $\Delta\mathbf{T}(A, E)$ has WEP (resp. LLP) if and only if $\Delta\mathbf{T}(A, E) \otimes \mathrm{C}^*(\mathbb{F}_\infty) = \Delta\mathbf{T}(A, E) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$ (resp. $\Delta\mathbf{T}(A, E) \otimes \mathbb{B}(\ell^2) = \Delta\mathbf{T}(A, E) \otimes_{\max} \mathbb{B}(\ell^2)$). Thus, the proof for nuclearity works as well. ☐

Since exactness passes to subalgebras, the next result due to Dykema [14] (see [16, 44] for alternative proofs) is an immediate consequence of Corollary 3.4.4. Note that our proof also says that any reduced amalgamated free product of nuclear C*-algebras is a subalgebra of a nuclear C*-algebra. Also, our result does not relay on the facts that exactness and nuclearity pass to quotients [8, 33, 34].

Corollary 3.4.5 (Dykema [14])**.** *Reduced amalgamated free products of exact* C*-*algebras are exact.*

Recall that a C*-algebra A is said to have the *completely bounded approximation property* (*CBAP*) if there exist a constant $C > 0$ and a net of finite rank CB maps φ_i on A such that $\|\varphi_i\|_{\mathrm{cb}} \le C$ and $\lim_i \|\varphi_i(x) - x\| = 0$ for $x \in A$. The *Haagerup constant* $\Lambda_{\mathrm{cb}}(A)$ is the infimum of those C for which $(\varphi_i)_i$ exists. When A does not have the CBAP, we set $\Lambda_{\mathrm{cb}}(A) = \infty$.

Lemma 3.4.6. *Let $D \subset A$ be a unital inclusion with conditional expectation $E\colon A \to D$ and (X, ϕ_X, ξ_0) be the associated GNS-representation. Consider the embedding maps $\mathbb{K}(X^\circ) \hookrightarrow \mathbb{K}(X)$ and $D \hookrightarrow De_D \subset \mathbb{K}(X)$ into corners. Then, we have $\Lambda_{\mathrm{cb}}(\mathbb{K}(X^\circ)) \leq \Lambda_{\mathrm{cb}}(D)$.*

Proof. Since $\mathbb{K}(X^\circ)$ is a hereditary subalgebra of $\mathbb{K}(X)$, we have $\Lambda_{\mathrm{cb}}(\mathbb{K}(X^\circ)) \leq \Lambda_{\mathrm{cb}}(\mathbb{K}(X))$. Take an approximate unit $(a_i \otimes p_i)$ of $\mathbb{K}(X) \otimes \mathbb{K}$. It suffices to show that for any i and any $\varepsilon > 0$, there exist CP contractions $\varphi_i\colon \mathbb{K}(X) \otimes \mathbb{K} \to D \otimes \mathbb{K}$ and $\psi_i\colon D \otimes \mathbb{K} \to \mathbb{K}(X) \otimes \mathbb{K}$ such that $\|(a_i \otimes p_i)x(a_i \otimes p_i) - \psi_i \circ \varphi_i(x)\| < \varepsilon\|x\|$ for $x \in \mathbb{K}(X) \otimes \mathbb{K}$. For each i, by [31, Lemma B.2] we find a separable closed subspace $X_i \subset X$ with $\xi_0 \in X_i$ which naturally forms a Hilbert C*-module over a separable C*-subalgebra $D_i \subset D$ such that $a_i \in \mathbb{K}(X_i)$. Since $D_i e_D \otimes \mathbb{K}$ is a full corner of $\mathbb{K}(X_i) \otimes \mathbb{K}$, it follows from [45, Lemma 2.5] that there exists $d_i \in \mathbb{K}(X_i) \otimes \mathbb{K}$ such that $\|d_i^* d_i - a_i \otimes p_i\| < \varepsilon$ and $d_i d_i^* \in De_D \otimes \mathbb{K}$. Then, CP contractions $\varphi_i(x) = d_i x d_i^*$ and $\psi_i(x) = d_i^* x d_i$ are the desired ones. □

Corollary 3.4.7. *It follows that $\Lambda_{\mathrm{cb}}(\Delta\mathbf{T}(A, E)) = \sup_k\{\Lambda_{\mathrm{cb}}(A_k) \mid k \in \mathcal{I}\}$.*

Proof. Since D is the range of the conditional expectation E_1, we have $\Lambda_{\mathrm{cb}}(D) \leq \Lambda_{\mathrm{cb}}(A_k)$. By [17] and Lemma 3.4.6, we obtain $\Lambda_{\mathrm{cb}}(B_k) = \Lambda_{\mathrm{cb}}(A_k)$. When $\mathcal{I}$ is finite, $B = \bigoplus_{k\in\mathcal{I}} B_k$; otherwise B is a split extension of D by $\bigoplus_k B_k$. Therefore, $\Lambda_{\mathrm{cb}}(\mathcal{T}(\mathfrak{X})) = \sup_k \Lambda_{\mathrm{cb}}(A_k)$ by [17] again. Since $\Delta\mathbf{T}(A, E) \cong \mathcal{O}(\mathfrak{X})$ is a quotient of $\mathcal{T}(\mathfrak{X})$ with UCP cross section, we obtain $\Lambda_{\mathrm{cb}}(\Delta\mathbf{T}(A, E)) \leq \sup_k \Lambda_{\mathrm{cb}}(A_k)$. The opposite implication follows from Proposition 3.3.3 (iii). □

The following generalizes Ozawa's result [38] for nuclearity.

Theorem 3.4.8. *Let $(A, E) = (A_1, E_1) \star_D (A_2, E_2)$ be a reduced amalgamated free product and assume that E_1 and E_2 are nondegenerate and the image of the GNS representation of E_1 contains the Jones projection. Then, the following hold true:*

(i) Let $(\star)$ be as in Corollary 3.4.4. Then, A has the property $(\star)$ if and only if so do both A_1 and A_2.

(ii) One has $\Lambda_{\mathrm{cb}}(A) = \max\{\Lambda_{\mathrm{cb}}(A_1), \Lambda_{\mathrm{cb}}(A_2)\}$.

Proof. Let $p \in A_1$ be a projection such that $\phi_{X_1}(p)$ is the Jones projection of E_1. A direct computation shows that $p = pe_{A_1} + P_2$ in $\Delta\mathbf{T}(A, E)$. Set $I_1 := \phi_{X_1}^{-1}(\mathbb{K}(X_1))$. Then, $\mathbb{K}(Y_1 I_1)$ is the ideal of $\Delta\mathbf{T}(A, E)$ generated by $I_1 e_{A_1}$. Since E_1 and E_2 are nondegenerate, the restriction of $\sigma_2\colon \Delta\mathbf{T}(A, E) \to X^{(2)}$ to A is faithful. Thus, we have $\mathbb{K}(Y_1 I_1) \cap A = \{0\}$ because $I_1 e_{A_1} \subset \ker \sigma_2$ (see Remark 3.3.2). Since $P_2 \in A + \mathbb{K}(Y_1 I_1)$, we have $\Delta\mathbf{T}(A, E) = A + \mathbb{K}(Y_1 I_1)$ and the split exact sequence

$$0 \longrightarrow \mathbb{K}(Y_1 I_1) \longrightarrow A + \mathbb{K}(Y_1 I_1) \longrightarrow A \longrightarrow 0.$$

Thus, the assertion follows from Corollary 3.4.4 and Corollary 3.4.7. $\square$

Recall that the *cb distance* of two finite dimensional operator spaces E and F is defined by

$$d_{\mathrm{cb}}(E,F) = \inf\{\|\varphi\|_{\mathrm{cb}}\|\varphi^{-1}\|_{\mathrm{cb}} \mid \varphi\colon E \to F \text{ linear bijection}\}.$$

When E and F are not isomorphic, we set $d_{\mathrm{cb}}(E,F) = \infty$. For any finite dimensional operator space E, Junge and Pisier introduced in [30] the following quantity

$$d_f(E) := \inf\{d_{\mathrm{cb}}(E,F) \mid F \subset \mathrm{C}^*(\mathbb{F}_\infty)\}.$$

If A is a C*-algebra, we define $d_f(A)$ by the supremum of $d_f(E)$ taken over all finite dimensional operator subspaces E of A.

Theorem 3.4.9. *For any reduced amalgamated free product $(A,E) = \bigstar_D(A_k,E_k)$, if $d_f(A_k) = 1$ holds for all $k \in \mathcal{I}$, then we have $d_f(\Delta\mathbf{T}(A,E)) = d_f(A) = 1$.*

Proof. Let $F \subset \Delta\mathbf{T}(A,E)$ be any finite dimensional operator subspace. Take a unital faithful representation $\Delta\mathbf{T}(A,E) \subset \mathbb{B}(\mathcal{H})$. Then, it was shown by Junge and Pisier [30] that

$$d_f(F) = \sup\left\{\frac{\|x\|_{\mathbb{B}(\mathcal{H})\otimes_{\max}\mathbb{B}(\ell^2)}}{\|x\|_{\min}} \mid x \in F \odot \mathbb{B}(\ell^2)\right\}.$$

Thus, it suffices to show that for any unital $*$-homomorphism $\pi \times \rho\colon \mathbb{B}(\mathcal{H}) \odot \mathbb{B}(\ell^2) \to \mathbb{B}(\mathcal{K})$, the restriction $\pi|_{\Delta\mathbf{T}(A,E)} \times \rho\colon \Delta\mathbf{T}(A,E) \odot \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ is min-bounded. Since $d_f(A_k) = 1$ holds, $\pi|_{A_k} \times \rho\colon A_k \odot \mathbb{B}(\ell^2) \to \mathbb{B}(\mathcal{K})$ is min-bounded for each $k \in \mathcal{I}$. Denote its bounded extension by $\tilde{\pi}_k$. Then $\tilde{\pi}_k$'s induce a $*$-representation of the full amalgamated free product $\bigstar_{D\otimes\mathbb{B}(\mathcal{H})}(A_k \otimes \mathbb{B}(\mathcal{H}))$, and the projections $\pi(P_k)$'s satisfy the assumption of Corollary 3.4.1. Hence, the assertion follows from the isomorphism $\Delta\mathbf{T}(A,E)\otimes\mathbb{B}(\mathcal{H}) \cong \Delta\mathbf{T}(A\otimes\mathbb{B}(\mathcal{H}), E\otimes \mathrm{id})$. $\square$

Theorem 3.4.10 (Blanchard–Dykema [3])**.** *Let $(A,E) = \bigstar_D(A_k,E_k)$ and $(\mathcal{A},\mathcal{E}) = \bigstar_D(\mathcal{A}_k,\mathcal{E}_k)$ be reduced amalgamated free products.*

(i) For any unital $*$-homomorphisms $\pi_k\colon A_k \to \mathcal{A}_k$, $k \in \mathcal{I}$ such that $\mathcal{E}_k \circ \pi_k = \pi_k \circ E_k$ there exists a unique $*$-homomorphism $\pi\colon A \to \mathcal{A}$ of which restriction to A_k equals π_k for every $k \in \mathcal{I}$. Moreover, if π_k is injective for every $k \in \mathcal{I}$, then so is π.

(ii) Assume that $D = \mathcal{D}$ and E_k and $\mathcal{E}_k$ are nondegenerate for $k \in \mathcal{I}$. For any UCP maps $\varphi_k\colon A_k \to \mathcal{A}_k$ such that $\varphi_k|_D = \mathrm{id}$, there exists a unique UCP map $\varphi\colon A \to \mathcal{A}$ such that

$$\varphi(a_1a_2\cdots a_n) = \varphi_{\iota(1)}(a_1)\varphi_{\iota(2)}(a_2)\cdots\varphi_{\iota(n)}(a_n)$$

for any reduced word $a_1\cdots a_n$ with $a_j \in A^\circ_{\iota(j)}$ and $\iota \in \mathcal{I}_n$.

Proof. We prove (i): By Corollary 3.4.1 the $*$-homomorphisms $\pi_k, k \in \mathcal{I}$ induce $\rho\colon \Delta\mathbf{T}(A, E) \to \Delta\mathbf{T}(\mathcal{A}, \mathcal{E})$. The covariant representation of $\mathfrak{X}$ corresponding to ρ admits a gauge action since $\Delta\mathbf{T}(\mathcal{A}, \mathcal{E})$ admits a gauge action, and is injective whenever π_k is injective for $k \in \mathcal{I}$. The restriction of ρ to A is the desired one.

We prove (ii): The argument here is essentially same as the proof of [7, Proposition 2.1]. Thus, we give only a sketch of the proof. Let $(\mathcal{X}_k, \phi_{\mathcal{X}_k}, \xi'_k)$ and $(\mathcal{X}, \phi_{\mathcal{X}}, \xi'_0)$ be the GNS representation of $\mathcal{E}_k$ and $\mathcal{E}$, respectively. For each $k \in \mathcal{I}$, by the Stinespring construction, there exists an A_k-D C*-correspondence (Z_k, π_k) and an isometry $w_k\colon \mathcal{X}_k \to Z_k$ such that $w_k^* \pi_k(\phi_{X_k}(a)) w_k = \phi_{\mathcal{X}_k}(\varphi_k(a))$ for $a \in A_k$. Let $E'_k\colon \mathbb{L}(Z_k) \to D$ be the conditional expectation given by the compression onto $w_k \xi'_k$. Consider the reduce amalgamated free product $(\mathcal{L}, E') = \bigstar_D(\mathbb{L}(Z_k), E'_k)$ and denote the GNS Hilbert C*-module associated with E' by X'. Then, there exists an isometry $w\colon \mathcal{X} \to X'$ such that

$$w(\zeta_1 \otimes \cdots \otimes \zeta_n) = w_{\iota(1)}\zeta_1 \otimes \cdots \otimes w_{\iota(n)}\zeta_n \quad \text{for} \quad n \geq 1, \iota \in \mathcal{I}_n, \zeta_k \in X^\circ_{\iota(k)} \ (1 \leq k \leq n).$$

By (i) there exists a $*$-homomorphism $\pi\colon A \to \mathcal{L}$ induced from $\pi_k, k \in \mathcal{I}$. Then, a direct computation shows that the UCP $\varphi\colon A \ni a \mapsto w^* \pi(a) w \in \mathbb{L}(\mathcal{X})$ is the desired one. $\square$

4 KK-theory of amalgamated free products

4.1 KK-equivalences

Theorem 4.1.1. *Let $\{(D \subset A_k, E_k)\}_{k\in\mathcal{I}}$ be any countable family of unital inclusions of separable* C*-*algebras with conditional expectations and $(A, E) = \bigstar_D(A_k, E_k)$ and $\mathfrak{A} := \bigstar_D A_k$ be the reduced and full amalgamated free products. Then, the canonical surjection $\lambda\colon \mathfrak{A} \to A$ is a KK-equivalence.*

We first deal with the case when $\mathcal{I} = \{1, 2\}$. Consider two A-$\mathfrak{A}$ C*-correspondences $(Z^0, \pi^0) := \bigoplus_{i=1}^2 (Y_i \otimes_{A_i} \mathfrak{A}, \phi_{Y_i} \otimes 1)$ and $(Z^1, \pi^1) := (X \otimes_D \mathfrak{A}, \phi_X \otimes 1)$. Let $S_k\colon X(r,k) \otimes_D A_k \to Y_k$ be as in Lemma 2.3.4 and define the isometry $S : Z^1 \to Z^0$ by

$$\begin{cases} S_1 \otimes 1 : X(r,1) \otimes_D \mathfrak{A} \to Y_1 \otimes_{A_1} \mathfrak{A}; \\ S_2 \otimes 1 : X(r,2)^\circ \otimes_D \mathfrak{A} \to Y_2^\circ \otimes_{A_2} \mathfrak{A}. \end{cases} \tag{4.1}$$

We set $\widetilde{\eta}_k := \eta_k \otimes 1_{\mathfrak{A}} \in Y_k \otimes_{A_k} \mathfrak{A} \subset Z^0$ for $k = 1, 2$.

Lemma 4.1.2 (cf. [52, Theorem 3.3 (2)])**.** *The operator S satisfies that* $\ker S^* = \widetilde{\eta}_1\mathfrak{A}$ *and $\pi^0(a)S - S\pi^1(a)$ is compact for all $a \in A$. Consequently, the triplet $(Z^0 \oplus Z^1, \pi^0 \oplus \pi^1, [\begin{smallmatrix} 0 & S^* \\ S & 0 \end{smallmatrix}])$ is a Kasparov A-$\mathfrak{A}$ bimodule.*

Proof. The first assertion is obvious. Thus, it suffices to show $\pi^0(x)S - S\pi^1(x)$ is compact for all $x \in A_1 \cup A_2$. In fact, since each $x \in A_2$ enjoys $xX(r,1) \subset X(r,1)$ and $xX(r,2)^\circ \subset X(r,2)^\circ$, one has $\pi^0(x)S = S\pi^1(x)$ for $x \in A_2$. If we define $S' : Z^0 \to Z^1$ by $S'\xi_0 \otimes a = \widetilde{\eta}_2 a$ for $a \in \mathfrak{A}$ and by S on $X^\circ \otimes_D \mathfrak{A}$, then S' intertwines the actions of A_1 by the above argument. Since S is a compact perturbation of S', we are done. $\square$

Remark 4.1.3. The construction of the above Kasparov bimodule is based on Julg and Valette's work [29] and its quantum analogue by Vergnioux [52]. Let us explain a "geometric" meaning of the operator S.

Let Γ be a locally compact second countable group action on a tree $\mathbf{T}$ and fix a base point $x_0 \in \mathbf{V}$. For any $x \in \mathbf{V} \setminus \{x_0\}$ we denote by $\beta(x)$ the unique edge in $\mathbf{E}$ relating x and some vertex in $[x_0, x]$. We define the co-isometry $V\colon \ell^2(\mathbf{V}) \to \ell^2(\mathbf{E})$ by $V\delta_{x_0} = 0$ and $V\delta_x = \delta_{\beta(x)}$ for $x \neq x_0$. Julg and Valette proved in [29] that the triplet $(\ell^2(\mathbf{V}), \ell^2(\mathbf{E}), V)$ implements the K-homology class of the trivial character. Now assume that Γ is discrete and all the stabilizer subgroups of the action $\Gamma \curvearrowright \mathbf{T}$ are amenable. Since the unitary representations of Γ on $\ell^2(\mathbf{V})$ and $\ell^2(\mathbf{E})$ are weakly contained in the regular representation, the above triplet defines an element γ in $K^1(\mathrm{C}^*_{\mathrm{red}}(\Gamma)) = KK(\mathrm{C}^*_{\mathrm{red}}(\Gamma), \mathbb{C})$. Let $d\colon \mathrm{C}^*_{\mathrm{red}}(\Gamma) \to \mathrm{C}^*_{\mathrm{red}}(\Gamma) \otimes \mathrm{C}^*(\Gamma)$ be the $*$-homomorphism given by $d(\lambda(g)) = \lambda(g) \otimes u(g)$, where u denotes the universal representation of Γ. Then, the Kasparov product of d and

$\gamma \otimes \mathrm{id} \in KK(\mathrm{C}^*_{\mathrm{red}}(\Gamma) \otimes \mathrm{C}^*(\Gamma), \mathrm{C}^*(\Gamma))$ gives the inverse of the canonical quotient $\mathrm{C}^*(\Gamma) \to \mathrm{C}^*_{\mathrm{red}}(\Gamma)$ (see [10] for details).

If Γ is an amalgamated free product of amenable groups Γ_1 and Γ_2 over Λ and the base point x_0 is $e\Gamma_1$, then one can check that the Kasparov bimodule representing this inverse element coincides with that we constructed in the previous lemma.

Theorem 4.1.4. *With the notation above, let α be the element in $KK(A, \mathfrak{A})$ implemented by $(Z^0 \oplus Z^1, \pi^0 \oplus \pi^1, [\begin{smallmatrix} 0 & S^* \\ S & 0 \end{smallmatrix}])$. Then, we have $\lambda \otimes_A \alpha = \mathrm{id}_{\mathfrak{A}}$ and $\alpha \otimes_{\mathfrak{A}} \lambda = \mathrm{id}_A$.*

Proof. We first prove that $\lambda \otimes_A \alpha = \mathrm{id}_{\mathfrak{A}}$ following the proof of [52, Theorem 3.3 (3)]. Set $\rho^0 := \pi^0 \circ \lambda$ and $\rho^1 := \pi^1 \circ \lambda$. Define the unitary $U : Z^1 \oplus \mathfrak{A} \to Z^0$ by S on Z^1 and by $U(0 \oplus a) := \widetilde{\eta}_2 a$ for $a \in \mathfrak{A}$. Since S is a compact perturbation of U, $\lambda \otimes_A \alpha - \mathrm{id}_{\mathfrak{A}}$ is implemented by

$$(Z^0 \oplus (Z^1 \oplus \mathfrak{A}), \rho^0 \oplus (\rho^1 \oplus L_{\mathfrak{A}}), [\begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix}])$$

(see §§2.5). Take a norm continuous path $(v_t)_{0 \le t \le 1}$ of unitaries in $\mathbb{M}_2(\mathbb{C})$ such that $v_0 = 1$ and $v_1 = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$. With the natural identification $\mathbb{M}_2(\mathbb{C}) \subset \mathbb{M}_2(\mathbb{C}) \otimes \mathfrak{A} = \mathbb{L}(\widetilde{\eta}_1 \mathfrak{A} \oplus \widetilde{\eta}_2 \mathfrak{A})$ we define the unitary $u_t \in \mathbb{L}(Z^1)$ by v_t on $\widetilde{\eta}_1 \mathfrak{A} \oplus \widetilde{\eta}_2 \mathfrak{A}$ and by the identity operator on $Z^1 \ominus (\widetilde{\eta}_1 \mathfrak{A} \oplus \widetilde{\eta}_2 \mathfrak{A})$. Since the restriction of $\pi^0(D)$ to $\widetilde{\eta}_1 \mathfrak{A} \oplus \widetilde{\eta}_2 \mathfrak{A}$ is just $\mathbb{C}1 \otimes D \subset \mathbb{M}_2(\mathbb{C}) \otimes A$ with the above identification, the family $(u_t)_{0 \le t \le 1}$ forms a norm continuous path of unitaries in $\pi^0(D)' \cap (\mathbb{C}1 + \mathbb{K}(Z^0))$ satisfying that $u_0 = 1$ and u_1 switches $\widetilde{\eta}_1 a$ and $\widetilde{\eta}_2 a$ for each $a \in \mathfrak{A}$. Let $j_k : A_k \hookrightarrow A$ be the inclusion map for $k = 1, 2$. Since $\mathrm{Ad}\, u_t \circ \pi^0 \circ j_1$ agrees with $\pi^0 \circ j_2$ on D, we have the natural $*$-homomorphism $\phi_t := (\mathrm{Ad}\, u_t \circ \pi^0 \circ j_1) \star (\pi^0 \circ j_2) : \mathfrak{A} \to \mathbb{L}(Z^0)$ thanks to the universality of $\mathfrak{A}$. Then, the Kasparov bimodules

$$\big(Z^0 \oplus (Z^1 \oplus \mathfrak{A}), \phi_t \oplus (\rho^1 \oplus L_{\mathfrak{A}}), [\begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix}]\big), \quad t \in [0, 1]$$

satisfy conditions (i), (ii) and (iii) in Lemma 2.5.3 (with $P = Q = \mathfrak{A}$), and its evaluation at $t = 0$ implements $\lambda \otimes_A \alpha - \mathrm{id}_{\mathfrak{A}}$. Thus, we need to show that $(Z^0 \oplus (Z^1 \oplus \mathfrak{A}), \phi_1 \oplus (\rho^1 \oplus L_{\mathfrak{A}}), [\begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix}])$ is degenerate, that is,

$$\phi_1(x) U = U(\rho^1(x) \oplus L_{\mathfrak{A}}(x)) \quad \text{for } x \in \mathfrak{A}. \tag{4.2}$$

Since U is unitary, we may assume that x is in $A_1^\circ \cup A_2$. When x is in A_2, the above equation is trivial because S intertwines $\pi^1(x)$ and $\pi^0(x)$. Let S' be as in the proof of the previous lemma. Then, we have $u_1 U = S'$ on Z^0 and $u_1 U(0 \oplus a) = \widetilde{\eta}_1 a$ for $a \in \mathfrak{A}$. Since S' intertwines the actions of A_1, we have $U(\pi^1(x) \oplus L_{\mathfrak{A}}(x)) = u_1 \pi^0(x) u_1^* U$ for every $x \in A_1$. Thus we obtain equation (4.2), and hence Lemma 2.5.3 shows $\lambda \otimes_A \alpha = \mathrm{id}_{\mathfrak{A}}$.

We next prove that $\alpha \otimes_{\mathfrak{A}} \lambda = \mathrm{id}_A$ in $KK(A, A)$. Note that $\alpha \otimes_{\mathfrak{A}} \lambda - \mathrm{id}_A$ is implemented by the Kasparov A-A bimodule

$$\big((Z^0 \otimes_\lambda A) \oplus (Z^1 \otimes_\lambda A \oplus A), (\pi^0 \otimes 1_A) \oplus (\pi^1 \otimes 1_A \oplus L_A), \left[\begin{smallmatrix} 0 & U^*\otimes 1 \\ U\otimes 1 & 0 \end{smallmatrix}\right]\big)$$

(see §§2.5). We observe that the family of Kasparov $\mathfrak{A}$-A bimodules

$$\big((Z^0 \otimes_\lambda A) \oplus (Z^1 \otimes_\lambda A \oplus A), (\rho^0 \otimes 1_A) \oplus (\phi_t \otimes 1_A \oplus \lambda) \left[\begin{smallmatrix} 0 & U^*\otimes 1 \\ U\otimes 1 & 0 \end{smallmatrix}\right]\big), \quad t \in [0, 1]$$

satisfies the conditions (i), (ii) and (iii) in Lemma 2.5.3 (with $P = \mathfrak{A}$ and $Q = A$) and its evaluations at endpoints implement $(\alpha \otimes_{\mathfrak{A}} \lambda - \mathrm{id}_A) \circ \lambda$ and 0. Thus, by Lemma 2.5.3 and the fact that $\phi_Y : A \to \mathbb{L}(Y)$ is faithful, it suffices to show that $\phi_t \otimes 1 : \mathfrak{A} \to \mathbb{L}(Z^0 \otimes_{\phi_Y \circ \lambda} Y)$ factors through $\lambda : \mathfrak{A} \to A$ for every $t \in [0, 1]$. We observe that $Z^0 \otimes_{\phi_Y \circ \lambda} Y \cong Y \otimes_A Y$. Thus, if we set $\phi' := \phi_Y \otimes 1 : A \to \mathbb{L}(Y \otimes_A Y)$ and $w_t := u_t \otimes 1 \in \mathbb{L}(Y \otimes_A Y)$, then $\phi_t \otimes 1_Y$ coincides with $\psi_t := (\mathrm{Ad}\, w_t \circ \phi' \circ \iota_1) \star (\phi' \circ \iota_2)$. Based on the decomposition $Y \otimes_A Y = \bigoplus_{k=1,2} \eta_k A_k \otimes_{A_k} Y \oplus \bigoplus_{l=1,2} Y_l^\circ \otimes_{A_l} Y$, we define the projection $R_1 \in \mathbb{L}(Y \otimes_A Y)$ by

$$R_1(Y \otimes_A Y) = \bigoplus_{k=1,2} \eta_k \otimes P_1 Y \oplus \bigoplus_{l=1,2} P_1 Y_l^\circ \otimes_{A_l} Y$$

and set $R_2 := 1 - R_1$. We observe that w commutes with R_1. Indeed, it follows from the definition of $w_t = u_t \otimes 1$ that $w_t(\eta_k \otimes P_1\xi)$ is a linear combination of $\eta_1 \otimes P_1\xi$ and $\eta_2 \otimes P_1\xi$ for $\xi \in Y$ and w_t is identical on $\bigoplus_{l=1,2} P_1 Y_l^\circ \otimes_{A_l} Y$. Thus, by Corollary 3.4.1 it suffices to show that $R_k^\perp \phi'(a) R_k^\perp = \phi'(E_k(a)) R_k^\perp$ for $a \in A_k$ and $k = 1, 2$. By symmetry, we may assume that $k = 2$ and $a \in A_2$. Since R_1 commutes with $\phi'(D)$, we also assume that $a \in A_2^\circ$. Then, for any $\zeta_j, \zeta_j'' \in Y$ and $\zeta_j' \in Y_j^\circ$ we have

$$\begin{aligned}
&R_1 \phi'(a) \left((\eta_1 \otimes P_1\zeta_1) \oplus (\eta_2 \otimes P_1\zeta_2) \oplus (P_1\zeta_1' \otimes \zeta_1'') \oplus (P_1\zeta_2' \otimes \zeta_2'')\right) \\
&\quad = R_1 \left(0 \oplus (\eta_2 \otimes aP_1\zeta_2) \oplus (a\eta_1 \otimes P_1\zeta_1 + aP_1\zeta_1' \otimes \zeta_1'') \oplus (aP_1\zeta_2' \otimes \zeta_2'')\right) \\
&\quad = 0 \oplus (\eta_2 \otimes P_1 a P_1\zeta_2) \oplus (P_1 a\eta_1 \otimes P_1\zeta_1 + P_1 a P_1\zeta_1' \otimes \zeta_1'') \oplus (P_1 a P_1\zeta_2' \otimes \zeta_2'') \\
&\quad = 0
\end{aligned}$$

by Proposition 3.3.3. □

Let $\mathcal{I}$ be a general countable set and let $\mathfrak{A} = \bigstar_D A_k$ and $(A, E) = \bigstar_D (A_k, E_k)$ be as in Theorem 4.1.1. We set $c_0 := c_0(\mathcal{I})$ and $\mathcal{K} := \mathbb{K}(\ell^2(\mathcal{I}))$.

Let $\{e_{kl}\}_{k,l\in\mathcal{I}}$ be the system of matrix units for the canonical basis $\{\delta_k\}_{k\in\mathcal{I}}$ of $\ell^2(\mathcal{I})$, and set $f_k := e_{kk}$. We realize $\sum_k A_k$ and $c_0 \otimes D$ inside $\mathcal{K} \otimes A$ as

$$\sum_k A_k = \mathrm{C}^*\{f_k \otimes a \mid k \in \mathcal{I}, a \in A_k\} \quad \text{and} \quad c_0 \otimes D = \mathrm{C}^*\{f_k \otimes d \mid k \in \mathcal{I}, d \in D\}.$$

Consider two conditional expectations $\sum_k E_k : \sum_k A_k \to c_0 \otimes D$ and $E_{c_0} \otimes \mathrm{id}_D : \mathcal{K} \otimes D \to c_0 \otimes D$ defined by $(\sum_k E_k)(f_l \otimes a) = f_l \otimes E_l(a)$ and $(E_{c_0} \otimes \mathrm{id}_D)(e_{kl} \otimes d) = \delta_{k,l} f_k \otimes d$ for $k, l \in \mathcal{I}$, $a \in A_k$ and $d \in D$. Set $\mathfrak{C} := (\sum_{n \geq 1} A_n) \star_{c_0 \otimes D} (\mathcal{K} \otimes D)$ and $(C, E_C) := (\sum_k A_k, \sum_k E_k) \star_{c_0 \otimes D} (\mathcal{K} \otimes D, E_{c_0} \otimes \mathrm{id}_D)$ and denote by $\Lambda : \mathfrak{C} \to C$ the canonical surjection. Here, when $\mathcal{I}$ is infinite, we define (C, E_C) by the C*-subalgebras of the reduced amalgamated free product $(C + \mathbb{C}1, E_C^{\sim})$ of $(\sum_k A_k + \mathbb{C}1, (\sum_n E_n)^{\sim})$ and $(\mathcal{K} \otimes D + \mathbb{C}1, E_{c_0}^{\sim})$ over $c_0 \otimes D + \mathbb{C}1$.

Proposition 4.1.5. *With the notation above, there exist isomorphisms $\pi : \mathfrak{C} \to \mathcal{K} \otimes \mathfrak{A}$ and $\pi_{\mathrm{red}} : C \to \mathcal{K} \otimes A$ such that the following diagram*

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\pi} & \mathcal{K} \otimes \mathfrak{A} \\ \Lambda \downarrow & & \downarrow \mathrm{id}_{\mathcal{K}} \otimes \lambda \\ C & \xrightarrow{\pi_{\mathrm{red}}} & \mathcal{K} \otimes A \end{array}$$

commutes.

Proof. If $\mathcal{I}$ is finite, we assume that $\mathcal{I} = \{1, 2, \ldots, |\mathcal{I}|\}$; otherwise we set $\mathcal{I} = \mathbb{N}$. The inclusion maps $\sum_k A_k \hookrightarrow \mathcal{K} \otimes \mathfrak{A}$ and $\mathcal{K} \otimes D \hookrightarrow \mathcal{K} \otimes \mathfrak{A}$ induce a $*$-homomorphism $\pi : \mathfrak{C} \to \mathcal{K} \otimes \mathfrak{A}$. For any $k, i, j \in \mathcal{I}$, $a \in A_k$, one has $e_{ij} \otimes a = \pi(e_{ik} \otimes 1)\pi(f_k \otimes a)\pi(e_{kj} \otimes 1) \in \pi(\mathfrak{C})$. Hence, π is surjective. Define $\sigma_k : A_k \to \mathfrak{C}$ by $\sigma_k(a) = (e_{1k} \otimes 1)(f_k \otimes a)(e_{k1} \otimes 1)$ for $a \in A_k$. We then obtain $\sigma = \bigstar_{k \in \mathcal{I}} \sigma_k : \mathfrak{A} \to \mathfrak{C}$. Define $\widetilde{\sigma} : \mathcal{K} \otimes \mathfrak{A} \to \mathfrak{C}$ by $\widetilde{\sigma}(e_{ij} \otimes a) = (e_{i1} \otimes 1)\sigma(a)(e_{1j} \otimes 1)$ for $a \in A$. Then, it is easy to see that $\widetilde{\sigma} \circ \pi = \mathrm{id}_{\mathfrak{C}}$, and hence π is bijective.

We next show that $(\mathrm{id}_{\mathcal{K}} \otimes \lambda) \circ \pi$ factors through $\Lambda : \mathfrak{C} \to C$. By adding infinitely many copies of (A_1, E_1) to $\{(A_k, E_k)\}_{k \in \mathcal{I}}$ we may assume that $\mathcal{I} = \mathbb{N}$. Take a unital faithful representation $\Delta\mathbf{T}(A, E) \subset \mathbb{B}(\mathcal{H})$. We may assume that $\mathcal{K} \otimes A \subset \mathcal{K} \otimes \Delta\mathbf{T}(A, E) \subset \mathbb{B}(\ell^2(\mathcal{I}) \otimes \mathcal{H})$. and set $Q_\Sigma := \sum_k e_{kk} \otimes P_k \in \mathbb{B}(\ell^2(\mathcal{I}) \otimes \mathcal{H})$ and $Q_{\mathcal{K}} = 1 - Q_\Sigma$. For any $e_{kk} \otimes a \in \mathbb{C}e_{kk} \otimes A_k$ we have

$$\begin{aligned} &(1 - Q_\Sigma)(e_{kk} \otimes a)(1 - Q_\Sigma) \\ &\quad = (e_{kk} \otimes a) - Q_\Sigma(e_{kk} \otimes a) - (e_{kk} \otimes a)Q_\Sigma + Q_\Sigma(e_{kk} \otimes a)Q_\Sigma \\ &\quad = (e_{kk} \otimes a) - (e_{kk} \otimes P_k a) - (e_{kk} \otimes a P_k) + (e_{kk} \otimes P_k a P_k) \\ &\quad = e_{kk} \otimes (1 - P_k)a(1 - P_k) = e_{kk} \otimes E_k(a). \end{aligned}$$

Also, for any $e_{ij} \otimes d \in \mathcal{K} \otimes D$ we have $(1 - Q_{\mathcal{K}})(e_{ij} \otimes d)(1 - Q_{\mathcal{K}}) = e_{ij} \otimes P_i d P_j = \delta_{i,j}(e_{ii} \otimes dP_i)$. Hence, applying Corollary 3.4.1 to the unitizations of $\sum_k A_k$ and $\mathcal{K} \otimes D$, we conclude that $(\mathrm{id}_{\mathcal{K}} \otimes \lambda) \circ \pi$ factors through $\Lambda : \mathfrak{C} \to C$.

Finally, we show that $\Lambda \circ \widetilde{\sigma}$ factors through $\mathrm{id}_{\mathcal{K}} \otimes \lambda : \mathcal{K} \otimes \mathfrak{A} \to \mathcal{K} \otimes A$. It is enough to show that $\Lambda \circ \sigma$ factors through λ. We assume that $C \subset \Delta\mathbf{T}(C + \mathbb{C}1, E_C^{\sim}) = \mathrm{C}^*(1, C, P_\Sigma, P_{\mathcal{K}})$. Set $e = e_{11} \otimes 1$ and $p_k = (e_{1k} \otimes 1)P_\Sigma(e_{k1} \otimes 1)$ in

$e\Delta\mathbf{T}(C + \mathbb{C}1, E_C^{\sim})e$. We then have $p_k p_l = \delta_{k,l} p_k$ and $e - p_k = (e_{1k} \otimes 1)(1 - P_\Sigma)(e_{k1} \otimes 1)$. Hence, for any $a \in A_k$ we have

$$\begin{aligned}(e - p_k)\sigma_k(a)(e - p_k) &= (e_{1k} \otimes 1)(f_k \otimes (1 - P_\Sigma)a(1 - P_\Sigma))(e_{k1} \otimes 1)\\ &= (e_{1k} \otimes 1)(f_k \otimes E_k(a)(1 - P_\Sigma))(e_{k1} \otimes 1)\\ &= \sigma_k(E_k(a))(e - p_k).\end{aligned}$$

Thus, $\Lambda \circ \sigma$ factors through λ by Corollary 3.4.1. □

The following general fact is well-known (see, e.g. [2, Proposition 17.8.7]).

Proposition 4.1.6. *Let $\mathcal{K}$ be as above and let $\iota : \mathcal{K} \hookrightarrow \mathbb{B}(\ell^2(\mathcal{I}))$ be the inclusion map. Fix a minimal projection $e \in \mathcal{K}$. For any separable* C*-*algebras $\mathcal{A}$ and $\mathcal{B}$, the mapping $\mathbb{E}(\mathcal{A}, \mathcal{B}) \ni (X, \phi, F) \mapsto (\mathcal{K} \otimes X, L_\mathcal{K} \otimes \phi, 1_\mathcal{K} \otimes F) \in \mathbb{E}(\mathcal{K} \otimes \mathcal{A}, \mathcal{K} \otimes \mathcal{B})$ induces an isomorphism $\tau : KK(\mathcal{A}, \mathcal{B}) \to KK(\mathcal{K} \otimes \mathcal{A}, \mathcal{K} \otimes \mathcal{B})$. The inverse of τ is given by the mapping $\mathbb{E}(\mathcal{K} \otimes \mathcal{A}, \mathcal{K} \otimes \mathcal{B}) \ni (Y, \psi, G) \mapsto (Y \otimes_{\iota \otimes L_\mathcal{B}} (\ell^2(\mathcal{I}) \otimes \mathcal{B}), (\psi \otimes_{\iota \otimes L_\mathcal{B}} 1) \circ \sigma, G \otimes_{\iota \otimes L_\mathcal{B}} 1) \in \mathbb{E}(\mathcal{A}, \mathcal{B})$, where $\sigma(a) = e \otimes a$ for $a \in \mathcal{A}$.*

We are now ready to prove Theorem 4.1.1 .

Proof of Theorem 4.1.1. We use the notation in the proof of Proposition 4.1.5. Applying Theorem 4.1.4 to the unitizations $\mathfrak{C} + \mathbb{C}1$ and $C + \mathbb{C}1$ together with Proposition 4.1.5, there exists $\beta \in KK(\mathcal{K} \otimes A, \mathcal{K} \otimes \mathfrak{A})$ such that $(\mathrm{id}_\mathcal{K} \otimes \lambda) \otimes_{\mathcal{K} \otimes A} \beta = \mathrm{id}_{\mathcal{K} \otimes \mathfrak{A}}$ and $\beta \otimes_{\mathcal{K} \otimes \mathfrak{A}} (\mathrm{id}_\mathcal{K} \otimes \lambda) = \mathrm{id}_{\mathcal{K} \otimes A}$. Let τ be as in Proposition 4.1.6. We then have $\mathrm{id}_\mathfrak{A} = \tau^{-1}(\mathrm{id}_{\mathcal{K} \otimes \mathfrak{A}}) = \tau^{-1}((\mathrm{id}_\mathcal{K} \otimes \lambda) \otimes_{\mathcal{K} \otimes A} \beta) = \lambda \otimes_A \tau^{-1}(\beta)$ and $\mathrm{id}_A = \tau^{-1}(\mathrm{id}_{\mathcal{K} \otimes A}) = \tau^{-1}(\beta \otimes_{\mathcal{K} \otimes \mathfrak{A}} (\mathrm{id}_\mathcal{K} \otimes \lambda)) = \tau^{-1}(\beta) \otimes_\mathfrak{A} \lambda$. Thus, λ gives a KK-equivalence. □

4.2 Six-term exact sequences

In this subsection, we give a new proof of the next theorem due to Fima and Germain [19]. Key ingredients of our proof are the right invertibility of the embedding $A \hookrightarrow \Delta\mathbf{T}(A, E)$ in KK-theory and exact sequences of KK-groups for Cuntz–Pimsner algebras [41].

Theorem 4.2.1 (Fima–Germain)**.** *Let $(A, E) = (A_1, E_1) \star_D (A_2, E_2)$ be a reduced amalgamated free product of unital separable* C*-*algebras and $i_k \colon D \hookrightarrow A_k$ and $j_k \colon A_k \hookrightarrow A$ be inclusion maps for $k = 1, 2$. Then, there are two cyclic exact sequences for any separable* C*-*algebra P:*

$$\begin{array}{ccccc} KK(P, D) & \xrightarrow{(i_{1*}, i_{2*})} & KK(P, A_1) \oplus KK(P, A_2) & \xrightarrow{j_{1*} - j_{2*}} & KK(P, A) \\ \uparrow & & & & \downarrow \\ KK^1(P, A) & \xleftarrow{j_{1*} - j_{2*}} & KK^1(P, A_1) \oplus KK^1(P, A_2) & \xleftarrow{(i_{1*}, i_{2*})} & KK^1(P, D) \end{array}$$

and

$$\begin{array}{ccccc} KK(D,P) & \xleftarrow{i_1^*-i_2^*} & KK(A_1,P)\oplus KK(A_2,P) & \xleftarrow{(j_1^*,j_2^*)} & KK(A,P) \\ \downarrow & & & & \uparrow \\ KK^1(A,P) & \xrightarrow{(j_1^*,j_2^*)} & KK^1(A_1,P)\oplus KK^1(A_2,P) & \xrightarrow{i_1^*-i_2^*} & KK^1(D,P). \end{array}$$

Let $\phi = \phi_Y \colon A \hookrightarrow \Delta\mathbf{T}(A,E)$ be the inclusion map and set $\rho_k := \pi \circ \Psi_k|_D \colon D \hookrightarrow DP_k \subset \Delta\mathbf{T}(A,E)$. For each $k \in \{1,2\}$, we denote by $\overline{k}$ the unique element in $\{1,2\} \setminus \{k\}$. We first show the right invertibility of $\phi \in KK(A, \Delta\mathbf{T}(A,E))$.

Lemma 4.2.2. *There exist $\beta \in KK(\Delta\mathbf{T}(A,E),A)$ and $\delta \in KK(\Delta\mathbf{T}(A,E),D)$ such that $(\phi \oplus \rho_1) \otimes_{\Delta\mathbf{T}(A,E)} (\beta \oplus \delta) = \mathrm{id}_A \oplus \mathrm{id}_D$.*

Proof. Let (Z,ϕ_Z) and $\Phi_Z \colon \Delta\mathbf{T}(A,E) \to \mathbb{L}(Z)$ be as in Eq. (2.1). Let $S \colon X \otimes_D A \to Z$ be as in Eq. (4.1). It follows from Lemma 4.1.2 that $S(\phi_X(a)\otimes 1) - \phi_Z(a)S$ is compact for $a \in A$. Since $\Phi_Z(P_1)S = S(\sigma_1(P_1) \otimes 1)$ holds, the triplet

$$(Z \oplus (X \otimes_D A), \Phi_Z \oplus (\sigma_1 \otimes 1), \left[\begin{smallmatrix} 0 & S \\ S^* & 0 \end{smallmatrix}\right])$$

is a $\Delta\mathbf{T}(A,E)$-A Kasparov bimodule and defines an element $\beta \in KK(\Delta\mathbf{T}(A,E),A)$. Since $\phi \otimes_{\Delta\mathbf{T}(A,E)} \beta$ is implemented by the A-A Kasparov bimodule

$$(Z \oplus (X \otimes_D A), \phi_Z \oplus (\phi_X \otimes 1), \left[\begin{smallmatrix} 0 & S \\ S^* & 0 \end{smallmatrix}\right]),$$

we have $\phi \otimes_{\Delta\mathbf{T}(A,E)} \beta = \mathrm{id}_A \in KK(A,A)$ by Theorem 4.1.1. Since $\Phi_Z(P_1) = S(\sigma_1(P_1)\otimes 1)S^*$ holds, we have $\rho_1 \otimes_{\Delta\mathbf{T}(A,E)} \beta = 0$. Let $\sigma_k \colon \Delta\mathbf{T}(A,E) \to \mathbb{L}(X^{(k)})$ be as in Remark 3.3.2. Since $\sigma_1 = \sigma_2 = \phi_X$ on A and $\sigma_1(P_1) - \sigma_2(P_1) = e_D$ hold, the triplet $(X^{(1)} \oplus X^{(2)}, \sigma_1 \oplus \sigma_2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$ is a $\Delta\mathbf{T}(A,E)$-D Kasparov bimodule and the corresponding element $\delta \in KK(\Delta\mathbf{T}(A,E),D)$ satisfies that $\rho_1 \otimes_{\Delta\mathbf{T}(A,E)} \delta = \mathrm{id}_D$ and $\phi \otimes_{\Delta\mathbf{T}(A,E)} \delta = 0$. □

To compute the K-theory of $\Delta\mathbf{T}(A,E) \cong \mathcal{O}(\mathfrak{X})$, we need to compute the K-theory of $J_{\mathfrak{X}}$. Thanks to the next technical lemma, we can assume that X_1° and X_2° are *full*, i.e., $\overline{\mathrm{span}}\{E(a^*b) \mid a,b \in A_k^\circ\} = D$ holds for each $k=1,2$. Note that X_k° is full whenever D is simple (e.g. $D = \mathbb{C}$).

Lemma 4.2.3. *Let φ be a nondegenerate state on D and set $\varphi_k := \varphi \circ E_k$ for $k=1,2$, $(\mathcal{T},\omega)$ be the Toeplitz algebra with the vacuum state, and $(\mathcal{A}_k, \widetilde{\varphi}_k) = (A_k,\varphi_k) \star (\mathcal{T},\omega)$ be the reduced free product. Denote by $F_k \colon \mathcal{A}_k \to D$ the composition of the canonical conditional expectation $\mathcal{A}_k \to A_k$ and $E_k \colon A_k \to D$ and by $\mathcal{X}_k$ the GNS Hilbert C^*-module of F_k. Set $(\mathcal{A},F) = (\mathcal{A}_1,F_1) \star_D (\mathcal{A}_2,F_2)$. Then $\mathcal{X}_k^\circ$ is full and the embedding maps $A_k \hookrightarrow \mathcal{A}_k$ and $A \hookrightarrow \mathcal{A}$ induce KK-equivalences.*

Proof. Let s be the unilateral shift generating $\mathcal{T}$. Then, one has $F_k(s) = 0$ and $F_k(s^*s) = 1$, and thus $\mathcal{X}_k^\circ$ is full. Let $\mathcal{H}_k$ and $\mathcal{H}$ be the C*-correspondences over A_k and A associated with the UCP maps $\varphi_k(\cdot)1$ and $\varphi(\cdot)1$ on A_k and A (see e.g. [6, Example 4.6.11]), respectively. It follows from [47, Theorem 2.3] that $(\mathcal{T}(\mathcal{H}_k), \varphi_k \circ E_{\mathcal{H}_k}) \cong (A_k, \varphi_k) \star (\mathcal{T}, \omega)$ (this can be shown for degenerate conditional expectations). Thus, the embedding $A_k \hookrightarrow \mathcal{A}_k$ induces a KK-equivalence by [41]. Similarly, by a result due to Speicher [50], we have

$$\begin{aligned}(\mathcal{A}, \varphi \circ F) &\cong (A, \varphi \circ E) \star (\mathcal{T}, \omega) \star (\mathcal{T}, \omega)\\ &\cong (\mathcal{T}(\mathcal{H}), E_{\mathcal{H}}) \star_A (\mathcal{T}(\mathcal{H}), E_{\mathcal{H}})\\ &\cong (\mathcal{T}(\mathcal{H} \oplus \mathcal{H}), E_{\mathcal{H} \oplus \mathcal{H}}),\end{aligned}$$

and thus $A \hookrightarrow \mathcal{A}$ gives a KK-equivalence by [41] again. $\square$

We may identify $\langle\!\langle A_k, E_k \rangle\!\rangle$ with C$^*(A_k, P_k)$. If X_1° and X_2° are full, then $\mathbb{K}(X_k^\circ) \subset \mathbb{K}(X_k)$ and $B_k \subset \langle\!\langle A_k, E_k \rangle\!\rangle$ are full corners, and thus all the horizontal embedding maps in the next commuting diagram induce KK-equivalences by [5].

$$\begin{array}{ccccc} \mathbb{K}(X_k^\circ) & \xhookrightarrow{\kappa_k} & \mathbb{K}(X_k) & \xhookleftarrow{\epsilon_k} & D \\ \big\downarrow & & \big\downarrow & \swarrow_{\rho_{\overline{k}}} & \\ B_k & \hookrightarrow & \langle\!\langle A_k, E_k \rangle\!\rangle & & \end{array} \tag{4.3}$$

Here the embedding $\epsilon_k \colon D \hookrightarrow \mathbb{K}(X_k)$ is given by $d \mapsto de_D$. We set $(X_k^\circ) := \kappa_k \otimes_{\mathbb{K}(X_k)} (\epsilon_k)^{-1} \in KK(\mathbb{K}(X_k^\circ), D)$. Since the inclusion map $\mu_k \colon A_k \hookrightarrow \langle\!\langle A_k, E_k \rangle\!\rangle$ gives a cross section of of the split extension $0 \longrightarrow \mathbb{K}(X_k) \longrightarrow \langle\!\langle A_k, E_k \rangle\!\rangle \longrightarrow A_k \longrightarrow 0$ (see §§3.2), the above commuting diagram implies that $\mu_k \oplus \rho_{\overline{k}} \in KK(A_k \oplus D, \langle\!\langle A_k, E_k \rangle\!\rangle)$ is a KK-equivalence.

By Theorem 3.3.5 and Theorem 3.3.6 we may use the identifications $\Delta\mathbf{T}(A, E) = \mathcal{O}(\mathfrak{X})$ and C$^*(\Pi, T) = \mathcal{T}(\mathfrak{X})$. Recall that the kernel of the quotient map $p \colon \mathcal{T}(\mathfrak{X}) \to \mathcal{O}(\mathfrak{X})$ is isomorphic to $\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})$, and that $\Pi \colon B \to \mathcal{T}(\mathfrak{X})$ and $\iota_\Omega := (\Pi - \psi_T \circ \phi_{\mathfrak{X}})|_{J_{\mathfrak{X}}} \colon J_{\mathfrak{X}} \to \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})$ induce KK-equivalences (see [41, 31]). By these KK-equivalences, one can rewrite the six-term exact sequence induced from the Toeplitz extension as

$$\begin{array}{ccccc} KK(P, J_{\mathfrak{X}}) & \xrightarrow[\iota_* - [\mathfrak{X}]]{} & KK(P, B) & \xrightarrow[\pi_*]{} & KK(P, \mathcal{O}(\mathfrak{X})) \\ \big\uparrow & & & & \big\downarrow \\ KK^1(P, \mathcal{O}(\mathfrak{X})) & \xleftarrow{\pi_*} & KK^1(P, B) & \xleftarrow{\iota_* - [\mathfrak{X}]} & KK^1(P, J_{\mathfrak{X}}) \end{array} \tag{4.4}$$

for any separable C*-algebra P (see [41, 31]). Here ι is the inclusion map $J_{\mathfrak{X}} \hookrightarrow B$ and $[\mathfrak{X}]$ is induced from $[(\mathfrak{X}, \phi_{\mathfrak{X}}|_{J_{\mathfrak{X}}}, 0)] \in KK(J_{\mathfrak{X}}, B)$. Let δ_p be the element corresponding in $KK^1(\mathcal{O}(\mathfrak{X}), \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}))$ to the Toeplitz extension (see [48, §1]). Then, the connecting map $KK(P, \mathcal{O}(\mathfrak{X})) \to KK^1(P, J_{\mathfrak{X}})$ in the above exact sequence is given by $\delta_p \otimes_{\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})} (\iota_\Omega)^{-1} \in KK^1(\mathcal{O}(\mathfrak{X}), J_{\mathfrak{X}})$. Note that $(X_1^\circ) \oplus (X_2^\circ) \in KK(J_{\mathfrak{X}}, D \oplus D)$ is a KK-equivalence.

Lemma 4.2.4. *Assume that X_1° and X_2° are full. Then there is a cyclic exact sequence*

$$\begin{array}{ccccc}
KK(P, D\oplus D) & \xrightarrow[\xi]{} & KK(P, A_1 \oplus A_2 \oplus D \oplus D) & \xrightarrow[\eta]{} & KK(P, \mathcal{O}(\mathfrak{X})) \\
\uparrow{\scriptstyle \partial} & & & & {\scriptstyle \partial}\downarrow \\
KK^1(P, \mathcal{O}(\mathfrak{X})) & \xleftarrow{\eta} & KK^1(P, A_1 \oplus A_2 \oplus D \oplus D) & \xleftarrow{\xi} & KK^1(P, D \oplus D),
\end{array}$$

where $\xi(x, y) = (-i_{1*}(y), -i_{2*}(x), x + y, x + y)$ and $\eta = \phi_* \circ j_{1*} + \phi_* \circ j_{2*} + \rho_{1*} + \rho_{2*}$. The map ∂ is induced from $\delta_p \otimes_{\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})} (\iota_\Omega)^{-1} \otimes_{J_{\mathfrak{X}}} ((X_1^\circ) \oplus (X_2^\circ)) \in KK^1(\mathcal{O}(\mathfrak{X}), D \oplus D)$.

Proof. The proof proceeds by rewriting the exact sequence in Eq. (4.4). Since X_k° is full, B_k is a full corner of $\langle\langle A_k, E_k \rangle\rangle$, and thus the inclusion $B_k \hookrightarrow \langle\langle A_k, E_k \rangle\rangle$ induces a KK-equivalence. Then, the next diagram commutes and all vertical arrows are isomorphisms:

$$\begin{array}{ccccc}
KK^p(P, \mathbb{K}(X_k^\circ)) & \xrightarrow[\iota_*]{} & KK^p(P, B_k) & \xrightarrow[\pi_*]{} & KK^p(P, \mathcal{O}(\mathfrak{X})) \\
{\scriptstyle (\kappa_k)_*}\downarrow & & \downarrow & & \| \\
KK^p(P, \mathbb{K}(X_k)) & \longrightarrow & KK^p(P, \langle\langle A_k, E_k \rangle\rangle) & \longrightarrow & KK^p(P, \mathcal{O}(\mathfrak{X})) \\
{\scriptstyle (\epsilon_k)_*}\uparrow & & \uparrow{\scriptstyle (\mu_k)_* + (\rho_{\overline{k}})_*} & & \| \\
KK^p(P, D) & \xrightarrow{(0,1)} & KK^p(P, A_k \oplus D) & \xrightarrow{\phi_* \circ j_{k*} + \rho_{\overline{k}*}} & KK^p(P, \mathcal{O}(\mathfrak{X})).
\end{array}$$

We next observe that $[\mathfrak{X}]$ is the direct sum of two maps $(\Psi_{\overline{k}} \circ i_{\overline{k}})_* \circ (X_k^\circ)_*$ from $KK^p(P, \mathbb{K}(X_k^\circ))$ to $KK^p(P, B_{\overline{k}})$, $k = 1, 2$. Thus, the assertion follows from next commuting diagram:

$$\begin{array}{ccc}
KK^p(P, \mathbb{K}(X_k^\circ)) & \xrightarrow{(\Psi_{\overline{k}} \circ i_{\overline{k}})_* \circ (X_k^\circ)_*} & KK^p(P, B_{\overline{k}}) \\
{\scriptstyle (X_k^\circ)_*}\downarrow & \nearrow{\scriptstyle (\Psi_{\overline{k}} \circ i_{\overline{k}})_*} & \downarrow \\
KK^p(P, D) & \xrightarrow[(i_{\overline{k}*},\ -(\cdot))]{} & KK^p(P, A_{\overline{k}} \oplus D).
\end{array} \tag{4.5}$$

□

We are now ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. We first show the exactness at $KK^p(P, A_1 \oplus A_2)$. By the previous lemma, we have $\operatorname{Im} \partial \circ \phi_* \subset \ker \xi = \{(x, -x) \mid x \in \ker(i_{1*}, i_{2*})\}$. Via the isomorphism $\ker \xi \ni (x, -x) \mapsto x \in \ker(i_{1*}, i_{2*}) \subset KK^p(P, D)$, we obtain a connecting map $\partial' \colon KK^p(P, A) \to KK^{p+1}(P, D)$. By Lemma 4.2.2, $\phi_* \colon KK^p(P, A) \to KK^p(P, \mathcal{O}(\mathfrak{X}))$ is injective. Thus, $j_{1*}(x) - j_{2*}(y) = 0$ if and only if $\eta(x, -y, 0, 0) = 0$ if and only if $(x, -y, 0, 0) = (i_{1*}(z), -i_{2*}(z))$ for some $z \in KK^p(P, D)$, and hence we obtain $\operatorname{Im}(i_{1*}, i_{2*}) = \ker(j_{1*} - j_{2*})$. Also, since ϕ_* is injective, we have $\ker \partial' = \ker \partial \circ \phi_* = \operatorname{Im}(j_{1*} + j_{2*})$, and thus the exactness at $KK^p(P, A)$ holds.

To see $\operatorname{Im} \partial' = \ker(i_{1*}, i_{2*})$, it is enough to see $\operatorname{Im} \partial \circ \phi_* \supset \ker \xi = \{(x, -x) \mid x \in \ker(i_{1*}, i_{2*})\}$. By the definition of ∂, this is equivalent to $\operatorname{Im}(\phi \otimes_{\mathcal{O}(\mathfrak{X})} \delta_p)_*$ contains $(\iota_\Omega)_* \circ ((X_1^\circ) \oplus (X_2^\circ))_*^{-1}(x, -x)$ for $x \in \ker(i_{1*}, i_{2*})$. Let $\Theta \colon \mathcal{O}(\mathfrak{X}) \to \mathcal{T}(\mathfrak{X})$ be as in Theorem 3.3.7 and put $\mathfrak{A} := \mathrm{C}^*(\Theta(A)) + \mathbb{K}(\mathcal{F}(\mathfrak{X}) J_{\mathfrak{X}})$. The next commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K}(\mathcal{F}(\mathfrak{X}) J_{\mathfrak{X}}) & \longrightarrow & \mathcal{T}(\mathfrak{X}) & \xrightarrow{p} & \mathcal{O}(\mathfrak{X}) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & \mathbb{K}(\mathcal{F}(\mathfrak{X}) J_{\mathfrak{X}}) & \xrightarrow{\nu} & \mathfrak{A} & \xrightarrow{q} & A & \longrightarrow & 0
\end{array}$$

and [48, Lemma 1.5] imply that $\phi \otimes_{\mathcal{O}(\mathfrak{X})} \delta_p \in KK^1(A, \mathbb{K}(\mathcal{F}(\mathfrak{X}) J_{\mathfrak{X}}))$ is the element corresponding to the semisplit extension $\mathfrak{A}$ of A. Hence it follows from the six-term exact sequence for $\mathfrak{A}$ that $\operatorname{Im}(\phi \otimes_{\mathcal{O}(\mathfrak{X})} \delta_p)_* = \ker \nu_*$. Therefore, it suffices to show that $\nu_* \circ (\iota_\Omega)_* \circ ((X_1^\circ) \oplus (X_2^\circ))_*^{-1}(x, -x) = 0$ for $x \in \ker(i_{1*}, i_{2*})$. Let $\theta_k \colon B_k \to \mathrm{C}^*(\Theta_k(A_k))$ be the inverse of the isomorphism in Remark 3.3.8. Note that $\iota_\Omega \otimes_{\mathbb{K}(\mathcal{F}(\mathfrak{X}) J_{\mathfrak{X}})} \nu = \iota \otimes_B (\theta_1 + \theta_2)$ in $KK(J_{\mathfrak{X}}, \mathfrak{A})$. Also, it follows from the proof of the previous lemma that $((X_1^\circ) \oplus (X_2^\circ))_*^{-1}(x, -x) \in ((X_1^\circ) \oplus (X_2^\circ))_*^{-1}(\ker \xi) = \ker(\iota_* - [\mathfrak{X}])$. Since $\theta_1 \circ \Psi_1 \circ i_1 = \theta_2 \circ \Psi_2 \circ i_2$, we have

$$\begin{aligned}
&\nu_* \circ (\iota_\Omega)_* \circ ((X_1^\circ) \oplus (X_2^\circ))_*^{-1}(x, -x) \\
&\quad = (\theta_1 + \theta_2)_* \circ \iota_* \circ ((X_1^\circ) \oplus (X_2^\circ))_*^{-1}(x, -x) \\
&\quad = (\theta_1 + \theta_2)_* \circ (-[\mathfrak{X}]) \circ ((X_1^\circ) \oplus (X_2^\circ))_*^{-1}(x, -x) \\
&\quad = (\theta_1 + \theta_2)_* \circ (\Psi_1 \circ i_1, \Psi_2 \circ i_2)_*(-x, x) \\
&\quad = -(\theta_1 \circ \Psi_1 \circ i_1)_*(x) + (\theta_2 \circ \Psi_2 \circ i_2)_*(x) \\
&\quad = 0.
\end{aligned}$$

Here the third equality follows from the commuting diagram (4.5). Thus, we obtain the exact sequence for $KK^p(P, -)$.

For the exact sequence for $KK^p(-P)$, it is enough to show that $\phi \oplus \rho_1$ is a KK-equivalnce. Let η be as in Lemma 4.2.4. Since $\rho_1 + \rho_2 = j_1 \circ i_1$ holds in

$KK(D, \mathcal{O}(\mathfrak{X}))$, a simple diagram chasing shows that $\phi_* + \rho_{1*}\colon KK(\mathcal{O}(\mathfrak{X}), A \oplus D) \to KK(\mathcal{O}(\mathfrak{X}), \mathcal{O}(\mathfrak{X}))$ is surjective. We show that $\phi + \rho_1$ is a KK-equivalence by the following trick from [40]: Take $\gamma \in KK(\mathcal{O}(\mathfrak{X}), A \oplus D)$ such that $1_{\mathcal{O}(\mathfrak{X})} - (\beta \oplus \delta) \otimes_{A\oplus D} (\phi \oplus \rho_1) = \gamma \otimes_{A\oplus D} (\phi \oplus \rho_1)$. Since the left hand side is an idempotent in the ring $KK(\mathcal{O}(\mathfrak{X}), \mathcal{O}(\mathfrak{X}))$, it follows from Lemma 4.2.2 that

$$\begin{aligned}
&\gamma \otimes_{A\oplus D} (\phi \oplus \rho_1) \\
&\qquad = \gamma \otimes_{A\oplus D} (\phi \oplus \rho_1) - \gamma \otimes_{A\oplus D} (\phi \oplus \rho_1) \otimes_{\mathcal{O}(\mathfrak{X})} (\beta \oplus \delta) \otimes_{A\oplus D} (\phi \oplus \rho_1) \\
&\qquad = \gamma \otimes_{A\oplus D} (\phi \oplus \rho_1) - \gamma \otimes_{A\oplus D} (\phi \oplus \rho_1) = 0.
\end{aligned}$$

□

Remark 4.2.5. Our proof shows that if X_1° and X_2° are full, then the composition of the connecting map $\partial'\colon KK(P, A) \to KK^1(P, D)$, the diagonal embedding $D \to D \oplus D$, and the KK-equivalence $(X_1^\circ \oplus X_2^\circ)^{-1} \otimes_{J_{\mathfrak{X}}} \iota_\Omega \in KK(D \oplus D, \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}))$ is given by the semisplit extension

$$0 \longrightarrow \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}) \longrightarrow \mathrm{C}^*(\Theta(A)) + \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}) \longrightarrow A \longrightarrow 0.$$

In the original proof in [19], it was shown that a natural embedding of the mapping cone C_i of the diagonal embedding $D \to A_1 \oplus A_2$ into the suspension SA has an inverse $x \in KK(SA, C_i)$. Then, the connecting map is given by the Kasparov product of x and the evaluation map $C_i \to D$. It might be interesting to compare these two maps.

As a by-product, we obtain the following theorem:

Theorem 4.2.6. *The element $\phi \oplus \rho_1\colon KK(A \oplus D, \Delta\mathbf{T}(A, E))$ is a KK-equivalence. Therefore, the KK-class of $\Delta\mathbf{T}(A, E)$ does not depend on the choice of conditional expectations $E_k, k = 1, 2$.*

Proof. When X_1° and X_2° are full, the assertion follows from the last paragraph of the proof of Theorem 4.2.1. In the general case, one can check the surjectivity of $\phi_* + \rho_{1*}$ by applying Theorem 4.2.1 to the reduced amalgamated free products A and $\Delta\mathbf{T}(A, E)$ (cf. Corollary 3.4.2). The second assertion follows from Theorem 4.1.1. □

We close this paper by the following corollary about K-nuclearity introduced by Skandalis [49]. Note that this corollary also follows from the original proof in [19] since the mapping cone of the diagonal embedding $i\colon D \to A_1 \oplus A_2$ is a semisplit extension of D by $SA_1 \oplus SA_2$.

Corollary 4.2.7. *Reduced amalgamated free products of K-nuclear C^*-algebras are K-nuclear.*

Proof. Assume that A_1, A_2 and D are K-nuclear. We may assume that X_1° and X_2° are full by Lemma 4.2.3. Then, $\mathbb{K}(X_k^\circ)$ and D are KK-equivalent, and thus $\mathbb{K}(X_k^\circ), J_{\mathfrak{X}}$ and $\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})$ are K-nuclear. It follows from [49, Proposition 3.8] that B_k has the same property. Since Π induces a KK-equivalence, $\mathcal{T}(\mathfrak{X})$ is K-nuclear, and thus so is $\Delta\mathbf{T}(A, E) \cong \mathcal{O}(\mathfrak{X})$ by [49, Proposition 3.8]. Therefore, $\phi \otimes_{\Delta\mathbf{T}(A,E)} \beta$ is implemented by some nuclear Kasparov bimodule, and hence the K-nuclearity of A follows from Lemma 4.2.2. $\square$

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