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Milnor-Selberg zeta functions and zeta regularizations

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Abstract

By a similar idea for constructing Milnor's gamma functions, we study "higher depth determinants" of the Laplacian on a compact Riemann surface of genus greater than one. We prove that, as a generalization of the determinant expression of the Selberg zeta function, this higher depth determinant can be expressed as a product of multiple gamma functions and what we call a Milnor-Selberg zeta function. Moreover, it is shown that the Milnor-Selberg zeta function admits an analytic continuation, a functional equation and, remarkably, has an Euler product.

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Contents

1 Introduction

In 1983, Milnor ([M]) introduced a family of functions $\{\gamma_r(z)\}_{r\geq 1}$ what he thought as a kind of "higher depth gamma functions". These are defined as partial derivatives of the Hurwitz zeta function $\zeta(w,z) := \sum_{n=0}^{\infty}(n+z)^{-w}$ at non-positive integer points with respect to the variable w.

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We call $\gamma_r(z)$ a Milnor gamma function of depth r and will denote by $\boldsymbol{\Gamma}_r(z)$ (see [KOW2] for some analytic properties of $\boldsymbol{\Gamma}_r(z)$). The purpose of his study about such functions is to construct functions satisfying a modified version of the Kubert identity, which plays an important role for the study of Iwasawa theory.

The aim of the present paper is, as an analogue of the study of the Milnor gamma functions, to introduce and study "higher depth determinants" of the Laplacians on a compact Riemann surface of genus $g \geq 2$ with negative constant curvature. Let us give the definition of the higher depth determinant in more general situations. Let A be an operator on some space. We assume that A has only discrete spectra $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ and the multiplicity of each eigenvalue λ_j is finite. Define the spectral zeta function $\zeta_A(w,z)$ of Hurwitz's type by

$$\zeta_A(w,z) := \sum_{j=0}^{\infty} (\lambda_j + z)^{-w}.$$

We further assume that the series converges absolutely in some right half w-plane (uniformly for z on any compact set) and, for $r \in \mathbb{N}$, can be continued meromorphically to a region containing $w = 1 - r$. Moreover, we assume that it is holomorphic at $w = 1 - r$. In such a situation, we define a *higher depth determinant* of A of depth r by

$$\mathrm{Det}_r(A + z) := \exp\Bigl(-\frac{\partial}{\partial w}\zeta_A(w,z)\Big|_{w=1-r}\Bigr).$$

When $r = 1$, this gives the usual (zeta-regularized) determinant of A. Notice that if A is a finite-rank operator and $\lambda_1, \ldots, \lambda_N$ are their eigenvalues, then we have $\mathrm{Det}_r(A) = \prod_{j=1}^{N} \lambda_j^{\lambda_j^{r-1}}$, whence $\mathrm{Det}_r(A \oplus B) = (\mathrm{Det}_r A) \cdot (\mathrm{Det}_r B)$ if both A and B are finite-rank.

To state our main results, let us recall the case of the usual determinant of the Laplacian, that is, the case $r = 1$. Let $\mathbb{H}$ be the complex upper half plane with the Poincaré metric and Γ a discrete, co-compact torsion-free subgroup of $SL_2(\mathbb{R})$. Then, $\Gamma\backslash\mathbb{H}$ becomes a compact Riemann surface of genus $g \geq 2$. Let $\Delta_\Gamma = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ be the Laplacian on $\Gamma\backslash\mathbb{H}$ and λ_j the jth eigenvalue of Δ_Γ. We write $\lambda_j = r_j^2 + \frac{1}{4}$ where $r_j \in i\mathbb{R}_{>0}$ if $0 \leq \lambda_j \leq \frac{1}{4}$ and $r_j > 0$ otherwise. It is shown that the series $\zeta_{\Delta_\Gamma}(w,z)$ converges absolutely for $\mathrm{Re}\,(w) > 1$, admits a meromorphic continuation to the whole plane $\mathbb{C}$ and is in particular holomorphic at $w = 0$ (see, e.g., [DHP, S, Vo]). Moreover, the determinant $\det(\Delta_\Gamma - s(1-s)) := \mathrm{Det}_1(\Delta_\Gamma - s(1-s))$ can be calculated as

$$\det\bigl(\Delta_\Gamma - s(1-s)\bigr) = G_\Gamma(s) Z_\Gamma(s) = \phi(s)^{g-1} Z_\Gamma(s). \tag{1.1}$$

Here, $\phi(s)$ is a meromorphic function defined by

$$\begin{aligned}\phi(s) :&= e^{-2(s-\frac{1}{2})^2 - 4(s-\frac{1}{2})\zeta'(0) + 4\zeta'(-1)} \Gamma(s)^{-2} G(s)^{-4} \\ &= e^{-2(s-\frac{1}{2})^2} \Gamma_1(s)^{-2} \Gamma_2(s)^4\end{aligned}$$

with $\zeta(s)$, $\Gamma(s)$, $G(s)$ and $\Gamma_n(s)$ being the Riemann zeta function, the classical gamma function, the Barnes G-function (= a double gamma function ([B1])) and the Barnes multiple gamma function ([B2]), respectively. Another factor $Z_\Gamma(s)$ in (1.1) is the Selberg zeta function defined by the Euler product

$$Z_\Gamma(s) := \prod_{P \in \mathrm{Prim}(\Gamma)} \prod_{n=0}^{\infty} \bigl(1 - N(P)^{-s-n}\bigr) \qquad (\mathrm{Re}\,(s) > 1).$$

Here, $\mathrm{Prim}\,(\Gamma)$ is the set of all primitive hyperbolic conjugacy classes of Γ and $N(P)$ is the square of the larger eigenvalue of $P \in \mathrm{Prim}\,(\Gamma)$. It is known that $Z_\Gamma(s)$ can be continued analytically to the whole plane $\mathbb{C}$ and has the functional equation

$$Z_\Gamma(1-s) = \Big(S_1(s)^2 S_2(s)^{-4}\Big)^{g-1} Z_\Gamma(s), \tag{1.2}$$

where $S_n(s)$ is the normalized multiple sine function defined by (3.32) ([KK]). Notice that if we define the complete Selberg zeta function $\Xi_\Gamma(s)$ by

$$\Xi_\Gamma(s) := \big(\Gamma_2(s)^2\Gamma_2(s+1)^2\big)^{g-1} Z_\Gamma(s),$$

then, the functional equation (1.2) is equivalent to

$$\Xi_\Gamma(1-s) = \Xi_\Gamma(s). \tag{1.3}$$

Moreover, it is well known that $Z_\Gamma(s)$ has zeros at $s = 1,\ 0,\ -k$ for $k \geq 1$ with multiplicity 1, $2g-1$, $2(g-1)(2k+1)$, respectively (the latest are called the trivial zeros because they also come from the gamma factor as the Riemann zeta function) and at $s = \frac{1}{2} \pm ir_j$ with $j \geq 1$ (these are called the non-trivial zeros). In particular, $Z_\Gamma(s)$ satisfies an analogue of the Riemann hypothesis, i.e., all complex zeros of $Z_\Gamma(s)$ are located on the line $\mathrm{Re}\,(s) = \frac{1}{2}$. See [D] for arithmetic trial of a determinant expression for the Riemann zeta function.

In this paper, as generalizations of the case of $r = 1$, we study the function

$$\mathrm{D}_{\Gamma,r}(s) := \mathrm{Det}_r\big(\Delta_\Gamma - s(1-s)\big) := \exp\Big(-\frac{\partial}{\partial w}\zeta_{\Delta_\Gamma}(w, -s(1-s))\Big|_{w=1-r}\Big)$$

for general $r \in \mathbb{N}$. The following theorem describes our main results.

Theorem 1.1. *Let $r \geq 2$.*

(i) *The function $\mathrm{D}_{\Gamma,r}(s)$ is holomorphic in $\mathbb{C} \setminus [0,1]$ and satisfies $\mathrm{D}_{\Gamma,r}(1-s) = \mathrm{D}_{\Gamma,r}(s)$.*

(ii) *It can be written as*

$$\mathrm{D}_{\Gamma,r}(s) = G_{\Gamma,r}(s) Z_{\Gamma,r}(s) = \phi_r(s)^{g-1} Z_{\Gamma,r}(s). \tag{1.4}$$

Here, $\phi_r(s)$ is a holomorphic function in $\mathbb{C} \setminus (-\infty, 0]$ and can be written as a product of multiple gamma functions. Another factor $Z_{\Gamma,r}(s)$ is a holomorphic function in $\mathbb{C} \setminus (-\infty, 1]$ having a functional equation

$$Z_{\Gamma,r}(1-s) = \Big(\prod_{j=1}^{2r} S_j(s)^{-\alpha_{r,j}(s-\frac{1}{2})}\Big)^{g-1} Z_{\Gamma,r}(s) \qquad (s \in \mathbb{C} \setminus \mathbb{R}), \tag{1.5}$$

where $\alpha_{r,j}(t)$ is the polynomial defined in (3.28). Moreover, $Z_{\Gamma,r}(s)$ has an Euler product expression.

Similarly to the result of $r = 1$, we may call $G_{\Gamma,r}(s)$ (or $\phi_r(s)$) a gamma factor and $Z_{\Gamma,r}(s)$ a *Milnor-Selberg zeta function* of depth r, respectively. In fact, one can see that $\phi_1(s) = \phi(s)$, $Z_{\Gamma,1}(s) = Z_\Gamma(s)$, $\alpha_{1,1}(t) = -2$ and $\alpha_{1,2}(t) = 4$ (see Remark 3.10) and hence the equation (1.4) (resp. (1.5)) coincides with (1.1) (resp. (1.2)) when $r = 1$.

The organization of the paper is as follows. In Section 2, from the integral representation of the spectral zeta function, we first show the existence of $\mathrm{D}_{\Gamma,r}(s)$ and study its basic analytic properties. In particular, we obtain the claim (i) in Theorem 1.1 (Theorem 2.3). Next, using the Selberg trace formula, we derive the factorization (1.4). Then, in Section 3, we determine the explicit expression of the gamma factor $\phi_r(s)$ in terms of the Milnor gamma function $\mathbf{\Gamma}_r(z)$ (Proposition 3.6). Furthermore, we give other expressions by the Barnes multiple gamma functions $\Gamma_n(z)$ (Theorem 3.9) and the Vignéra multiple gamma functions $G_n(z)$ (Corollary 3.11), respectively. We notice that the latters take advantage of seeing analytic properties of $\phi_r(s)$. Employing ladder relations of the multiple gamma and multiple sine functions, we also give a functional equation of $\phi_r(s)$ (Theorem 3.13). In Section 4, via the expression (1.4), we first study analytic properties of the Milnor-Selberg zeta function $Z_{\Gamma,r}(s)$ such as an analytic continuation and a functional equation (Theorem 4.1). Moreover, we establish an Euler product expression of $Z_{\Gamma,r}(s)$ in the following sense; we define a "poly-Selberg zeta function" $Z_\Gamma^{(m)}(s)$, which is another generalization of $Z_\Gamma(s)$ defined by an Euler product. Then, it is shown that $Z_{\Gamma,r}(s)$ can be expressed as a product and quotient of $Z_\Gamma^{(m)}(s)$ (Theorem 4.12).

We find that, from the equation (1.4), remarkable cancellation of the singularities on $(-\infty,0)$ occurs (though both functions $G_{\Gamma,r}(s)$ and $Z_{\Gamma,r}(s)$ have singularities on the above interval, their product $\mathrm{D}_{\Gamma,r}(s)$ does not). Moreover, we notice that cancellation of the singularities of the poly-Selberg zeta functions $Z_\Gamma^{(m)}(s)$ also occurs in the above Euler product expression of $Z_{\Gamma,r}(s)$. In fact, although we show that $Z_\Gamma^{(m)}(s)$ has singularities on $(-\infty \pm ir_j, \frac{1}{2} \pm ir_j]$ for all j with $r_j > 0$ (Proposition 4.9), it follows from Theorem 1.1 that $Z_{\Gamma,r}(s)$ is holomorphic on these intervals. See also Remark 4.7.

In the course of the explicit determination of the gamma factor $\phi_r(s)$, we will encounter the following series involving the Hurwitz zeta function;

$$R_m(t,z) := \sum_{j=1}^{\infty} \frac{\zeta(2j+1,z)}{2j+m+1} t^{2j+m+1}.$$

This type of series has been studied in several places (see, e.g., [A, KS]). It will be shown that the exponential of such a series (with $z=\frac{1}{2}$) is expressed as a product of the Milnor gamma function (Proposition 3.4). We also note that a similar discussion developed in the paper yields explicit expressions of the higher depth determinants of the Laplacian on the higher dimensional spheres ([Y]). See also [WY] for number theoretic analogues of the present topic.

In this paper, as usual, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ are respectively denoted by the field of all complex, real and rational numbers. We also use the notations $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ to denote the set of all rational, positive and non-negative integers, respectively.

2 Higher depth determinants

2.1 Existence and basic properties

Let $\Delta_\Gamma = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ be the Laplacian on the compact Riemann surface $\Gamma\backslash\mathbb{H}$ of genus $g \geq 2$ and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ the spectra of Δ_Γ. For $T \geq 0$, define $\theta_{\Delta_\Gamma}^{(T)}(t) := \sum_{\lambda_j \geq T} e^{-\lambda_j t}$. It is known that $\theta_{\Delta_\Gamma}^{(T)}(t)$ converges absolutely for $\mathrm{Re}\,(t) > 0$ and has the asymptotic formula $\theta_{\Delta_\Gamma}^{(T)}(t) \sim \sum_{n=-1}^{\infty} c_n^{(T)} t^n$ for some $c_n^{(T)} \in \mathbb{R}$ with $c_{-1}^{(T)} \neq 0$ as $t \downarrow 0$ (see, e.g., [R]). Define $\zeta_{\Delta_\Gamma}^{(T)}(w,z) := \sum_{\lambda_j \geq T} (\lambda_j + z)^{-w}$. Here, we let $-\pi < \arg(z) \leq \pi$ and take $\arg(\lambda_j + z)$ to be $-\pi < \arg(\lambda_j + z) \leq \pi$ for all $j \in \mathbb{N}_0$. In particular, let $\zeta_{\Delta_\Gamma}(w,z) := \zeta_{\Delta_\Gamma}^{(0)}(w,z)$. It is also known that the series $\zeta_{\Delta_\Gamma}^{(T)}(w,z)$ converges absolutely and uniformly in any compact set in $\{w \in \mathbb{C} \,|\, \mathrm{Re}\,(w) > 1\} \times \{z \in \mathbb{C} \,|\, \mathrm{Re}\,(z) > -T\}$ and hence defines a holomorphic function in the region. One easily sees the following

Lemma 2.1. *The function $\zeta^{(T)}_{\Delta_\Gamma}(w,z)$ admits a meromorphic continuation to the region $\mathbb{C}\times\{z\in\mathbb{C}\,|\,\mathrm{Re}\,(z)>-T\}$ with a simple poles at $w=1$.*

Proof. Let $\mathrm{Re}\,(w)>1$ and $\mathrm{Re}\,(z)>-T$. Then, from the integral expression

$$\zeta^{(T)}_{\Delta_\Gamma}(w,z)=\frac{1}{\Gamma(w)}\int_0^\infty t^w e^{-zt}\theta^{(T)}_{\Delta_\Gamma}(t)\frac{dt}{t}, \tag{2.1}$$

for any $N\in\mathbb{N}_0$, we have

$$\begin{aligned}\zeta^{(T)}_{\Delta_\Gamma}(w,z)&=\frac{1}{\Gamma(w)}\int_0^1 t^w e^{-zt}\Big(\theta^{(T)}_{\Delta_\Gamma}(t)-\sum_{n=-1}^{N}c^{(T)}_n t^n\Big)\frac{dt}{t}\\&\quad+\frac{1}{\Gamma(w)}\int_0^1 t^w e^{-zt}\Big(\sum_{n=-1}^{N}c^{(T)}_n t^n\Big)\frac{dt}{t}+\frac{1}{\Gamma(w)}\int_1^\infty t^w e^{-zt}\theta^{(T)}_{\Delta_\Gamma}(t)\frac{dt}{t}\\&=\frac{1}{\Gamma(w)}\int_0^1 t^w e^{-zt}\Big(\theta^{(T)}_{\Delta_\Gamma}(t)-\sum_{n=-1}^{N}c^{(T)}_n t^n\Big)\frac{dt}{t}\\&\quad+\frac{1}{\Gamma(w)}\sum_{n=-1}^{N}c^{(T)}_n\int_0^1 t^{w+n-1}\Big(\sum_{m=0}^{\infty}\frac{(-zt)^m}{m!}\Big)dt+\frac{1}{\Gamma(w)}\int_1^\infty t^w e^{-zt}\theta^{(T)}_{\Delta_\Gamma}(t)\frac{dt}{t},\end{aligned}$$

whence

$$\begin{aligned}\zeta^{(T)}_{\Delta_\Gamma}(w,z)&=\frac{1}{\Gamma(w)}\int_0^1 t^w e^{-zt}\Big(\theta^{(T)}_{\Delta_\Gamma}(t)-\sum_{n=-1}^{N}c^{(T)}_n t^n\Big)\frac{dt}{t}\\&\quad+\frac{1}{\Gamma(w)}\sum_{n=-1}^{N}\sum_{m=0}^{\infty}\frac{c^{(T)}_n(-z)^m}{m!(w+n+m)}+\frac{1}{\Gamma(w)}\int_1^\infty t^w e^{-zt}\theta^{(T)}_{\Delta_\Gamma}(t)\frac{dt}{t}.\end{aligned} \tag{2.2}$$

Since $\theta^{(T)}_{\Delta_\Gamma}(t)-\sum_{n=-1}^{N}c^{(T)}_n t^n=O(t^{N+1})$ as $t\downarrow 0$, the first integral in the righthand-side of (2.2) converges absolutely for $\mathrm{Re}\,(w)>-(N+1)$. The second term defines a meromorphic function on $\mathbb{C}$ having a simple poles at $w=1$ (notice that the points $w=0,-1,-2,\dots$ are not poles of $\zeta^{(T)}_{\Delta_\Gamma}(w,z)$ because of the gamma factor). Moreover, since $\mathrm{Re}\,(z)>-T$, the last integral converges absolutely for all $w\in\mathbb{C}$, whence it defines an entire function as a function of w. Therefore, letting $N\to\infty$, we obtain a meromorphic continuation of $\zeta^{(T)}_{\Delta_\Gamma}(w,z)$ to $\mathbb{C}\times\{z\in\mathbb{C}\,|\,\mathrm{Re}\,(z)>-T\}$. $\square$

We now study the higher depth determinants. From Lemma 2.1, the function

$$\mathrm{Det}^{(T)}_r(\Delta_\Gamma+z):=\exp\Big(-\frac{\partial}{\partial w}\zeta^{(T)}_{\Delta_\Gamma}(w,z)\Big|_{w=1-r}\Big)$$

is well-defined and is holomorphic in $\{z\in\mathbb{C}\,|\,\mathrm{Re}\,(z)>-T\}$. Notice that, since $\frac{\partial}{\partial w}\zeta^{(T)}_{\Delta_\Gamma}(w,z)\big|_{w=1-r}$ is holomorphic, by the definition, $\mathrm{Det}^{(T)}_r(\Delta_\Gamma+z)$ has no zeros. Put $\mathrm{Det}_r(\Delta_\Gamma+z):=\mathrm{Det}^{(0)}_r(\Delta_\Gamma+z)$. We claim that $\mathrm{Det}_r(\Delta_\Gamma+z)$ admits an analytic continuation. Actually, let $\mathrm{Re}\,(z)>0$. Then, for any $T>0$, we have

$$\zeta_{\Delta_\Gamma}(w,z)=\sum_{0\le\lambda_j<T}(\lambda_j+z)^{-w}+\zeta^{(T)}_{\Delta_\Gamma}(w,z).$$

From Lemma 2.1, the righthand-side is holomorphic at $w=1-r$. Hence, differentiating the both hand sides at $w=1-r$, we have

$$\mathrm{Det}_r(\Delta_\Gamma+z)=\prod_{0\le\lambda_j<T}\exp\Big((\lambda_j+z)^{r-1}\log(\lambda_j+z)\Big)\cdot\mathrm{Det}^{(T)}_r(\Delta_\Gamma+z). \tag{2.3}$$

When $r=1$, since $\exp(\log(\lambda_j+z))=\lambda_j+z$, the first factor in the righthand-side of (2.3) is a polynomial and hence is entire. This shows that $\mathrm{Det}_1(\Delta_\Gamma+z)$ can be extended analytically to $\{z\in\mathbb{C}\,|\,\mathrm{Re}\,(z)>-T\}$. When $r\ge 2$, the first factor in this case is holomorphic in $\mathbb{C}\setminus(-\infty,0]$ (notice that the points $z=-\lambda_j$ for j with $0\le\lambda_j<T$ are branch points), whence $\mathrm{Det}_r(\Delta_\Gamma+z)$ can be also to $\{z\in\mathbb{C}\,|\,\mathrm{Re}\,(z)>-T\}\setminus(-T,0]$. Therefore, letting $T\to\infty$, one obtains the following

Proposition 2.2. *The function* $\mathrm{Det}_r(\Delta_\Gamma+z)$ *can be continued analytically to*

(i) *an entire function with zeros at* $z=-\lambda_j$ *when* $r=1$.

(ii) *a holomorphic function in* $\mathbb{C}\setminus(-\infty,0]$ *with no zeros when* $r\ge 2$. □

Based on the above discussions, we next study the case $z=-s(1-s)$. Consider the function $\zeta^{(T)}_{\Delta_\Gamma}(w,-s(1-s))=\sum_{\lambda_j\ge T}(\lambda_j-s(1-s))^{-w}$. Here, we also let $-\pi<\arg(s)\le\pi$ and take $\arg(\lambda_j-s(1-s))$ to be $-\pi<\arg(\lambda_j-s(1-s))\le\pi$ for all $j\in\mathbb{N}_0$. Write $\lambda_j=r_j^2+\frac14$ where $r_j\in i\mathbb{R}_{>0}$ if $0\le\lambda_j\le\frac14$ and $r_j>0$ otherwise. From an easy observation, for all j with $\lambda_j\ge T$, we see that the argument $\arg(\lambda_j-s(1-s))$ is continuous in $W^{(T)}$ where

$$W^{(T)}:=\begin{cases}\{s\in\mathbb{C}\,|\,\mathrm{Re}\,(s)>\frac12\}\setminus(\frac12,\alpha^{(T)}] & (0\le T\le\frac14),\\ \{s\in\mathbb{C}\,|\,\mathrm{Re}\,(s)>\frac12\} & (T>\frac14).\end{cases}$$

Here, we put $\alpha^{(T)}:=\frac12-ir_{j^{(T)}}$ for $0\le T\le\frac14$ with $j^{(T)}:=\min\{j\,|\,\lambda_j\ge T\}$ (for example, $\alpha^{(0)}=1$).

Let $U^{(T)}:=\{s\in\mathbb{C}\,|\,\mathrm{Re}\,(-s(1-s))>-T\}$ (see Figures 1, 2 and 3 for $T=0$, $T=\frac14$ and $T>\frac14$, respectively). Notice that $U^{(T)}$ has two connected components if $0\le T\le\frac14$ and is connected otherwise. Clearly, $\lim_{T\to\infty}U^{(T)}=\mathbb{C}$. From Lemma 2.1, the function $\zeta^{(T)}_{\Delta_\Gamma}(w,-s(1-s))$ is meromorphic in $\mathbb{C}\times(W^{(T)}\cap U^{(T)})$ and is in particular holomorphic at $w=1-r$ for all $r\in\mathbb{N}$. Moreover, for $T>\frac14$, we see that it is also continuous at $\mathrm{Re}\,(s)=\frac12$ because, for all j with $\lambda_j\ge T$, $\arg(\lambda_j-s(1-s))=0$ if $\mathrm{Re}\,(s)=\frac12$.

Let $D^{(T)}_{\Gamma,r}(s):=\mathrm{Det}^{(T)}_r(\Delta_\Gamma-s(1-s))$, which is holomorphic in $W^{(T)}\cap U^{(T)}$. Similarly to the previous discussion, an analytic continuation of $D_{\Gamma,r}(s):=D^{(0)}_{\Gamma,r}(s)$ is given as follows. Let $s\in W\cap U$ where $W:=W^{(0)}$ and $U:=U^{(0)}$. Then, for any $T>\frac14$, we have

$$\mathrm{D}_{\Gamma,r}(s)=e^{(T)}_{\Gamma,r}(s)\cdot\mathrm{D}^{(T)}_{\Gamma,r}(s),\tag{2.4}$$

where

$$e^{(T)}_{\Gamma,r}(s):=\prod_{0\le\lambda_j<T}\exp\Big(\big(\lambda_j-s(1-s)\big)^{r-1}\log\big(\lambda_j-s(1-s)\big)\Big).$$

Since $-s(1-s)$ is symmetric with respect to s and $1-s$, $\mathrm{D}^{(T)}_{\Gamma,r}(s)$ can be analytically continued to $U^{(T)}$ via $\mathrm{D}^{(T)}_{\Gamma,r}(s)=\mathrm{D}^{(T)}_{\Gamma,r}(1-s)$ (we also use the continuity of $\mathrm{D}^{(T)}_{\Gamma,r}(s)$ at $\mathrm{Re}\,(s)=\frac12$). Similarly, $e^{(T)}_{\Gamma,r}(s)$ admits an analytic continuation. Indeed, when $r=1$, since $e^{(T)}_{\Gamma,1}(s)$ is again a polynomial, it is naturally extended to $\mathbb{C}$ as an entire function satisfying $e^{(T)}_{\Gamma,1}(s)=e^{(T)}_{\Gamma,1}(1-s)$. Now let $r\ge 2$. Notice that $e^{(T)}_{\Gamma,r}(s)$ has branch points at $s=\frac12\pm ir_j$ for j with $0\le\lambda_j<T$. By the same idea as above, via $e^{(T)}_{\Gamma,r}(s)=e^{(T)}_{\Gamma,r}(1-s)$, it can be extended to $\mathbb{C}\setminus([0,1]\cup\{\frac12\pm ir_j\,|\,j \text{ with } \frac14<\lambda_j<T\})$ Moreover, since $\lim_{s\to0}s^{r-1}\log s=0$, we see that the points $s=\frac12\pm ir_j$ are removable. Therefore, one can conclude that $e^{(T)}_{\Gamma,r}(s)$ is eventually continued to $\mathbb{C}\setminus[0,1]$. As a consequence, from (2.4), $\mathrm{D}_{\Gamma,r}(s)$ is continued to $U^{(T)}$ if $r=1$ and $U^{(T)}\setminus[0,1]$ otherwise. Therefore, letting $T\to\infty$, one obtains the following

Theorem 2.3. *The function* $\mathrm{D}_{\Gamma,r}(s)$ *can be continued analytically to*

(i) *an entire function with zeros at* $s = \frac{1}{2} \pm i r_j$ *when* $r = 1$.

(ii) *a holomorphic function in* $\mathbb{C} \setminus [0,1]$ *with no zeros when* $r \geq 2$.

Moreover, $\mathrm{D}_{\Gamma,r}(s)$ *satisfies the functional equation* $\mathrm{D}_{\Gamma,r}(1-s) = \mathrm{D}_{\Gamma,r}(s)$. □

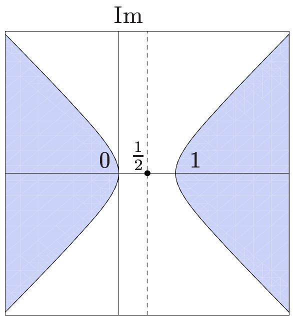

Figure 1: $U = U^{(0)}$.

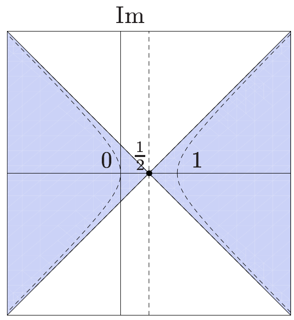

Figure 2: $U^{(\frac{1}{4})}$.

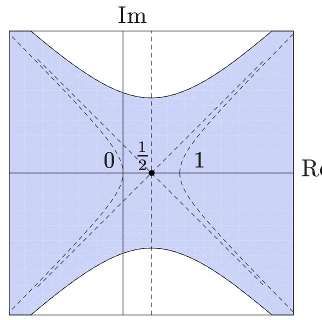

Figure 3: $U^{(T)}$ with $T > \frac{1}{4}$.

2.2 Initial calculations via the trace formula

In this subsection, we study the function $\zeta_{\Delta_\Gamma}(w, -s(1-s))$ by using the Selberg trace formula

$$\sum_{j=0}^{\infty} \hat{f}(r_j) = \sum_{\gamma \in \mathrm{Hyp}\,(\Gamma)} \frac{\log N(\delta_\gamma)}{N(\gamma)^{\frac{1}{2}} - N(\gamma)^{-\frac{1}{2}}} f\big(\log N(\gamma)\big) + (g-1) \int_{-\infty}^{\infty} \hat{f}(r) r \tanh(\pi r) dr. \tag{2.5}$$

Here, $\mathrm{Hyp}\,(\Gamma)$ is the set of all hyperbolic conjugacy classes in Γ, $\delta_\gamma \in \mathrm{Prim}\,(\Gamma)$ for $\gamma \in \mathrm{Hyp}\,(\Gamma)$ is the unique element satisfying $\gamma = \delta_\gamma^k$ for some $k \geq 1$ and f is a test function whose Fourier transform $\hat{f}(r) := \int_{-\infty}^{\infty} f(x) e^{-irx} dx$ satisfies the conditions $\hat{f}(-r) = \hat{f}(r)$, $\hat{f}$ is holomorphic in the band $\{r \in \mathbb{C} \,|\, |\mathrm{Im}\, r| < \frac{1}{2} + \delta\}$ and $\hat{f}(r) = O(|r|^{-2-\delta})$ as $|r| \to \infty$ for some $\delta > 0$,

For $r \in \mathbb{N}$, let $\mathrm{Re}\,(w) > r$ and $s \in W \cap U$. Throughout the present paper, we always denote by $t = t(s) := s - \frac{1}{2}$. Notice that if $s \in W \cap U$, then $\mathrm{Re}\,(t^2) > \frac{1}{4}$. From (2.1) with $T = 0$, we have

$$\zeta_{\Delta_\Gamma}\big(w+1-r, -s(1-s)\big) = \frac{1}{\Gamma(w+1-r)} \int_0^{\infty} \xi^{w+1-r} e^{-t^2 \xi} \left(\sum_{j=0}^{\infty} e^{-r_j^2 \xi} \right) \frac{d\xi}{\xi}.$$

Applying the trace formula (2.5) with the test function $f(x) = \frac{1}{2\sqrt{\pi\xi}} e^{-\frac{x^2}{4\xi}}$ (then $\hat{f}(r) = e^{-r^2 \xi}$) to the inner sum in the above integral, changing the order of the integrations, we have

$$\begin{aligned} \zeta_{\Delta_\Gamma}\big(w+1-r, -s(1-s)\big) &= \Theta_{\Gamma,r}(w,s) + (g-1) \int_{-\infty}^{\infty} \big(x^2 + t^2\big)^{-w+r-1} x \tanh(\pi x) dx \\ &= \Theta_{\Gamma,r}(w,s) + 2(g-1) \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} t^{2(r-1-\ell)} J_{2\ell+1}(w,t), \end{aligned} \tag{2.6}$$

where

$$\Theta_{\Gamma,r}(w,s) := \frac{1}{\Gamma(w+1-r)} \int_0^{\infty} \xi^{w+1-r} \theta_\Gamma(\xi, s) \frac{d\xi}{\xi}$$

with

$$\theta_\Gamma(\xi, s) := \frac{1}{2\sqrt{\pi\xi}} \sum_{\gamma \in \mathrm{Hyp}\,(\Gamma)} \frac{\log N(\delta_\gamma)}{N(\gamma)^{\frac{1}{2}} - N(\gamma)^{-\frac{1}{2}}} e^{-t^2 \xi - \frac{(\log N(\gamma))^2}{4\xi}}$$

and

$$J_m(w,t) := \int_0^\infty \varphi_m(x;t,w)dx \tag{2.7}$$

with

$$\varphi_m(x;t,w) := (x^2+t^2)^{-w} x^m \tanh(\pi x).$$

Notice that, when $s \in W \cap U$, since there exists a constant $\varepsilon_\Gamma > 0$ such that $\log N(\gamma) > \varepsilon_\Gamma$ for all $\gamma \in \mathrm{Hyp}\,(\Gamma)$, the integral in $\Theta_{\Gamma,r}(w,s)$ converges absolutely for all $w \in \mathbb{C}$ and hence $\Theta_{\Gamma,r}(w,s)$ defines an entire function as a function of w. Moreover, we will show in the next section that $J_m(w,t)$ can be continued meromorphically to the whole plane $\mathbb{C}$ and is in particular holomorphic at $w=0$. Therefore, differentiating the both hand sides of (2.6) at $w=0$, we see that the higher depth determinant

$$\begin{aligned}
\mathrm{D}_{\Gamma,r}(s) &= \exp\Bigl(-\frac{\partial}{\partial w}\zeta_{\Delta_\Gamma}(w,-s(1-s))\Big|_{w=1-r}\Bigr) \\
&= \exp\Bigl(-\frac{\partial}{\partial w}\zeta_{\Delta_\Gamma}(w+1-r,-s(1-s))\Big|_{w=0}\Bigr)
\end{aligned}$$

can be expressed as

$$\mathrm{D}_{\Gamma,r}(s) = G_{\Gamma,r}(s) Z_{\Gamma,r}(s), \tag{2.8}$$

where $G_{\Gamma,r}(s) := \phi_r(s)^{g-1}$ with

$$\phi_r(s) := \prod_{\ell=0}^{r-1} \exp\Bigl(-\frac{\partial}{\partial w} J_{2\ell+1}(w,t)\Big|_{w=0}\Bigr)^{2\binom{r-1}{\ell} t^{2(r-1-\ell)}} \tag{2.9}$$

and

$$Z_{\Gamma,r}(s) : = \exp\Bigl(-\frac{\partial}{\partial w}\Theta_{\Gamma,r}(w,s)\Big|_{w=0}\Bigr). \tag{2.10}$$

Our next task is to examine each factor $\phi_r(s)$ and $Z_{\Gamma,r}(s)$ more precisely.

3 Gamma factors

3.1 Calculation of the derivative of $J_m(w,t)$ at $w=0$

To obtain an explicit expression of the gamma factor $\phi_r(s)$, in this subsection, we study the function $J_m(w,t)$ defined by (2.7). Notice that the integral $J_m(w,t)$ converges absolutely and uniformly in any compact set in $\{w \in \mathbb{C} \,|\, \mathrm{Re}\,(w) > \frac{m+1}{2}\} \times \{s \in \mathbb{C} \,|\, \mathrm{Re}\,(s) > \frac{1}{2}\}$ and hence define a holomorphic function in the region. In what follows, for simplicity, we assume that $\frac{1}{2} < s < 1$, which implies that $0 < t < \frac{1}{2}$. Moreover, from the definition (2.9), we may assume that m is odd.

Suppose $\mathrm{Re}\,(w) > \frac{m+1}{2}$. Let R be a sufficiently large positive integer and ε a small real number. Consider the counterclockwise contour integral with the integrand $\varphi_m(z;t,w)$ on the z-plane (we set $-\pi < \arg(z) \le \pi$) along the path

$$\mathcal{C}(t;\varepsilon,R) := [+0,R] \sqcup C^+(0;R) \sqcup [-R,-0] \sqcup L_-(t;\varepsilon) \sqcup C(it;\varepsilon) \sqcup L_+(t;\varepsilon),$$

where $C^+(0;R)$ is the semi-circle in the upper half plane from R to $-R$, $C(it;\varepsilon)$ is the circle with radius ε centered at it and $L_+(t;\varepsilon)$ (resp. $L_-(t;\varepsilon)$) is the right (resp. left) side of the segment

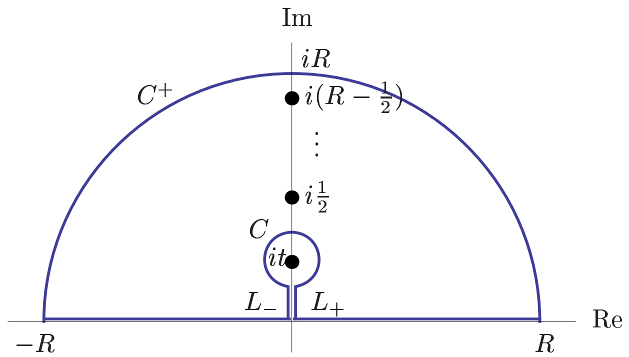

Figure 4: $\mathcal{C}(t;\varepsilon,R)$.

connecting 0 and $(1-\varepsilon)it$. We take ε so small that the circle $C(it;\varepsilon)$ contains no points of the form $i(k+\frac{1}{2})$, that is, the poles of $\tanh(\pi z)$, for all $k\in\mathbb{Z}_{\geq 0}$ (Figure 4).

The residue theorem yields

$$\int_{\mathcal{C}(t;\varepsilon,R)}\varphi_m(z;t,w)dz = 2i^{m+1}e^{-\pi i w}\sum_{k=0}^{R-1}\Big(k+\frac{1}{2}\Big)^{-2w+m}\Big(1-t^2\Big(k+\frac{1}{2}\Big)^{-2}\Big)^{-w}. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned}\int_{\mathcal{C}(t;\varepsilon,R)}\varphi_m(z;t,w)dz = &\int_{C^+(0;R)}\varphi_m(z;t,w)dz + \Big(\int_{[-R,-0]}+\int_{[+0,R]}\Big)\varphi_m(z;t,w)dz \\ &+ A_m(w;t,\varepsilon)+B_m(w;t,\varepsilon),\end{aligned} \tag{3.2}$$

where

$$A_m(w;t,\varepsilon) := \Big(\int_{L_+(t;\varepsilon)}+\int_{L_-(t;\varepsilon)}\Big)\varphi_m(z;t,w)dz, \quad B_m(w;t,\varepsilon) := \int_{C(it;\varepsilon)}\varphi_m(z;t,w)dz.$$

Now, let us calculate each integral in the righthand-side of (3.2). At first, it is easy to see that the integral on $C^+(0;R)$ converges to 0 as $R\to+\infty$ because it is $O(R^{m+1-2\mathrm{Re}(w)})$. Next, since $\arg(z^2+t^2)=0$ (resp. 2π) if $z\in[+0,R]$ (resp. $z\in[-R,-0]$), we have

$$\begin{aligned}\Big(\int_{[-R,-0]}+\int_{[+0,R]}\Big)\varphi_m(z;t,w)dz &= (1+e^{-2\pi i w})\int_0^R\varphi_m(x;t,w)dx. \\ &= 2e^{-\pi i w}\cos(\pi w)\int_0^R\varphi_m(x;t,w)dx.\end{aligned}$$

Here, we have used the fact that m is odd. This shows that

$$\lim_{R\to+\infty}\Big(\int_{[-R,-0]}+\int_{[+0,R]}\Big)\varphi_m(z;t,w)dz = 2e^{-\pi i w}\cos(\pi w)J_m(w,t).$$

Finally, let us calculate $A_m(w;t,\varepsilon)$ and $B_m(w;t,\varepsilon)$. Notice that both functions are entire as functions of w and, by the Cauchy integral theorem, do not depend on the choice of ε. Since

$\arg(z^2+t^2)=0$ (resp. 2π) if $z\in L_+(t;\varepsilon)$ (resp. $z\in L_-(t;\varepsilon)$), we have

$$\begin{aligned} A_m(w;t,\varepsilon) &= (1-e^{-2\pi i w})\int_0^{(1-\varepsilon)t}\varphi_m(i\xi;t,w)id\xi \\ &= 2i^{m+1}t^{-2w}e^{-\pi i w}\sin(\pi w)\int_0^{(1-\varepsilon)t}\Big(1-\frac{\xi^2}{t^2}\Big)^{-w}\xi^m\tan(\pi\xi)d\xi, \end{aligned}$$

whence

$$A_m(w;t,\varepsilon)=\lim_{\varepsilon\to 0}A_m(w;t,\varepsilon)=2i^{m+1}t^{-2w}e^{-\pi i w}\sin(\pi w)\int_0^{t}\Big(1-\frac{\xi^2}{t^2}\Big)^{-w}\xi^m\tan(\pi\xi)d\xi.$$

Moreover, by straightforward calculations, we have $B_m(w;t,\varepsilon)=O(\varepsilon^{1-\mathrm{Re}\,(w)})$ as $\varepsilon\to 0$ and hence $B_m(w;t,\varepsilon)=\lim_{\varepsilon\to 0}B_m(w;t,\varepsilon)=0$ for $\mathrm{Re}\,(w)<1$. This shows that $B_m(w;t,\varepsilon)=0$ for any $w\in\mathbb{C}$ by the identity theorem. Therefore, letting $R\to+\infty$ in the formulas (3.1) and (3.2), we have

$$J_m(w,t)=\frac{i^{m+1}}{\cos(\pi w)}\Big(J_{m,1}(w,t)+J_{m,2}(w,t)\Big), \tag{3.3}$$

where

$$J_{m,1}(w,t):=-t^{-2w}\sin(\pi w)\int_0^t\Big(1-\frac{\xi^2}{t^2}\Big)^{-w}\xi^m\tan(\pi\xi)d\xi, \tag{3.4}$$

$$J_{m,2}(w,t):=\sum_{k=0}^{\infty}\Big(k+\frac{1}{2}\Big)^{-2w+m}\Big(1-t^2\Big(k+\frac{1}{2}\Big)^{-2}\Big)^{-w}. \tag{3.5}$$

The function $J_{m,1}(w,t)$ is clearly entire as a function of w. Let us calculate the derivative of $J_{m,1}(w,t)$ at $w=0$. To do that, we need the following basic multiple trigonometric functions ([KOW1], see also [KK]). Put $P_1(u):=(1-u)$ and $P_n(u):=(1-u)\exp(u+\frac{u^2}{2}+\cdots+\frac{u^n}{n})$ for $n\geq 2$. Then, the basic multiple sine $\mathcal{S}_n(z)$ and cosine function $\mathcal{C}_n(z)$ are respectively defined by

$$\mathcal{S}_n(z):=\begin{cases} 2\pi z\displaystyle\prod_{\substack{m\in\mathbb{Z}\\ m\neq 0}}P_1\Big(\frac{z}{m}\Big)=2\pi z\prod_{m=1}^{\infty}\Big(1-\frac{z^2}{m^2}\Big)=2\sin(\pi z) & (n=1), \\ \exp\Big(\dfrac{z^{n-1}}{n-1}\Big)\displaystyle\prod_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\Big(P_n\Big(\frac{z}{m}\Big)P_n\Big(-\frac{z}{m}\Big)^{(-1)^{n-1}}\Big)^{m^{n-1}} & (n\geq 2), \end{cases}$$

$$\mathcal{C}_n(z):=\prod_{\substack{m\in\mathbb{Z}\\ m:\mathrm{odd}}}P_n\Big(\frac{z}{(\frac{m}{2})}\Big)^{(\frac{m}{2})^{n-1}}=\mathcal{S}_n(2z)^{2^{1-n}}\mathcal{S}_n(z)^{-1}.$$

When $n\geq 2$, we have $\mathcal{S}_n(0)=\mathcal{C}_n(0)=1$ and, for $z\neq 0$, the integral expressions

$$\mathcal{S}_n(z)=\exp\Big(\int_0^z\pi\xi^{n-1}\cot(\pi\xi)d\xi\Big),\quad \mathcal{C}_n(z)=\exp\Big(-\int_0^z\pi\xi^{n-1}\tan(\pi\xi)d\xi\Big), \tag{3.6}$$

where the paths lie in $\mathbb{C}\backslash\{\pm1,\pm2,\ldots\}$.

Proposition 3.1. *We have*

$$\frac{\partial}{\partial w}J_{m,1}(w,t)\Big|_{w=0}=\log\mathcal{C}_{m+1}(t). \tag{3.7}$$

Proof. From (3.4), we have

$$\begin{aligned}J_{m,1}(w,t) &= -\big(1+O(w)\big)\big(\pi w+O(w^3)\big)\int_0^t \big(1+O(w)\big)\xi^m \tan(\pi\xi)d\xi \\ &= \Big(-\int_0^t \pi\xi^m \tan(\pi\xi)d\xi\Big)w+O(w^2).\end{aligned}$$

Hence, from (3.6), one obtains the desired claim. □

We next study the function $J_{m,2}(w,t)$. We first show the following

Lemma 3.2. *It holds that*

$$J_{m,2}(w,t)=\sum_{j=0}^{\infty}\binom{w+j-1}{j}t^{2j}\zeta\big(2w+2j-m,\frac{1}{2}\big). \tag{3.8}$$

This gives a meromorphic continuation to the whole plane $\mathbb{C}$ as a function of w with possible simple poles at $w=\frac{m+1}{2}-j$ for $j\in\mathbb{Z}_{\geq 0}$. In particular, it is holomorphic at $w=0$.

Proof. Notice that, since $0<t<\frac{1}{2}$ (recall that we assume that $\frac{1}{2}<s<1$), we have $|t^2(k+\frac{1}{2})^{-2}|<1$ for all $k\geq 0$. Therefore, from (3.5), using the binomial theorem, we have

$$\begin{aligned}J_{m,2}(w,t)&=\sum_{k=0}^{\infty}\big(k+\frac{1}{2}\big)^{-2w+m}\sum_{j=0}^{\infty}\binom{w+j-1}{j}t^{2j}\big(k+\frac{1}{2}\big)^{-2j}\\ &=\sum_{j=0}^{\infty}\binom{w+j-1}{j}t^{2j}\zeta\big(2w+2j-m,\frac{1}{2}\big).\end{aligned}$$

Hence one obtains the expression (3.8). Moreover, since $\zeta(2w+2j-m,\frac{1}{2})$ is uniformly bounded with respect to j, this gives a meromorphic continuation to $\mathbb{C}$. Now the rest of assertions are clear because the Hurwitz zeta function $\zeta(w,z)$ has a simple pole at $w=1$. Notice that the origin is not a pole of $J_{m,2}(w,t)$ since, when $j=\frac{m+1}{2}\geq 1$, $\binom{w+j-1}{j}=O(w)$ as $w\to 0$. □

Before calculating the derivative of $J_{m,2}(w,t)$ at $w=0$, let us recall the multiple gamma functions, which is obtained from the Barnes multiple zeta function ([B2]) defined by

$$\zeta_n(w,z):=\sum_{m_1,\ldots,m_n\geq 0}\frac{1}{(m_1+\cdots+m_n+z)^w}\qquad (\mathrm{Re}\,(w)>n).$$

This clearly gives a generalization of the Hurwitz zeta function; $\zeta_1(w,z)=\zeta(w,z)$. It is known that $\zeta_n(w,z)$ can be continued meromorphically to the whole plane $\mathbb{C}$ with possible simple poles at $w=1,2,\ldots,n$. The multiple gamma function $\Gamma_{n,r}(z)$ of depth r is defined by

$$\Gamma_{n,r}(z):=\exp\Big(\frac{\partial}{\partial w}\zeta_n(w,z)\Big|_{w=1-r}\Big).$$

In particular, we put $\Gamma_n(z):=\Gamma_{n,1}(z)$ and $\mathbf{\Gamma}_r(z):=\Gamma_{1,r}(z)$. These are respectably called the Barnes multiple gamma function ([B2]) and the Milnor gamma function of depth r ([M], see also [KOW2]). From the Lerch formula $\frac{\partial}{\partial w}\zeta(w,z)\big|_{w=0}=\log\frac{\Gamma(z)}{\sqrt{2\pi}}$, we have $\Gamma_{1,1}(z)=\Gamma_1(z)=\mathbf{\Gamma}_1(z)=\frac{\Gamma(z)}{\sqrt{2\pi}}$, whence these in fact give generalizations of the classical gamma function. We remark that $\Gamma_n(z)^{-1}$ is an entire function with zeros at $z=-k$ of order $\binom{k+n-1}{n-1}$ for $k\in\mathbb{N}_0$.

From the expression (3.8), it can be written as

$$\begin{aligned} J_{m,2}(w,t) &= \zeta\big(2w-m,\frac{1}{2}\big) + \sum_{j=1}^{\frac{m-1}{2}} \binom{w+j-1}{j} t^{2j} \zeta\big(2w+2j-m,\frac{1}{2}\big) \\ &\quad + \binom{w+\frac{m+1}{2}-1}{\frac{m+1}{2}} t^{m+1} \zeta\big(2w+1,\frac{1}{2}\big) + \sum_{j=\frac{m+3}{2}}^{\infty} \binom{w+j-1}{j} t^{2j} \zeta\big(2w+2j-m,\frac{1}{2}\big) \\ &=: T_1(w,t)+T_2(w,t)+T_3(w,t)+T_4(w,t). \end{aligned}$$

Then, using the expansions $\binom{w+j-1}{j} = \frac{1}{j}(w + H(j-1)w^2 + O(w^3))$ as $w \to 0$ for $j \geq 1$ where $H(m) := \sum_{k=1}^{m} \frac{1}{k}$ and $\zeta(w,z) = \frac{1}{w-1} - \psi(z) + O(w-1)$ as $w \to 1$ where $\psi(z) := \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'}{\Gamma}(z)$ is the digamma function and employing the formula $\zeta(1-m,z) = -\frac{B_m(z)}{m}$ for $m \in \mathbb{N}$ where $B_m(z)$ is the Bernoulli polynomial defined by $\frac{xe^{zx}}{e^x-1} = \sum_{m=0}^{\infty} B_m(z)\frac{x^m}{m!}$, we have

$$\begin{aligned} T_1(w,t) &= \zeta\big(-m,\frac{1}{2}\big) + \frac{\partial}{\partial w}\zeta\big(2w-m,\frac{1}{2}\big)\Big|_{w=0} w + O(w^2) \\ &= -\frac{B_{m+1}(\frac{1}{2})}{m+1} + \left(2\log\mathbf{\Gamma}_{m+1}\big(\frac{1}{2}\big)\right)w + O(w^2), \end{aligned} \tag{3.9}$$

$$\begin{aligned} T_2(w,t) &= \sum_{j=1}^{\frac{m-1}{2}} \frac{1}{j}\big(w+O(w^2)\big)t^{2j}\Big(\zeta\big(2j-m,\frac{1}{2}\big)+O(w)\Big) \\ &= \left(-\sum_{j=1}^{\frac{m-1}{2}} \frac{B_{m+1-2j}(\frac{1}{2})}{j(m+1-2j)} t^{2j}\right) w + O(w^2), \end{aligned} \tag{3.10}$$

$$\begin{aligned} T_3(w,t) &= \frac{t^{m+1}}{\frac{m+1}{2}}\Big(w + H\big(\frac{m-1}{2}\big)w^2 + O(w^3)\Big)\Big(\frac{1}{2w} - \psi\big(\frac{1}{2}\big) + O(w^2)\Big) \\ &= \frac{1}{m+1}t^{m+1} + \left(\frac{2}{m+1}\Big(\frac{1}{2}H\big(\frac{m-1}{2}\big) - \psi\big(\frac{1}{2}\big)\Big)t^{m+1}\right)w + O(w^2), \end{aligned} \tag{3.11}$$

$$\begin{aligned} T_4(w,t) &= \sum_{j=\frac{m+3}{2}}^{\infty} \frac{1}{j}\big(w+O(w^2)\big)t^{2j}\Big(\zeta\big(2j-m,\frac{1}{2}\big)+O(w)\Big) \\ &= \big(2R_m(t)\big)w + O(w^2) \end{aligned} \tag{3.12}$$

as $w \to 0$. Here, $R_m(t)$ is defined by

$$R_m(t) := \sum_{j=1}^{\infty} \frac{\zeta\big(2j+1,\frac{1}{2}\big)}{2j+m+1} t^{2j+m+1}. \tag{3.13}$$

This yields the following

Proposition 3.3. *We have*

$$\begin{aligned} \frac{\partial}{\partial w}J_{m,2}(w,t)\Big|_{w=0} &= 2\sum_{k=1}^{m+1}(-1)^k\binom{m}{k-1}t^{m+1-k}\log\mathbf{\Gamma}_k\big(t+\frac{1}{2}\big) \\ &\quad - \log\mathcal{C}_{m+1}(t) + \frac{1}{m+1}\Big(H\big(\frac{m-1}{2}\big) - 2H(m)\Big)t^{m+1}. \end{aligned} \tag{3.14}$$

Proof. This follows from the identity $\frac{\partial}{\partial w}J_{m,2}(w,t)\big|_{w=0}=\sum_{j=1}^{4}\frac{\partial}{\partial w}T_j(w,t)\big|_{w=0}$ together with (3.9), (3.10), (3.11), (3.12) and the following proposition, which will be proved in Subsection 3.4. □

Proposition 3.4. *Let $m \geq 1$ be an odd integer. Then, we have*

$$R_m(t) = \sum_{k=1}^{m+1}(-1)^k\binom{m}{k-1}t^{m+1-k}\log\mathbf{\Gamma}_k\big(t+\frac{1}{2}\big) - \frac{1}{2}\log\mathcal{C}_{m+1}(t) \\ - \frac{1}{m+1}\Big(H(m)-\psi\big(\frac{1}{2}\big)\Big)t^{m+1} + \frac{1}{2}\sum_{j=1}^{\frac{m-1}{2}}\frac{B_{m+1-2j}(\frac{1}{2})}{j(m+1-2j)}t^{2j} - \log\mathbf{\Gamma}_{m+1}\big(\frac{1}{2}\big). \tag{3.15}$$

From the equation (3.3), we have $\frac{\partial}{\partial w}J_m(w,t)\big|_{w=0}= i^{m+1}(\frac{\partial}{\partial w}J_{m,1}(w,t)\big|_{w=0}+\frac{\partial}{\partial w}J_{m,2}(w,t)\big|_{w=0})$ (notice that $\cos(\pi w)^{-1}=1+O(w^2)$ as $w\to 0$). Therefore, substituting (3.7) and (3.14) into this equation, one finally obtains the derivatives of $J_m(w,t)$ at $w=0$.

Proposition 3.5. *It holds that*

$$\frac{\partial}{\partial w}J_m(w,t)\Big|_{w=0} = \frac{i^{m+1}}{m+1}\Big(H\big(\frac{m-1}{2}\big)-2H(m)\Big)t^{m+1} \\ + 2i^{m+1}\sum_{k=1}^{m+1}(-1)^k\binom{m}{k-1}t^{m+1-k}\log\mathbf{\Gamma}_k\big(t+\frac{1}{2}\big). \tag{3.16}$$

□

3.2 Explicit expressions of $\phi_r(s)$

We obtain the following expression of $\phi_r(s)$ by the Milnor gamma functions.

Proposition 3.6. *We have*

$$\phi_r(s) = e^{-\frac{(2r)!!}{r^2(2r-1)!!}(s-\frac{1}{2})^{2r}}\prod_{k=r}^{2r}\mathbf{\Gamma}_k(s)^{\binom{r}{k-r}\frac{2k}{r}(-1)^{k+r-1}(2s-1)^{2r-k}}. \tag{3.17}$$

To prove this, we need the following lemmas about sums of the binomial coefficients.

Lemma 3.7. *Let $r\in\mathbb{N}$. Then, the following equality holds;*

$$C_r := \sum_{\ell=0}^{r-1}\binom{r-1}{\ell}\frac{(-1)^\ell}{\ell+1}\big(H(\ell)-2H(2\ell+1)\big) = -\frac{(2r)!!}{r^2(2r-1)!!}. \tag{3.18}$$

Proof. Since $H(\ell)-2H(2\ell+1) = -2\sum_{k=0}^{\ell}\frac{1}{2k+1}$ (we understand that $H(\ell)=0$ for $\ell\leq 0$), changing the order of the summations, we have

$$C_r = -2\sum_{k=0}^{r-1}\frac{1}{2k+1}\sum_{\ell=k}^{r-1}\binom{r-1}{\ell}\frac{(-1)^\ell}{\ell+1} = -\frac{2}{r}\sum_{k=0}^{r-1}\binom{r-1}{k}\frac{(-1)^k}{2k+1}. \tag{3.19}$$

Here, we have used the formula $\sum_{\ell=k}^{r-1}\binom{r-1}{\ell}\frac{(-1)^\ell}{\ell+1}=\frac{(-1)^k}{r}\binom{r-1}{k}$, which is easily obtained by induction on k. Therefore, since the sum on the rightmost-hand side of (3.19) is equal to the beta integral

$$\int_0^1(1-x^2)^{r-1}dx = \frac{1}{2}B\big(\frac{1}{2},r\big) = \frac{\Gamma(\frac{1}{2})\Gamma(r)}{2\Gamma(r+\frac{1}{2})} = \frac{(2r)!!}{2r(2r-1)!!},$$

one obtains the desired formula. □

Lemma 3.8. (i) *Let $r\in\mathbb{N}$ and $1\le k\le 2r$. Then, the following equality holds;*

$$D_r(k):=\sum_{\ell=\lfloor\frac{k-1}{2}\rfloor}^{r-1}4(-1)^{\ell+k}\binom{r-1}{\ell}\binom{2\ell+1}{k-1}=\binom{r}{k-r}\frac{2k}{r}(-1)^{k+r-1}2^{2r-k}. \tag{3.20}$$

In particular, $D_r(k)=0$ if $1\le k\le r-1$.

(ii) *For $1\le p\le 2r$, we have*

$$\widetilde{D}_{r,p}:=\sum_{k=p}^{2r}\binom{k-1}{k-p}D_r(k)=\begin{cases}0 & (p:\text{odd}),\\ 4\binom{r-1}{\frac{p}{2}-1}(-1)^{\frac{p}{2}-1} & (p:\text{even}).\end{cases} \tag{3.21}$$

Proof. From the binomial expansion $(1-t^2)^{r-1}=\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}t^{2\ell}$, one sees that

$$\begin{aligned}D_r(k)&=\frac{4(-1)^k}{(k-1)!}\frac{d^{k-1}}{dt^{k-1}}\Big(t(1-t^2)^{r-1}\Big)\Big|_{t=1}\\&=\frac{4(-1)^k}{(k-1)!}\left[\frac{d^{k-1}}{dt^{k-1}}(1-t^2)^{r-1}\Big|_{t=1}+(k-1)\frac{d^{k-2}}{dt^{k-2}}(1-t^2)^{r-1}\Big|_{t=1}\right].\end{aligned} \tag{3.22}$$

This shows that $D_r(k)=0$ if $1\le k\le r-1$. Now, let $r\le k\le 2r$ and $j\ge r-1$. Then, by the Leibniz rule, we have

$$\begin{aligned}\frac{d^j}{dt^j}(1-t^2)^{r-1}\Big|_{t=1}&=\frac{d^j}{dt^j}\Big((1-t)^{r-1}(1+t)^{r-1}\Big)\Big|_{t=1}\\&=\sum_{m=0}^{j}\binom{j}{m}\Big(\frac{d^m}{dt^m}(1-t)^{r-1}\Big)\Big|_{t=1}\cdot\Big(\frac{d^{j-m}}{dt^{j-m}}(1+t)^{r-1}\Big)\Big|_{t=1}\\&=j!\binom{r-1}{j-r+1}(-1)^{r-1}2^{2r-2-j}.\end{aligned}$$

Hence, from (3.22), using this formula with $j=k-1$ and $k-2$, one obtains (3.20). Moreover, it is clear that the equation (3.20) is also valid for $1\le k\le r-1$ because $\binom{r}{k-r}=0$ for such a k. Hence the claim (i) follows. We next show the claim (ii). Consider the generating function $\sum_{p=1}^{2r}\widetilde{D}_{r,p}x^p$. Changing the order of the summations and using the formula (3.20), we see that this equals

$$\begin{aligned}\sum_{p=1}^{2r}\sum_{k=p}^{2r}\binom{k-1}{k-p}D_r(k)x^p&=\sum_{k=1}^{2r}D_r(k)x(1+x)^{k-1}\\&=4x^2(1-x^2)^{r-1}\\&=4\sum_{p=1}^{r}\binom{r-1}{p-1}(-1)^{p-1}x^{2p}.\end{aligned}$$

This shows the claim. □

We now give the proof of Proposition 3.6.

Proof of Proposition 3.6. From the equation (3.16), changing the order of the products, we have

$$\begin{aligned}\phi_r(s)&=e^{C_rt^{2r}}\prod_{\ell=0}^{r-1}\prod_{k=1}^{2\ell+2}\mathbf{\Gamma}_k\big(t+\frac{1}{2}\big)^{4(-1)^{\ell+k}\binom{r-1}{\ell}\binom{2\ell+1}{k-1}t^{2r-k}}\\&=e^{C_rt^{2r}}\prod_{k=1}^{2r}\mathbf{\Gamma}_k(s)^{D_r(k)t^{2r-k}}.\end{aligned} \tag{3.23}$$

Therefore, one immediately obtains the formula from (3.18) and (3.20). □

To clarify the analytic properties of $\phi_r(s)$, let us rewrite the expression (3.17) in terms of the Barnes multiple gamma functions. The following expression is obtained in [KOW2];

$$\mathbf{\Gamma}_r(z) = \prod_{j=1}^{r} \Gamma_j(z)^{c_{r,j}(z)}, \tag{3.24}$$

where, for $r \geq 1$ and $j \geq 1$, $c_{r,j}(z)$ is the polynomial in z defined by

$$c_{r,j}(z) := \sum_{l=0}^{j-1} \binom{j-1}{l} (-1)^l (z-l-1)^{r-1}. \tag{3.25}$$

For example, we have $c_{r,r}(z) = (r-1)!$, $c_{r,r-1}(z) = \frac{1}{2}(2z-r)(r-1)!$, $\ldots$ and $c_{r,1}(z) = (z-1)^{r-1}$. It is easy to see that the polynomial $c_{r,j}(z)$ satisfies the recursion formula $c_{r,j}(z) = (z-1)c_{r-1,j}(z) + (j-1)c_{r-1,j-1}(z-1)$. From this, noting that $c_{1,j}(z) = (1-1)^{j-1} = 0$ for $j \geq 2$, we see that $c_{r,j}(z) = 0$ for $1 \leq r \leq j-1$. We also notice that it is given by the generating function

$$(T+z)^{r-1} = \sum_{j=1}^{r} c_{r,j}(z) \binom{T+j-1}{j-1}. \tag{3.26}$$

For $x \in \mathbb{R}$, let us denote $\lfloor x \rfloor$ by the largest integer not exceeding x. Then, we have the following expression of $\phi_r(s)$ in terms of the Barnes multiple gamma functions.

Theorem 3.9. *We have*

$$\phi_r(s) = e^{-\frac{(2r)!!}{r^2(2r-1)!!}(s-\frac{1}{2})^{2r}} \prod_{j=1}^{2r} \Gamma_j(s)^{\alpha_{r,j}(s-\frac{1}{2})}, \tag{3.27}$$

where $\alpha_{r,j}(t)$ is the even polynomial defined by

$$\alpha_{r,j}(t) := 4 \sum_{\ell=\lfloor\frac{j+1}{2}\rfloor}^{r} \binom{r-1}{\ell-1} (-1)^{\ell-1} c_{2\ell,j}\Big(\frac{1}{2}\Big) t^{2r-2\ell}. \tag{3.28}$$

This gives a meromorphic continuation of $\phi_r(s)$ to the whole plane $\mathbb{C}$ with poles at $s=-k$ of order $2(2k+1)$ if $r=1$ and an analytic continuation to the region $\mathbb{C} \setminus (-\infty, 0]$ otherwise.

Proof. From the equations (3.23) and (3.24), it suffices to show that

$$\alpha_{r,j}(t) = \sum_{k=j}^{2r} c_{k,j}\Big(t+\frac{1}{2}\Big) D_r(k) t^{2r-k}. \tag{3.29}$$

Actually, from the equation (3.25), the righthand-side of (3.29) is rewritten as

$$\begin{aligned} &\sum_{k=j}^{2r} \left(\sum_{l=0}^{j-1} \binom{j-1}{l} (-1)^l \Big(t-\frac{1}{2}-l\Big)^{k-1} \right) D_r(k) t^{2r-k} \\ &= \sum_{l=0}^{j-1} \binom{j-1}{l} (-1)^l \left(\sum_{k=j}^{2r} \sum_{m=0}^{k-1} \binom{k-1}{m} \Big(-\frac{1}{2}-l\Big)^{(k-m)-1} D_r(k) t^{2r-(k-m)} \right). \end{aligned}$$

Putting $k-m=p$ in the inner sums, one sees that this is equal to

$$\begin{aligned}&\sum_{p=1}^{2r}\Bigg(\sum_{k=\max\{p,j\}}^{2r}\binom{k-1}{k-p}D_r(k)\Bigg)\Bigg(\sum_{l=0}^{j-1}\binom{j-1}{l}(-1)^l\big(\frac{1}{2}-l-1\big)^{p-1}\Bigg)t^{2r-p}\\&=\sum_{p=1}^{2r}\Bigg(\sum_{k=\max\{p,j\}}^{2r}\binom{k-1}{k-p}D_r(k)\Bigg)c_{p,j}\big(\frac{1}{2}\big)t^{2r-p}.\end{aligned}$$

Moreover, the summand for $p=1,2,\ldots,j-1$ vanishes because $c_{p,j}(z)=0$, whence, consequently, this can be written as $\sum_{p=j}^{2r}\widetilde{D}_{r,p}c_{p,j}(\frac{1}{2})t^{2r-p}$. Therefore, from (3.21), one obtains the equation (3.29). The rest of the assertion follows from the equations $\alpha_{1,1}(t)=-2$ and $\alpha_{1,2}(t)=4$ and the fact that the point $z=-k$ for $k\in\mathbb{N}_0$ is a zero of $\Gamma_j(z)^{-1}$ of order $\binom{k+j-1}{j-1}$. This ends the proof. □

Remark 3.10. From the equations $c_{r,r-1}(z)=\frac{1}{2}(2z-r)(r-1)!$ and $c_{r,r}(z)=(r-1)!$, one sees that both $\alpha_{r,2r-1}(t)$ and $\alpha_{r,2r}(t)$ are integers respectively given by

$$\begin{aligned}\alpha_{r,2r-1}(t)&=4(-1)^{r-1}c_{2r,2r-1}\big(\frac{1}{2}\big)=(-1)^r(4r-2)(2r-1)!,\\ \alpha_{r,2r}(t)&=4(-1)^r c_{2r,2r}\big(\frac{1}{2}\big)=4(-1)^{r-1}(2r-1)!.\end{aligned}$$

We next give an expression of $\phi_r(s)$ via the Vignéras multiple gamma function $G_n(z)$, which are characterized by a generalization of the Bohr-Mollerup theorem ([Vi]). Notice that $G_1(z)=\Gamma(z)$ and $G_2(z)=G(z)$ where $G(z)$ is the Barnes G-function studied in [B1]. From (27) in p. 87, [SC]), we know that $G_n(z)$ is essentially equal to $\Gamma_n(z)$;

$$G_n(z)=e^{(-1)^n\sum_{j=0}^{n-1}b_{n,j}(z)\zeta'(-j)}\cdot\Gamma_n(z)^{(-1)^{n-1}}. \tag{3.30}$$

Here, $b_{n,j}(z)$ is the polynomial of degree $n-1-j$ defined by the generating function $\binom{j+n-1}{n-1}=\sum_{k=0}^{n-1}b_{n,k}(z)(j+z)^k$. To be more precise, let $s(n,m)$ be the Stirling number of the first kind defined by $(z)_n=\sum_{m=0}^{n}(-1)^{n+m}s(n,m)z^m$ where $(z)_n:=\frac{\Gamma(z+n)}{\Gamma(z)}=z(z+1)\cdots(z+n-1)$. Then, it is given by $b_{n,k}(z)=\frac{(-1)^{n-1-k}}{(n-1)!}\sum_{m=k}^{n-1}\binom{m}{k}s(n,m+1)z^{m-k}$. The following expression is immediately obtained from (3.27) together with the identity (3.30).

Corollary 3.11. *We have*

$$\phi_r(s)=e^{-\frac{(2r)!!}{r^2(2r-1)!!}(s-\frac{1}{2})^{2r}+\sum_{\ell=0}^{2r-1}\beta_{r,\ell}(s-\frac{1}{2})\zeta'(-\ell)}\prod_{j=1}^{2r}G_j(s)^{(-1)^{j-1}\alpha_{r,j}(s-\frac{1}{2})}. \tag{3.31}$$

Here, $\beta_{r,\ell}(t)$ is the polynomial defined by

$$\beta_{r,\ell}(t):=\sum_{j=\ell+1}^{2r}b_{j,\ell}\big(t+\frac{1}{2}\big)\alpha_{r,j}(t).$$

□

Example 3.12. Let $t = s - \frac{1}{2}$. Then, from (3.27) and (3.31), we have

$$\begin{aligned}
\phi_1(s) &= e^{-2t^2}\Gamma_1(s)^{-2}\Gamma_2(s)^4 \\
&= e^{-2t^2-4t\zeta'(0)+4\zeta'(-1)}G_1(s)^{-2}G_2(s)^{-4}, \\
\phi_2(s) &= e^{-\frac{2}{3}t^4}\Gamma_1(s)^{-2t^2+\frac{1}{2}}\Gamma_2(s)^{4t^2-13}\Gamma_3(s)^{36}\Gamma_4(s)^{-24} \\
&= e^{-\frac{2}{3}t^4-8t^2\zeta'(-1)+12t\zeta'(-2)-4\zeta'(-3)}G_1(s)^{-2t^2+\frac{1}{2}}G_2(s)^{-4t^2+13}G_3(s)^{36}G_4(s)^{24}, \\
\phi_3(s) &= e^{-\frac{16}{45}t^6}\Gamma_1(s)^{-2t^4+t^2-\frac{1}{8}}\Gamma_2(s)^{4t^4-26t^2+\frac{121}{4}}\Gamma_3(s)^{72t^2-330}\Gamma_4(s)^{-48t^2+1020}\Gamma_5(s)^{-1200}\Gamma_6(s)^{480} \\
&= e^{-\frac{16}{45}t^6-16t^3\zeta'(-2)+32t^2\zeta'(-3)-20t\zeta'(-4)+4\zeta'(-5)} \\
&\quad \times G_1(s)^{-2t^4+t^2-\frac{1}{8}}G_2(s)^{-4t^4+26t^2-\frac{121}{4}}G_3(s)^{72t^2-330}G_4(s)^{48t^2-1020}G_5(s)^{-1200}G_6(s)^{-480}.
\end{aligned}$$

Notice that the expression of $\phi(s) = \phi_1(s)$ is obtained in [Vo].

3.3 Functional equations of $\phi_r(s)$

In this subsection, we establish a functional equation of $\phi_r(s)$. To do that, we first recall the normalized multiple sine function $S_n(z)$ studied in [KK];

$$S_n(z) := \Gamma_n(z)^{-1}\Gamma_n(n-z)^{(-1)^n}. \tag{3.32}$$

Notice that, from the reflection formula, we have $S_1(z) = 2\sin(\pi z) = \mathcal{S}_1(z)$. We remark that it is shown in [KK, Theorem 2.14] that $\mathcal{S}_n(z)$ can be expressed as a product of $S_j(z)$ for $j = 1, 2, \dots, n$ and vice versa.

Theorem 3.13. *We have*

$$\phi_r(1-s) = \Big(\prod_{k=1}^{2r} S_k(s)^{\alpha_{r,k}(s-\frac{1}{2})}\Big)\phi_r(s) \tag{3.33}$$

for all $s \in \mathbb{C}$ *if* $r = 1$ *and* $s \in \mathbb{C} \setminus \mathbb{R}$ *otherwise.*

To obtain the functional equation, we need the following lemmas.

Lemma 3.14. *We have*

$$\Gamma_n(1-z) = \prod_{j=1}^{n}\big(S_j(z)\Gamma_j(z)\big)^{(-1)^j\binom{n-1}{j-1}}. \tag{3.34}$$

Proof. Let $E_n(z) := S_n(z)\Gamma_n(z)$. We first notice that, from the ladder relations

$$\Gamma_n(z+1) = \Gamma_n(z)\Gamma_{n-1}(z)^{-1}, \tag{3.35}$$

$$S_n(z+1) = S_n(z)S_{n-1}(z)^{-1}, \tag{3.36}$$

we have $E_n(z+1) = E_n(z)E_{n-1}(z)^{-1}$. Here, we put $\Gamma_0(z) := z^{-1}$ and $S_0(z) := -1$. By the definition (3.32), we have $E_n(z)^{(-1)^n} = \Gamma_n(n-z)$. Hence, using the relation (3.35) repeatedly, we have

$$\begin{aligned}
E_n(z)^{(-1)^n} = \Gamma_n(n-z) &= \Gamma_n(1-z)\prod_{m=0}^{n-2}\Gamma_{n-1}\big(n-1-(z+m)\big)^{-1} \\
&= \Gamma_n(1-z)\prod_{m=0}^{n-2}E_{n-1}(z+m)^{(-1)^n}.
\end{aligned}$$

Therefore, using the equation $E_{n-1}(z+m)=\prod_{j=n-1-m}^{n-1}E_j(z)^{(-1)^{n-1-j}\binom{m}{n-1-j}}$ which is obtained from the ladder relation of $E_n(z)$ and the formula $\sum_{m=a}^{b}\binom{m}{a}=\binom{b+1}{b-a}$, we have

$$\begin{aligned}\Gamma_n(1-z)&=E_n(z)^{(-1)^n}\Big(\prod_{m=0}^{n-2}E_{n-1}(z+m)^{(-1)^n}\Big)^{-1}\\&=E_n(z)^{(-1)^n}\Big(\prod_{m=0}^{n-2}\prod_{j=n-1-m}^{n-1}E_j(z)^{(-1)^{j+1}\binom{m}{n-1-j}}\Big)^{-1}\\&=E_n(z)^{(-1)^n}\prod_{j=1}^{n-1}E_j(z)^{(-1)^j\sum_{m=n-1-j}^{n-2}\binom{m}{n-1-j}}\\&=\prod_{j=1}^{n}E_j(z)^{(-1)^j\binom{n-1}{j-1}}.\end{aligned}$$

This shows the claim. □

Lemma 3.15. *For $k=1,2,\ldots,2r$, we have*

$$\alpha_{r,k}(t)=(-1)^k\sum_{j=k}^{2r}\binom{j-1}{k-1}\alpha_{r,j}(t). \tag{3.37}$$

Proof. Write the righthand-side of (3.37) as $\widetilde{\alpha}_{r,k}(t)$. It suffices to show that $\sum_{k=1}^{2r}\alpha_{r,k}(t)\binom{T+j-1}{j-1}=\sum_{k=1}^{2r}\widetilde{\alpha}_{r,k}(t)\binom{T+j-1}{j-1}$. In fact, from (3.28), using the formula (3.26), we have

$$\sum_{k=1}^{2r}\alpha_{r,k}(t)\binom{T+j-1}{j-1}=4t^{2r-2}\big(T+\frac{1}{2}\big)\Big(1-\frac{(T+\frac{1}{2})^2}{t^2}\Big)^{r-1}.$$

On the other hand, using the identity $\sum_{k=1}^{j}(-1)^k\binom{j-1}{k-1}\binom{T+k-1}{k-1}=-\binom{-T+j-2}{j-1}$, we have

$$\begin{aligned}\sum_{k=1}^{2r}\widetilde{\alpha}_{r,k}(t)\binom{T+j-1}{j-1}&=-\sum_{j=1}^{2r}\alpha_{r,j}(t)\binom{(-T-1)+j-1}{j-1}\\&=-4t^{2r-2}\big(-T-\frac{1}{2}\big)\Big(1-\frac{(-T-\frac{1}{2})^2}{t^2}\Big)^{r-1}\\&=4t^{2r-2}\big(T+\frac{1}{2}\big)\Big(1-\frac{(T+\frac{1}{2})^2}{t^2}\Big)^{r-1}.\end{aligned}$$

Hence the claim follows. □

We now give a proof of Theorem 3.13.

Proof of Theorem 3.13. We first notice that both functions t^{2r} and $\alpha_{r,j}(t)$ are invariant under the transform $s\mapsto 1-s$ because they are even polynomials. Therefore, replacing s with $1-s$ in (3.27) and using the formulas (3.34) and (3.37), we have

$$\begin{aligned}\phi_r(1-s) &= e^{-\frac{(2r)!!}{r^2(2r-1)!!}t^{2r}}\prod_{j=1}^{2r}\left(\prod_{k=1}^{j}\big(S_k(s)\Gamma_k(s)\big)^{(-1)^k\binom{j-1}{k-1}}\right)^{\alpha_{r,j}(t)}\\ &= e^{-\frac{(2r)!!}{r^2(2r-1)!!}t^{2r}}\prod_{k=1}^{2r}\big(S_k(s)\Gamma_k(s)\big)^{(-1)^k\sum_{j=k}^{2r}\binom{j-1}{k-1}\alpha_{r,j}(t)}\\ &= \Big(\prod_{k=1}^{2r}S_k(s)^{\alpha_{r,k}(t)}\Big)\phi_r(s).\end{aligned}$$

This completes the proof. □

3.4 A proof of Proposition 3.4

Let $m\in\mathbb{N}_0$. The aim of this subsection is to give a proof of Proposition 3.4. Let

$$R_m(t,z) := \sum_{j=1}^{\infty}\frac{\zeta(2j+1,z)}{2j+m+1}t^{2j+m+1}\qquad (|t|<|z|).$$

Note that $R_m(t)=R_m(t,\frac{1}{2})$. We start from the identity ([SC, p.159 (4)])

$$R_0(t,z)=\sum_{j=1}^{\infty}\frac{\zeta(2j+1,z)}{2j+1}t^{2j+1}=\frac{1}{2}\Big(\log\Gamma(z-t)-\log\Gamma(z+t)\Big)+t\psi(z)\qquad (|t|<|z|).$$

Letting $z=\frac{1}{2}$, we have

$$\begin{aligned}R_m(t) &= \int_0^t \xi^m\frac{d}{d\xi}R_0\big(\xi,\frac{1}{2}\big)d\xi\\ &= -\frac{1}{2}\int_0^t\xi^m\Big(\psi\big(\frac{1}{2}-\xi\big)+\psi\big(\frac{1}{2}+\xi\big)\Big)d\xi+\frac{1}{m+1}\psi\big(\frac{1}{2}\big)t^{m+1}\\ &= -\Phi_m\big(t,\frac{1}{2}\big)-\frac{1}{2}\log\mathcal{C}_{m+1}(t)+\frac{1}{m+1}\psi\big(\frac{1}{2}\big)t^{m+1},\end{aligned}\tag{3.38}$$

where

$$\Phi_m(t,z) := \int_0^t\xi^m\psi(\xi+z)d\xi.$$

Notice that, in the last equality, we have used the formula $\psi(\frac{1}{2}-\xi)=\psi(\frac{1}{2}+\xi)-\pi\tan(\pi\xi)$ and (3.6). Hence, we have to evaluate the integral $\Phi_m(t,z)$. Define the polynomials ${}_nA_{-k}(z)$ for $1\le k\le n-1$ and ${}_nB_m(z)$ for $m\in\mathbb{Z}_{\ge 0}$ by the generating function

$$\frac{te^{(n-z)t}}{(e^t-1)^n}=\sum_{k=1}^{n-1}(-1)^k{}_nA_{-k}(z)t^{-k}+\sum_{m=0}^{\infty}(-1)^m{}_nB_m(z)\frac{t^m}{m!}.$$

These are called the Barnes multiple Bernoulli polynomials ([B2]). Notice that the degree of ${}_nB_m(z)$ is $m+n-1$. Since ${}_1B_m(z)=B_m(z)$, ${}_nB_m(z)$ gives a generalization of the Bernoulli polynomial. In fact, using the polynomial ${}_nB_m(z)$, one can evaluate the special values of the Barnes multiple zeta function $\zeta_n(w,z)$ at non-positive integer points;

$$\zeta_n(1-m,z)=-\frac{{}_nB_m(z)}{m}\qquad (m\in\mathbb{N}).\tag{3.39}$$

To obtain an explicit expression of $\Phi_m(t,z)$, we first show the following

Lemma 3.16. *It holds that*

$$(3.40)\quad (m+1)\int_0^t \xi^m \log\Gamma_{n,r}(\xi+z)d\xi = \sum_{k=1}^{m+1}(-1)^{k-1}\binom{m+1}{k}\binom{r+k-1}{k}^{-1} t^{m+1-k}\log\Gamma_{n,r+k}(t+z) + P_{n,r}(t,z;m),$$

where $P_{n,r}(t,z;m)$ is the polynomial in t of degree $n+r+m$ defined by

$$(3.41)\quad P_{n,r}(t,z;m) := \sum_{k=1}^{m+1}\frac{(-1)^k H(r,r+k-1)}{r+k}\binom{m+1}{k}\binom{r+k-1}{k}^{-1} t^{m+1-k}{}_nB_{r+k}(t+z) + (-1)^{m+1}\binom{m+r}{m+1}^{-1}\Big(\log\Gamma_{n,m+r+1}(z) - \frac{H(r,m+r)}{m+r+1}{}_nB_{m+r+1}(z)\Big)$$

with $H(m,n) := \sum_{k=m}^{n}\frac{1}{k} = H(n) - H(m-1)$ for $n \geq m$.

Proof. Integration by parts yields

$$\int_0^t \xi^m(\xi+\alpha)^{-w}d\xi = \sum_{k=1}^{m+1}\frac{(-1)^{k-1}m!}{(m+1-k)!}\frac{t^{m+1-k}(t+\alpha)^{-w+k}}{(-w+1)_k} + (-1)^{m+1}m!\frac{\alpha^{-w+m+1}}{(-w+1)_{m+1}}.$$

Hence, for $\mathrm{Re}\,(w) > m+n+1$, changing the order of the integral and summation, we have

$$\int_0^t \xi^m\zeta_n(w,\xi+z)d\xi = \sum_{m_1,\dots,m_n\geq 0}\int_0^t \xi^m\big(\xi+(m_1+\cdots+m_n+z)\big)^{-w}d\xi = \sum_{k=1}^{m+1}\frac{(-1)^{k-1}m!}{(m+1-k)!}\frac{t^{m+1-k}\zeta_n(w-k,t+z)}{(-w+1)_k} + (-1)^{m+1}m!\frac{\zeta_n(w-m-1,z)}{(-w+1)_{m+1}}.$$

This gives a meromorphic continuation of the lefthand-side to the whole plane $\mathbb{C}$. Now the desired formula (3.40) immediately follows from the above equation by differentiating the both hand sides at $w = 1-r$ together with the following equation obtained from (3.39);

$$\frac{\partial}{\partial w}\Big(\frac{\zeta_n(w-k,z)}{(-s+1)_k}\Big)\Big|_{w=1-r} = \frac{1}{(r)_k}\Big(\log\Gamma_{n,r+k}(z) - \frac{1}{r+k}H(r,r+k-1)_nB_{r+k}(z)\Big).$$

□

Corollary 3.17. *It holds that*

$$(3.42)\quad \Phi_m(t,z) = \sum_{k=1}^{m+1}(-1)^{k+1}\binom{m}{k-1}t^{m+1-k}\log\mathbf{\Gamma}_k(t+z) + P_m(t,z),$$

where $P_m(t,z)$ is the polynomial in t of degree $m+1$ defined by

$$(3.43)\quad P_m(t,z) := \frac{1}{m+1}H(m)t^{m+1} + \sum_{l=1}^{m}\frac{(-1)^{l+m}B_{m+1-l}(z)}{l(m+1-l)}t^l + (-1)^{m+1}\log\mathbf{\Gamma}_{m+1}(z).$$

Proof. From the formula (3.40), we have

$$\begin{aligned}\Phi_m(t,z) &= t^m \log \mathbf{\Gamma}_1(t+z) - m\int_0^t \xi^{m-1}\log \Gamma_{1,1}(\xi+z)d\xi \\ &= \sum_{k=1}^{m+1}(-1)^{k+1}\binom{m}{k-1}t^{m+1-k}\log \mathbf{\Gamma}_k(t+z) - P_{1,1}(t,z;m-1).\end{aligned}$$

Here, we have used the identity $\Gamma_{1,1}(z) = \mathbf{\Gamma}_1(z) = \frac{\Gamma(z)}{\sqrt{2\pi}}$. Hence it suffices to show that

$$-P_{1,1}(t,z;m-1) = P_m(t,z). \tag{3.44}$$

From the definition (3.41), using the identity $B_m(t+z) = \sum_{l=0}^{m}\binom{n}{l}B_l(z)t^{m-l}$ and changing the order of the summations, we have

$$\begin{aligned}-P_{1,1}(t,z;m-1) = {}& \frac{(-1)^{m+1}}{m+1}\sum_{l=0}^{m+1}\binom{m+1}{l}d_l(m)B_{m+1-l}(z)t^l \\ &+ \frac{(-1)^m H(m)}{m+1}B_{m+1}(z) + (-1)^{m+1}\log \mathbf{\Gamma}_{m+1}(z),\end{aligned}$$

where $d_l(m) := \sum_{k=0}^{l}\binom{l}{k}(-1)^k H(m-k)$. Therefore, to obtain the equation (3.44), it is enough to show that

$$d_l(m) = \begin{cases} H(m) & (l=0), \\ (-1)^{l-1}\dfrac{(m-l)!(l-1)!}{m!} & (1\le l\le m), \\ (-1)^{m+1}H(m) & (l=m+1). \end{cases}$$

Actually, the case $l=0$ is clear. Let $1 \le l \le m+1$ and $F_l(s) := \sum_{m=0}^{\infty} d_l(m)s^m$ a generating function of $d_l(m)$. Then, from the identity $\sum_{m=1}^{\infty} H(m)s^m = -(1-s)^{-1}\log(1-s)$, we have

$$F_l(s) = \sum_{k=0}^{l}\binom{l}{k}(-s)^k\sum_{m=1}^{\infty}H(m)s^m = -(1-s)^{l-1}\log(1-s).$$

Hence, by the Leibniz rule, we have

$$\begin{aligned}d_l(m) = \frac{1}{m!}\frac{d^m}{ds^m}F_l(s)\Big|_{s=0} &= -\frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{d^{m-k}}{ds^{m-k}}\left((1-s)^{l-1}\right)\frac{d^k}{ds^k}\left(\log(1-s)\right)\Big|_{s=0} \\ &= \sum_{k=\max\{m+1-l,1\}}^{m}\frac{(-1)^{m-k}}{k}\binom{l-1}{m-k}.\end{aligned} \tag{3.45}$$

When $1 \le l \le m$, this is equal to

$$\sum_{k=0}^{l-1}\frac{(-1)^k}{m-k}\binom{l-1}{k} = \int_0^1 x^{m-1}(1-x^{-1})^{l-1}dx = (-1)^{l-1}\frac{(m-l)!(l-1)!}{m!}.$$

On the other hand, when $l=m+1$, using the formula $\psi(m) = -\gamma + H(m-1)$ for $m\in\mathbb{N}$ and the equation (see, e.g., [SC, p.15 (13)])

$$\int_0^1 \frac{(1-\xi)^{z-1}-1}{\xi}d\xi = -\gamma - \psi(z) \qquad (\mathrm{Re}\,(z)>0),$$

where $\gamma = 0.57721\ldots$ is the Euler constant, we see that the rightmost-hand side of (3.45) equals

$$(-1)^m \sum_{k=1}^{m} \frac{(-1)^k}{k} \binom{m}{k} = (-1)^m \int_0^1 \frac{(1-x)^m - 1}{x} dx = (-1)^{m+1} H(m).$$

This completes the proof. □

Now we give the proof of Proposition 3.4.

Proof of Proposition 3.4. Notice that, since m is odd, $B_{m+1-l}(\frac{1}{2}) = 0$ if l is odd. Hence one obtains the formula (3.15) from (3.38) together with the formulas (3.42) and (3.43) with $z = \frac{1}{2}$. □

4 Milnor-Selberg zeta functions

In this section, we study the Milnor-Selberg zeta function $Z_{\Gamma,r}(s)$ defined by (2.10).

4.1 Analytic properties of $Z_{\Gamma,r}(s)$

Theorem 4.1. *The function $Z_{\Gamma,r}(s)$ can be continued analytically to the whole plane $\mathbb{C}$ when $r = 1$ and $\mathbb{C} \setminus (-\infty, 1]$ otherwise. Moreover, it satisfies the functional equation*

$$Z_{\Gamma,r}(1-s) = \Big(\prod_{j=1}^{2r} S_j(s)^{-\alpha_{r,j}(s-\frac{1}{2})}\Big)^{g-1} Z_{\Gamma,r}(s) \tag{4.1}$$

for all $s \in \mathbb{C}$ if $r = 1$ and $s \in \mathbb{C} \setminus \mathbb{R}$ otherwise.

Proof. From (2.8), we have

$$Z_{\Gamma,r}(s) = \phi_r(s)^{-(g-1)} \mathrm{D}_{\Gamma,r}(s). \tag{4.2}$$

This gives an analytic continuation from Theorem 2.3 and Theorem 3.9. The functional equation (4.1) immediately follows from Theorem 2.3 and (3.33). □

To study a "complete Milnor-Selberg zeta function", we need the following lemma.

Lemma 4.2. *For $0 \le l \le 2r-1$, let*

$$\hat{\alpha}_{r,l}(t) := (-1)^l \sum_{j=1}^{2r-l} \binom{2r-j}{l} \alpha_{r,j}(t).$$

Then, for $1 \le m \le 2r$, we have

$$\sum_{l=2r-m}^{2r-1} \binom{l}{2r-m} \hat{\alpha}_{r,l}(t) = (-1)^m \alpha_{r,m}(t). \tag{4.3}$$

Proof. Using the equation $\sum_{l=a}^{b} (-1)^l \binom{l}{a}\binom{b}{l} = \delta_{a,b}(-1)^a$ where $\delta_{a,b}$ is the Kronecker delta, we have

$$\sum_{l=2r-m}^{2r-1} \binom{l}{2r-m} \hat{\alpha}_{r,l}(t) = \sum_{j=1}^{m} \Bigg(\sum_{l=2r-m}^{2r-j} (-1)^l \binom{l}{2r-m} \binom{2r-j}{l} \Bigg) \alpha_{r,j}(t) = (-1)^m \alpha_{r,m}(t).$$

□

The following gives a generalization of the functional equation (1.3).

Corollary 4.3. *Define the complete Milnor-Selberg zeta function by*

$$\Xi_{\Gamma,r}(s) := \Big(\prod_{l=0}^{2r-1} \Gamma_{2r}(s+l)^{\hat{\alpha}_{r,l}(s-\frac{1}{2})}\Big)^{g-1} Z_{\Gamma,r}(s). \tag{4.4}$$

Then, we have

$$\Xi_{\Gamma,r}(s) = e^{\frac{(2r)!!}{r^2(2r-1)!!}(g-1)(s-\frac{1}{2})^{2r}} \cdot \mathrm{D}_{\Gamma,r}(s). \tag{4.5}$$

In particular, $\Xi_{\Gamma,r}(s)$ is an entire function when $r=1$ and is holomorphic function in $\mathbb{C}\setminus[0,1]$ otherwise. Moreover, it satisfies the functional equation

$$\Xi_{\Gamma,r}(1-s) = \Xi_{\Gamma,r}(s).$$

Proof. We prove the equation (4.5). To do that, from (2.8) with (3.27), it is sufficient to show that $\prod_{l=0}^{2r-1}\Gamma_{2r}(s+l)^{\hat{\alpha}_{r,l}(t)} = \prod_{m=1}^{2r}\Gamma_m(s)^{\alpha_{r,m}(t)}$. Actually, from the ladder relation (3.35), one can show that $\Gamma_{2r}(s+l) = \prod_{j=0}^{l}\Gamma_{2r-j}(s)^{(-1)^j\binom{l}{j}}$. This yields

$$\begin{aligned}\prod_{l=0}^{2r-1}\Gamma_{2r}(s+l)^{\hat{\alpha}_{r,l}(t)} &= \prod_{l=0}^{2r-1}\prod_{j=0}^{l}\Gamma_{2r-j}(s)^{(-1)^j\binom{l}{j}\hat{\alpha}_{r,l}(t)}\\ &= \prod_{m=1}^{2r}\Gamma_m(s)^{(-1)^m\sum_{l=2r-m}^{2r-1}\binom{l}{2r-m}\hat{\alpha}_{r,l}(t)}\\ &= \prod_{m=1}^{2r}\Gamma_m(s)^{\alpha_{r,m}(t)}.\end{aligned}$$

In the last equality, we have used the equation (4.3). This completes the proof. □

Example 4.4. Let $t = s-\frac{1}{2}$. Then, we have

$$\begin{aligned}\Xi_{\Gamma,1}(s) &= \big(\Gamma_2(s)^2\Gamma_2(s+1)^2\big)^{g-1}Z_{\Gamma,1}(s) = \Xi_\Gamma(s),\\ \Xi_{\Gamma,2}(s) &= \big(\Gamma_4(s)^{-\frac{1}{2}+2t^2}\Gamma_4(s+1)^{-\frac{23}{2}-2t^2}\Gamma_4(s+2)^{-\frac{23}{2}-2t^2}\Gamma_4(s+3)^{-\frac{1}{2}+2t^2}\big)^{g-1}Z_{\Gamma,2}(s),\\ \Xi_{\Gamma,3}(s) &= \big(\Gamma_6(s)^{\frac{1}{8}-t^2+2t^4}\Gamma_6(s+1)^{\frac{237}{8}-21t^2-6t^4}\Gamma_6(s+2)^{\frac{841}{4}+22t^2+4t^4}\\ &\quad\times\Gamma_6(s+3)^{\frac{841}{4}+22t^2+4t^4}\Gamma_6(s+4)^{\frac{237}{8}-21t^2-6t^4}\Gamma_6(s+5)^{\frac{1}{8}-t^2+2t^4}\big)^{g-1}Z_{\Gamma,3}(s).\end{aligned}$$

Remark 4.5. Using the ladder relations (3.35) and (3.36), one can prove that

$$\prod_{j=1}^{2r} S_j(s)^{\alpha_{r,j}(t)} = \frac{\prod_{l=0}^{2r-1}\Gamma_{2r}(1-s+l)^{\hat{\alpha}_{r,j}(t)}}{\prod_{l=0}^{2r-1}\Gamma_{2r}(s+l)^{\hat{\alpha}_{r,j}(t)}}$$

This reads the definition (4.4) of the complete Milnor-Selberg zeta function.

Remark 4.6. From the expression (4.5), we see that the Milnor-Selberg zeta function $Z_{\Gamma,r}(s)$ has no "non-trivial zeros" because $\mathrm{D}_{\Gamma,r}(s)$ does have no zeros.

Remark 4.7. Let $f(s)$ be a function on $\mathbb{C}$ and $m(s)$ a polynomial. Suppose that it can be written as $f(s)=(s-a)^{m(s)}g(s)$ around $s=a$ where $g(s)$ is a holomorphic function at $s=a$ with $g(a)\neq 0$. In this case, let us say that $f(s)$ has a "multiplicity polynomial $m(s)$ at $s=a$" and write $m(s)=m(s;f,a)$. It is clear that this is a generalization of the multiplicity (or order) of zeros or poles of meromorphic functions (they are the case $m(s)\in\mathbb{Z}$). For example, if $f(s)$ has a zero (resp. a pole) of order n at $s=a$, then the multiplicity polynomial of $f(s)^{q(s)}$ for a polynomial $q(s)$ at $s=a$ is given by $m(s;f^q,a)=nq(s)$ (resp. $-nq(s)$).

From the expression (4.2), for $k\in\mathbb{N}$, one can see that

$$m(s;Z_{\Gamma,r},-k)=2(g-1)(2k+1)(s-k)^{r-1}(s-k-1)^{r-1}. \tag{4.6}$$

In particular, when $r=1$, this coincides with the multiplicity $2(g-1)(2k+1)$ of the trivial zero $s=-k$ of the Selberg zeta function $Z_\Gamma(s)$. The equation (4.6) is obtained as follows; from the equation (4.2) and the expression (3.27) together with the fact that $\Gamma_j(s)^{-1}$ has a zero at $s=-k$ of order $\binom{k+j-1}{j-1}$, it can be written as

$$Z_{\Gamma,r}(s)=\left(\prod_{j=1}^{2r}(s+k)^{(g-1)\binom{k+j-1}{j-1}\alpha_{r,j}(t)}\right)\cdot z_{\Gamma,r}(s),$$

where $z_{\Gamma,r}(s)$ is some holomorphic function at $s=-k$ with $z_{\Gamma,r}(-k)\neq 0$. This shows that

$$\begin{aligned}
m(s;Z_{\Gamma,r},-k)&=(g-1)\sum_{j=1}^{2r}\binom{k+j-1}{j-1}\alpha_{r,j}(t)\\
&=4(g-1)\sum_{l=1}^{r}\left[\sum_{j=1}^{2\ell}c_{2\ell,j}\big(\tfrac{1}{2}\big)\binom{k+j-1}{j-1}\right]\binom{r-1}{\ell-1}(-1)^{\ell-1}t^{2r-2\ell}\\
&=2(g-1)(2k-1)t^{2r-2}\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}\Big(-\big(k+\tfrac{1}{2}\big)^2t^{-2}\Big)^{\ell}\\
&=2(g-1)(2k-1)\Big(t^2-\big(k+\tfrac{1}{2}\big)^2\Big)^{r-1}.
\end{aligned}$$

Note that, in the third equality, we have employed the formula (3.26).

In the next subsection, to establish an Euler product expression of $Z_{\Gamma,r}(s)$, we introduce a poly-Selberg zeta function.

4.2 Poly-Selberg zeta functions

For $m\in\mathbb{N}$, define the function $Z_\Gamma^{(m)}(s)$ by the following Euler product.

$$Z_\Gamma^{(m)}(s):=\prod_{P\in\mathrm{Prim}\,(\Gamma)}\prod_{n=0}^{\infty}H_m\big(N(P)^{-s-n}\big)^{(\log N(P))^{-m+1}}, \tag{4.7}$$

where $H_m(z):=\exp(-Li_m(z))$ with $Li_m(z):=\sum_{k=1}^{\infty}\frac{z^k}{k^m}$ being the polylogarithm of degree m. This infinite product converges absolutely for $\mathrm{Re}\,(s)>1$. We call $Z_\Gamma^{(m)}(s)$ a *poly-Selberg zeta function* of degree m. Notice that this gives a generalization of the Selberg zeta function $Z_\Gamma(s)$. In fact, since $Li_1(z)=-\log(1-z)$ and hence $H_1(z)=1-z$, we have $Z_\Gamma^{(1)}(s)=Z_\Gamma(s)$.

To give an analytic continuation of $Z_\Gamma^{(m)}(s)$, we first show the following

Lemma 4.8. *It holds that*

$$-\log Z_\Gamma^{(m)}(s)=\sum_{\gamma\in \mathrm{Hyp}\,(\Gamma)}\frac{\log N(\delta_\gamma)}{N(\gamma)^{\frac12}-N(\gamma)^{-\frac12}}\frac{N(\gamma)^{-s+\frac12}}{(\log N(\gamma))^m}\qquad (\mathrm{Re}\,(s)>1). \tag{4.8}$$

Proof. Notice that $N(P^k)=N(P)^k$ and hence $\log N(P^k)=k\log N(P)$ for $P\in \mathrm{Prim}\,(\Gamma)$ and $k\in\mathbb{N}$. Hence, from the definition, we have

$$\begin{aligned}
-\log Z_\Gamma^{(m)}(s)&=\sum_{P\in \mathrm{Prim}\,(\Gamma)}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{N(P)^{-(s+n)k}}{k^m(\log N(P))^{m-1}}\\
&=\sum_{P\in \mathrm{Prim}\,(\Gamma)}\sum_{k=1}^{\infty}\frac{N(P)^{-ks}}{k^m(\log N(P))^{m-1}}\frac{1}{1-N(P)^{-k}}\\
&=\sum_{P\in \mathrm{Prim}\,(\Gamma)}\sum_{k=1}^{\infty}\frac{\log N(P)}{N(P^k)^{\frac12}-N(P^k)^{-\frac12}}\frac{N(P^k)^{-s+\frac12}}{(\log N(P^k))^m}.
\end{aligned}$$

Writing $P^k=\gamma$, one obtains the desired expression. □

The poly-Selberg zeta function satisfies a differential ladder relation. As a consequence, one can obtain an iterated integral representation of $Z_\Gamma^{(m)}(s)$ which gives an analytic continuation. To see this, define the following region (see Figure 5);

$$\Omega_\Gamma:=\mathbb{C}\setminus\Big(\bigcup_{r_j>0}(-\infty+ir_j,\frac12+ir_j]\cup(-\infty,1]\cup\bigcup_{r_j>0}(-\infty-ir_j,\frac12-ir_j]\Big).$$

Notice that the points $s=\frac12\pm ir_j$ with $r_j>0$ are non-trivial zeros of the Selberg zeta function $Z_\Gamma(s)$.

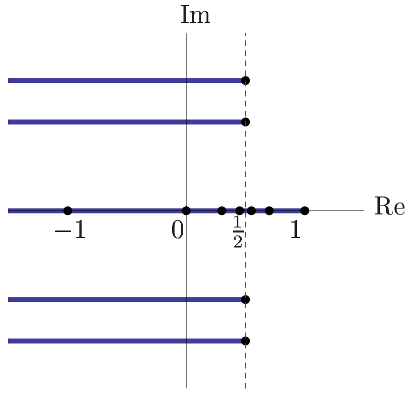

Figure 5: Ω_Γ.

Proposition 4.9. (i) *It holds that*

$$\frac{d^{m-1}}{ds^{m-1}}\log Z_\Gamma^{(m)}(s)=(-1)^{m-1}\log Z_\Gamma(s). \tag{4.9}$$

(ii) *Fix $a\in\mathbb{C}$ with* $\mathrm{Re}\,(a)>1$. *Then, for $m\ge 2$, we have*

$$Z_\Gamma^{(m)}(s)=Q_\Gamma^{(m)}(s,a)\exp\Bigg(\underbrace{\int_a^s\int_a^{\xi_{m-1}}\cdots\int_a^{\xi_2}}_{m-1}\log Z_\Gamma(\xi_1)d\xi_1\cdots d\xi_{m-1}\Bigg)^{(-1)^{m-1}}, \tag{4.10}$$

where $Q_\Gamma^{(m)}(s,a) := \prod_{k=0}^{m-2} Z_\Gamma^{(m-k)}(a)^{\frac{(-1)^k}{k!}(s-a)^k}$ *and the path for each integral lies in* Ω_Γ*. This gives an analytic continuation of* $Z_\Gamma^{(m)}(s)$ *to the region* Ω_Γ*.*

Proof. We first show the equation (4.9). The case $m=1$ is clear because $Z_\Gamma^{(1)}(s) = Z_\Gamma(s)$. Let $m \geq 2$. From (4.8) (or the differential equation $\frac{d}{dz}Li_m(z) = \frac{1}{z}Li_{m-1}(z)$), we have

$$\frac{d}{ds}\log Z_\Gamma^{(m)}(s) = -\log Z_\Gamma^{(m-1)}(s). \tag{4.11}$$

Using this equation repeatedly, one obtains the equation (4.9). We next show (ii) by induction on m. Let $m=2$. Then, integrating the equation (4.9) with $m=2$, we have

$$\log Z_\Gamma^{(2)}(s) = \log Z_\Gamma^{(2)}(a) - \int_a^s \log Z_\Gamma(\xi)d\xi. \tag{4.12}$$

Here, we take the path in $\mathrm{Re}\,(s) > 1$. This immediately shows the equation (4.10) with $m=2$ for $\mathrm{Re}\,(s) > 1$. Here, in the righthand-side of (4.12), one can move s freely in the region in where $\log Z_\Gamma(s)$ is holomorphic, that is, Ω_Γ. Hence the equation (4.10) gives an analytic continuation of $Z_\Gamma^{(2)}(s)$ to Ω_Γ. Now, assume that the claim (ii) holds for $m-1$. Then, we have

$$\begin{aligned}\log Z_\Gamma^{(m-1)}(\xi) = &\sum_{k=0}^{m-1}\frac{(-1)^k}{k!}(\xi-a)^k \log Z_\Gamma^{(m-1-k)}(a) \\ &+ (-1)^{m-2}\underbrace{\int_a^\xi\int_a^{\xi_{m-2}}\cdots\int_a^{\xi_2}}_{m-2}\log Z_\Gamma(\xi_1)d\xi_1\cdots d\xi_{m-2}.\end{aligned} \tag{4.13}$$

Taking the integral of the equation (4.13) together with the formula (4.11), we have

$$\begin{aligned}\log Z_\Gamma^{(m)}(s) = \log Z_\Gamma^{(m)}(a) + &\sum_{k=0}^{m-3}\frac{(-1)^{k+1}}{(k+1)!}(s-a)^{k+1}\log Z_\Gamma^{(m-(k+1))}(a) \\ &+ (-1)^{m-2}\int_a^s\underbrace{\int_a^\eta\int_a^{\xi_{m-2}}\cdots\int_a^{\xi_1}}_{m-2}\log Z_\Gamma(\xi_1)d\xi_1\cdots d\xi_{m-2}d\xi.\end{aligned}$$

Therefore, by the same discussion as above, one shows the claim (ii) for all m. □

Remark 4.10. Since $Z_\Gamma(s)$ has zeros at $s = 1, 0, -k$ for $k \in \mathbb{N}$ and $\frac{1}{2} \pm ir_j$ for $j \in \mathbb{N}$, the poly-Selberg zeta function $Z_\Gamma^{(m)}(s)$ with $m \geq 2$ is in general a multi-valued function in $\mathbb{C}$.

Remark 4.11. The discussion above can be also applied to the case of the Hecke L-functions; in [WY], we show that "higher depth regularized products" of the non-trivial zeros of the Hecke L-function can be evaluated as a product of the Milnor gamma functions and "poly-Hecke L-functions", which are similarly defined by an Euler product like the poly-Selberg zeta function.

4.3 Euler product expressions of $Z_{\Gamma,r}(s)$

Via the poly-Selberg zeta functions, one obtains the following Euler product expression of $Z_{\Gamma,r}(s)$ (remark that the poly-Selberg zeta function is defined by the Euler product (4.7)).

Theorem 4.12. *It holds that*

$$Z_{\Gamma,r}(s)=\left(\prod_{m=0}^{r-1} Z_{\Gamma}^{(r+m)}(s)^{\frac{(r-1+m)!}{m!(r-1-m)!}(2s-1)^{r-1-m}}\right)^{(r-1)!(-1)^{r-1}} \qquad (\mathrm{Re}\,(s)>1). \tag{4.14}$$

Proof. We start from the definition (2.10). Let $s\in W\cap U$. Then, using the formula

$$\int_0^\infty \xi^w e^{-(a\xi+\frac{b}{\xi})}\frac{d\xi}{\xi}=2\Big(\frac{b}{a}\Big)^{\frac{w}{2}}K_w(2a^{\frac12}b^{\frac12}) \qquad (\mathrm{Re}\,(a)>0,\ \mathrm{Re}\,(b)>0),$$

where $K_\nu(x)$ is the modified Bessel function of the second kind, we have

$$\begin{aligned}\Theta_{\Gamma,r}(w,s)&=\frac{1}{\sqrt{\pi}\Gamma(w+1-r)}\\&\quad\times\sum_{\gamma\in\mathrm{Hyp}\,(\Gamma)}\frac{\log N(\delta_\gamma)}{N(\gamma)^{\frac12}-N(\gamma)^{-\frac12}}\Big(\frac{\log N(\gamma)}{2t}\Big)^{w+\frac12-r}K_{w+\frac12-r}\big(t\log N(\gamma)\big).\end{aligned}$$

Here, employing the asymptotic formulas

$$\begin{aligned}\frac{1}{\Gamma(w+1-r)}&=(-1)^{r-1}(r-1)!w+O(w^2),\\ \Big(\frac{\log N(\gamma)}{2t}\Big)^{w+\frac12-r}&=\Big(\frac{\log N(\gamma)}{2t}\Big)^{\frac12-r}+O(w)\end{aligned}$$

and

$$K_{w+\frac12-r}\big(t\log N(\gamma)\big)=K_{\frac12-r}\big(t\log N(\gamma)\big)+O(w)$$

as $w\to 0$ together with the identity (see, e.g., [EMOT])

$$K_{\frac12-r}(y)=K_{r-\frac12}(y)=\Big(\frac{\pi}{2y}\Big)^{\frac12}e^{-y}\sum_{m=0}^{r-1}(2y)^{-m}\frac{(r-1+m)!}{m!(r-1-m)!},$$

one can see that

$$\begin{aligned}\log Z_{\Gamma,r}(s)&=-\frac{\partial}{\partial w}\Theta_{\Gamma,r}(w,s)\Big|_{w=0}\\&=(-1)^{r-1}(r-1)!\sum_{m=0}^{r-1}\frac{(r-1+m)!}{m!(r-1-m)!}(2t)^{r-1-m}\log Z_{\Gamma}^{(r+m)}(s)\end{aligned} \tag{4.15}$$

and hence obtain the expression (4.14). Notice that we have used the equation (4.8). □

Example 4.13. Let $t=s-\frac12$. Then, for $\mathrm{Re}\,(s)>1$, we have

$$\begin{aligned}Z_{\Gamma,1}(s)&=Z_{\Gamma}^{(1)}(s)=Z_\Gamma(s),\\ Z_{\Gamma,2}(s)&=Z_{\Gamma}^{(2)}(s)^{-(2t)}Z_{\Gamma}^{(3)}(s)^{-2},\\ Z_{\Gamma,3}(s)&=Z_{\Gamma}^{(3)}(s)^{2(2t)^2}Z_{\Gamma}^{(4)}(s)^{12(2t)}Z_{\Gamma}^{(5)}(s)^{24}.\end{aligned}$$

Remark 4.14. We remark that, when $r\geq 2$, from Theorem 4.1, we have already known that $Z_{\Gamma,r}(s)$ can be continued analytically to the region $\mathbb{C}\setminus(-\infty,1]$. From Proposition 4.9, though the equation (4.14) also gives an analytic continuation of $Z_{\Gamma,r}(s)$ to Ω_Γ, it is not clarified that it further gives an analytic continuation (beyond Ω_Γ) to $\mathbb{C}\setminus(-\infty,1]$.

Finally, as is the case of $Z_\Gamma^{(m)}(s)$, we show that the Milnor-Selberg zeta function also satisfies a differential ladder relation and has an iterated integral representation.

Corollary 4.15. (i) *It holds that*

$$\Big(\frac{1}{2s-1}\frac{d}{ds}\Big)^{r-1}\log Z_{\Gamma,r}(s)=(r-1)!\log Z_\Gamma(s). \tag{4.16}$$

(ii) *Fix* $a\in\mathbb{C}$ *with* $\mathrm{Re}\,(a)>1$. *Then, for* $r\geq 2$, *we have*

$$Z_{\Gamma,r}(s)=Q_{\Gamma,r}(s,a)\exp\Bigg(\underbrace{\int_a^s\int_a^{\xi_{r-1}}\cdots\int_a^{\xi_2}}_{r-1}\Big(\prod_{j=1}^{r-1}(2\xi_j-1)\Big)\log Z_\Gamma(\xi_1)d\xi_1\cdots d\xi_{r-1}\Bigg)^{(r-1)!}, \tag{4.17}$$

where $Q_{\Gamma,r}(s,a):=\prod_{k=0}^{r-2}Z_{\Gamma,r-k}(a)^{\binom{r-1}{k}(s-a)^k(s+a-1)^k}$ *and the path for each integral lies in* Ω_Γ.

Proof. To prove the equation (4.16), it is sufficient to show that

$$\Big(\frac{1}{2s-1}\frac{d}{ds}\Big)\log Z_{\Gamma,r}(s)=(r-1)\log Z_{\Gamma,r-1}(s)\qquad (r\geq 2). \tag{4.18}$$

In fact, from the expression (4.15) together with (4.11), one sees that

$$\begin{aligned}\frac{d}{ds}\log Z_{\Gamma,r}(s)&=(-1)^r(r-1)!\sum_{m=0}^{r-2}\frac{(r+m-2)!}{m!(r-m-2)!}(2s-1)^{r-m-1}\log Z_\Gamma^{(m+r-1)}(s)\\&=(2s-1)(r-1)\log Z_{\Gamma,r-1}(s).\end{aligned}$$

This shows the equation (4.18). The iterated integral representation (4.17) can be easily derived by the similar discussion performed in Proposition 4.9. ☐

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