A Darboux transformation for discrete s-isothermic surfaces

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Abstract. We give an overview on the discretization of isothermic surfaces, with special emphasis on the so-called s-isothermic surfaces, which are in some sense a nonlinear deformation of the classical discrete isothermic surfaces. For s-isothermic surfaces we give a way to define surfaces of constant mean curvature (cmc surfaces for short) without actually defining an a priori notion of curvature itself. We will compute discrete versions of rotational symmetric cmc surfaces (Delaunay surfaces) as an example. Finally, we give a discrete equivalent of the Sinh-Gordon equation, solutions of which describe – in complete analogy to the smooth case – discrete s-isothermic cmc surfaces.

Keywords. mathematics, discrete differential geometry, s-isothermic nets, constant mean curvature

1. Introduction

Isothermic surfaces are surfaces that allow conformal parametrization by curvature lines. This parametrization is called isothermic as well.

The class of isothermic surfaces includes surfaces of revolution and quadrics as well as minimal surfaces and surfaces of constant mean curvature (cmc surfaces). It can be shown that there is an integrable system underlying these surfaces. In the case of cmc surfaces this is the Sinh-Gordon equation. The integrable nature allows to study these surfaces using methods from soliton theory and the last part of this paper will be concerned with finding a discrete analog of this Sinh-Gordon equation by looking at certain discretizations of cmc surfaces.

Since conformal curvature line parametrization is preserved by Möbius transformations, any Möbius transform of an isothermic surface is isothermic again.

The following definition of an isothermic surface in isothermic parametrization is the classical one:

Definition 1. $f : \mathbb{R}^2 \to \mathbb{R}^n$ is called an isothermic net if $f_{xy} \in \text{span}(f_x, f_y)$, $f_x \perp f_y$, and $\|f_x\| = s = \|f_y\|$ with $s : \mathbb{R}^2 \to \mathbb{R}_+$.

Here the subindices x and y denote the partial derivatives wrt. the two parameter directions.

Yet another way to characterize isothermic surfaces (in isothermic parametrization) is the existence of a dual surface (or Christoffel transform): If f is an isothermic net, then

$$
\begin{aligned}
f_x^* &:= \frac{f_x}{\|f_x\|^2} \\
f_y^* &:= -\frac{f_y}{\|f_y\|^2}
\end{aligned}
\tag{1}
$$

can be integrated to give a new isothermic net f^*. f^* is only defined up to translation and scaling. One should keep in mind here that the definition of a dual surface is a euclidean construction (it involves the choice of ∞, since one has to measure length). However the notion of isothermicity is a conformal one.

There are many more characterizations of isothermic surfaces, but the one given in section 3 is of special interest for us: There isothermic nets are characterized as solutions to the Moutard equation

$$
f_{xy} = \lambda f
\tag{2}
$$

in the light cone of a Minkowski space. The corresponding model of Möbius geometry is introduced in section 2.

Section 4 prepares the basis for the two major known discretizations of isothermic surfaces which follow in section 5. Section 6 will give some examples, including in particular rotational symmetric ones. In section 7 we will finally present a s-isothermic (discrete) version of cmc surfaces, give cmc surfaces of revolution as examples (some of the calculations here are postponed to an appendix) and derive a discrete version of the Sinh-Gordon equation.

2. The classical model

We will identify points in $\mathbb{R}^n$ (and S^n) with lines in the light cone $\mathbb{L}^{n+1}$ in Minkowski $\mathbb{R}_1^{n+2}$:

$$
p \in \mathbb{R}^n \mapsto \hat{p} = \left(\frac{1 + |p|^2}{2}, p, \frac{1 - |p|^2}{2}\right) \in \mathbb{L}^{n+1} \subset \mathbb{R}_1^{n+2}
$$

$$
q \in S^n \mapsto \hat{q} = (1, q) \in \mathbb{L}^{n+1} \subset \mathbb{R}_1^{n+2}
$$

Moreover one can identify (oriented) spheres in $\mathbb{R}^n$ and S^n with points in the spacelike unit sphere S_1^{n+1} in Minkowski

$\mathbb{R}_1^{n+2}$: A sphere s with center c and radius r in $\mathbb{R}^n$ maps to

$$\hat{s} = \frac{1}{r}\left(\frac{1 + (|c|^2 - |r|^2)}{2}, c, \frac{1 - (|c|^2 - |r|^2)}{2}\right)$$

in this picture. Changing the orientation of the sphere s corresponds to sending $\hat{s}$ to its negative.

The geometry behind this identification can be thought of as follows: After projecting R^n stereographically into $S^n \subset \mathbb{R}^{n+1}$ one can identify a sphere s in S^n with the tip of the cone that touches S^n in s (the polar to the point with respect to S^n). Now one embeds $\mathbb{R}^{n+1}$ in $\mathbb{R}_1^{n+2}$ via $p \mapsto (1, p)$ and projectivizes. Thus a point in S^n gets mapped to a line in the light cone $\mathbb{L}^{n+1}$ and a sphere is identified with a spacelike line. There are two length one representatives in such a spacelike line. They correspond to the two possible orientations of the sphere.

There are several advantages to this representation. First of all, the Möbius transformations are now linear maps (they become orthogonal transformations of the Minkowski space). Second, the intersection angle of two spheres s_1 and s_2 is given by the arccos of the (Lorentz-) scalar product $\langle \hat{s}_1, \hat{s}_2 \rangle$ of their (normalized) representations in Minkowski space. This angle might become imaginary, if the spheres do not intersect, but the following formula will hold in any case:

$$\text{(3)} \qquad \|c_2 - c_1\|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \langle \hat{s}_1, \hat{s}_2 \rangle.$$

In case of intersecting spheres this is the known cosine formula. Another interpretation of that quantity is that it is the cross-ratio[1] of the four distinct points at which a circle that intersects the two spheres orthogonally hits the spheres.

However the scalar product between a point and a sphere is only meaningful if it is 0. In this case the point lies on the sphere. The Lorentz scalar product between two points in the lift given above is $-1/2$ times the squared distance of the points.

$$\langle \hat{p}_1, \hat{p}_2 \rangle = -\frac{1}{2}\|p_2 - p_1\|^2.$$

A circle in $\mathbb{R}^n$ or S^n can be identified with a time-like 3-space in $\mathbb{R}_1^{n+2}$: Its intersection with the light cone gives the points on the circle and the unit vectors in the (space-like) $(n-1)$-dimensional orthogonal complement give the spheres that contain the circle (they are orthogonal to all the points of the circle and thus contain them). Space-like vectors in that 3-space represent spheres that intersect the circle orthogonally. Thus three spheres have an orthogonal

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[1] The cross-ratio of four complex numbers z_1, z_2, z_3, and z_4 is given by

$$\text{(4)} \qquad cr(z_1, z_2, z_3, z_4) := \frac{z_1 - z_2}{z_2 - z_3} \frac{z_3 - z_4}{z_4 - z_1}.$$

This quantity is invariant with respect to Möbius transformations.

Since four points with real cross-ratio are known to lie on a common circle, one can extend the notion of real cross-ratios to points in arbitrary dimension: A cross-ratio can be defined in a Clifford algebra setup as well but for our needs real cross-ratios are sufficient.

circle iff their span is timelike. A very good treatment of Möbius geometry can be found in [7].

From now on we will no longer distinguish between the spheres and their representations as spacelike unit vectors or points and lightlike lines.

2.1. THE MOUTARD EQUATION

Definition 2 (Moutard equation). A map $f : \mathbb{R}^2 \to \mathbb{R}^n$ is said to solve a Moutard equation if there is a function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\text{(5)} \qquad f_{xy} = \lambda f$$

holds.

Definition 3 (Moutard transformation). Let f and g be solutions to the Moutard equation with functions λ_f and λ_g.

Then g is a Moutard transform of $f \Leftrightarrow$ there exist ν and μ such that

$$g_x + f_x = \nu(g + f), \quad g_y - f_y = \mu(g - f)$$

with

$$\nu_y = \mu_x = -\lambda_f \nu \mu, \quad \lambda_g = -\lambda + 2\mu\nu.$$

3. ISOTHERMIC NETS

Isothermic surfaces in $\mathbb{R}^n$ or S^n are surfaces that allow conformal parametrization by curvature lines (this parametrization is also called isothermic parametrization). As noted in the introduction, this class of surfaces include surfaces of revolution and quadrics as well as minimal surfaces and surfaces of constant mean curvature (cmc surfaces).

It can be shown that there is an integrable system underlying these surfaces. This allows to study them using methods from soliton theory.

The following definition is a slightly relaxed definition of an isothermic parametrized isothermic surface (see also Definition 1)

Definition 4. $f : \mathbb{R}^2 \to \mathbb{R}^n$ is called an isothermic net if $f_{xy} \in \text{span}(f_x, f_y)$, $f_x \perp f_y$, and $\|f_x\| = \alpha s, \|f_y\| = \beta s$ with $\alpha, \beta, s : \mathbb{R}^2 \to \mathbb{R}_+$ $\alpha_y = \beta_x = 0$.

The characterization by existence of a dual surface generalizes accordingly: If f is an isothermic net, then

$$\text{(6)} \qquad \begin{aligned} f_x^* &:= \alpha^2 \frac{f_x}{\|f_x\|^2} \\ f_y^* &:= -\beta^2 \frac{f_y}{\|f_y\|^2} \end{aligned}$$

can be integrated to give a new isothermic net f^*. This is only defined up to translation and scaling.

The crucial step for the discretization is the following strong link between isothermic surfaces and particular solutions to the Moutard equation.

Theorem 1. $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a surface and $\hat{f} : \mathbb{R}^2 \to \mathbb{R}_1^{n+2}$ its lift into $\mathbb{L}^{n+1}$. Then f is an isothermic net iff $\hat{f}$ can be scaled so that $F = \frac{1}{s}\hat{f}$ solves a Moutard equation (5).

Proof. Let f be an isothermic net. Define $e^u := s = \frac{\|f_x\|}{\alpha} = \frac{\|f_y\|}{\beta}$. Then $\hat{f}_{xy}$ can be calculated to be

$$(7) \qquad \hat{f}_{xy} = u_y \hat{f}_x + u_x \hat{f}_y,$$

and with that one finds for $F = \frac{1}{s}\hat{f}$

$$(8) \qquad F_{xy} = (u_x u_y - u_{xy})F.$$

If, on the other hand, F solves a Moutard equation we can define $s = 1/|F_1 + F_{n+2}|$ and set $f = s(F_2, \ldots, F_{n+1})$. Now one can compute f_{xy} to be

$$f_{xy} = \frac{s_y}{s}f_x + \frac{s_x}{s}f_y$$

Moreover $f_x \perp f_y$ is easy to check: Since $\langle F, F \rangle = 0$ and the Moutard equation imply that $\langle sF_x, sF_y \rangle = 0$. This together with the fact that $(sF_1)_x(sF_{n+2})_y = (sF_1)_y(sF_{n+2})_x$ implies $f_x \perp f_y$.

The fact that α and β depend on x and y respectively follows. $\qquad\square$

3.1. Darboux transformations

Classically Darboux transformations have been defined as follows: If $f : \mathbb{R}^2 \to \mathbb{R}^n$ is isothermic then f and $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^n$ form a Darboux pair if they both envelop a sphere congruence that maps curvature lines onto curvature lines and is conformal. $\tilde{f}$ is then said to be a Darboux transform of f. One can phrase this in a Riccatti type differential equation [8]:

$$\tilde{f}_x = \alpha^2 \lambda (f - \tilde{f}) \frac{f_x}{\|f_x\|^2}(f - \tilde{f}),$$

$$\tilde{f}_y = -\beta^2 \lambda (f - \tilde{f}) \frac{f_y}{\|f_y\|^2}(f - \tilde{f}).$$

Here again x and y denote the isothermic parameters and the multiplication has to be understood in a Clifford algebra. However we will instead define a Darboux transform as follows:

Definition 5 (Darboux transformation). Let f and g be isothermic nets and F and G their corresponding solutions to the Moutard equation as stated in Theorem 1. Then f and g form a Darboux pair iff F and G are related by a Moutard transformation.

We will omit a proof of the equivalence of the two definitions here and refer to [6].

3.2. Special cases

As mentioned before minimal, and cmc surfaces in $\mathbb{R}^3$ are isothermic. One can find the following characterizations for them using their isothermic properties:

Definition 6. The isothermic net $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a minimal surface iff its dual is contained in a sphere.

The dual surface is in fact the Gauß map of f.

Definition 7. The isothermic net $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a cmc surface iff its (properly scaled and placed) dual surface is a Darboux transform of f as well.

4. The discrete Moutard equation

The following definition of the discrete Moutard equation can be found in [12] and it turned out to be the key ingredient for discretizing isothermicity.

Definition 8. A map $F : \mathbb{Z}^2 \to \mathbb{R}^n$ is said to solve the discrete Moutard equation if there is a field $\lambda : \mathbb{Z}^2 \to \mathbb{R}$ such that
$$(9)$$
$$f(m,n) + f(m+1, n+1) = \lambda(m,n)(f(m+1,n) + f(m,n+1)).$$

The field λ is defined on the faces of the quadrilateral mesh.

There are some simple observations about this discrete Moutard equation that we will need later:

Each four points that form an elementary quadrilateral of a solution are linearly dependent.

One can restrict solutions to constant length

$$\langle f(m,n), f(m,n) \rangle = c$$

(including $c = 0$). In this case $\lambda(m,n)$ is fixed: Assume the four points of an elementary quadrilateral are p, p_1, p_{12}, and p_2. Then the Moutard equation reads

$$(10) \qquad p_{12} = \lambda(p_1 + p_2) - p$$

and one gets from the condition $\langle p_{12}, p_{12} \rangle = c$

$$c = \langle p_{12}, p_{12} \rangle = \lambda^2 \langle p_1 + p_2, p_1 + p_2 \rangle - 2\lambda \langle p_1 + p_2, p \rangle + c.$$

If we assume that $\langle p_1 + p_2, p_1 + p_2 \rangle \neq 0$, this determines λ to be

$$\lambda = 2 \frac{\langle p_1 + p_2, p \rangle}{\langle p_1 + p_2, p_1 + p_2 \rangle} = \frac{\langle p_1, p \rangle + \langle p_2, p \rangle}{c + \langle p_1, p_2 \rangle}.$$

Inserting this in equation (10) and multiplying with p_1 or p_2 gives that pairs of points along opposite edges have equal scalar products. Thus for the whole lattice
$$(11)$$
$$\langle f(m,n), f(m+1,n) \rangle = \langle f(m,n+1), f(m+1,n+1) \rangle$$
$$\langle f(m,n), f(m,n+1) \rangle = \langle f(m+1,n), f(m+1,n+1) \rangle$$

holds.

Conversely, $\langle f(m,n), f(m,n) \rangle = c$ together with equation (11) and the condition that the four points of an elementary quadrilateral are linearly dependent will yield two choices: the Moutard equation or
$$(12)$$
$$f(m,n) - f(m+1,n+1) = \lambda(m,n)(f(m+1,n) - f(m,n+1)).$$

This equation is often called a Moutard equation as well. In complete analogy of the smooth Moutard transformation we can formulate a discrete one:

Definition 9. Two solutions to the discrete Moutard equation f and $\tilde{f}$ are said to be Moutard transforms of each other iff there exist $\nu, \mu : \mathbb{Z}^2 \to \mathbb{R}$ with

$$\tilde{f}_1 + f = \nu(f_1 + \tilde{f}),$$

$$\tilde{f}_2 - f = \mu(f_2 - \tilde{f}).$$

The existence of Moutard transforms is a consequence of the 3D consistency of the discrete Moutard equation [12, 6]. The fields ν and μ are not free but have to satisfy a linear difference equation.

5. Discrete isothermic nets

In this section we will define discrete isothermic nets and s-isothermic nets using a discretization of the Moutard equation (5) and then see that they coincide with the known definitions.

Discrete isothermic surfaces have played an important role in the development of the discrete integrable geometry. They form the common framework for discrete minimal and discrete cmc surfaces [3] and the study of their transformations gave deeper insight into the structure of discrete integrable systems [6]. Let us briefly give the basic facts and definitions for discrete isothermic surfaces that we will need later. A more complete treatment (with proofs) can be found in [1, 3]

Definition 10. A discrete isothermic net is a map $F : \mathbb{Z}^2 \to \mathbb{R}^3$ such that all elementary quadrilaterals have cross-ratio $c(m, n) = -\alpha(m)^2/\beta(n)^2$.

The indices m and n denote the two lattice indices of $\mathbb{Z}^2$. Thus α and β both depend on one direction only.

Given Theorem 1 that characterizes smooth isothermic nets via solutions to the (smooth) Moutard equation in the light cone $\mathbb{L}^{n+1}$, the following definition is quite natural:

Definition 11. Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a map and $\hat{f} : \mathbb{Z}^2 \to \mathbb{L}^{n+1}$ its lift into the light cone. f is called a discrete isothermic net if there is a field $s : \mathbb{Z}^2 \to \mathbb{R}$ such that $\frac{1}{s}\hat{f}$ solves the discrete Moutard equation.

The next lemma shows that this definition is equivalent to the one given by Bobenko and Pinkall [1, 3]:

Lemma 1. $f : \mathbb{Z}^2 \to \mathbb{R}^n$ *is a discrete isothermic net iff for all* $(m, n) \in \mathbb{Z}^2$
(13)
$$cr(f(m, n), f(m+1, n), f(m+1, n+1), f(m, n+1))$$
$$= -\frac{\alpha^2(m)}{\beta^2(n)}$$

holds with $\alpha, \beta : \mathbb{Z} \to \mathbb{R}$.

Proof. Let us start with four points of an elementary quadrilateral. The lifts of the four points are linearly dependent since they solve the Moutard equation. Therefore they are all contained in a light-like 3-space. This in turn says that the points are concircular and thus their cross-ratio must be real.

As we have noted before, $\langle \hat{p}_1, \hat{p}_2 \rangle = -\frac{1}{2}\|p_2 - p_1\|^2$ holds for the lifts $\hat{p}_1$ and $\hat{p}_2$ of two points p_1 and p_2. Hence

$$\frac{\langle \hat{p}_1, \hat{p}_2 \rangle \langle \hat{p}_3, \hat{p}_4 \rangle}{\langle \hat{p}_2, \hat{p}_3 \rangle \langle \hat{p}_4, \hat{p}_1 \rangle} = \frac{\alpha^4(m)}{\beta^4(n)} = q^2$$

gives the squared absolute value of the cross-ratio q of the four points p_1, p_2, p_3, and p_4 for *any* lift, since numerator and denominator are linear in all four points.

So it only remains to show that the cross-ratio is negative. For this we compute

$$\frac{\langle \hat{p}_1, \hat{p}_3 \rangle \langle \hat{p}_2, \hat{p}_4 \rangle}{\langle \hat{p}_3, \hat{p}_2 \rangle \langle \hat{p}_4, \hat{p}_1 \rangle} = (1 - q)^2$$

and find that $(1 - q)^2 = (1 + |q|)^2$ which implies that q is in fact negative.

If, on the other hand, f is a map in $\mathbb{R}^n$ with cross-ratio $-\frac{\alpha_k}{\beta_l}$ we can scale its lift $\hat{f} \in \mathbb{R}^{n+2}$ point-wise in such a way that $\left\langle \hat{f}(k, l), \hat{f}(k+1, l) \right\rangle = \alpha_k$ and $\left\langle \hat{f}(k, l), \hat{f}(k, l+1) \right\rangle = \beta_l$. Now given $\hat{f}, \hat{f}_1, \hat{f}_{12}$, and $\hat{f}_2$ with $\left\langle \hat{f}, \hat{f}_1 \right\rangle = \left\langle \hat{f}_2, \hat{f}_{12} \right\rangle = \alpha$ and $\left\langle \hat{f}, \hat{f}_2 \right\rangle = \left\langle \hat{f}_1, \hat{f}_{12} \right\rangle = \beta$ we know that they are linearly dependent (since the corresponding f's are concircular) and

$$\mu \hat{f}_{12} = \hat{f} + \nu \hat{f}_1 + \eta \hat{f}_2$$

must hold for some μ, ν, and η.

$$
\begin{array}{rclcl}
0 &=& \mu^2 \left\langle \hat{f}_{12}, \hat{f}_{12} \right\rangle &=& 2(\nu\alpha + \eta\beta + \nu\eta \left\langle \hat{f}_1, \hat{f}_2 \right\rangle) \\
&\Rightarrow& -\nu\eta \left\langle \hat{f}_1, \hat{f}_2 \right\rangle &=& \nu\alpha + \eta\beta \\
\alpha &=& \left\langle \hat{f}_{12}, \hat{f}_2 \right\rangle &=& \frac{1}{\mu}(\beta + \nu \left\langle \hat{f}_1, \hat{f}_2 \right\rangle) \\
&\Rightarrow& \frac{\eta\alpha}{\mu} &=& -\nu\alpha \\
\beta &=& \left\langle \hat{f}_{12}, \hat{f}_1 \right\rangle &=& \frac{1}{\mu}(\alpha + \eta \left\langle \hat{f}_1, \hat{f}_2 \right\rangle) \\
&\Rightarrow& \frac{\nu\beta}{\mu} &=& -\eta\beta
\end{array}
$$

So together one concludes

$$\eta = -\mu\nu \text{ and } \nu = -\mu\eta \Rightarrow \mu = \pm 1 \text{ and } \nu = \mp\eta$$

and $\hat{f}_{12} - \hat{f} = \lambda(\hat{f}_1 - \hat{f}_2)$ or $\hat{f}_{12} + \hat{f} = \lambda(\hat{f}_1 + \hat{f}_2)$. But the second solution corresponds to a negative cross-ratio of f as we have already seen. Thus the first would give a positive cross-ratio. Therefore we can conclude that the scaled $\hat{f}$ solves the Moutard equation. $\square$

We identified solutions to the discrete Moutard equation in the light cone with the well known discretization of isothermic nets, but, as we have seen, we are allowed to restrict the Moutard equation to constant length solutions $\langle f, f \rangle = c$ not necessarily zero and in fact solutions in the spacelike unit sphere can be interpreted as discretizations of isothermic nets as well (more on the intimate connection between isothermicity and the Moutard equation can be found in [5]): The corresponding discrete geometry is known as s-isothermic.

Classically, s-isothermic surfaces have been defined as discrete surfaces build from touching spheres with the additional condition that the four spheres that form an elementary quadrilateral should have a common orthogonal circle (it is easy to show that there is always a circle through the four touching points, so the condition is in fact that the circle intersects the spheres perpendicularly). They first appeared in the literature in [3].

From what we have discussed so far it is clear, that this implies that the lifts of the spheres that form an s-isothermic surface solve the discrete Moutard equation. Therefore it is natural to relax the notion of s-isothermic to general solutions of the discrete Moutard equation in the spacelike unit sphere in Minkowski R^n: The spheres no longer need to touch, but neighboring spheres in one direction should have scalar products that are constant in the other direction.

Definition 12. A map $f : \mathbb{Z}^2 \to S_1^{n+1}$ is called an s-isothermic net iff f solves the discrete Moutard equation.

The original (narrow) definition corresponds to the special case

$$\langle f(m,n), f(m+1,n)\rangle = \langle f(m,n), f(m,n+1)\rangle = 1.$$

Note that the four spheres of an elementary quadrilateral do not necessarily have a common orthogonal circle anymore, since the 3-space they span need not be timelike. In case it is spacelike the orthogonal complement is a timelike $(n-1)$-space and the spheres contain a common orthogonal $(n-2)$ sphere (in case of spheres in $\mathbb{R}^3$ this would be a 1-sphere: a point pair).

Definition 13. Let f be an s-isothermic net. Let $r(m,n)$ and $c(m,n)$ denote the radii and centers of the corresponding spheres. Then the dual net f^* is defined up to translation by the following conditions for the centers and radii of its spheres:
(14)

$$r^*(m,n) = \frac{1}{r(m,n)}$$

$$c^*(m+1,n) - c^*(m,n) = \frac{c(m+1,n) - c(m,n)}{r(m+1,n)r(m,n)}$$

$$c^*(m,n+1) - c^*(m,n) = -\frac{c(m,n+1) - c(m,n)}{r(m,n+1)r(m,n)}.$$

Proof. We have to show that f^* is well defined and s-isothermic.

If f is s-isothermic then – denoting the radii of the spheres for one elementary quadrilateral by $r := r(n,m), r_1 := r(n+1,m), r_{12} := r(n+1,m+1)$, and $r_2 := r(n,m+1)$ (and analogously for the centers c) – we have

$$\frac{1}{r} + \frac{1}{r_{12}} = \lambda\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \quad \frac{c}{r} + \frac{c_{12}}{r_{12}} = \lambda\left(\frac{c_1}{r_1} + \frac{c_2}{r_2}\right)$$

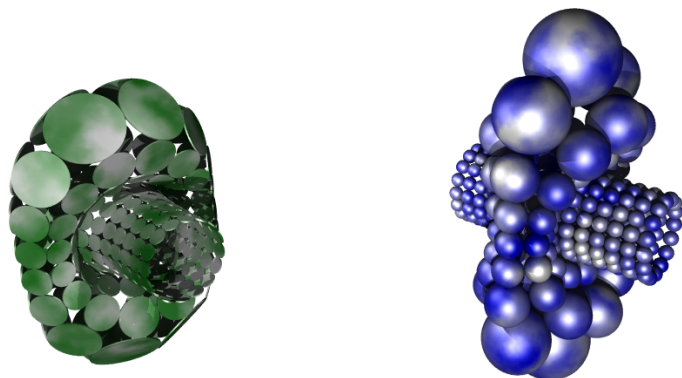and $|c|^2 + r^2 + |c_{12}|^2 + r_{12}^2 = \lambda\left(|c_1|^2 + r_1^2 + |c_2|^2 + r_2^2\right)$ which

Figure 1: A Darboux transform of an s-isothermic cylinder. The circles and the touching spheres are shown separately.

fixes λ. Now

$$0 = (c_1^* - c^*) + (c_{12}^* - c_1^*) - (c_{12}^* - c_2^*) - (c_2^* - c^*)$$

$$\Leftrightarrow \quad 0 = \frac{c_1 - c}{r_1 r} - \frac{c_{12} - c_1}{r_{12}r_1} + \frac{c_{12} - c_2}{r_{12}r_2} - \frac{c_2 - c}{r_2 r}$$

$$\Leftrightarrow \quad \left(\frac{1}{r} + \frac{1}{r_{12}}\right)\frac{c_1}{r_1} - \frac{1}{r_1}\left(\frac{c}{r} + \frac{c_{12}}{r_{12}}\right)$$

$$= -\left(\frac{1}{r_{12}} + \frac{1}{r}\right)\frac{c_2}{r_2} + \frac{1}{r_2}\left(\frac{c_{12}}{r_{12}} + \frac{c}{r}\right)$$

$$\Leftrightarrow \quad \left(\left(\frac{1}{r} + \frac{1}{r_{12}}\right) - \lambda\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right)\frac{c_1}{r_1}$$

$$= -\left(\left(\frac{1}{r_{12}} + \frac{1}{r}\right) - \lambda\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right)\frac{c_2}{r_2}$$

$$\Leftrightarrow \quad (0)\frac{c_1}{r_1} = -(0)\frac{c_2}{r_2}.$$

So f^* is well defined. Since f and f^* are dual to each other, the condition that the edges of f sum to 0 is equivalent to f^* being a solution to the Moutard equation. ☐

Another way of proving the dualizability of s-isothermic nets is by showing that they are in fact Koenigs nets (see again [6]).

5.1. DISCRETE DARBOUX TRANSFORMATIONS

For Darboux transformations we now can simply adapt our definition from before:

Definition 14. Let f and g be s-isothermic nets. Then f and g are said to form a Darboux pair if they are related by a Moutard transformation.

Figure 1 shows a Darboux transform of an s-isothermic cylinder. Note, however, that they do not necessarily need to be cylinders again.

By construction, corresponding elementary quadrilaterals of f and g always lie in a 4-space. If that 4-space is timelike, the orthogonal complement is 1-dimensional spacelike, which gives that the eight spheres from the two quadrilaterals have a common orthogonal sphere. These spheres play the role of the Darboux sphere congruence.

A Darboux transformation for discrete (non s-) isothermic surfaces is discussed in [8]. This notion coincides with the Moutard transformation for the lifted nets as well.

6. Examples

6.1. Simple examples

Examples of discrete s-isothermic nets are easily derived from Schramm's circle patterns with combinatorics of the square grid [13] in $\mathbb{R}^2$ or S^2, once one has replaced one half of the circles by spheres intersecting $\mathbb{R}^2$ (or S^2) orthogonally in these circles[2]. In case of Schramm patterns in $\mathbb{R}^2$, the dual is again a Schramm pattern (in $\mathbb{R}^2$). However, in case of Schramm patterns in S^2, this is not true anymore. Instead the resulting surface will be a discrete s-minimal surface (a surface with vanishing mean curvature). The motivation for this definition is the following: In the continuous case minimal surfaces can be characterized by the fact that they are isothermic surfaces for which the dual surface is contained in a sphere. Discrete s-minimal surfaces are studied in great detail in [2].

s-minimal surfaces have only been defined for the case of s-isothermic nets with touching spheres. However, we will generalize the notion in the obvious way.

Figure 2 shows an s-minimal catenoid. It can be constructed from the discrete version of the exponential map.

6.2. Surfaces of revolution

Other examples that can be easily constructed are s-isothermic surfaces of revolution in R^3 (The above mentioned catenoid is of course an example for those as well).
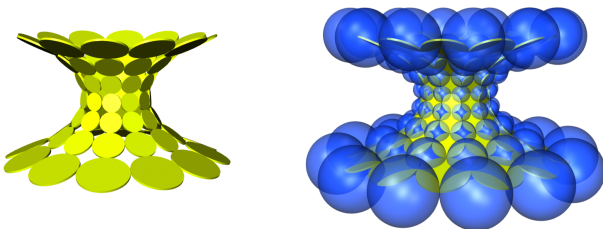

Figure 2: An s-minimal catenoid. The circles are shown as disks.

It is a well known fact that every rotational symmetric surface in $\mathbb{R}^3$ can be parametrized by isothermic coordinates and thus constitutes an isothermic surface.

A discrete s-isothermic surface of revolution in $\mathbb{R}^3$ is an s-isothermic net that has a discrete rotational symmetry: The radii r of the spheres depend on one variable only

$(r(k,l) = r(k))$, and if the axis of rotation is the x-axis, then the centers c have the form

$$c(k,l) = (c_1(k), \cos(l\phi)c_2(k), \sin(l\phi)c_2(k)).$$

Lemma 2. *Let (s_k) be a sequence of spheres with centers $c_k = (x_k, y_k, 0)$ and radii r_k. Then the condition for (s_k) to serve as "meridian curve" for a discrete rotational s-isothermic net (with the x-axis being the axis of rotation) is*

$$\frac{y_k^2}{r_k^2} = const.$$

If the angle for the discrete rotation is ϕ, then

$$\alpha = 2\sin\frac{\phi}{2}\frac{y_k^2}{r_k^2} - 1 \quad \text{and} \quad \beta_k = \frac{r_k^2 + r_{k+1}^2 - \|c_{k+1} - c_k\|^2}{2r_k r_{k+1}}.$$

In the special case of $\alpha = \beta \equiv -1$ (touching spheres), this reduces to

$$\|c_{k+1} - c_k\|^2 = \sin\frac{\phi}{2}(y_k + y_{k+1})^2.$$

Proof. Let ϕ be the angle of rotation for the discrete net of revolution. Let d_k denote the distance between the two sphere centers $c(k,l)$ and $c(k, l+1)$. Then one can read from the isosceles triangle formed by the centers and their projection on the x-axis $(x_k, 0, 0)$ that

$$d_k = 2y_k \sin\frac{\phi}{2}$$

must hold. On the other hand, we know that the angle between the spheres is given by

$$\begin{aligned}
\alpha_k &= \frac{r^2(k,l) + r^2(k, l+1 - d_k^2)}{2r(k,l)r(k, l+1)} = 1 - \frac{d_k^2}{2r^2(k)} \\
&= 1 - \sin\frac{\phi}{2}\frac{y^2(k)}{r^2(k)}.
\end{aligned}$$

Since α must not depend on k, the first claim and the formula for α are shown. The formula for $\beta(k)$ is just the formula for the angle of spheres (3) rephrased. Inserting the formula for α in the case $\alpha = -1$ gives the last claim. $\square$

Note that in the special case $\alpha = \beta \equiv -1$ scaling the height with $\cos\frac{\phi}{2}$ gives a polygon

$$\hat{c}_k = (x_k, \hat{y}_k) = (x_k, \cos\frac{\phi}{2}y_k)$$

that can be interpreted as the polygon of the touching points. For this polygon the relation reads

$$\|\hat{c}_{k+1} - \hat{c}_k\|^2 = \tan\frac{\phi}{2}\hat{y}_k\hat{y}_{k+1}.$$

A similar relation holds for discrete rotational symmetric (non s-) isothermic surfaces[3] [9].

Figure 3 shows a rotational symmetric s-isothermic surface. It is in fact even a discrete s-cmc net (a surface of constant mean curvature). We will study its construction in greater detail in section 7.

---

[2]Since one may choose which half of the circles one replaces, a Schramm pattern gives rise to two *different* s-isothermic surfaces.

[3]In fact, the same polygon can serve in both cases, up to different angles of rotation.
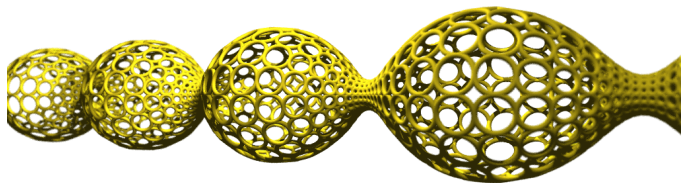
Figure 3: An s-isothermic Delaunay surface. Only the orthogonal circles are shown here.

7. S-CMC NETS

An important class of isothermic nets are *surfaces of constant mean curvature* (or short *cmc surfaces*) in $\mathbb{R}^3$. Cmc surfaces come in pairs: The centers of the mean curvature spheres of a cmc surface f form another cmc surface f^*. This correspondence is a duality, meaning that $f^{**} = f$. It turns out that f^* is the (properly scaled and placed) dual surface in the isothermic sense and–in addition–is a Darboux transform of f. One may even define cmc surfaces (without umbillics) to be isothermic surfaces for which there is a dual surface that is a Darboux transform as well. This provides the necessary tools to define discrete s-cmc surfaces:

Definition 15. A discrete s-isothermic surface is called a *discrete surface of constant mean curvature* or, for short, *s-cmc surface* if its (properly scaled and placed) dual surface is a Darboux transform too.

For simplicity we will restrict ourselves from now on to the narrow definition of discrete s-isothermic (i. e. the spheres of the surfaces touch).

Before we can give the first examples we have to discuss the geometry of a s-cmc net and its dual/Darboux transform. Since the spheres s, s_1, s_{12}, and s_2 form an elementary quadrilateral and their dual ones s^*, s_1^*, s_{12}^*, and s_2^* (note that this dual is actually scaled by some factor λ we will calculate in a moment) have a common orthogonal sphere (since the two are Darboux transforms of each other) and since corresponding edges of the quadrilaterals formed by the centers $(c, c_1, \ldots, c^*, c_1^*, \ldots)$ have parallel edges that touch this sphere (in points $p_1, p_1^*, p_2, p_2^*, \ldots$ – see Fig 4), the edge $\overline{p_i^* p_i}$ is perpendicular to both the edge $\overline{c_i c}$ and the line $\overline{c_i^* c^*}$, $i \in \{1, 2\}$. We will set $\tau_1 := \|p_1^* - p_1\|$ and $\tau_2 := \|p_2^* - p_2\|$. In particular, the distance $d := \|c^* - c\|$ is not constant: Looking once more at Fig. 4 one finds

$$\begin{array}{rcccl} d^2 & = & (r - r^*)^2 + \tau_1^2 & = & r^2 - 2rr^* + r^{*2} + \tau_1^2 \\ d^2 & = & (r + r^*)^2 + \tau_2^2 & = & r^2 + 2rr^* + r^{*2} + \tau_2^2. \end{array}$$

With $r^* = \lambda/r$ this results in

$$4\lambda = 4rr^* = \tau_2^2 - \tau_1^2.$$

Now it is obvious that the two lengths τ_1 and τ_2 (as well as the scaling factor λ) are constant for the whole net. Thus there is not one constant distance between the net and its dual (as in the smooth case and in the discrete isothermic case) but two.
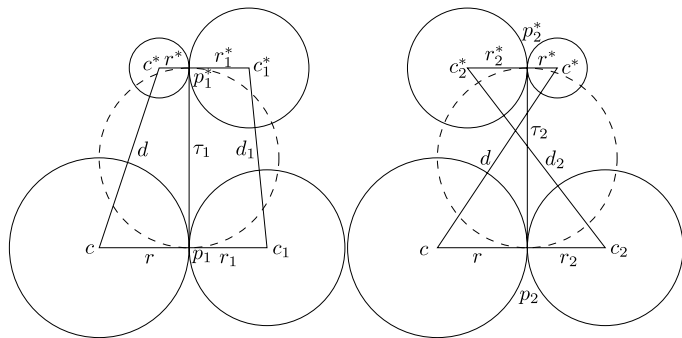

Figure 4: The two distances between an s-cmc net and its dual.

The simplest example of a cmc surface one can think of is the cylinder and one easily convinces oneself that its natural s-isothermic discretization is in fact s-cmc: Let $\phi = 2\pi/N$ for some $N \in \mathbb{N}$. The radii of all spheres are the same $r = \sin(\phi/2)$ and the centers are given by $(2mr, \cos(n\phi), \sin(n\phi), r)$, $n, m \in \mathbb{Z}$. It is easy to see that if N is even, the dual surface of the cylinder is the cylinder itself (in the case that N is odd, one has to rotate it by $\phi/2$ around its axis).

We will now construct non-trivial examples by exploiting what we already found for surfaces of revolution.

7.1. DELAUNAY SURFACES

Delaunay surfaces are cmc surfaces of revolution. There is a classical construction for them: Take an ellipse (or hyperbola) and roll it without friction or sliding on a straight line. Then a focal point of the ellipse will generate a curve under this movement. It can be shown that theis curve constitutes the meridian curve for a cmc surface and its dual surface. The round cylinder is the limiting case when the ellipse becomes a circle. The other limit case is the ellipse degenerating to the line connecting the foci which leads to a chain of spheres.

There is an analogous construction in the discrete case. For discrete isothermic surfaces this was given in [9]. A general treatment for a discrete curvature theory derived from parallel surfaces can be found in [4]. Here we give the corresponding treatment tailored to our definition of s-isothermic cmc.

We will start by explaining a suitable notion of "rolling" a polygon on a line and identify the correct discretization of an ellipse. Let $q : \mathbb{Z} \to \mathbb{R}^2$ be a polygon, $c \in \mathbb{R}^2$ a fixed point. Think of this as being a set of triangles $\triangle(q_n, q_{n+1}, c)$. Now take these triangles and place them with the edges $[q_n, q_{n+1}]$ on a straight line, e.g. the x-axis. The result is a sequence of points p_n. This new polygon p is the discrete trace of c when unrolling the polygon q along the x-axis (Fig. 5).

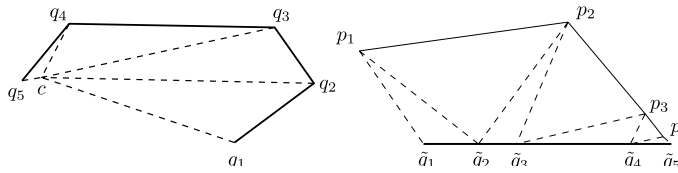The condition (2) for p to give rise to an s-isothermic net

Figure 5: Unrolling a polygon

reads in terms of q as follows:

$$(15) \qquad \frac{1}{\tau} + 1 = \frac{1 + \cos(\alpha_n - \beta_n)}{1 + \cos(\alpha_n + \beta_n)},$$

where $\alpha_n = \angle(q_{n-1}, q_n, p)$, $\beta_n = \angle(p, q_n, q_{n+1})$ and $\tau = |cr| \sin^2 \frac{\phi}{2}$. Or equivalently:

$$(16) \qquad \tan \frac{\alpha_n}{2} \tan \frac{\beta_n}{2} = const.$$

Let E be a given ellipse. Choose a starting point q_0 on E and a starting direction in q_0 pointing to the inner region of E. Shooting a ball in that direction will give a new point q_1 where it hits the ellipse again. The new direction in q_1 is given by the usual reflection law: incoming angle = outgoing angle. This leads to a sequence $q_0, q_1, q_2, \ldots$ (see Fig. 7 upper left). If one thinks of these points as the vertices of a polygon, it is again a well known fact that the edges are tangential to either a confocal hyperbola or a confocal ellipse, depending on whether the first shot goes between the two foci or not [10]. A proof of that can be found in the appendix. In a hyperbola the situation is similar with the one difference that one has to change the branch if the shooting line doesn't hit the first branch twice (Fig. 6).
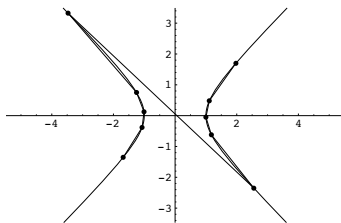
Figure 6: Billiard in a hyperbola

Lemma 3. *The polygon obtained by playing the standard billiard in an ellipse together with one focus satisfies condition (15).*

Proof. A proof is given in the appendix, in Lemma 4. □

Theorem 2. *The discrete rotational surface obtained by rotating the unrolled trace of a ball in an elliptic (or hyperbolic) billiard is a discrete rotational cmc surface.*

Proof. To proof this, one has to find a dual surface with edges in two constant distances (one for each lattice direction). If we trace both foci when evolving the ellipse, we get
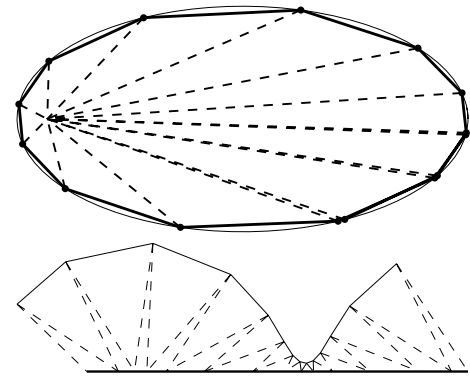
Figure 7: Billiard in an ellipse and the meridian curve generated from it

a second polygon $\hat{p}$. Now mirror it at the axis. Corollary 2 shows that the distance $|p_n - \hat{p}_n| = \tilde{\tau}$ is constant. From the reflection law one gets that $|p_n - \hat{p}_{n+1}| = |p_{n+1} - \hat{p}_n| = \tau$ and therefore is constant. So $[p_n, p_{n+1}]$ and $[\hat{p}_n, \hat{p}_{n+1}]$ are parallel and

$$|p_n - p_{n+1}|^2 = \frac{2(c^2 - \tilde{c}^2)}{|\hat{p}_n - \hat{p}_{n+1}|^2}.$$

This in turn leads to

$$|p_n - p_{n+1}| = \frac{\lambda}{|\hat{p}_n - \hat{p}_{n+1}|}$$

with $\lambda = \frac{\tau^2 - \tilde{\tau}^2}{4}$. □

7.2. A DISCRETE SINH-GORDON EQUATION

We will now derive a difference equation for the radii of the spheres and orthogonal circles in an s-cmc net. The reason to call this a discrete Sinh-Gordon equation is the following: If f is a smooth cmc surface (with mean curvature H and without umbilics) in isothermic parametrization, there is a scalar field u given by $e^u := \|f_x\| = \|f_y\|$ and it can be shown that u then solves the Sinh-Gordon equation

$$u_{xx} - u_{yy} + H \sinh u = 0.$$

Now for s-isothermic nets the radii of the spheres play the role of the metric factor u in the smooth case, since they give the local contribution of a vertex to the edge length of the net, which can be viewed as the length of the partial derivatives.

We start by looking at a circle together with its four neighboring spheres. Let R denote the radius of the circle and $r_1, r_2, r_3,$ and r_4 the radii of the spheres. For the angles ϕ_k made by the center of the circle and the two points where the k-th sphere intersects the circle, we find $\tan \frac{\phi_k}{2} = \frac{r_k}{R}$, and since $e^{i\phi_k} = \frac{1 + i \tan \frac{\phi_k}{2}}{1 - i \tan \frac{\phi_k}{2}}$, we get

$$\prod_{k=1}^{4} \frac{1 + i \tan \frac{\phi_k}{2}}{1 - i \tan \frac{\phi_k}{2}} = \prod_{k=1}^{4} \frac{R + ir_k}{R - ir_k} = 1.$$
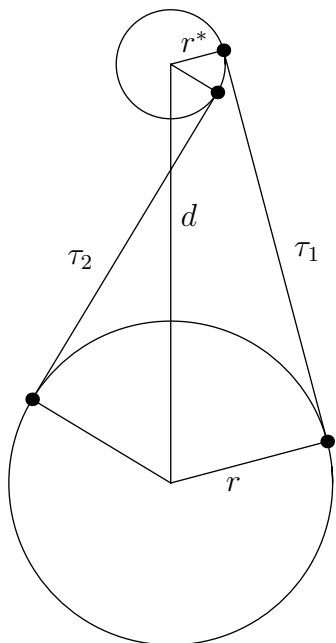
Figure 8: A sphere and its dual.

This can be solved for, say, r_4 giving

$$(17) \qquad r_4 = \frac{r_1 r_2 r_3 - R^2(r_1 + r_2 + r_3)}{R^2 - r_1 r_2 - r_1 r_3 - r_2 r_3},$$

or for the central R:

$$(18) \qquad R^2 = \frac{r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4}{r_1 + r_2 + r_3 + r_4}.$$

This is of course the same equation as one gets for the radii of a Schramm type circle pattern (see [13]).

The situation for a central sphere with radius r and four neighboring circles with radii R_1, R_2, R_3 and R_4 is more difficult. Since the configuration is not planar anymore, the angles made by the intersection points p and p' of a circle with the sphere do not sum to 2π anymore. Still we define $\tilde{\phi}_k$ via $\tan\frac{\tilde{\phi}_k}{2} = \frac{R_k}{r}$. Now look at the spherical triangle made by the two intersection points p and $p'$ and the point q where the line connecting the centers of the sphere and its dual hits the sphere. After scaling the sphere to be the unit sphere, one has

$$(19) \qquad \begin{aligned} \sin a &= \frac{\tau_1}{d} & \cos a &= \frac{r - r^*}{d} \\ \sin b &= \frac{\tau_2}{d} & \cos b &= \frac{r + r^*}{d} \\ c &= \tilde{\phi}_k \end{aligned}$$

for the sides $a, b,$ and c of that triangle. Using $A = \frac{\tau_1 - \tau_2}{2}$ and $B = \frac{\tau_1 + \tau_2}{2}$ and some spherical trigonometry one can find for the angle γ_k at q:

$$\tan\frac{\gamma_k}{2} = \sqrt{\frac{A^2 - R_k^2}{B^2 - R_k^2} \frac{B^2 + r^2}{A^2 + r^2}}.$$

Now we can write again

$$\begin{aligned} e^{i\gamma_k} &= \frac{1 + i\tan\frac{\gamma_k}{2}}{1 - i\tan\frac{\gamma_k}{2}} \\ &= \frac{\sqrt{(B^2 - R_k^2)(A^2 + r^2)} + i\sqrt{(A^2 - R_k^2)(B^2 + r^2)}}{\sqrt{(B^2 - R_k^2)(A^2 + r^2)} - i\sqrt{(A^2 - R_k^2)(B^2 + r^2)}} \end{aligned}$$

and use $\prod_{k=1}^{4} e^{i\gamma_k} = 1$ to get

$$(20) \qquad \prod_{k=1}^{4} \frac{\sqrt{(B^2 - R_k^2)(A^2 + r^2)} + i\sqrt{(A^2 - R_k^2)(B^2 + r^2)}}{\sqrt{(B^2 - R_k^2)(A^2 + r^2)} - i\sqrt{(A^2 - R_k^2)(B^2 + r^2)}} = 1.$$

Equations (20) and (18) furnish an evolution of the radii R and r. Note that both are invariant under the change $r \to \alpha/r$ and $R \to \alpha/R$. One can eliminate the circle radii from the system by substituting equation (17) into equation (20). This gives an equation for the radii of 9 neighboring spheres.

Setting

$$H(r_1, r_2, r_3, r_4) = \frac{r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4}{r_1 + r_2 + r_3 + r_4}$$

and

$$G(r, H) = \frac{\sqrt{(B^2 - H)(A^2 + r^2)} + i\sqrt{(A^2 - H)(B^2 + r^2)}}{\sqrt{(B^2 - H)(A^2 + r^2)} - i\sqrt{(A^2 - H)(B^2 + r^2)}},$$

we get the equation

$$(21) \qquad \begin{aligned} 1 =\ & G(r, H(r, r_1, r_2, r_3)) G(r, r_5, r, r_3, r_4)) \\ & \times G(r, H(r_6, r_7, r, r_5)) G(r, H(r_7, r_8, r_1, r)) \end{aligned}$$

for the radii of the spheres numbered as follows:

$$\begin{array}{ccccc} & & r_6 & & \\ & r_7 & & r_5 & \\ r_8 & & r & & r_4 \\ & r_1 & & r_3 & \\ & & r_2 & & \end{array}$$
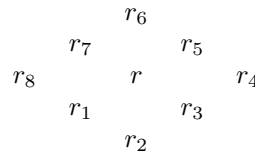
Theorem 3. *Given a positive solution to equations (20) and (18) (after choosing the constants A and B or equivalently τ_1 and τ_2), there is an s-isothermic cmc surface having sphere and circle radii as given by the solution. This surface is unique up to Euclidean motions.*

Proof. We start with a solution R and r to equations (20) and (18). The τ_i can be calculated from A and B as $\tau_1 = B + A$ and $\tau_2 = B - A$. Combinatorially we do have a $\mathbb{Z}^2$ lattice and we want to find circles of radii $r(n, m)$ associated to its faces and spheres of radii $R(n, m)$ at its vertices such that neighboring spheres do touch and the spheres at the corners of a quadrilateral intersect the corresponding circle orthogonally. Now start with the circle corresponding to a given face (n, m). Equation (20) ensures that one can place spheres of radii $R(n, m), R(n+1, m), R(n+1, m+1)$, and $R(n, m + 1)$ around it, satisfying the touching and intersecting property. Knowing τ_1 and τ_2, we can place the

dual circle of radius $R^*(n, m) := (\tau_2^2 - \tau_1^2)/(4R(0,0))$ and the corresponding intersecting spheres above it at height $h = (\tau_1^2 - (R(n, m) - R^*(n, m))^2)^{\frac{1}{2}}$, making them a quadrilateral of a s-cmc net and its dual. The only remaining obstruction is that one needs to be able to place four of these combinatorial cubes consistently around a vertex (sphere) and its dual, but the condition that this is possible is exactly the equation (18). Thus we can build a s-cmc net and its dual with the given radii, and this net is unique up to the choice of how to place the first circle – which is the claimed freedom of a Euclidean motion.

□

Since equation (18) lets us compute the circle radii from the neighboring sphere radii, we have the following

Corollary 1. *Given a positive solution to equation (21) (after choosing the constants A and B or equivalently c_1 and c_2), there is an s-isothermic cmc surface having sphere and circle radii as given by the solution. This surface is unique up to Euclidean motions.*

8. Conclusion

We have shown how solutions to the discrete Moutard equation give discrete s-isothermic surfaces and how to define s-isothermic cmc nets using the Darboux/Moutard transformation for them. From the geometry of these surfaces we were able to derive a discrete version of the Sinh-Gordon equation.

Interesting open problems include the notion of s-isothermic cmc-1 surfaces in hyperbolic space, which can be defined by Darboux transforms of s-isothermic surfaces in S^2 and the question of how to define a mean curvature sphere for s-isothermic surfaces that is compatible with our definition (meaning that its radius is constant for cmc surfaces).

A. Billiards in an ellipse

The motion of a free particle in a bounded region that is reflected elastically at the boundary is called a *billiard*. Billiards in two-dimensional convex regions in the plane are called *Birkhoff billiards*. A Birkhoff billiard is called *smooth* if the boundary is described by an infinitely differentiable function. Here we will derive the evolution equation for the billiard in an ellipse, since the trajectory of the billiard map serves as a discretization of the ellipse itself.

We first recall some basic facts about ellipses:

Definition 16. *Let m_1 and m_2 be two points in $\mathbb{C}$ and $L \geq |m_1 - m_2|$. The set E of all points p for which $|m_1 - p| + |m_2 - p| = L$ is called an ellipse. m_1 and m_2 are called foci of E.*

It is easy to see that E is the trace of a smooth regular curve.

Proposition 1. *Let E be an ellipse with foci m_1 and m_2, and let $p \in E$. The bisector of the complementary angle*

made by the line segments (m_1, p) and (m_2, p) is tangential to E at p.

Figure 9: The tangent of an ellipse

Proof. Denote by m_2' the point obtained by reflecting m_2 at T. Then $|m_1 - m_2'| = |m_1 - p| + |m_2 - p|$ holds. Choose any point $x \in T$, $x \neq p$. Using the triangle inequality for the triangle (m_1, x, m_2') one gets

$$
\begin{aligned}
|m_1 - x| + |m_2 - x| &= |m_1 - x| + |m_2' - x| \\
&> |m_1 - m_2'| = |m_1 - p| + |m_2 - p| + L.
\end{aligned}
$$

(22)

Accordingly, $x \notin E$ and hence the bisector T touches the ellipse E at p. □

Without loss of generality, we may now assume that the foci of an ellipse are located at $m_1 = -c$ and $m_2 = c$ with $c \in \mathbb{R}^+$.

The ellipse $E : x = x(\phi)$ may therefore be parametrized by $x = a\cos\phi + ib\sin\phi$, where a and b are real numbers. Defining $A = (a + b)/2$ and $B = (a - b)/2$, this can be rewritten as

$$ x = Ae^{i\phi} + Be^{-i\phi}. $$

Differentiation with respect to ϕ then gives the direction of the tangent of E at a point x:

$$ x' = i(Ae^{i\phi} - Be^{-i\phi}). \tag{23} $$

The Birkhoff billiard in an ellipse is completely determined by the sequence of points where the particle hits the bounding ellipse.

To determine this sequence, one prescribes an initial point $x(0)$ on the ellipse together with an initial direction $e^{i\alpha(0)}$ pointing inwards and computes the line through $x(0)$ with direction $e^{i\alpha(0)}$. Its second point of intersection with the ellipse defines $x(1)$ and the direction $e^{i\alpha(1)}$ is given by the condition that $e^{i\alpha(1)} - e^{i\alpha(0)}$ is perpendicular to the tangent at $x(1)$. Iteration of this process then generates the entire Birkhoff billiard.

The following can be found in [11]

Theorem 4. *The equations of motion for the billiard in an ellipse are given by*

(24)
$$e^{i(\phi_1+\phi)} = \frac{\frac{A}{B}e^{2i\alpha}-1}{e^{2i\alpha}-\frac{A}{B}}$$
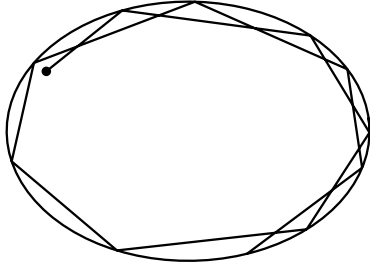$$e^{i(\alpha+\alpha_{\bar 1})} = \frac{\frac{A}{B}e^{2i\phi}-1}{e^{2i\phi}-\frac{A}{B}}.$$

Figure 10: The free motion of a particle bounded by an ellipse

Proof. We begin with a technical remark which will be exploited frequently:

(25)
$$e^{i\gamma}-e^{i\beta} = 2\sin(\frac{\gamma-\beta}{2})e^{i\frac{\gamma+\beta+\pi}{2}}.$$

Using this we can compute that

(26)
$$\begin{aligned} e^{2i\alpha} &= \frac{(x_1-x)^2}{|x_1-x|^2} = \frac{A(e^{i\phi_1}-e^{i\phi})+B(e^{-i\phi_1}-e^{-i\phi})}{A(e^{-i\phi_1}-e^{-i\phi})+B(e^{i\phi_1}-e^{i\phi})} \\ &= \frac{Ae^{i(\phi_1+\phi)}-B}{Be^{i(\phi_1+\phi)}-A}, \end{aligned}$$

which is equivalent to $(24)_1$. Furthermore it is noted that $e^{i\alpha_1}-e^{i\alpha}$ is perpendicular to x' and hence

(27)
$$(e^{i\alpha_1}-e^{i\alpha})^2 = (ix')^2.$$

Expansion by means of (25) and (23) then yields $(24)_2$. □

The similarity of the evolution equations for $e^{i\phi}$ and $e^{i\alpha}$ suggests combining ϕ and α into one variable: We define a new field ω by $\omega(2k) = 2\phi(k)$ and $\omega(2k+1) = 2\alpha(k)$. Then, ω obeys the following equation:

(28)
$$e^{i(\omega_1-2\omega+\omega_{\bar 1})} = \left(\frac{Ae^{i\omega}-B}{Ae^{-i\omega}-B}\right)^2.$$

The latter constitutes a well-known discrete version of the mathematical pendulum equation

$$\omega'' = \rho\sin\omega.$$

It is known that the edges of the billiard in an ellipse are tangent to a confocal quadric. The nature of the quadric Q depends on the initial conditions $x(0)$ and $e^{i\alpha(0)}$. If the latter are such that the first edge $\Delta x(0) = (x(1) -$ $x(0))$ and the line segment (m_1, m_2) connecting the foci do not intersect, then Q constitutes an ellipse. If $\Delta x(0)$ passes through one of the foci, then Q degenerates to two points, while if $\Delta x(0)$ and (m_1, m_2) intersect, then Q is a hyperbola. Here, we focus on the first case. The third case may be proven in an analogous manner.

Lemma 4. *Let $x(k)$ be the vertices of a billiard in an ellipse E with foci m_1 and m_2 and set $\alpha(k) = \angle(m_2 - x(k), x(k-1)-x(k))$ and $\beta(k) = \angle(x(k+1)-x, m_2-x(k))$. Then*

(29)
$$\tan\frac{\alpha(k)}{2}\tan\frac{\beta(k)}{2} = \text{const.}$$

Proof. Since the tangent and normal to the ellipse bisect the angles made by $m_1 - x$ and $m_2 - x$, the angle $\angle(x_1 - x, m_1-x)$ equals α and $\angle(m_1-x, x_{\bar 1}-x)$ must be β. Let r_1 be the radius of the circumferencing circle of the triangle
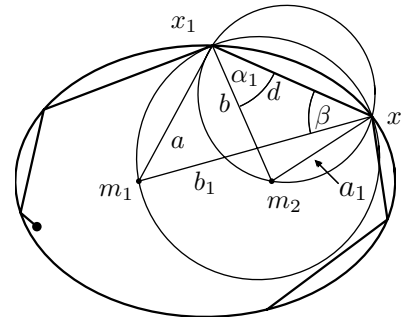

Figure 11: The edges and some of the angles as used in the adjacent proof.

(x, x_1, m_1) and let r_2 be the one for (x, x_1, m_2). Moreover, set $d = |x-x_1|, a = |m_1-x_1|, b = |m_2-x_1|, a_1 = |m_2-x|$ and $b_1 = |m_1-x|$. Then, elementary trigonometry gives for the angles in the two triangles:

(30)
$$\begin{aligned} \tan\frac{\alpha}{2} &= \frac{2r_1}{d+b_1-a} & \tan\frac{\beta}{2} &= \frac{2r_2}{d+a_1-b} \\ \tan\frac{\alpha_1}{2} &= \frac{2r_2}{d+b-a_1} & \tan\frac{\beta_1}{2} &= \frac{2r_1}{d+a-b_1}. \end{aligned}$$

Since, by definition of the ellipse, $a+b = a_1+b_1$, we deduce that

(31)
$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \tan\frac{\alpha_1}{2}\tan\frac{\beta_1}{2}.$$

□

Note that the sign of the constant determines whether or not $\Delta x(0)$ and (m_1, m_2) intersect. Thus relation (29) ensures that either all or none of the edges of the billiard pass inbetween the foci.

In order to show that the billiard is indeed tangent to a confocal ellipse, it is convenient to bring the conserved quantity (29) into the form

(32)
$$\frac{1+\cos(\alpha(k)-\beta(k))}{1+\cos(\alpha(k)+\beta(k))} = \text{const.}$$

Theorem 5. *If the first edge of a billiard in an ellipse E does not intersect the line segment between the two foci m_1 and m_2, then the edges of the billiard are tangent to a confocal ellipse.*
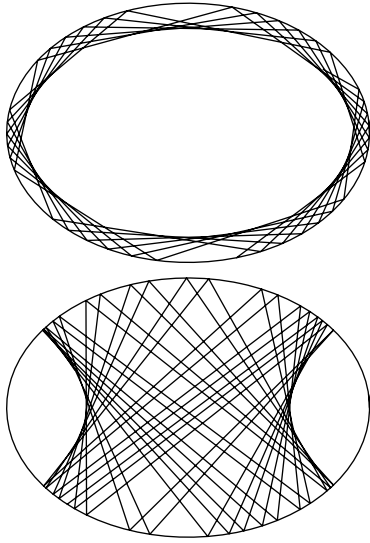

Figure 12: The billiard is tangent to a confocal quadric

Proof. As mentioned before, relation (29) ensures that none of the edges intersect (m_1, m_2). Thus the following geometric construction is valid for all edges. If $x(k)$ is a vertex of the billiard, then the angle made by $m_1 - x(k)$ and $m_2 - x(k)$ is $\alpha(k) - \beta(k)$ (with α and β defined as above). If we set $2c = |m_1 - m_2|$ and $l(k) = |m_1 - x(k)|$ and $\tilde{l}(k) = |m_2 - x(k)|$, then we obtain

$$4c^2 = l^2 + \tilde{l}^2 - 2l\tilde{l}\cos(\alpha - \beta) = (l + \tilde{l})^2 - 2l\tilde{l}(1 + \cos(\alpha - \beta)).$$

Since $l(k) + \tilde{l}(k)$ is constant, we conclude that

$$(33) \qquad 2l\tilde{l}(1 + \cos(\alpha - \beta)) = 2l_1\tilde{l}_1(1 + \cos(\alpha_1 - \beta_1)).$$

Let m_1' be m_1 reflected in the straight line through $x_{\bar{1}}$ and x. Then $|m_1' - x| = l$ and the angle between $m_2 - x$ and $m_1' - x$ is $\alpha + \beta$. Let p be the intersection point of (m_1', m_2) and $(x_{\bar{1}}, x)$. Then $d = |m_1' - m_2| = |m_1 - p| + |m_2 - p|$.

Since $(x_{\bar{1}}, x)$ bisects the angle in p it is tangent to an ellipse through p with foci m_1 and m_2. What is left to show is that d is constant, since in this case all the ellipses (for all the edges of the billiard) coincide. For the triangle m_2, m_1', x, one has

$$d^2 = l^2 + \tilde{l}^2 - 2l\tilde{l}\cos(\alpha + \beta) = (l + \tilde{l})^2 - 2l\tilde{l}(1 + \cos(\alpha + \beta)),$$

and therefore we get for two time steps and with use of equations (33) and (32)

$$(34) \qquad \frac{(l + \tilde{l})^2 - d^2}{(l + \tilde{l})^2 - d_1^2} = \frac{l\tilde{l}(1 + \cos(\alpha + \beta))}{l_1\tilde{l}_1(1 + \cos(\alpha_1 + \beta_1))} = 1.$$

This shows the constancy of d. □

Corollary 2. *Let E be an ellipse with foci m_1 and m_2 and l a line not intersecting the line segment between the two foci. Moreover, let x and x_1 be the points where l hits E. If m_1' and m_2' are the foci mirrored at l, then*

$$\|m_1 - m_2'\| = \|m_2 - m_1'\|$$

holds.

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